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Analysis of a SEIR-KS Mathematical Model For Computer Virus Propagation in a Periodic Environment

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Abstract: In this work we develop a study of positive periodic solutions for a mathematical model of the dynamics of computer virus propagation. We propose a generalized compartment model of SEIR-KS type, since we consider that the population is partitioned in five classes: susceptible (S); exposed (E); infected (I); recovered (R); and kill signals (K), and assume that the rates of virus propagation are time dependent functions. Then, we introduce a sufficient condition for the existence of positive periodic solutions of the generalized SEIR-KS model. The proof of the main results are based on a priori estimates of the SEIR-KS system solutions and the application of coincidence degree theory. Moreover, we present an example of a generalized system satisfying the sufficient condition.

Keywords: periodic solutions; positive solutions; SEIR-KS model; computer virus model

MSC: 34K13; 34D23

1. Introduction

1.1. Scope

In the last decades, due to its theoretical and practical importance and significance, the mathematical models for dynamics of propagation for epidemics have been extensively studied, see for instance [1–10] and references in those works. In particular, mathematical models are powerful tools since it permits to explain, estimate and simulate the spread of infectious disease propagation, and consequently help to design and test control strategies like an optimal time of vaccination.

From the historical point of view, the earliest mathematical models in epidemiology were introduced in 1927 [11]. Following the presentation given in [12], we have that the basic idea considered in [11], in order to describe the dynamics of a virus, was the partition of the total population N in three classes: the susceptible class S formed for those individuals capable of contracting the disease and becoming themselves infectives; the infective class I formed for those individuals capable of transmitting the disease to susceptibles; the removed or recovered class R formed for those individuals which having contracted the disease, have died or, are permanently immune, or have been isolated, thus being unable to further transmit the disease. Moreover, they consider the three assumptions:

the period of the epidemic is too short such that N is constant, the transfer process from S to I is modeled by the mass action law and the transfer process from I to R is of exponential decay type. Then, the called SIR model is given by the following system

$$\frac{dS(t)}{dt} = -kI(t)S(t), \quad \frac{dI(t)}{dt} = kI(t)S(t) - \lambda I(t), \quad \frac{dR(t)}{dt} = \lambda I(t),$$

where k and λ are some positive constants. A particular case of SIR model is the well known SIS model, which is deduced by considering the partition of the population in two classes of individuals: susceptible and infected. Afterwards, numerous generalizations are given by several authors, who have improved the SIR mathematical model by incorporating for instance the vital dynamics, a generalized transmission forces, other classes of individuals and vaccination.

It is well known that the outbreaks of parasite population, which generate the epidemics occur around the same time of each year. Then becomes natural to study the periodicity or model these diseases by incorporating periodic functions into the epidemic models. For instance, in the case of the SIR model the periodic models are introduced by considering the facts that k and λ are time dependent periodic functions.

On the other hand, the compartmental models were introduced for biological epidemics. However, by the newest observation that the diffusion of biological virus is analogous to several processes in other areas, the ideas have been widely adapted and used to describe other phenomenon. For instance, the computer virus propagation in a network [13–18]. In particular, in this paper our aim is to study the periodicity of the mathematical model for virus propagation introduced in [18].

1.2. The Generalized SEIR-KS Mathematical Model

In [18] the authors construct a compartmental model for computer virus propagation. They consider that the population of individuals is given by computers or nodes in a network which are in corresponding communications all the time. The population is partitioned in five classes: the susceptible class S formed by the nodes which are virus-free uninfected; the exposed class E formed by the nodes which are infected, but the virus is latent; the infected class I formed by the nodes which are infected and the virus is breaking out; the recovered class R formed by the nodes which have recovered from virus infection and acquired immunization; and the kill signals class K formed by special nodes, which are a sort of anti-virus epidemic riding on the back of the virus propagation and all of them constitute a new compartment, which is generated among the infectious nodes then they can spontaneously transmit it to their neighboring nodes. The dynamic of computer virus transmission is studied by considering the following list of assumptions:

- (A1) The network at time t is formed by a total of $N(t)$ nodes. Then, we have the following relation $N(t) = S(t) + E(t) + I(t) + R(t) + K(t)$ at each time t .
- (A2) There is a behavior similar to vital dynamics of biological virus. More specifically, related with births and deaths, there is two characteristics in the process: (i) the new nodes are connected to the network at constant rate b and a fraction p are of susceptible type and the remaining fraction $q = 1 - p$ are of exposed type; and (ii) each node, by system crash or network interruption, are disconnected from the network at constant rate μ .
- (A3) The dynamics of exposed nodes are characterized by three facts: (i) the susceptible nodes are transformed in exposed nodes with probability per unit time $\beta E(t)$ with β a constant; (ii) the exposed nodes are converted into infected ones at constant rate α ; and (iii) the exposed nodes are converted into kill signals ones at constant rate χ .
- (A4) The infected nodes are converted into kill signals nodes or recovered ones at constant rates γ and ε , respectively.
- (A5) The kill signal nodes satisfy two additional premises: (i) the susceptible nodes receive the kill signal and converted into recovered ones with probability $\phi K(t)$; and (ii) the infected nodes receives and relays the kill signal nodes with probability $\delta K(t)$. Here ϕ and δ are constants.

Then, the following ordinary differential equation system

$$\frac{dS(t)}{dt} = pb - \beta S(t)E(t) - \phi S(t)K(t) - \mu S(t), \quad (1a)$$

$$\frac{dE(t)}{dt} = qb + \beta S(t)E(t) - \alpha E(t) - \chi E(t) - \mu E(t), \quad (1b)$$

$$\frac{dI(t)}{dt} = \alpha E(t) - \delta I(t)K(t) - \gamma I(t) - \varepsilon I(t) - \mu I(t), \quad (1c)$$

$$\frac{dK(t)}{dt} = \delta I(t)K(t) + \gamma I(t) + \chi E(t) - \mu K(t), \quad (1d)$$

$$\frac{dR(t)}{dt} = \phi S(t)K(t) + \varepsilon I(t) - \mu R(t), \quad (1e)$$

is introduced as the mathematical model for computer virus propagation.

In this work, with the purpose to study the existence of periodic solutions for systems of Equation (1), we consider a more general model by assuming that constants on the assumptions (A2)–(A5) are time dependent real functions, i.e., the parameters $b, p, q, \alpha, \beta, \gamma, \chi, \phi, \delta, \mu$ and ε are time dependent real functions. More precisely, we are motivated by the analysis of the following generalized model:

$$\frac{dS(t)}{dt} = p(t)b(t) - \beta(t)S(t)E(t) - \phi(t)S(t)K(t) - \mu(t)S(t), \quad (2a)$$

$$\frac{dE(t)}{dt} = q(t)b(t) + \beta(t)S(t)E(t) - \alpha(t)E(t) - \chi(t)E(t) - \mu(t)E(t), \quad (2b)$$

$$\frac{dI(t)}{dt} = \alpha(t)E(t) - \delta(t)I(t)K(t) - \gamma(t)I(t) - \varepsilon(t)I(t) - \mu(t)I(t), \quad (2c)$$

$$\frac{dK(t)}{dt} = \delta(t)I(t)K(t) + \gamma(t)I(t) + \chi(t)E(t) - \mu(t)K(t), \quad (2d)$$

$$\frac{dR(t)}{dt} = \phi(t)S(t)K(t) + \varepsilon(t)I(t) - \mu(t)R(t). \quad (2e)$$

We observe that the system in Equation (2) can be uncoupled in the study of the system in Equation (2)a–e. Indeed, it is the strategy considered in [18] to analyze the stability. However, to study the existence of periodic solutions is more convenient to consider the full system, since it is not straightforward the fact that the existence of positive periodic solutions for Equation (2)a–d implies the existence of positive periodic solution for Equation (2)e.

1.3. Reformulation of System in Equation (2) as Operator Equation

Firstly, we introduce a change of variable such that the system in Equation (2) is replaced by an equivalent system. Then, we reformulate the new system as seen in Equation (4) as an operator equation which will be analyzed by the topological degree theory.

For S, E, I, K and R satisfying the system in Equation (2), we consider the new functions S^*, E^*, I^*, K^* and R^* defined explicitly by the relation

$$(S, E, I, K, R)(t) = \left(\exp(S^*(t)), \exp(E^*(t)), \exp(I^*(t)), \exp(K^*(t)), \exp(R^*(t)) \right). \quad (3)$$

Then, by differentiation in Equation (3) and using the fact that (S, E, I, K, R) satisfy the mathematical model in Equation (2), we deduce that $(S^*, E^*, I^*, K^*, R^*)$ is a solution of the system

$$\frac{dS^*(t)}{dt} = p(t)b(t)\exp(-S^*(t)) - \beta(t)\exp(E^*(t)) - \phi(t)\exp((K^* - S^*)(t)) - \mu(t), \quad (4a)$$

$$\frac{dE^*(t)}{dt} = q(t)b(t)\exp(-E^*(t)) + \beta(t)\exp(S^*(t)) - \alpha(t) - \chi(t) - \mu(t), \quad (4b)$$

$$\frac{dI^*(t)}{dt} = \alpha(t)\exp((E^* - I^*)(t)) - \delta(t)\exp(K^*(t)) - \gamma(t) - \varepsilon(t) - \mu(t), \quad (4c)$$

$$\frac{dK^*(t)}{dt} = \delta(t)\exp(I^*(t)) + \gamma(t)\exp((I^* - K^*)(t)) + \chi(t)\exp((E^* - K^*)(t)) - \mu(t). \quad (4d)$$

$$\frac{dR^*(t)}{dt} = \phi(t)\exp((K^* + S^* - R^*)(t)) + \varepsilon(t)\exp((I^* - R^*)(t)) - \mu(t). \quad (4e)$$

Thus, our aim is to study the positive periodic solutions of Equation (2) equivalently replaced by the analysis of positive periodic solution of the new system (4).

