mathematics

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Geometric Inequalities of Warped Product Submanifolds and Their Applications

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Received: 17 April 2020; Accepted: 7 May 2020; Published: 11 May 2020

Abstract: In the present paper, we prove that if Laplacian for the warping function of complete warped product submanifold $\mathbb{M}^m = \mathbb{B}^p \times_h \mathbb{F}^q$ in a unit sphere $\mathbb{S}^{m+k}$ satisfies some extrinsic inequalities depending on the dimensions of the base $\mathbb{B}^p$ and fiber $\mathbb{F}^q$ such that the base $\mathbb{B}^p$ is minimal, then $\mathbb{M}^m$ must be diffeomorphic to a unit sphere $\mathbb{S}^m$. Moreover, we give some geometrical classification in terms of Euler–Lagrange equation and Hamiltonian of the warped function. We also discuss some related results.

Keywords: warped product; sphere theorem; Laplacian; inequalities; diffeomorphic

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1. Introduction and Main Results

We will use the following acronyms throughout the paper: 'WP' for Warped product, 'WF' for warping function, 'RM' for Riemannian manifold, and 'SFF' for second fundamental form. The idea of the warped product was initiated by Bishop and O'Neil [1] when they gave an example of complete Riemannian manifold with negative curvature. If $(\mathbb{B}, g_{\mathbb{B}})$ and $(\mathbb{F}, g_{\mathbb{F}})$ are two Riemannian manifolds (RMs), and h is a positive differentiable function defined on the base manifold $\mathbb{B}$, then we define the metric $g = \pi^* g_{\mathbb{B}} + h^2 \sigma^* g_{\mathbb{F}}$ on the product manifold $\mathbb{B} \times \mathbb{F}$, where π and σ are the projection maps on B and F, respectively. Under such stipulations, the product manifold is referred to as warped product (WP) of $\mathbb{B}$ and $\mathbb{F}$, and written as $\mathbb{M} = \mathbb{B} \times_h \mathbb{F}$. Here, h is referred to as warping function (WF).

We observe that $\mathbb{M}$ is a Riemannian product, or trivial warped product, when h is constant. Notice that there has been a great interest in the study of warped products over the recent years. For example, S. Nolker [2] derived the decompositions of the standard spaces of an isometric immersion of warped products and D.K. Kim and Y.H. Kim in [3] proved that if the scalar is non-constant then there is no non-trivial compact Einstein warped product. Recently, an interesting fundamental result proved by Djaczer in [4] showed that an isometric immersion of warped products into space forms must be product of isometric immersions under extrinsic conditions. Moreover, by using DDVV conjecture, Roth [5] obtained an inequality for submanifold of WP $I \times_h \mathbb{M}^m(c)$ where I is an interval and $\mathbb{M}^m(c)$ is a real space form and also provided some rigidity results based on submanifolds of $\mathbb{R} \times_{e^{\lambda t}} \mathbb{H}^m(c)$, where λ is a real constant. Salavessa in [6] obtained that the Heinz mean curvature $m\|\mathbb{H}\|^2 \leq \frac{A_\Psi(\partial\mathcal{D})}{V_\Psi(\mathcal{D})}$ holds in WP spaces of type $M \times_{e^\Psi} N$ in case that a graph of submanifold $(x, h(x))$ of Riemannian WP $M \times_{e^\Psi} N$ is immersed with parallel mean curvature, where $A_\Psi(\partial\mathcal{D})$ and $V_\Psi(\mathcal{D})$ are $\Psi-$weighted area and volume, respectively.

On the other hand, the investigation of the relations between curvature invariants and topology is an important problem in Riemannian geometry as well as in global differential geometry. For instance, a beautiful and classical theorem established by Myers [7] states that "if $\mathbb{M}$ is a complete Riemannian manifold with Ricci curvature $Ric(\mathbb{M}) > 1$, then the diameter $d(\mathbb{M})$ of $\mathbb{M}$ is not greater than π, and, therefore, $\mathbb{M}$ is compact and its fundamental group $\pi_1((\mathbb{M})$ is finite". Due to the distinctive work of Rauch [8], Berger [9] proved the rigidity theorem for a simply connected and complete manifold $\mathbb{M}$ of even dimension and the sectional curvature satisfying $\frac{1}{4} \leq K_{\mathbb{M}} \leq 1$. Furthermore, Grove and Shiohama in [10] has generalized the sphere theorem. There are lots of interesting and well-known results regarding the topology of complete manifolds of positive Ricci curvature. The curvature and topology of manifolds play a substantial role in global differential geometry. Later on, a splitting theorem, resulting from the work of Cheeger and Gromoll in [11], states that "if $\mathbb{M}$ is a complete non-compact manifold of non-negative Ricci curvature and if $\mathbb{M}$ contains a straight line, then M is isometric to the Riemannian product $\mathbb{M} \times \mathbb{R}$". In the sequel, Schoen and Yau [12] proved that a complete non-compact $\mathbb{M}$ of dimension 3 and positive Ricci curvature is diffeomorphic to $\mathbb{R}^3$. Using the first eigenvalue of the Laplacian operator, the result stating that "if $\mathbb{M}$ is complete such that if $Ric(\mathbb{M}) > 1$ and if $d(\mathbb{M}) = \pi$, then $\mathbb{M}$ is isometric to the standard unit sphere" has been proven by Cheng in [13].