Theorem 1. Consider the sets of functions $\{S, E, I, K, R\}$ and $\{S^*, E^*, I^*, K^*, R^*\}$ are related by Equation (3). Then, the functions S, E, I, K and R are a solution of the system in Equation (2) if and only if the functions S^*, E^*, I^*, K^* and R^* are a solution of the system in Equation (4). In particular, we have that the following two assertions are valid: (a) If S^*, E^*, I^*, K^* and R^* satisfying the system in Equation (4) are ω -periodic functions, then the functions S, E, I, K and R satisfying the system in Equation (2) are ω -periodic; and (b) The existence of a solution for the system in Equation (4) imply the existence of a positive solution for the system in Equation (2).

Proof. The proof fact that $\{S, E, I, K, R\}$ is a solution of the system in Equation (2) if and only if $\{S^*, E^*, I^*, K^*, R^*\}$ is straightforward by the change of variable (3), differentiation and algebraic rearrangements. Now, we get the proof of item (a) by using the change of variable (3), for illustration, we consider the case of function S and we have that $S(t + \omega) = \exp(S^*(t + \omega)) = \exp(S^*(t)) = S(t)$. The item (b) is a straightforward consequence of the definition of the functions S^*, E^*, I^*, K^* and R^* given in Equation (3). $\square$

In order to define the operator equation, we consider the normed vector spaces X and Y and introduce the operators $L : \text{Dom } L \subset X \rightarrow Y$ and $N : X \rightarrow Y$ explicitly defined by the relations

$$L \left((x_1, x_2, x_3, x_4, x_5)^T \right) = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}, \frac{dx_4}{dt}, \frac{dx_5}{dt} \right)^T \quad (5)$$

$$N \left((x_1, x_2, x_3, x_4, x_5)^T \right) = (\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4, \mathcal{N}_5)^T, \quad (6)$$

where

$$\mathcal{N}_1(t) = p(t)b(t)\exp(-x_1(t)) - \beta(t)\exp(x_2(t)) - \phi(t)\exp((x_4 - x_1)(t)) - \mu(t), \quad (7)$$

$$\mathcal{N}_2(t) = q(t)b(t)\exp(-x_2(t)) + \beta(t)\exp(x_1(t)) - \alpha(t) - \chi(t) - \mu(t), \quad (8)$$

$$\mathcal{N}_3(t) = \alpha(t)\exp((x_2 - x_3)(t)) - \delta(t)\exp(x_4(t)) - \gamma(t) - \varepsilon(t) - \mu(t), \quad (9)$$

$$\mathcal{N}_4(t) = \delta(t)\exp(x_3(t)) + \gamma(t)\exp((x_3 - x_4)(t)) + \chi(t)\exp((x_2 - x_4)(t)) - \mu(t), \quad (10)$$

$$\mathcal{N}_5(t) = \phi(t)\exp((x_1 + x_4 - x_5)(t)) + \varepsilon(t)\exp((x_3 - x_5)(t)) - \mu(t). \quad (11)$$

The operator notation implies that the system in Equation (4) can be rewritten as the following operator equation

$$L \left((S^*, E^*, I^*, K^*, R^*)^T \right) = N \left((S^*, E^*, I^*, K^*, R^*)^T \right), \quad (S^*, E^*, I^*, K^*, R^*) \in \text{Dom } L \subset X, \quad (12)$$

where the appropriate Banach spaces X and Y are defined by

$$X = Y = \left\{ \mathbf{x}^T \in C(\mathbb{R}, \mathbb{R}^5) : \mathbf{x}(t + \omega) = \mathbf{x}(t), \quad \|\mathbf{x}\| = \sum_{i=1}^5 \max_{t \in [0, \omega]} |x_i(t)| < \infty \right\}. \quad (13)$$

Hereinafter we use the bold notation $\mathbf{x}^T := (x_1, x_2, x_3, x_4, x_5)^T$. We notice that the spaces in Equation (13) are the more convenient, since we are concerned with the analysis of ω -periodic solutions. However, if the interest is to analyze other properties we should be consider a suitable definition of X and Y .

1.4. Main Results

By convenience of presentation, we introduce the notation

$$\bar{f} = \frac{1}{\omega} \int_0^\omega f(t) dt, \quad f^\perp = \min_{x \in [0, \omega]} f(x), \quad \text{and} \quad f^\top = \max_{x \in [0, \omega]} f(x), \quad (14)$$

for any positive real valued bounded function f defined on $[0, \omega]$.

Let us consider the following assumption

$$\left. \begin{array}{l} \text{The initial condition } (S(0), E(0), I(0), K(0), R(0)) \in \mathbb{R}_+^5; \text{ the coefficient functions } b, p, \\ q, \alpha, \beta, \gamma, \chi, \phi, \delta, \mu \text{ and } \varepsilon \text{ are positive, continuous, } \omega\text{-periodic on } [0, \omega]; \text{ and there are the} \\ \text{strictly positive constants } \kappa_1 \text{ and } \kappa_2 \text{ such that} \\ (pb)^\perp - \phi^\top (\phi^\perp)^{-1} \bar{b} \exp(2\omega\bar{\mu}) \geq \kappa_1 > 0 \\ 1 - \frac{\varepsilon^\top \alpha^\top \bar{b}}{\mu^\top (\gamma + \varepsilon + \mu)^\perp (\alpha + \chi + \mu)^\perp} \exp\left(\omega\left[(\gamma + \varepsilon + \mu)^\top + \phi^\top (\phi^\perp)^{-1} \bar{b} + \mu^\top\right]\right) \geq \kappa_2 > 0 \end{array} \right\} \quad (15)$$

Then, the main result of the paper are given by the following three theorems.

Theorem 2. Let X and Y the spaces defined on Equation (13); $Q : Y \rightarrow Y$ defined by $Q(\mathbf{x}^T) = \omega^{-1} \int_0^\omega \mathbf{x}(\tau)^T d\tau$; and the operators $L : X \rightarrow Y$ and $N : X \rightarrow Y$ defined on Equations (5) and (6), respectively. Moreover, assume that the hypothesis in Equation (15) is satisfied. Then, there are the positive constants $\rho_1, \rho_2, \rho_3, d_1, d_2, d_3, \delta_1, \delta_2$ and δ_3 , such that the following two assertions are valid

(a) If $\lambda \in]0, 1[$ and $\mathbf{x} \in \text{Dom } L$ are such that $L(\mathbf{x}) = \lambda N(\mathbf{x})$, the following inequalities

$$x_i(t) < \ln(\rho_i/\omega) + d_i, \quad i = 1, \dots, 5 \quad (16)$$

$$\ln(\delta_i) < x_i(t), \quad i = 1, \dots, 5, \quad (17)$$

holds for all $t \in [0, \omega]$.

(b) If $\mathbf{x} \in \text{Ker } L$ are such that $QN\mathbf{x} = 0$, the following inequalities

$$x_i(t) < \ln(\rho_i/\omega), \quad i = 1, \dots, 5, \quad (18)$$

$$\ln(\delta_i) < x_i(t), \quad i = 1, \dots, 5, \quad (19)$$

holds for all $t \in [0, \omega]$.

Theorem 3. If the hypothesis in Equation (15) is satisfied, there exists at least one ω -periodic solution of Equation (4).

Theorem 4. Consider that the hypothesis in Equation (15) is satisfied. Then, the system in Equation (2) has at least one positive ω -periodic solution.

1.5. Related Works

There are several works where the study of positive periodic solutions is developed, for instance in [13,19–33]. In particular, recently in [13] was proved the existence of at one positive periodic solution of the following system modeling the dynamics of a computer virus

$$\begin{aligned}\frac{dS(t)}{dt} &= b(t) - \mu_1(t)S(t) - \beta_1(t)S(t)L(t) - \beta_2(t)S(t)A(t) + \gamma_1(t)L(t) + \gamma_2(t)A(t), \\ \frac{dL(t)}{dt} &= \beta_1(t)S(t)L(t) + \beta_2(t)S(t)A(t) + \alpha_2(t)A(t) - [\mu_2(t) + \alpha_1(t) + \gamma_1(t)]L(t), \\ \frac{dA(t)}{dt} &= \alpha_1(t)L(t) - [\mu_3(t) + \alpha_1(t) + \gamma_2(t)]A(t),\end{aligned}$$

by assuming that $S(0)$, $L(0)$ and $A(0)$ are strictly positive and the functions $b, \mu_1, \mu_2, \mu_3, \beta_1, \beta_2, \gamma_1, \gamma_2, \alpha_1$ and α_2 are positive, continuous, ω -periodic on $[0, \omega]$ and

$$\left(\frac{\alpha_1}{\alpha_1 + \mu_2}\right)^\top (\alpha_2 + \gamma_2)^\top < (\mu_3 + \alpha_2 + \gamma_2)^\perp$$

We observe that S, L and A denotes the susceptible computers, the latent computers and the infectious computers, respectively.

1.6. Outline of the Paper

The paper is organized as follows. In Section 2, we introduce some terminology related to the coincidence degree theory and some useful results. In Sections 3–5 we develop the proof of Theorems 2–4, respectively. Finally, in Section 6, we present an examples of a system with coefficients satisfying Equation (15).

2. Preliminaries

In this paper, we utilize the standard notation and terminology of topological degree theory. However, for self-contained presentation, we recall some notation, concepts and results related to the statement of Mawhin's theorem, [34]. Moreover, we prove some properties for the operators L and N defining on the operator Equation (12).

2.1. The Mawhin's Continuation Theorem

Definition 1. Let X and Y be normed vector spaces and $L : \text{Dom } L \subset X \rightarrow Y$ a linear operator. Then, L is called a Fredholm operator of index zero, if the following assertions

$$\dim(\text{Ker } L) = \text{codim}(\text{Im } L) < \infty \quad \text{and} \quad \text{Im } L \text{ is closed in } Y, \quad (20)$$

are valid.

Proposition 1. Let X and Y be normed vector spaces and $L : \text{Dom } L \subset X \rightarrow Y$ a linear operator. If L is a Fredholm mapping of index zero, then

- (i) There are two continuous projectors $P : X \rightarrow X$ and $Q : Y \rightarrow Y$ such that $\text{Im } P = \text{Ker } L$ and $\text{Im } L = \text{Ker } Q = \text{Im } (I - Q)$.
- (ii) $L_P := L|_{\text{Dom } L \cap \text{Ker } P} : (I - P)X \rightarrow \text{Im } L$ is invertible and its inverse is denoted by K_P .
- (iii) There is an isomorphism $J : \text{Im } Q \rightarrow \text{Ker } L$.

Definition 2. Let X and Y be normed vector spaces and $L : \text{Dom } L \subset X \rightarrow Y$ a Fredholm mapping of index zero. Let $P : X \rightarrow X$ and $Q : Y \rightarrow Y$ be two continuous projectors such that $\text{Im } P = \text{Ker } L$ and $\text{Im } L = \text{Ker } Q = \text{Im } (I - Q)$. Let us consider $N : X \rightarrow Y$ a continuous operator and $\Omega \subset X$ an open

bounded set. Then, N is called L -compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is a bounded set and the operator $K_P(I - Q)N$ is compact on $\overline{\Omega}$.

Definition 3. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, $f \in C^1(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$ and $y \in \mathbb{R}^n \setminus f(\partial\Omega \cup N_f)$, i.e., y is a regular value of f . Here, $N_f = \{x \in \Omega : J_f(x) = 0\}$ the critical set of f and J_f the Jacobian of f at x . Then, the degree $\deg\{f, \Omega, y\}$ is defined by $\deg\{f, \Omega, y\} = \sum_{x \in f^{-1}(y)} \text{sgn} J_f(x)$, with the agreement that $\sum_{\emptyset} = 0$.

Theorem 5. Assume that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two Banach spaces and Ω is an open bounded set. Consider that $L : \text{Dom } L \subset X \rightarrow Y$ be a Fredholm mapping of index zero and $N : X \rightarrow Y$ be L -compact on $\overline{\Omega}$. If the following hypotheses

- (C₁) $Lx \neq \lambda Nx$ for each $(\lambda, x) \in]0, 1[\times (\partial\Omega \cap \text{Dom } L)$.
- (C₂) $QNx \neq 0$ for each $x \in \partial\Omega \cap \text{Ker } L$.
- (C₃) $\deg(JQN, \Omega \cap \text{Ker } L, 0) \neq 0$.

are valid. Then the operator equation $Lx = Nx$ has at least one solution in $\text{Dom } L \cap \overline{\Omega}$.

2.2. L Is a Fredholm Operator of Index Zero

Lemma 1. The operator $L : \text{Dom } L \subset X \rightarrow Y$ defined on Equation (5), with X and Y the Banach spaces given on Equation (13), is a Fredholm operator of index zero. Moreover the sets $\text{Ker } L$ and $\text{Im } L$ are characterized by $\text{Ker } L \cong \mathbb{R}^5$ and $\text{Im } L = \left\{ y \in Y : \int_0^\omega y(\tau)^T d\tau = 0 \right\}$, respectively.

Proof. In order to prove the Lemma we apply the Definition 1 or more precisely we prove that L satisfy Equation (20).

The left condition in Equation (20) is proved as follows. Let $(s_0, l_0, i_0, k_0, r_0) \in \mathbb{R}^5$ such that $x(t_0) = (s_0, l_0, i_0, k_0, r_0)$, we observe that $x^T \in \text{Ker } L$ is equivalent to $x(t) = (s_0, l_0, i_0, k_0, r_0)$ for all $t \geq t_0$. Then, we have that $\text{Ker } L \cong \mathbb{R}^5$. Now, if we select arbitrarily $y^T \in \text{Im } L$, we have that there is $x \in \text{Dom } L$ such that $Lx^T = y^T$. Then, from Equation (5) and ω -periodic behavior of x , we deduce that $\int_t^{t+\omega} y(\tau)^T d\tau = 0$ for each $t \geq t_0$ or equivalently $\text{Im } L = \left\{ y \in Y : \int_0^\omega y(\tau)^T d\tau = 0 \right\}$. Now, by linear algebra results, we recall the existence of isomorphisms $X \cong \text{Im } L \oplus (X/\text{Im } L)$, $X \cong \text{Ker } L \oplus (X/\text{Ker } L)$, and $\text{Im } L \cong X/\text{Ker } L$. Thus, we have that $\text{Ker } L \cong X/\text{Im } L$ and we get that $\dim(\text{Ker } L) = \text{codim}(\text{Im } L) = 4$.

To prove the left condition in Equation (20) we introduce the linear continuous mapping $F : \text{Im } L \subset Y \rightarrow \mathbb{R}^5$ defined by $F(x^T) = \int_0^\omega x^T(\tau) d\tau$ and observe that $F^{-1}(0) = \text{Im } L$. Thus, clearly $\text{Im } L$ is a closed set of the space Y . $\square$

2.3. Construction of the Projectors P, Q and the Operator K_P

We remark that the existence of three abstract projectors P, Q and K_P associated to L , is guaranteed by Proposition 1. However, by convenience of some calculus in the following sections we introduce explicitly the definitions of P and Q given by

$$P : X \rightarrow X, \quad Q : Y \rightarrow Y, \quad P(x^T) = Q(x^T) = \frac{1}{\omega} \int_0^\omega x(\tau)^T d\tau \quad (21)$$

and notice that satisfy the relations in Proposition 1. More precisely, we have that

- (a) $\text{Ker } L = \text{Im } P$. We prove that $\text{Ker } L \subset \text{Im } P$ as follows: from the isomorphism $\text{Ker } L \cong \mathbb{R}^5$ given on Lemma 1, we observe that $x^T \in \text{Ker } L$ is equivalent to the fact that $x(t)$ is constant for all $t \geq t_0$, which at the same time implies that $x \in \text{Im } P$, since for $x(t)$ constant we have that $P(x^T) = x^T$. Conversely, the proof of the inclusion $\text{Im } P \subset \text{Ker } L$ is deduced by the following

- facts: for $\mathbf{y}^T \in \text{Im } P$ there is $\mathbf{z} \in X$ such that $P(\mathbf{z}^T) = \mathbf{y}^T$ and from Equation (21) we obtain that $\omega^{-1} \int_0^\omega \mathbf{z}(\tau)^T d\tau = \mathbf{y}^T$ which implies by differentiation the fact that $L(\mathbf{y}^T) = 0$ or $\mathbf{y} \in \text{Ker } L$.
- (b) $\text{Ker } Q = \text{Im } L$. From the definition of Q given in Equation (21) we have that $\mathbf{y}^T \in \text{Ker } Q$ is equivalent to $\int_0^\omega \mathbf{y}(\tau)^T d\tau = 0$ and from the characterization of $\text{Im } L$ given on Lemma 1 is at the same time equivalent to $\mathbf{y}^T \in \text{Im } L$.
- (c) $\text{Im } (I - Q) = \text{Im } L$. Let $\mathbf{y}^T \in \text{Im } (I - Q)$, then there is $\mathbf{z} \in X$ such that $(I - Q)(\mathbf{z}^T) = \mathbf{y}^T$, which implies that

$$\int_0^\omega \mathbf{y}(\tau)^T d\tau = \int_0^\omega \left(\mathbf{z}(\tau)^T - \frac{1}{\omega} \int_0^\omega \mathbf{z}(m)^T dm \right) d\tau = (0, 0, 0, 0)$$

and, from the characterization of $\text{Im } L$ given on Lemma 1, we get that $\mathbf{y}(\tau)^T \in \text{Im } L$. Thus, we obtain that $\text{Im } (I - Q) \subset \text{Im } L$. By analogous arguments, we can prove the inclusion $\text{Im } L \subset \text{Im } (I - Q)$.