The non-existence of a compact stable minimal submanifold or stable currents is sharply associated with the topology and geometric function theory on Riemannian structure of the whole manifold. Recently, it has been shown in [14] that if the sectional curvature of a compact oriented minimal submanifold $\mathbb{M}$ of dimension m in the unit sphere $\mathbb{S}^{m+k}$ with codimension p satisfies some pinching condition $K_{\mathbb{M}} \geq \frac{p.sign(p-1)}{2(p+1)}$, then $\mathbb{M}$ is either a totally geodesic sphere, one of the Clifford minimal hypersurface $\mathbb{S}^k(\frac{k}{m}) \times \mathbb{S}^{m-k}(\frac{m-k}{m})$ in $\mathbb{S}^{m+1}$ for $k = 1, \ldots, m-1$, or the Veronese surface in $\mathbb{S}^4$. Later on, some new results for the non-existence of the stable currents, vanishing homology groups, topological and differential theorems are well known (see [15–23] and references therein). Therefore, it was an objective for mathematicians to understand geometric function theory and topological invariant of Riemannian submanifolds as well as in Riemannian space forms. Surely, this is a fruitful problem in Riemannian geometry. Using the result of Lawson and Simon [24] and following Leung [20] homotopic sphere theorem for compact oriented submanifolds in a sphere, also motivated by the idea of complete Riemannian manifold and without assumption that $\mathbb{M}^m$ is simply connected, Xu and Zao (Theorem 1.2 in [21]) concluded the following result:

Theorem 1. *[21] Let $\mathbb{M}^m$ be an oriented complete submanifold of dimension m in the unit sphere $\mathbb{S}^{m+k}$ satisfying the following inequality*

$$\|\mathbf{B}(\mathbb{X}, \mathbb{X})\|^2 < \frac{1}{3}, \ \ \forall \mathbb{X} \in \Gamma(TM), \tag{1}$$

where $\mathbb{X}$ is a unit vector at any point of $\mathbb{M}^m$ and **B** *is SFF, the second fundamental form. Then $\mathbb{M}^m$ is diffeomorphic to the sphere $\mathbb{S}^m$.*

This is one of the motivations to study—the differential and topological manifolds, and their direct relations with warped product submanifolds theory. In this way, a natural question arises: Is it possible to extend Theorem 1 to the warped product submanifolds to the cases with base manifold is minimal in a sphere? What is the best pinching constant for the differentiable rigidity sphere theorem of complete minimal warped product submanifold in a unite sphere under pinching conditions using the Laplace operator for the warping function?

The main goal of this note is to extend the rigidity Theorem 1 to a complete warped product submanifolds and find the solution for our proposed problem where motivation comes from the Nash embedding theorem [25] which states that "every Riemannian manifold has an isometric immersion into Euclidean space of sufficient high codimension". To prove our findings we shall use the technique of Chen [26] for an isometric minimal immersion from warped products to the ambient manifold, where he proved the following relation as:

$$\sum_{\alpha=1}^{p}\sum_{\beta=1}^{q} K(e_\alpha \wedge e_\beta) = \frac{q\Delta h}{h}. \tag{2}$$

Therefore, using Theorem 1 and formula (2), we announce our main finding of this study as follows:

Theorem 2. *Let $\ell : \mathbb{M}^{p+q} = \mathbb{B}^p \times_h \mathbb{F}^q \longrightarrow \mathbb{S}^{p+q+k}$ be an isometric immersion from a WP submanifold $\mathbb{M}^{p+q}$ of dimension $(p+q)$ into a unit sphere $\mathbb{S}^{p+q+k}$ of dimension $(p+q+k)$ such that the base manifold $\mathbb{B}^p$ is minimal. Assume that $\mathbb{M}^{p+q}$ is an oriented complete WP submanifold satisfying the following inequality*

$$\frac{\Delta h}{h} > \left(\frac{2(3pq-1)}{3q}\right), \tag{3}$$

where Δh is the Laplace operator for the warping function h defined on base manifold $\mathbb{B}^p$. Then $\mathbb{M}^{p+q}$ is diffeomorphic to a sphere $\mathbb{S}^{p+q}$.

In particular, if we follows the statement of Theorem D in [21], then we give another topological sphere theorem which is a consequence of Theorem 2, i.e.,

Theorem 3. *Let $\ell : \mathbb{M}^{p+q} = \mathbb{B}^p \times_h \mathbb{F}^q \longrightarrow \mathbb{S}^{p+q+k}$ be an isometric immersion from an $(p+q)$-dimensional oriented complete WP submanifold $\mathbb{M}^{p+q}$ into a $(p+q+k)$-dimensional unit sphere $\mathbb{S}^{p+q+k}$ such that the base manifold $\mathbb{B}^p$ is minimal. If the following inequality holds*

$$\frac{\Delta h}{h} > \left(\frac{2(3pq-1)}{3q}\right),$$

where Δf is the Laplace of f defined on base manifold $\mathbb{B}^p$, then $\mathbb{M}^{p+q}$ is homeomorphic to the sphere $\mathbb{S}^{p+q}$.

Hence, we noticed that Theorems 2 and 3 are differentiable sphere theorems for complete warped product submanifolds without assumption that $\mathbb{M}^n$ is simply connected.

2. Preliminaries and Notations

Let $\mathbb{S}^{m+k}$ denote the sphere with constant sectional curvature $c = 1 > 0$ and dimension $(m+k)$. We use the fact that $\mathbb{S}^{m+k}$ admits a canonical isometric embedding in $\mathbb{R}^{m+k+1}$ as

$$\mathbb{S}^{m+k} = \{\mathbb{X} \in \mathbb{R}^{m+k+1} \ : ||\mathbb{X}||^2 = 1\}.$$

Thus, the Riemannian curvature tensor $\widetilde{R}$ of a sphere $\mathbb{S}^{m+k}$ fulfils

$$\widetilde{R}(Z_1, Z_2, Z_3, Z_4) = g(Z_1, Z_4)g(Z_2, Z_3) - g(Z_2, Z_4)g(Z_1, Z_4), \tag{4}$$

$\forall\ Z_1, Z_2, Z_3, Z_4 \in \Gamma(T\widetilde{M})$, where $T\widetilde{M}$ is a tangent bundle of $\mathbb{S}^{m+k}$. Hence, $\mathbb{S}^{m+k}$ is a manifold with constant sectional curvature 1 and codimension k.

Let $\nabla^\perp$ and ∇ be the induced connections on normal bundle $T^\perp\mathbb{M}$ and the tangent bundle $T\mathbb{M}$ of $\mathbb{M}$, respectively, where $\mathbb{M}$ is a m-dimensional RM in a Riemannian $\widetilde{M}^n$ of dimension n with induced metric g. The Weingarten and Gauss formulae are defined as

$$\widetilde{\nabla}_{Z_1}\xi = -A_\xi Z_1 + \nabla^\perp_{Z_1}\xi,$$

and

$$\widetilde{\nabla}_{Z_1} Z_2 = \nabla_{Z_1} Z_2 + \mathbf{B}(Z_1, Z_2),$$

$\forall\, Z_1, Z_2 \in \Gamma(T\mathbb{M})$ and $\xi \in \Gamma(T^{\perp}\mathbb{M})$, where A_ξ and $\mathbf{B}$ are, respectively, shape operator (corresponding to ξ) and the second fundamental form as $\mathbb{M}^m$ immersed into $\widetilde{M}$, and they verify the relation

$$g(\mathbf{B}(Z_1, Z_2), \xi) = g(A_\xi Z_1, Z_2).$$

If the curvature tensors of $\widetilde{M}^n$ and $\mathbb{M}^m$ are denoted by $\widetilde{R}$ and R, then the Gauss equation is given by

$$R(Z_1, Z_2, Z_3, Z_4) = \widetilde{R}(Z_1, Z_2, Z_3, Z_4) + g\big(\mathbf{B}(Z_1, Z_4), \mathbf{B}(Z_2, Z_3)\big) - g\big(\mathbf{B}(Z_1, Z_3), \mathbf{B}(Z_2, Z_4)\big), \quad (5)$$

$\forall\, Z_1, Z_2, Z_3, Z_4 \in \Gamma(T\widetilde{M})$.

Let $\{e_1, \cdots e_m\}$ be an orthonormal basis of $T_x\mathbb{M}$ and $e_s = (e_{m+1}, \cdots e_{m+k})$ belongs to an orthonormal basis of $T^{\perp}\mathbb{M}$, then the squared norm of $\mathbf{B}$ is

$$\mathbf{B}^s_{\alpha\beta} = g(\mathbf{B}(e_\alpha, e_\beta), e_s), \quad (6)$$

and

$$||\mathbf{B}(e_\alpha, e_\beta)||^2 = \sum_{s=m+1}^{m+k} \sum_{\alpha=1}^{p} \sum_{\beta=1}^{q} \left(\mathbf{B}^s_{\alpha\beta}\right)^2. \quad (7)$$

The squared norm of the mean curvature vector $\mathbb{H}$ of a Riemannian submanifold $\mathbb{M}^m$ is defined by

$$||\mathbb{H}||^2 = \frac{1}{m^2} \sum_{s=m+1}^{m+k} \left(\sum_{\alpha=1}^{m} \mathbf{B}^s_{\alpha\alpha} \right)^2. \quad (8)$$

A submanifold $\mathbb{M}^m$ of a RM, $\widetilde{M}^{m+k}$, is referred to as *totally geodesic* and *totally umbilical* if

$$\mathbf{B}(Z_1, Z_2) = 0 \quad \text{and} \quad \mathbf{B}(Z_1, Z_2) = g(Z_1, Z_2)\mathbb{H},$$

$\forall\, Z_1, Z_2 \in \Gamma(T\mathbb{M})$, respectively, where $\mathbb{H}$ is the mean curvature vector of $\mathbb{M}^m$. Moreover, if $\mathbb{H} = 0$, then $\mathbb{M}^m$ is *minimal* in $\widetilde{M}^{m+k}$.

Now, we give a definition of the scalar curvature of Riemannian submanifold $\mathbb{M}^m$, which is denoted by $\tau(T_x\mathbb{M}^m)$, at some x in $\mathbb{M}^m$, as

$$\tau(T_x M^m) = \sum_{1 \le \alpha < \beta \le m} K_{\alpha\beta}, \quad (9)$$

where $K_{\alpha\beta} = K\big(e_\alpha \wedge e_\beta\big)$. The first equality (9) is equal to the following equation:

$$2\tau(T_x M^m) = \sum_{1 \le \alpha < \beta \le m} K_{\alpha\beta}, \;\; 1 \le \alpha, \beta \le m.$$

The above equation will be considerably used in subsequent proofs throughout the paper. In similar way, the scalar curvature $\tau(L_x)$ of an $L-$plane is defined as

$$\tau(L_x) = \sum_{1 \le \alpha < \beta \le m} K_{\alpha\beta}.$$

If the plane section spanned by e_α and e_β at x, then the sectional curvatures of the submanifold $\mathbb{M}^m$ and Riemannian manifold $\widetilde{M}^{m+k}$ are denoted by $K_{\alpha\beta}$ and $\widetilde{K}_{\alpha\beta}$, respectively. Thus, $\widetilde{K}_{\alpha\beta}$ and $K_{\alpha\beta}$ are considered to be the extrinsic and intrinsic sectional curvature of the span $\{e_\alpha, e_\beta\}$ at x. Using Gauss Equation (5), and using (9), we conclude that

$$\sum_{1\leq\alpha<\beta\leq m+k} K_{\alpha\beta} = \sum_{1\leq\alpha<\beta\leq m+k} \widetilde{K}_{\alpha\beta} + \sum_{r=m+1}^{n+k} \Big(\mathbf{B}^r_{\alpha\alpha}\mathbf{B}^r_{\beta\beta} - (\mathbf{B}^r_{\alpha\beta})^2\Big). \tag{10}$$

Now, we provide the proofs of the main findings of the study.