- (d) Operators K_P and L_P . The notation L_P is introduced for the restriction of L to $\text{Dom } L \cap \text{Ker } P$, i.e., L_P is the operator defined from $\text{Dom } L \cap \text{Ker } P$ to $\text{Im } L$ and $L_P = L$ on $\text{Dom } L \cap \text{Ker } P$. The symbol K_P is used to denote the inverse of L_P , and is precisely defined as the operator such that

$$K_P(\mathbf{x}^T)(t) = \int_0^t \mathbf{x}(\tau)^T d\tau - \frac{1}{\omega} \int_0^\omega \int_0^\eta \mathbf{x}(m)^T dmd\eta. \quad (22)$$

We notice that, we can prove that the operator K_P is the inverse of the operator L_P by application of the following identity

$$\int_0^t \frac{d}{ds} \mathbf{x}(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^t \frac{d}{dm} \mathbf{x}(m) dmdt = \mathbf{x}(t),$$

which is valid only for all $\mathbf{x}^T \in \text{Dom } L \cap \text{Ker } P$.

Thus, the projectors P and Q defined on Equation (21) satisfy the Proposition 1, since we can follow (i) and (ii) are satisfied from (a)–(c) and (d), respectively.

2.4. N Defined on Equation (6) Is a Continuous Operator

Lemma 2. The operator $N : X \rightarrow Y$ defined on Equation (5), with X and Y the Banach spaces given on Equation (13), is a continuous operator.

Proof. Let us choose arbitrarily the sequence $\{\mathbf{x}_n\} \subset X$ which converges to $\bar{\mathbf{x}}$ in the norm induced topology of X . By the definition of N given on Equation (6) and applying componentwise the inequality

$$|\exp(z_2) - \exp(z_1)| = \left| \int_{z_2}^{z_1} \exp(s) ds \right| \leq \max \{ \exp(z_1), \exp(z_2) \} |z_2 - z_1|, \quad \forall z_1, z_2 \in \mathbb{R},$$

we get the existence of $C > 0$ depending only on $b, \mu_1, \beta_1, \beta_2, \gamma_1, \gamma_2, \alpha_1$ and α_2 such that $\|N(\mathbf{x}_n) - N(\bar{\mathbf{x}})\| \leq C \|\mathbf{x}_n - \bar{\mathbf{x}}\|$. Thus, the sequence $\{N(\mathbf{x}_n)\} \subset X$ converges to $N(\bar{\mathbf{x}})$ in the topology of X induced by the norm. Hence, we can deduce that N is a continuous operator. $\square$

2.5. N Defined on Equation (6) Is L -Compact on any Ball of X Centered at $(0, 0, 0, 0, 0)$.

Lemma 3. Assuming that $h \in \mathbb{R}^+$ is an arbitrary and fix number defining the radius h of an open ball of X centered at $(0, 0, 0, 0, 0)$, denoted by $\Omega \subset X$, i.e.,

$$\Omega = \left\{ (x_1, x_2, x_3, x_4, x_5) \in X : \| (x_1, x_2, x_3, x_4, x_5) \| < h \right\}. \quad (23)$$

Moreover, consider L and N defined on Equations (5) and (6), respectively. If the assumption in Equation (15) is satisfied, the operator N is L -compact on $\overline{\Omega}$.

Proof. The proof is focused in the verification of the fact that L satisfy the two requirements of Definition 2: $QN(\overline{\Omega})$ is a bounded set and $K_P(I - Q)N$ is a compact operator on $\overline{\Omega}$, since Ω is an open bounded set by the its definition given on Equation (23) and L is a Fredholm operator of index zero by application of Lemma 1.

To prove that $QN(\overline{\Omega})$ is bounded we proceed as follows. We observe that

$$QN(\mathbf{x}^T) = \frac{1}{\omega} \int_0^\omega N(\tau)^T d\tau. \quad (24)$$

Then, for $\mathbf{x} \in \overline{\Omega}$ we have that $\|QN(\mathbf{x}^T)\| \leq \frac{1}{\omega} \int_0^\omega \|N\| d\tau = \|N\|$, which implies that $QN(\overline{\Omega})$ is bounded.

In order to prove that $K_P(I - Q)N$ is a compact operator on $\overline{\Omega}$, we observe that from Equations (6), (21) and (22) we get

$$(K_P(I - Q)N)(\mathbf{x}^T)(t) = \int_0^t N(\tau)^T d\tau + \left(\frac{1}{2} - \frac{t}{\omega}\right) \int_0^\omega N(\tau)^T d\tau - \frac{1}{\omega} \int_0^\omega \int_0^\eta N(m)^T dm d\eta.$$

Then, we deduce that $\|K_P(I - Q)N\| \leq 2\omega\|N\|$, as a result we have that $(K_P(I - Q)N)(\overline{\Omega})$ is a bounded, since the operator N is bounded on $\overline{\Omega}$. Moreover, we can prove the bound

$$|(K_P(I - Q)N)(\mathbf{x}^T)(t) - (K_P(I - Q)N)(\mathbf{x}^T)(s)| \leq 2\|N\| |t - s|, \quad \forall t, s \in [t_0, \infty[.$$

i.e., $K_P(I - Q)N$ is an equicontinuous operator. Hence, by Arzela Ascoli's theorem we get that $K_P(I - Q)N$ is a compact operator on $\overline{\Omega}$. $\square$

2.6. A Useful Auxiliary Result

Proposition 2. [13] Let $\psi : [0, \omega] \subset \mathbb{R}^+ \rightarrow \mathbb{R}$ be an absolutely continuous function satisfying the differential inequality

$$\frac{d}{dt}\psi(t) + m(t)\psi(t) \geq 0, \quad \forall t \in [0, \omega], \quad (25)$$

with $m \in L^1([0, \omega])$ such that $0 < m_1 \leq m(t) \leq m_2$ for all $t \in [0, \omega]$ and for some positive constants m_1 and m_2 . Then, if $\psi(0) > 0$ we have that $\psi(t) \geq \psi(0) \exp(-m_2\omega) > 0$ for all $t \in [0, \omega]$.

3. Proof of Theorem 2

3.1. Four Useful Lemmata

We introduce four Lemmata related with some estimates for the operator equation $Lx = \lambda Nx$, which is equivalent to the following system

$$\frac{dx_1}{dt} = \lambda [pb \exp(-x_1) - \beta \exp(x_2) - \phi \exp(x_4 - x_1) - \mu], \quad (26a)$$

$$\frac{dx_2}{dt} = \lambda [qb \exp(-x_2) + \beta \exp(x_1) - \alpha - \chi - \mu], \quad (26b)$$

$$\frac{dx_3}{dt} = \lambda [\alpha \exp(x_2 - x_3) - \delta \exp(x_4) - \gamma - \varepsilon - \mu], \quad (26c)$$

$$\frac{dx_4}{dt} = \lambda [\delta \exp(x_3) + \gamma \exp(x_3 - x_4) + \chi \exp(x_2 - x_4) - \mu], \quad (26d)$$

$$\frac{dx_5}{dt} = \lambda [\phi \exp(x_1 + x_4 - x_5) + \varepsilon \exp(x_3 - x_4) - \mu], \quad (26e)$$

and also can be rewritten as the system

$$\frac{d}{dt} \exp(x_1) + \lambda \mu \exp(x_1) = \lambda [pb - \beta \exp(x_1 + x_2) - \phi \exp(x_4)], \quad (27a)$$

$$\frac{d}{dt} \exp(x_2) + \lambda (\alpha + \chi + \mu) \exp(x_2) = \lambda [qb + \beta \exp(x_1 + x_2)], \quad (27b)$$

$$\frac{d}{dt} \exp(x_3) + \lambda (\gamma + \varepsilon + \mu) \exp(x_3) = \lambda [\alpha \exp(x_2) - \delta \exp(x_3 + x_4)], \quad (27c)$$

$$\frac{d}{dt} \exp(x_4) + \lambda \mu \exp(x_4) = \lambda [\delta \exp(x_3 + x_4) + \gamma \exp(x_3) + \chi \exp(x_2)], \quad (27d)$$

$$\frac{d}{dt} \exp(x_5) + \lambda \mu \exp(x_5) = \lambda [\phi \exp(x_1 + x_4) + \varepsilon \exp(x_3 - x_4 + x_5)]. \quad (27e)$$

We notice that to deduce Equation (27) we multiply the i -th equation of the system in Equation (26) by $\exp(x_i)$. Thus, the proof of estimates for $Lx = \lambda Nx$ is focused in to get the estimates of the solutions of Equation (26) (or equivalently of Equation (27)).