3. Proof of Main Findings

3.1. Proof of Theorem 2

Assume that $\mathbb{M}^m = \mathbb{B}^p \times_h \mathbb{F}^q \to \mathbb{S}^{m+k}$ is a warped product in which the base $\mathbb{B}^p$ is minimal. Let $\{e_1 \dots e_p, e_{p+1} \dots e_m\}$ be a local orthonormal frame fields of $\mathbb{M}^m$ such that $\{e_1 \dots e_p\}$ are tangents to $\mathbb{B}^p$ and $\{e_{p+1} \dots e_m\}$ are tangents to $\mathbb{F}^q$. First, we define the two unit vectors $\mathbb{X}$ and $\mathbb{Y}$ to estimate the upper bound of the terms $||\mathbf{B}(e_\alpha, e_\beta)||^2$. We can define these two unit vectors as follows:

$$\mathbb{X} = \frac{1}{\sqrt{2}}\big(e_\alpha + e_\beta\big), \;\; and \;\; \mathbb{Y} = \frac{1}{\sqrt{2}}\big(e_\alpha - e_\beta\big), \;\; 1 \leq \alpha \leq p \;\&\; 1 \leq \beta \leq q.$$

Eliminating e_α and e_β from the above equation, one obtains:

$$e_\alpha = \frac{1}{\sqrt{2}}\big(\mathbb{X} + \mathbb{Y}\big), \;\; and \;\; e_\beta = \frac{1}{\sqrt{2}}\big(\mathbb{X} - \mathbb{Y}\big), \;\; 1 \leq \alpha \leq p \;\&\; 1 \leq \beta \leq q.$$

Then we derive

$$\begin{aligned} ||\mathbf{B}(e_\alpha, e_\beta)||^2 =& \|\mathbf{B}\big(\frac{\mathbb{X}+\mathbb{Y}}{\sqrt{2}}, \frac{\mathbb{X}-\mathbb{Y}}{\sqrt{2}}\big)\|^2 \\ =& \frac{1}{4}\|\mathbf{B}(\mathbb{X},\mathbb{X}) - \mathbf{B}(\mathbb{Y},\mathbb{Y})\|^2 \\ =& \frac{1}{4}\Big\{\|\mathbf{B}(\mathbb{X},\mathbb{X})\|^2 + \|\mathbf{B}(\mathbb{Y},\mathbb{Y})\|^2 - 2g\big(\mathbf{B}(\mathbb{X},\mathbb{X}), \mathbf{B}(\mathbb{Y},\mathbb{Y})\big)\Big\}. \end{aligned}$$

Using the Cauchy–Schwartz inequality for orthonormal vector fields, we conclude that

$$\|\mathbf{B}(e_\alpha, e_\beta)\|^2 \leq \frac{1}{4}\Big\{\|\mathbf{B}(\mathbb{X},\mathbb{X})\|^2 + \|\mathbf{B}(\mathbb{Y},\mathbb{Y})\|^2 + 2||\mathbf{B}(\mathbb{X},\mathbb{X})||\,||\mathbf{B}(\mathbb{Y},\mathbb{Y})||\Big\}.$$

In virtue of (1), the above equation implies that

$$||\mathbf{B}(e_\alpha, e_\beta)||^2 < \frac{1}{4}\left(\frac{1}{3} + \frac{2}{3} + \frac{1}{3}\right) = \frac{1}{3}. \tag{11}$$

Next, from curvature tensor Equation (4) of the sphere $\mathbb{S}^{m+k}$ and the Gauss Equation (5), we find that

$$m^2||\mathbb{H}||^2 + m(m-1) = ||\mathbf{B}||^2 + \sum_{1\leq A<B\leq m} K(e_A \wedge e_B).$$

The above equation can be written for warped product manifold $\mathbb{M}^n$ and from the viewpoint of (8) and (6) as:

$$\sum_{s=m+1}^{m+k}\left(\sum_{A=1}^{m}\mathbf{B}^{s}_{AA}\right)^{2}+m(m-1)=\sum_{s=m+1}^{m+k}\sum_{i,j=1}^{p}(\mathbf{B}^{s}_{ij})^{2}+\sum_{s=m+1}^{m+k}\sum_{a,b=1}^{q}(\mathbf{B}^{s}_{ab})^{2}$$
$$+2\sum_{s=m+1}^{m+k}\sum_{\alpha=1}^{p}\sum_{\beta=1}^{q}(\mathbf{B}^{s}_{\alpha\beta})^{2}+\sum_{\alpha=1}^{p}\sum_{\beta=1}^{q}K(e_{\alpha}\wedge e_{\beta})$$
$$+\sum_{1\leq i<j\leq p}K(e_i\wedge e_j)+\sum_{1\leq a<b\leq q}K(e_a\wedge e_b).$$