Lemma 4. Assume that $(S(0), E(0), I(0), K(0), R(0)) \in \mathbb{R}_+^5$; the coefficient functions $b, p, q, \alpha, \beta, \gamma, \chi, \phi, \delta, \mu$ and ε are positive, continuous and ω -periodic on $[0, \omega]$; and the operators $L : \text{Dom } L \subset X \rightarrow Y$ and defined on Equations (5) and (6), with X and Y the Banach spaces given on Equation (13). Then, the solution of the operator equation $Lx = \lambda Nx$ with $\lambda \in]0, 1[$ satisfy the following inequalities

$$\exp(x_2(t)) \geq \exp \left(E(0) - (\alpha + \chi + \mu)^\top \omega \right), \quad (28)$$

$$\exp(x_4(t)) \geq \exp \left(K(0) - \mu^\top \omega \right), \quad (29)$$

$$\exp(x_5(t)) \geq \exp \left(R(0) - \mu^\top \omega \right), \quad (30)$$

$$(pb)^\perp \leq \left[\mu^\top + \beta^\top \max_{t \in [0, \omega]} \exp(x_2(t)) \right] \exp(x_1(t)) + \phi^\top \max_{t \in [0, \omega]} \exp(x_4(t)), \quad (31)$$

$$\alpha^\perp \exp \left(E(0) - (\alpha + \chi + \mu)^\top \omega \right) \leq \left[(\gamma + \varepsilon + \mu)^\top + \delta^\top \max_{t \in [0, \omega]} \exp(x_4(t)) \right] \exp(x_3(t)), \quad (32)$$

for any $t \in [0, \omega]$.

Proof. By the continuity of the coefficient functions and the fact that $\lambda \in]0, 1[$, we have that $\lambda(\alpha + \chi + \mu)(t) \in [\lambda(\alpha + \chi + \mu)^\perp, (\alpha + \chi + \mu)^\top] \subset \mathbb{R}^+$ and $\lambda\mu(t) \in [\lambda\mu^\perp, \mu^\top] \subset \mathbb{R}^+$, for any $t \in [0, \omega]$. Then, we can prove Equations (28)–(30), by straightforward application of Proposition 2 to Equations (27)b,d,e, respectively, since we have that

$$\begin{aligned}\exp(x_2(t)) &\geq \exp(x_2(0)) \exp(-(\alpha + \chi + \mu)^\top \omega) = \exp(E(0) - (\alpha + \chi + \mu)^\top \omega), \\ \exp(x_4(t)) &\geq \exp(x_4(0)) \exp(-\mu^\top \omega) = \exp(K(0) - \mu^\top \omega), \\ \exp(x_5(t)) &\geq \exp(x_5(0)) \exp(-\mu^\top \omega) = \exp(R(0) - \mu^\top \omega),\end{aligned}$$

for any $t \in [0, \omega]$. Now, to prove Equations (31) and (32), for $i = 1, 3$, we introduce the notation $\tau_i \in [0, \omega]$ for the points where x_i has a minimum. Then, using the notation in Equation (14), from Equations (27)a,c, and (28) we get

$$\begin{aligned}(pb)^\perp &\leq (pb)(\tau_1) \\ &= \mu(\tau_1) \exp(x_1(\tau_1)) + \beta(\tau_1) \exp((x_1 + x_2)(\tau_1)) + \phi(\tau_1) \exp(x_4(\tau_1)) \\ &= [\mu(\tau_1) + \beta(\tau_1) \exp(x_2(\tau_1))] \exp(x_1(\tau_1)) + \phi(\tau_1) \exp(x_4(\tau_1)) \\ &\leq [\mu^\top + \beta^\top \max_{t \in [0, \omega]} \exp(x_2(t))] \exp(x_1(t)) + \phi^\top \max_{t \in [0, \omega]} \exp(x_4(t)), \\ \alpha^\perp \exp(E(0) - (\alpha + \chi + \mu)^\top \omega) &\leq \alpha(\tau_3) \exp(x_2(\tau_3)) \\ &= (\gamma + \varepsilon + \mu)(\tau_3) \exp(x_3(\tau_3)) + \delta(\tau_3) \exp((x_3 + x_4)(\tau_3)) \\ &\leq [(\gamma + \varepsilon + \mu)^\top + \delta^\top \max_{t \in [0, \omega]} \exp(x_4(t))] \exp(x_3(t)),\end{aligned}$$

for any $t \in [0, \omega]$. $\square$

Lemma 5. Assume that hypotheses of Lemma 4. Then, the solution of the operator equation $Lx = \lambda Nx$ with $\lambda \in]0, 1[$ satisfy the integral inequalities

$$\int_0^\omega \exp(x_1(t)) dt \leq \frac{\omega \bar{b}}{\mu^\perp}, \quad (33)$$

$$\int_0^\omega \exp(x_2(t)) dt \leq \frac{\omega \bar{b}}{(\alpha + \chi + \mu)^\perp}, \quad (34)$$

$$\int_0^\omega \exp(x_3(t)) dt \leq \frac{\omega \alpha^\top \bar{b}}{(\alpha + \chi + \mu)^\perp (\gamma + \varepsilon + \mu)^\perp}, \quad (35)$$

$$\int_0^\omega \exp(x_4(t)) dt \leq \frac{\omega \bar{b}}{\phi^\perp}, \quad (36)$$

$$\int_0^\omega \exp(x_5(t)) dt \leq \frac{\omega \bar{b} \phi^\top}{\mu^\perp \phi^\perp} \max_{t \in [0, \omega]} \exp(x_1(t)) + \frac{\varepsilon^\top \max_{t \in [0, \omega]} \exp(x_3(t))}{\mu^\perp \exp(K(0) - \mu^\top \omega)} \int_0^\omega \exp(x_5(t)) dt. \quad (37)$$

Proof. We integrate the equations of the system in Equation (27) on $[0, \omega]$ and using the ω -periodicity of $\mathbf{x}$ we deduce the following identities

$$\int_0^\omega p(t)b(t)dt = \int_0^\omega [\beta(t)\exp((x_1 + x_2)(t)) + \phi(t)\exp(x_4(t)) + \mu(t)\exp(x_1(t))] dt, \quad (38a)$$

$$\int_0^\omega q(t)b(t)dt = \int_0^\omega [-\beta(t)\exp((x_1 + x_2)(t)) + (\alpha + \chi + \mu)(t)\exp(x_2(t))] dt, \quad (38b)$$

$$\int_0^\omega \alpha(t)\exp(x_2(t))dt = \int_0^\omega [\delta(t)\exp((x_3 + x_4)(t)) + (\gamma + \varepsilon + \mu)(t)\exp(x_3(t))] dt, \quad (38c)$$

$$\int_0^\omega \mu(t)\exp(x_4(t))dt = \int_0^\omega [\delta(t)\exp((x_3 + x_4)(t)) + \gamma(t)\exp(x_3(t)) + \chi(t)\exp(x_2(t))] dt, \quad (38d)$$

$$\int_0^\omega \mu(t)\exp(x_5(t))dt = \int_0^\omega [\phi(t)\exp((x_1 + x_4)(t)) + \varepsilon(t)\exp((x_3 - x_4 + x_5)(t))] dt. \quad (38e)$$

Then, adding Equation (38)a,b, using the ω -periodicity of x_1 and x_2 , and the fact that $p(t) + q(t) = 1$, we deduce that

$$\int_0^\omega b(t)dt = \int_0^\omega [\mu(t)\exp(x_1(t)) + \{\alpha(t) + \chi(t) + \mu(t)\}\exp(x_2(t)) + \phi(t)\exp(x_4(t))] dt,$$

which implies Equations (33), (34) and (36), since, by the positivity of α, χ, μ and ϕ and the notation in Equation (14), we get the inequalities

$$\begin{aligned} \int_0^\omega \exp(x_1(t))dt &\leq \frac{1}{\min_{t \in [0, \omega]} \mu(t)} \int_0^\omega \mu(t)\exp(x_1(t))dt \\ &\leq \frac{1}{\mu^\perp} \int_0^\omega [\mu(t)\exp(x_1(t)) + \{\alpha(t) + \chi(t) + \mu(t)\}\exp(x_2(t)) + \phi(t)\exp(x_4(t))] dt \\ &= \frac{1}{\mu^\perp} \int_0^\omega b(t)dt = \frac{\omega \bar{b}}{\mu^\perp}, \\ \int_0^\omega \exp(x_2(t))dt &\leq \frac{1}{\min_{t \in [0, \omega]} (\alpha + \chi + \mu)(t)} \int_0^\omega (\alpha + \chi + \mu)(t)\exp(x_2(t))dt \\ &\leq \frac{1}{(\alpha + \chi + \mu)^\perp} \int_0^\omega [\mu(t)\exp(x_1(t)) + \{\alpha(t) + \chi(t) + \mu(t)\}\exp(x_2(t)) + \phi(t)\exp(x_4(t))] dt \\ &= \frac{1}{(\alpha + \chi + \mu)^\perp} \int_0^\omega b(t)dt = \frac{\omega \bar{b}}{(\alpha + \chi + \mu)^\perp}, \\ \int_0^\omega \exp(x_4(t))dt &\leq \frac{1}{\min_{t \in [0, \omega]} \phi(t)} \int_0^\omega \phi(t)\exp(x_4(t))dt \\ &\leq \frac{1}{\phi^\perp} \int_0^\omega [\mu(t)\exp(x_1(t)) + \{\alpha(t) + \chi(t) + \mu(t)\}\exp(x_2(t)) + \phi(t)\exp(x_4(t))] dt \\ &= \frac{1}{\phi^\perp} \int_0^\omega b(t)dt = \frac{\omega \bar{b}}{\phi^\perp}. \end{aligned}$$

The inequality in Equation (35) is a consequence of Equations (38)c and (34), since

$$\begin{aligned} \int_0^\omega \exp(x_3(t)) dt &\leq \frac{1}{\min_{t \in [0, \omega]} (\gamma + \varepsilon + \mu)(t)} \int_0^\omega (\gamma + \varepsilon + \mu)(t) \exp(x_3(t)) dt \\ &\leq \frac{1}{(\gamma + \varepsilon + \mu)^\perp} \int_0^\omega [\delta(t) \exp((x_3 + x_4)(t)) + (\gamma + \varepsilon + \mu)(t) \exp(x_3(t))] dt \\ &= \frac{1}{(\gamma + \varepsilon + \mu)^\perp} \int_0^\omega \alpha(t) \exp(x_2(t)) dt \\ &\leq \frac{\omega \alpha^\top \bar{b}}{(\gamma + \varepsilon + \mu)^\perp (\alpha + \chi + \mu)^\perp}. \end{aligned}$$

Now, from Equations (38)e, (36) and (32), we deduce the following estimate

$$\begin{aligned} \mu^\perp \int_0^\omega \exp(x_5(t)) dt &\leq \int_0^\omega \mu(t) \exp(x_5(t)) dt \\ &= \int_0^\omega [\phi(t) \exp((x_1 + x_4)(t)) + \varepsilon(t) \exp((x_3 - x_4 + x_5)(t))] dt \\ &\leq \frac{\omega \bar{b} \phi^\top}{\phi^\perp} \max_{t \in [0, \omega]} \exp(x_1(t)) + \varepsilon^\top \frac{\max_{t \in [0, \omega]} \exp(x_3(t))}{\min_{t \in [0, \omega]} \exp(x_4(t))} \int_0^\omega \exp(x_5(t)) dt \\ &\leq \frac{\omega \bar{b} \phi^\top}{\phi^\perp} \max_{t \in [0, \omega]} \exp(x_1(t)) + \varepsilon^\top \frac{\max_{t \in [0, \omega]} \exp(x_3(t))}{\exp(K(0) - \mu^\top \omega)} \int_0^\omega \exp(x_5(t)) dt, \end{aligned}$$

which implies Equation (37). $\square$

Lemma 6. Assume that hypotheses of Lemma 4. Then, the solution of the operator equation $Lx = \lambda Nx$ with $\lambda \in]0, 1[$ satisfy the integral inequalities

$$\int_0^\omega \left| \frac{dx_1}{dt}(t) \right| dt \leq 2\omega \bar{b} \max_{t \in [0, \omega]} \exp(-x_1(t)), \quad (39)$$

$$\int_0^\omega \left| \frac{dx_2}{dt}(t) \right| dt < 2\omega (\alpha + \chi + \mu)^\top, \quad (40)$$

$$\int_0^\omega \left| \frac{dx_3}{dt}(t) \right| dt < 2\omega \left((\gamma + \varepsilon + \mu)^\top + \delta^\top \bar{b} (\phi^\top)^{-1} \right), \quad (41)$$

$$\int_0^\omega \left| \frac{dx_4}{dt}(t) \right| dt < 2\omega \bar{\mu}, \quad (42)$$

$$\int_0^\omega \left| \frac{dx_5}{dt}(t) \right| dt < 2\omega \bar{\mu} \max_{t \in [0, \omega]} \exp(x_4(t) - x_5(t)). \quad (43)$$

Proof. We integrate the system in Equation (26) on $[0, \omega]$ and by using the ω -periodicity behavior of x , we have that

$$\int_0^\omega p(t) b(t) \exp(-x_1(t)) dt = \int_0^\omega [\beta(t) \exp(x_2(t)) - \phi(t) \exp(x_4(t) - x_1(t)) - \mu(t)] dt, \quad (44a)$$

$$\int_0^\omega [q(t) b(t) \exp(-x_2(t)) + \beta(t) \exp(x_1(t))] dt = \int_0^\omega [\alpha(t) + \chi(t) + \mu(t)] dt, \quad (44b)$$

$$\int_0^\omega [\alpha(t) \exp(x_2(t) - x_3(t)) - \delta(t) \exp(x_4(t))] dt = \int_0^\omega [\gamma(t) + \varepsilon(t) + \mu(t)] dt, \quad (44c)$$

$$\int_0^\omega [\delta(t) \exp(x_3(t)) + \gamma(t) \exp(x_3(t) - x_4(t)) + \chi(t) \exp(x_2(t) - x_4(t))] dt = \int_0^\omega \mu(t) dt, \quad (44d)$$

$$\int_0^\omega \phi(t) \exp(x_1(t) + x_4(t) - x_5(t)) + \varepsilon(t) \exp(x_3(t) - x_4(t)) - \mu(t) \exp(x_4(t) - x_5(t)) dt = 0. \quad (44e)$$

Then, taking the modulus of the each equations defining the system in Equation (26); integrating each resulting equations on $[0, \omega]$; using the information that $\lambda \in]0, 1[$; employing the relations of Equation (44); and applying the inequalities on Lemmas 4 and 5, we obtain the following estimates

$$\begin{aligned} \int_0^\omega \left| \frac{dx_1}{dt}(t) \right| dt &< 2 \int_0^\omega p(t)b(t) \exp(-x_1(t)) dt \leq 2\omega \bar{b} \max_{t \in [0, \omega]} \exp(-x_1(t)), \\ \int_0^\omega \left| \frac{dx_2}{dt}(t) \right| dt &< 2 \int_0^\omega (\alpha + \chi + \mu)(t) dt \leq 2\omega(\alpha + \chi + \mu)^\top, \\ \int_0^\omega \left| \frac{dx_3}{dt}(t) \right| dt &< 2 \int_0^\omega [(\gamma + \varepsilon + \mu)(t) + \delta(t) \exp(x_4(t))] dt \\ &\leq 2\omega \left((\gamma + \varepsilon + \mu)^\top + \delta^\top \bar{b}(\phi^\top)^{-1} \right), \\ \int_0^\omega \left| \frac{dx_4}{dt}(t) \right| dt &< 2 \int_0^\omega \mu(t) dt = 2\omega \bar{\mu}, \\ \int_0^\omega \left| \frac{dx_5}{dt}(t) \right| dt &< 2 \int_0^\omega \mu(t) \exp(x_4(t) - x_5(t)) dt \\ &\leq 2\omega \bar{\mu} \max_{t \in [0, \omega]} \exp(x_4(t) - x_5(t)), \end{aligned}$$

which conclude the proof of lemma. $\square$

Lemma 7. Assume that hypotheses of Lemma 4. Moreover consider that the hypotheses (15) and $\mathbf{x}$ is the solution of the operator equation $L\mathbf{x} = \lambda N\mathbf{x}$ with $\lambda \in]0, 1[$ the following estimates

$$\text{there exists } \delta_i > 0 \text{ such that } \exp(x_i(t)) > \delta_i, \quad t \in [0, \omega], \quad i = 1, \dots, 5, \quad (45)$$

$$\text{there exists } \rho_i > 0 \text{ such that } \int_0^\omega \exp(x_i(t)) dt < \rho_i, \quad i = 1, \dots, 5, \quad (46)$$

$$\text{there exists } d_i > 0 \text{ such that } \int_0^\omega \left| \frac{dx_i}{dt}(t) \right| dt < d_i, \quad i = 1, \dots, 5, \quad (47)$$

are satisfied. In particular, $\max_{t \in [0, \omega]} \exp(x_i(t)) \leq \rho_i(\omega)^{-1} \exp(d_i)$ and $x_i(t) < \ln(\rho_i/\omega) + d_i$ for $t \in [0, \omega]$ and $i = 1, \dots, 5$.