Using (10) and (2) in the above equation, we derive

$$\sum_{s=m+1}^{m+k}\left(\sum_{A=1}^{m}\mathbf{B}^{s}_{AA}\right)^{2}+m(m-1)=\sum_{s=m+1}^{m+k}\sum_{i,j=1}^{p}(\mathbf{B}^{s}_{ij})^{2}+\sum_{s=m+1}^{m+k}\sum_{a,b=1}^{q}(\mathbf{B}^{s}_{ab})^{2}$$
$$+2\sum_{s=m+1}^{m+k}\sum_{\alpha=1}^{p}\sum_{\beta=1}^{q}(\mathbf{B}^{s}_{\alpha\beta})^{2}+\frac{q\Delta f}{f}$$
$$+\sum_{1\leq i<j\leq p}\widetilde{K}(e_i\wedge e_j)+\sum_{1\leq a<b\leq q}\widetilde{K}(e_a\wedge e_b)$$
$$+\sum_{s=m+1}^{m+k}\sum_{1\leq i<j\leq p}\left(\mathbf{B}^{s}_{ii}\mathbf{B}^{s}_{jj}-(\mathbf{B}^{s}_{ij})^{2}\right)$$
$$+\sum_{s=m+1}^{m+k}\sum_{1\leq a<b\leq q}\left(\mathbf{B}^{s}_{aa}\mathbf{B}^{s}_{bb}-(\mathbf{B}^{s}_{ab})^{2}\right).$$

Thus, from (4) and some rearrangements in the last equation, one obtains:

$$\sum_{s=m+1}^{m+k}\left(\sum_{A=1}^{m}\mathbf{B}^{s}_{AA}\right)^{2}=\sum_{s=m+1}^{m+k}\sum_{i,j=1}^{p}(\mathbf{B}^{s}_{ij})^{2}+\sum_{s=m+1}^{m+k}\sum_{a,b=1}^{q}(\mathbf{B}^{s}_{ab})^{2}-2pq$$
$$+2\sum_{s=m+1}^{m+k}\sum_{\alpha=1}^{p}\sum_{\beta=1}^{q}(\mathbf{B}^{s}_{\alpha\beta})^{2}+\frac{q\Delta h}{h}-\sum_{s=m+1}^{m+k}\sum_{1\leq i<j\leq p}(\mathbf{B}^{s}_{ij})^{2}$$
$$+\sum_{s=m+1}^{m+k}\sum_{1\leq i<j\leq p}\mathbf{B}^{s}_{ii}\mathbf{B}^{s}_{jj}+\sum_{s=m+1}^{m+k}\left((\mathbf{B}^{s}_{11})^{2}+\cdots+(\mathbf{B}^{s}_{pp})^{2}\right)$$
$$-\sum_{s=m+1}^{m+k}\left((\mathbf{B}^{s}_{11})^{2}+\cdots+(\mathbf{B}^{s}_{pp})^{2}\right)+\sum_{s=m+1}^{m+k}\sum_{1\leq a<b\leq q}\mathbf{B}^{s}_{aa}h^{s}_{bb}$$
$$-\sum_{s=m+1}^{m+k}\sum_{1\leq a<b\leq q}(\mathbf{B}^{s}_{ab})^{2}+\sum_{s=m+1}^{m+k}\left((\mathbf{B}^{s}_{p+1p+1})^{2}+\cdots+(\mathbf{B}^{s}_{mm})^{2}\right)$$
$$-\sum_{s=m+1}^{m+k}\left((\mathbf{B}^{s}_{p+1p+1})^{2}+\cdots+(\mathbf{B}^{s}_{mm})^{2}\right).$$

This can take the form

$$\sum_{s=m+1}^{m+k}\left(\sum_{A=1}^{m}\mathbf{B}^{s}_{AA}\right)^{2}=\sum_{s=m+1}^{m+k}\sum_{i,j=1}^{p}(\mathbf{B}^{s}_{ij})^{2}+\sum_{s=m+1}^{m+k}\sum_{a,b=1}^{q}(\mathbf{B}^{s}_{ab})^{2}-2pq$$
$$+2\sum_{s=m+1}^{m+k}\sum_{\alpha=1}^{p}\sum_{\beta=1}^{q}(\mathbf{B}^{s}_{\alpha\beta})^{2}+\frac{q\Delta h}{h}$$
$$+\sum_{s=m+1}^{m+k}\left\{\sum_{1\leq i<j\leq p}\mathbf{B}^{s}_{ii}\mathbf{B}^{s}_{jj}+(\mathbf{B}^{s}_{11})^{2}+\cdots+(\mathbf{B}^{s}_{pp})^{2}\right\}$$
$$-\sum_{s=m+1}^{m+k}\left\{\sum_{1\leq i<j\leq p}(\mathbf{B}^{s}_{ij})^{2}+(\mathbf{B}^{s}_{11})^{2}+\cdots+(\mathbf{B}^{s}_{pp})^{2}\right\}$$
$$+\sum_{s=m+1}^{m+k}\left\{\sum_{1\leq a<b\leq q}\mathbf{B}^{s}_{aa}\mathbf{B}^{s}_{bb}+(\mathbf{B}^{s}_{p+1p+1})^{2}+\cdots+(\mathbf{B}^{s}_{mm})^{2}\right\}$$
$$-\sum_{s=m+1}^{m+k}\left\{\sum_{1\leq a<b\leq q}(\mathbf{B}^{s}_{ab})^{2}+(\mathbf{B}^{s}_{p+1p+1})^{2}+\cdots+(\mathbf{B}^{s}_{mm})^{2}\right\}$$

Using the binomial theorem and the fact that the base manifold $\mathbb{B}^p$ is minimal, then it not hard to check that

$$\sum_{s=m+1}^{m+k}\left(\sum_{A=p+1}^{m}\mathbf{B}^{s}_{AA}\right)^{2}=\sum_{s=m+1}^{m+k}\sum_{i,j=1}^{p}(\mathbf{B}^{s}_{ij})^{2}+\sum_{s=m+1}^{m+k}\sum_{a,b=1}^{q}(\mathbf{B}^{s}_{ab})^{2}-2pq$$
$$+2\sum_{s=m+1}^{m+k}\sum_{\alpha=1}^{p}\sum_{\beta=1}^{q}(\mathbf{B}^{s}_{\alpha\beta})^{2}+\frac{q\Delta h}{h}$$
$$+\sum_{s=m+1}^{m+k}\left((\mathbf{B}^{s}_{11})^{2}+\cdots+(\mathbf{B}^{s}_{pp})^{2}\right)-\sum_{s=m+1}^{m+k}\sum_{i,j=1}^{p}(\mathbf{B}^{s}_{ij})^{2}$$
$$+\sum_{s=m+1}^{m+k}\left((\mathbf{B}^{s}_{p+1p+1})^{2}+\cdots+(\mathbf{B}^{s}_{mm})^{2}\right)-\sum_{s=m+1}^{m+k}\sum_{a,b=1}^{q}(\mathbf{B}^{s}_{ab})^{2}. \tag{12}$$

From the hypothesis of the theorem, we know that $\mathbb{B}^p$ is minimal and using this, we get that the fifth term of the right hand side in Equation (12) is equal to zero and seventh the term is equal to the first term of left hand side. Thus, we have:

$$2pq=2\sum_{s=m+1}^{m+k}\sum_{\alpha=1}^{p}\sum_{\beta=1}^{q}(\mathbf{B}^{s}_{\alpha\beta})^{2}+\frac{q\Delta h}{h}.$$

From (7), it implies that

$$||\mathbf{B}(e_\alpha,e_\beta)||^2=\frac{q}{2}\Big(-\frac{\Delta h}{h}\Big)+pq. \tag{13}$$

From assumption(3), we find that

$$-\frac{\Delta h}{h}<\Big(\frac{2-6pq}{3q}\Big) \tag{14}$$

Combining (13) with (14), one obtains:

$$
\begin{aligned}
||\mathbf{B}(e_\alpha, e_\beta)||^2 &< \frac{q}{2}\Big(\frac{2-6pq}{3q}\Big) + pq = \frac{1}{3} - \frac{3pq}{3} + pq \\
&< \frac{1}{3}.
\end{aligned}
\tag{15}
$$

Therefore, the proof follows from Theorem 1 and pinching condition (1) together with (15).

Remark 1. *The proofs of Theorems 2 and 3 follow easily using the same technique.*

3.2. Some Applications

Assume that $\{e_1, \dots, e_m\}$ is a local orthonormal basis of vector field $\mathbb{M}^m$. Then the gradient of function φ and its squared norm is defined as:

$$
\nabla\varphi = \sum_{i=1}^{m} e_i(\varphi)e_i,
$$

and

$$
\|\nabla\varphi\|^2 = \sum_{i=1}^{m} \big(e_i(\varphi)\big)^2. \tag{16}
$$

Let φ be a differentiable function defined on $\mathbb{M}^m$ such that $\varphi \in \mathcal{F}(\mathbb{M}^m)$, then the *Lagrangian* of the function φ is given in (p. 44, [27]).

$$
L_\varphi = \frac{1}{2}||\nabla\varphi||^2. \tag{17}
$$

The Euler–Lagrange formula of the *Lagrangian* (17) satisfies

$$
\Delta\varphi = 0. \tag{18}
$$

At point $x \in \mathbb{M}^n$ in a local orthonormal basis, the Hamiltonian would take the form (see [27] for more details):

$$
H(p, x) = \frac{1}{2}\sum_{i=1}^{m} p(e_i)^2.
$$

Put $p = d\varphi$, where d is a differential operator, and using (16), we get:

$$
H(d\varphi, x) = \frac{1}{2}\sum_{i=1}^{m} d\varphi(e_i)^2 = \frac{1}{2}\sum_{i=1}^{m} e_i(\varphi)^2 = \frac{1}{2}||\nabla\varphi||^2. \tag{19}
$$

Assuming that $\mathbb{M}^m = \mathbb{B}^p \times_f \mathbb{F}^q$ is a warped product, then $\forall\ Z_1 \in \Gamma(T\mathbb{B})$ and $Z_2 \in \Gamma(T\mathbb{F})$, we have

$$
\nabla_{Z_2} Z_1 = \nabla_{Z_1} Z_2 = (Z_1 \ln h)Z_2.
$$

Using the unit vector fields X and Z which are tangents to $\Gamma(T\mathbb{B})$ and $\Gamma(T\mathbb{F})$, resp.; then one obtains:

$$
\begin{aligned}
K(Z_1 \wedge Z_2) =& g(R(Z_1, Z_2)Z_1, Z_2) = (\nabla_{Z_1} Z_1)\ln h g(Z_2, Z_2) - g\big(\nabla_{Z_1}((Z_1 \ln h)Z_2), Z_2\big) \\
=& (\nabla_{Z_1} Z_1)\ln h g(Z_2, Z_2) - g\big(\nabla_{Z_1}(Z_1 \ln h)Z_2 + (Z_1 \ln h)\nabla_{Z_1} Z_2, Z_2\big) \\
=& (\nabla_{Z_1} Z_1)\ln h g(Z_2, Z_2) - (Z_1 \ln h)^2 - Z_1(Z_1 \ln h).
\end{aligned}
$$