Proof. We get the proof by application of Lemmas 4, 5 and 6, and the hypotheses in Equation (15). We notice that we can prove some relations in Equations (45)–(47) by a straightforward consequence of Lemmas 4, 5 and 6. More precisely, we can deduce

$$(45) \text{ for } i = 2, 4, 5, \text{ with } \delta_2 = \exp \left(-(\alpha + \chi + \mu)^\top \omega \right), \delta_4 = \exp \left(-\mu^\top \omega \right), \delta_5 = \delta_4; \quad (48)$$

$$(46) \text{ for } i = 1, 2, 3, 4, \text{ with } \rho_1 = \frac{\omega \bar{b}}{\mu^\perp}, \rho_2 = \frac{\omega \bar{b}}{(\alpha + \chi + \mu)^\perp}, \rho_3 = \frac{\alpha^\top \rho_2}{(\gamma + \varepsilon + \mu)^\perp}, \rho_4 = \frac{\omega \bar{b}}{\phi^\perp}; \quad (49)$$

$$(47) \text{ for } i = 2, 3, 4, \text{ with } d_2 = 2\omega(\alpha + \chi + \mu)^\top, d_3 = 2 \left(\omega(\gamma + \varepsilon + \mu)^\top + \delta^\top \rho_4 \right), d_4 = 2\omega \bar{\mu}; \quad (50)$$

from Equations (28)–(30); (33)–(36); and (40)–(42); respectively. Meanwhile, to prove the remaining inequalities we proceed as follows:

- (i) we prove that $\max_{t \in [0, \omega]} \exp(x_i(t)) \leq \rho_i(\omega)^{-1} \exp(d_i)$ for $i = 2, 3, 4$;
- (ii) we prove Equation (45) for $i = 1, 3$;
- (iii) we prove Equation (47) for $i = 1$;

- (iv) we prove Equation (46) for $i = 5$;
 (v) we prove Equation (47) for $i = 5$.

Proof of (i). From Equation (49) and the intermediate value for integrals we can deduce that there exist $\xi_i \in [0, \omega]$ satisfying the inequality $x_i(\xi_i) < \ln(\rho_i/\omega)$ for $i = 2, 3, 4$. Then, by the fundamental theorem of calculus and Equation (50), we deduce that

$$x_i(t) = x_i(\xi_i) + \int_{\xi_i}^t \frac{dx_i}{dt}(t)dt < \ln(\rho_i/\omega) + \int_{\xi_i}^t \frac{dx_i}{dt}(t)dt < \ln(\rho_i/\omega) + d_i, \quad i = 2, 3, 4,$$

for any $t \in [0, \omega]$, which clearly implies (i).

Proof of (ii). We notice that the assertion proved in (i) for $i = 2, 4$ and Equation (31) imply that

$$(pb)^\perp \leq [\mu^\top + \beta^\top \rho_2(\omega)^{-1} \exp(d_2)] \exp(x_1(t)) + \phi^\top \rho_4(\omega)^{-1} \exp(d_4), \quad (51)$$

for any $t \in [0, \omega]$. By hypotheses in Equation (15) we have that $(pb)^\perp - \phi^\top \rho_4(\omega)^{-1} \exp(d_4) \geq \kappa_1$, then Equation (51) implies Equation (45) for $i = 1$ with $\delta_1 = \kappa_1 [\mu^\top + \beta^\top \rho_2(\omega)^{-1} \exp(d_2)]^{-1}$. Now, from the assertion proved in (i) for $i = 4$ and Equation (32) we can deduce Equation (45) for $i = 3$ with $\delta_3 = \alpha^\perp [(\gamma + \varepsilon + \mu)^\top + \delta^\top \rho_4(\omega)^{-1} \exp(d_4)]^{-1}$.

Proof of (iii). From Equation (48) and Lemma 6, we can follow that Equation (47) for $i = 1$ is satisfied with $d_1 = 2\omega\bar{b}/\delta_1$.

Proof of (iv). Form similar arguments and notation to the proof of step (i), Equation (47) and Equation (47) for $i = 1$, we can deduce that

$$x_1(t) = x_1(\xi_1) + \int_{\xi_1}^t \frac{dx_1}{dt}(t)dt < \ln(\rho_1/\omega) + \int_{\xi_1}^t \frac{dx_1}{dt}(t)dt < \ln(\rho_1/\omega) + d_1,$$

for some $\xi_1 \in [0, \omega]$ and any $t \in [0, \omega]$. Then, $\max_{t \in [0, \omega]} \exp(x_1(t)) \leq \rho_1(\omega)^{-1} \exp(d_1)$. Now, from Equation (37) and the assertion proved in (i) for $i = 3$ we deduce that

$$\begin{aligned} \int_0^\omega \exp(x_5(t))dt &\leq \frac{\omega\bar{b}\phi^\top}{\mu^\perp\phi^\perp} \max_{t \in [0, \omega]} \exp(x_1(t)) + \frac{\varepsilon^\top \max_{t \in [0, \omega]} \exp(x_3(t))}{\mu^\perp \exp(K(0) - \mu^\top \omega)} \int_0^\omega \exp(x_5(t))dt \\ &\leq \frac{\omega\bar{b}\phi^\top \rho_1}{\mu^\perp \phi^\perp \omega} \exp(d_1) + \frac{\varepsilon^\top \rho_3 \exp(d_3)}{\mu^\perp \omega \exp(-\mu^\top \omega)} \int_0^\omega \exp(x_5(t))dt. \end{aligned} \quad (52)$$

Thus, the hypotheses in Equation (15) implies

$$\kappa_2 \int_0^\omega \exp(x_5(t))dt \leq \left(1 - \frac{\varepsilon^\top \rho_3 \exp(d_3)}{\mu^\perp \omega \exp(-\mu^\top \omega)}\right) \int_0^\omega \exp(x_5(t))dt \quad (53)$$

$$\leq \frac{\omega\bar{b}\phi^\top \rho_1}{\mu^\perp \phi^\perp \omega} \exp(d_1), \quad (54)$$

which implies Equation (46) for $i = 5$ with $\rho_5 = \omega\bar{b}\phi^\top \rho_1 \exp(d_1) [\mu^\perp \phi^\perp \omega \kappa_2]^{-1}$.

Proof of (iv). From (i) with $i = 4$ and Equation (48)

$$\begin{aligned} \int_0^\omega \left| \frac{dx_5}{dt}(t) \right| dt &< 2\omega\bar{\mu} \max_{t \in [0, \omega]} \exp(x_4(t) - x_5(t)) \\ &\leq \frac{2\bar{\mu}\rho \exp(d_4)}{d_5} := d_5. \end{aligned}$$

Then, Equation (47) for $i = 5$ is satisfied.

Summarizing we have that Equation (45) is followed by Equation (48) and (ii); Equation (46) is a consequence of Equation (49) and (iv); and Equation (47) is proved from Equation (50), and (iii) and (v). Moreover, we observe that a sequence of similar arguments and notation to the proof of step (i), Equations (47) and (47) for $i = 5$, implies that

$$x_5(t) = x_5(\xi_5) + \int_{\xi_5}^t \frac{dx_5}{dt}(t) dt < \ln(\rho_5/\omega) + \int_{\xi_5}^t \frac{dx_5}{dt}(t) dt < \ln(\rho_5/\omega) + d_5,$$

for some $\xi_5 \in [0, \omega]$ and any $t \in [0, \omega]$. Then, $\max_{t \in [0, \omega]} \exp(x_5(t)) \leq \rho_5(\omega)^{-1} \exp(d_5)$. Then, we get the additional and particular inequalities are followed from (i) and (iv). $\square$

3.2. Proof of (a)

We can prove the estimate in Equation (17) by application of Lemma (7).

3.3. Proof of (b)

If $\mathbf{x} \in \text{Ker } L$, then by the results of Section 2.3, we have that $\mathbf{x}(t) \in \mathbb{R}^5$ is constant for any $t \in [0, \omega]$. By notational convenience we consider that $\mathbf{x}(t) = (S_0, E_0, I_0, K_0, R_0)$. Then, from Equation (24) the condition $QN(\mathbf{x}^T) = QN((S_0, E_0, I_0, K_0, R_0)^T) = 0$ implies that

$$0 = \bar{p}\bar{b} \exp(-S_0) - \bar{\beta} \exp(E_0) - \bar{\phi} \exp(K_0 - S_0) - \bar{\mu}, \quad (55a)$$

$$0 = \bar{q}\bar{b} \exp(-E_0) + \bar{\beta} \exp(S_0) - \bar{\alpha} - \bar{\chi} - \bar{\mu}, \quad (55b)$$

$$0 = \bar{\alpha} \exp(E_0 - I_0) - \bar{\delta} \exp(K_0) - \bar{\gamma} - \bar{\varepsilon} - \bar{\mu}, \quad (55c)$$

$$0 = \bar{\delta} \exp(I_0) + \bar{\gamma} \exp(I_0 - K_0) + \bar{\chi} \exp(E_0 - K_0) - \bar{\mu}, \quad (55d)$$

$$0 = \bar{\phi} \exp(S_0 + K_0 - R_0) + \bar{\varepsilon} \exp(I_0 - R_0) - \bar{\mu}. \quad (55e)$$

Then, from Equation (55) and following similar arguments to the proof of Lemma 7, we can deduce that in this case an inequality of the type in Equation (46) is also valid, i.e.,

$$\exp(S_0) < \frac{\rho_1}{\omega}, \quad \exp(E_0) < \frac{\rho_2}{\omega}, \quad \exp(I_0) < \frac{\rho_3}{\omega}, \quad \exp(K_0) < \frac{\rho_4}{\omega} \quad \text{and} \quad \exp(R_0) < \frac{\rho_5}{\omega}.$$

which implies Equation (18). Moreover, from Lemma 7 and the fact that $\text{Ker } L \subset \text{Dom } L$, we can deduce that

$$\exp(S_0) > \delta_1, \quad \exp(E_0) > \delta_2, \quad \exp(I_0) > \delta_3, \quad \exp(K_0) > \delta_4, \quad \text{and} \quad \exp(R_0) > \delta_5.$$

Thus, the inequality in Equation (19) is also satisfied.