If $\{e_1, \cdots e_m\}$ is an orthonormal basis for $\mathbb{M}^m$, then we can take a sum over the vector fields as follows

$$\sum_{\alpha=1}^{p}\sum_{\beta=1}^{q} K(e_\alpha \wedge e_\beta) = \sum_{\alpha=1}^{p}\sum_{\beta=1}^{q}\left((\nabla_{e_\alpha} e_\alpha) \ln h - e_\alpha (e_\beta \ln h) - (e_\alpha \ln f)^2 \right)$$
$$= q\Big(\Delta(\ln h) - ||\nabla(\ln h)||^2 \Big). \tag{20}$$

Thus, from (20) and (2), it follows that

$$\frac{\Delta h}{h} = \Delta(\ln h) - ||\nabla(\ln h)||^2. \tag{21}$$

Here, motivated by the historical development on the study of Lagrangian and Hamiltonian, we will give the following theorems as

Theorem 4. *Let $\ell : \mathbb{M}^m = \mathbb{B}^p \times_h \mathbb{F}^q \longrightarrow \mathbb{S}^{m+k}$ be an isometric immersion from an oriented complete WP submanifold $\mathbb{M}^m$ of dimension m into a sphere $\mathbb{S}^{m+k}$ of dimension $(m+k)$ such that the base manifold $\mathbb{B}^p$ is minimal and the function h satisfies the Euler–Lagrange equation with following inequality*

$$L_h < \left(\frac{1 - 3pq}{3q} \right) 2h^2, \tag{22}$$

where L_h is the Lagrangian of h. Then $\mathbb{M}^m$ is diffeomorphic to $\mathbb{S}^m$.

Proof. Using the fact that the warping function $\ln h$ satisfies the Euler–Lagrange equation, from the hypothesis of the theorem, and using (18), we have

$$\Delta \ln h = 0. \tag{23}$$

From (21) and (15), we derive

$$\Delta \ln h - \frac{\|\nabla h\|^2}{h^2} > 2p - \frac{2}{3q}. \tag{24}$$

It follows from (23) and (24) that

$$\|\nabla h\|^2 < 2ph^2 - \frac{2h^2}{3q}.$$

Using (17), we get desired result (22) which ends the proof. □

Theorem 5. *Suppose that $\ell : \mathbb{M}^m = \mathbb{B}^p \times_h \mathbb{F}^q \longrightarrow \mathbb{S}^{m+k}$ is an isometric immersion from an oriented complete WP submanifold $\mathbb{M}^m$ of dimension m into a sphere $\mathbb{S}^{m+k}$ of dimension $(m+k)$ such that the base manifold $\mathbb{B}^p$ is minimal and satisfies the relation*

$$H(dh, x) < \left\{ \frac{\Delta(lnh)}{2} + \left(\frac{1}{3q} - p \right) \right\} h^2. \tag{25}$$

Then $\mathbb{M}^m$ is diffeomorphic to $\mathbb{S}^m$.

Proof. Using Equation (19) in (24), we get required pinching condition (25). □

4. Conclusion Remark

We provide the characterization of a complete warped manifold to be diffeomorphically a unit sphere and some geometric classifications using Euler Lagrange formula along with Hamiltonian of the warping function. The topology of warped products and main extrinsic and intrinsic curvature invariants are emphatically related. Hence, our results may be seen as topological and differential sphere theorems from the viewpoint of warped product submanifolds theory. This paper shows the relation between the notion of warped product manifold and homotopy-homology theory. Therefore, we hope that this paper will be of great interest with respect to the topology of Riemannian geometry [28–35] which may find possible applications in physics.

Author Contributions: Conceptualization, A.A., F.M. and N.A.; methodology, A.A.; software, N.A.; validation, W.A.M.O., N.A. and F.M.; formal analysis, W.A.M.O.; investigation, A.A.; resources, N.A.; data curation, F.M.; writing–original draft preparation, N.M.; writing–review and editing, A.A. and F.M.; visualization, W.A.M.O.; supervision, A.A.; project administration, F.M.; funding acquisition, F.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Deanship of Scientific Research at Princess Nourah bint Abdulrahman University through the Fast-track Research Funding Program and The APC was funded by Fast-track Research Funding Program.

Acknowledgments: This research was funded by the Deanship of Scientific Research at Princess Nourah bint Abdulrahman University through the Fast-track Research Funding Program.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Bishop, R.L.; O'Neil, B. Manifolds of negative curvature. *Trans. Am. Math. Soc.* **1969**, *145*, 1–9. [CrossRef]
2. Nolker, S. Isometric immersions of warped products. *Differ. Geom. Appl.* **1996**, *6*, 1–30. [CrossRef]
3. Kim, D.K.; Kim, Y.H. Compact Einstein warped product spaces with nonpositive scalar curvature. *Proc. Am. Math. Soc.* **2003**, *131*, 2573–2576. [CrossRef]
4. Dajczer, D.; Vlachos, T. Isometric immersions of warped products. *Proc. Am. Math. Soc.* **2013**, *141*, 1795–1803. [CrossRef]
5. Roth, J. A DDVV inequality for submanifold of warped products. *Bull. Aust. Math. Soc.* **2017**, *95*, 495–499. [CrossRef]
6. Salavessa, I.M.C. Heinz mean curvature estimates in warped product spaces $M \times_{e^{\Psi}} N$. *Ann. Glob. Anal. Geom.* **2018**, *53*, 265. [CrossRef]
7. Myers, S.B. Riemannian manifolds with positive mean curvature. *Duke Math. J.* **1941**, *8*, 401–404. [CrossRef]
8. Rauch, H. A contribution to differential geometry in the large. *Ann. Math.* **1951**, *54*, 38–55. [CrossRef]
9. Berger, M. Les variétés riemanniennes $(\frac{1}{4})$-pincées. *Ann. Scuola Norm. Sup. Pisa Cl. Sei.* **1960**, *14*, 161–170.
10. Grove, K.; Shiohama, K. A generalized sphere theorem. *Ann. Math.* **1977**, *106*, 201–211. [CrossRef]
11. Cheeger, J.; Gromoll, D. The splitting theorem for manifolds of nonnegative Ricci curvature. *J. Differ. Geom.* **1971**, *6*, 119–129. [CrossRef]
12. Schoen, B.; Yau, S.T. Complete three dimensional manifolds with positive Ricci curvature and scalar curvature. In *Seminar on Differential Geometry*; Princeton University Press: Princeton, NJ, USA, 1982; Volume 102, pp. 209–228.
13. Cheng, S.Y. Eigenvalue comparison theorem and its geometric applications. *Math. Z.* **1975**, *143*, 289–297. [CrossRef]
14. Gu, J.R.; Xu, X.W. On Yau rigidity theorem for minimal submanifolds in spheres. *Math. Res. Lett.* **2012**, *19*, 511–523. [CrossRef]
15. Andrews, B.; Baker, C. Mean curvature flow of pinching submanifolds in spheres. *J. Differ. Geom.* **2010**, *58*, 357–395. [CrossRef]
16. Ali, A.; Laurian, P.I.; Alkhaldi, A.H. Ricci curvature on warped product submanifolds in spheres with geometric applications. *J. Geom. Phys.* **2019**, *146*, 103510. [CrossRef]
17. Ali, A.; Alkhaldi, A.H.; Laurian, P.I. Stable currents and homology groups in a compact CR-warped product submanifold with negative constant sectional curvature. *J. Geom. Phys.* **2020**, *148*, 103566. [CrossRef]

18. Gauchman, H. Minimal submanifold of a sphere with bounded second fundamental form. *Trans. Am. Math. Soc.* **1986**, *298*, 779–791. [CrossRef]
19. Leung, P.F. On a relation between the topology and the intrinsic and extrinsic geometries of a compact submanifold. *Proc. Edinb. Math. Soc.* **1985**, *28*, 305–311. [CrossRef]
20. Leung, P.F. On the topology of a compact submanifold of a sphere with bounded second fundmental form. *Manuscr. Math.* **1993**, *79*, 183–185. [CrossRef]
21. Xu, X.W.; Zhao, E.T. Topological and differentiable sphere theorems for complete submanifolds. *Commun. Anal. Geom.* **2009**, *17*, 565–585. [CrossRef]
22. Xu, X.W.; Ye, F. Differentiable sphere theorems for submanifolds of positive *k*-th Ricci curvature. *Manuscr. Math.* **2012**, *138*, 529–543. [CrossRef]
23. Xu, H.W.; Gu, J.R. Geometric, topological and differentiable rigidity of submanifolds in space forms. *Geom. Funct. Anal.* **2013**, *23*, 1684. [CrossRef]
24. Lawson, H.W.; Simons, J. On stable currents and their application to global problems in real and complex geometry. *Ann. Math.* **1973**, *98*, 427–450. [CrossRef]
25. Nash, J. The imbedding problem for Riemannian manifolds. *Ann. Math.* **1956**, *63*, 20–63. [CrossRef]
26. Chen, B.Y. On isometric minimal immersions from warped products into real space forms. *Proc. Edinb. Math. Soc.* **2002**, *45*, 579–587. [CrossRef]
27. Calin, O.; Chang, D.C. *Geometric Mechanics on Riemannian Manifolds: Applications to Partial Differential Equations*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2006.
28. Alkhaldi, A.H.; Ali, A. Classification of warped product submanifolds in Kenmotsu space forms admitting gradient Ricci solitons. *Mathematics* **2019**, *7*, 112. [CrossRef]
29. Aquib, M.; Boyom, B.N.; Shahid, M.H.; Vîlcu, G.E. The first fundamental equation and generalized wintgen-type inequalities for submanifolds in generalized space forms. *Mathematics* **2019**, *7*, 1151. [CrossRef]
30. Ali, R.; Mofarreh, F.; Alluhaibi, N.; Ali, A.; Ahmad, I. On differential equations characterizing Legendrian submanifolds of Sasakian space forms. *Mathematics* **2020**, *8*, 150. [CrossRef]
31. Ali, A.; Alkhaldi, A.H.; Laurian, P.I. Ali. R (2020) Eigenvalue inequalities for the p-Laplacian operator on C-totally real submanifolds in Sasakian space forms. *Appl. Anal.* **2020**. [CrossRef]
32. Decu, S.; Haesen, S.; Verstraelen, L. Inequalities for the Casorati Curvature of Statistical Manifolds in Holomorphic Statistical Manifolds of Constant Holomorphic Curvature. *Mathematics* **2020**, *8*, 251. [CrossRef]
33. Macsim, G.; Mihai, A.; Mihai, I. $delta(2,2)$-Invariant for Lagrangian Submanifolds in Quaternionic Space Forms. *Mathematics* **2020**, *8*, 480. [CrossRef]
34. Siddiqui, A.N.; Ali, A.; Alkhaldi, A.H. Chen optimal inequalities of CR-warped products of generalized Sasakian space forms. *J. Taibah Univ. Sci.* **2020**, *14*, 322–330. [CrossRef]
35. Siddiqui, A.N.; Chen, B.-Y.; Bahadir, O. Statistical solitons and inequalities for statistical warped product submanifolds. *Mathematics* **2019**, *7*, 797. [CrossRef]