4. Proof of Theorem 3

4.1. A Previous Lemma

Lemma 8. Let X and Y be the spaces defined on Equation (13); $\Omega \subset X$ the open ball centered at $(0, 0, 0, 0, 0)$ with radius

$$h = \sum_{i=1}^3 \max \left\{ \left| \ln(\delta_i) \right|, \left| \ln \left(\frac{\rho_i}{\omega_i} \right) \right| + d_i \right\}, \quad (56)$$

where δ_i , ρ_i and d_i are defined in the proof of Lemma 7; and L , N and Q the operators defined on Equations (5), (6) and (21), respectively. If Equation (15) is satisfied, the operators L and N satisfy the properties (C_1) – (C_3) of Theorem 5.

Proof. We prove (C_1) and (C_2) by contradiction argument and we prove (C_3) by application of invariance property of the topological degree. Indeed, we have that

- (C_1) Let us assume that there are $\delta \in]0, 1[$ and $\mathbf{x} \in \partial\Omega \cap \text{Dom } L$ such that $L\mathbf{x} = \delta N\mathbf{x}$. Then, by application of Theorem 2-(a) we deduce that $\mathbf{x} \in \text{Int } \Omega$ which is a contradiction to the assumption that $\mathbf{x} \in \partial\Omega$.
- (C_2) Let us assume that there is $\mathbf{x} \in \partial\Omega \cap \text{Ker } L$ such that $QN\mathbf{x} = 0$. Then, by application of Theorem 2-(b) we deduce that $\mathbf{x} \in \text{Int } \Omega$ which is a contradiction to the assumption that $\mathbf{x} \in \partial\Omega$.
- (C_3) Let us define the mapping $\Phi : \text{Dom } L \times [0, 1] \rightarrow X$ by the following relation

$$\Phi(\mathbf{x}, v) = \begin{bmatrix} \overline{pb} \exp(-x_1) - \overline{\beta} \exp(x_2) - \overline{\phi} \exp(x_4 - x_1) - \overline{\mu} \\ \overline{qb} \exp(-x_2) - \overline{\alpha} + \overline{\chi} + \overline{\mu} \\ \overline{a} \exp(x_2 - x_3) - \overline{\delta} \exp(x_4) - \overline{\gamma} + \overline{\varepsilon} + \overline{\mu} \\ \overline{\delta} \exp(x_3) + \overline{\chi} \exp(x_2 - x_4) - \overline{\mu} \\ \exp(x_1 + x_4 - x_5) - \overline{\mu} \end{bmatrix} + v \begin{bmatrix} 0 \\ \overline{\beta} \exp(x_1) \\ 0 \\ \overline{\gamma} \exp(x_3 - x_4) \\ \overline{\varepsilon} \exp(x_3 - x_5) \end{bmatrix}.$$

We prove that $\Phi(\mathbf{x}, v) \neq 0$ when $\mathbf{x}^T \in \partial\Omega \cap \text{Ker } L$ and $v \in [0, 1]$. From Lemma 1 we recall that $\mathbf{x}^T(t) = (S_0, E_0, I_0, K_0, R_0) \in \mathbb{R}^5$ is a constant. Let us consider that the conclusion is false, then the constant vector $(S_0, E_0, I_0, K_0, R_0)^T$ with $\|(S_0, E_0, I_0, K_0, R_0)\| = h$ satisfies $\Phi(S_0, E_0, I_0, K_0, R_0, v) = 0$, that is,

$$\begin{aligned} 0 &= \overline{pb} \exp(-S_0) - \overline{\beta} \exp(E_0) - \overline{\phi} \exp(K_0 - S_0) - \overline{\mu}, \\ 0 &= \overline{qb} \exp(-E_0) - \overline{\alpha} + \overline{\chi} + \overline{\mu} + v \overline{\beta} \exp(S_0), \\ 0 &= \overline{a} \exp(E_0 - I_0) - \overline{\delta} \exp(K_0) - \overline{\gamma} + \overline{\varepsilon} + \overline{\mu}, \\ 0 &= \overline{\delta} \exp(I_0) + \overline{\chi} \exp(E_0 - K_0) - \overline{\mu} + v \overline{\gamma} \exp(I_0 - K_0), \\ 0 &= \overline{\phi} \exp(S_0 + K_0 - R_0) - \overline{\mu} + v \overline{\varepsilon} \exp(I_0 - R_0). \end{aligned}$$

Then, by following similar reasoning steps to the proof of Theorem 2-(a) we get that $\|(S_0, E_0, I_0, K_0, R_0)^T\| < h$, which contradicts to the assumption that $\|(S_0, E_0, I_0, K_0, R_0)^T\| = h$.

Let us consider $J = I : \text{Im } Q \rightarrow \text{Ker } L$ such that $\mathbf{x}^T \mapsto \mathbf{x}^T$, then by applying the Homotopy Invariance Theorem of Topology Degree, using the fact that the system

$$\begin{aligned} 0 &= \overline{pb} \exp(-x_1(t)) - \overline{\beta} \exp(x_2(t)) - \overline{\phi} \exp(x_4(t) - x_1(t)) - \overline{\mu}, \\ 0 &= \overline{qb} \exp(-x_2(t)) + \overline{\beta} \exp(x_1(t)) - \overline{\alpha} + \overline{\chi} + \overline{\mu}, \\ 0 &= \overline{a} \exp(x_2(t) - x_3(t)) - \overline{\delta} \exp(x_4(t)) - \overline{\gamma} + \overline{\varepsilon} + \overline{\mu}, \\ 0 &= \overline{\delta} \exp(x_3(t)) + \overline{\chi} \exp(x_2(t) - x_4(t)) + \overline{\gamma} \exp(x_3(t) - x_4(t)) - \overline{\mu}, \\ 0 &= \overline{\phi} \exp(x_1(t) + x_4(t) - x_5(t)) + \overline{\varepsilon} \exp(x_3(t) - x_5(t)) - \overline{\mu}. \end{aligned}$$

has a unique solution $\mathbf{x}^{*T} \in \partial\Omega \cap \text{Ker } L$, noticing that the determinant of the Jacobian of Φ at $\mathbf{x}^{*T}$ is given by

$$\left| J_{\Phi}(\mathbf{x}^{*T}) \right| = - \left[\bar{\phi} \exp(x_1^* + x_4^* - x_5^*) + \bar{\epsilon} \exp(x_3^* - x_5^*) \right] (\Pi_1 + \Pi_2),$$

with Π_1 and Π_2 the positive functions

$$\begin{aligned} \Pi_1 &= \left[-\bar{\beta} \exp(x_2^*) - \mu \right] \left[-\bar{q}\bar{b} \exp(x_2^*) \right] \left[\bar{\alpha} \exp(x_2^* - x_3^*) \left(\bar{\chi} \exp(x_2^* - x_4^*) + \bar{\gamma} \exp(x_3^* - x_4^*) \right) \right. \\ &\quad \left. + \bar{\delta} \exp(x_4^*) \left(\bar{\delta} \exp(x_3^*) + \bar{\gamma} \exp(x_3^* - x_4^*) \right) \right] \\ \Pi_2 &= -\bar{\beta} \exp(x_1^*) \left\{ \left[-\bar{\beta} \exp(x_2^*) \right] \left[\bar{\alpha} \exp(x_2^* - x_3^*) \left(\bar{\chi} \exp(x_2^* - x_4^*) + \bar{\gamma} \exp(x_3^* - x_4^*) \right) \right. \right. \\ &\quad \left. \left. + \bar{\delta} \exp(x_4^*) \left(\bar{\delta} \exp(x_3^*) + \bar{\gamma} \exp(x_3^* - x_4^*) \right) \right] + \left[-\bar{\phi} \exp(x_2^* - x_1^*) \right] \right. \\ &\quad \left. \left[\bar{\alpha} \exp(x_2^* - x_3^*) \left(\bar{\delta} \exp(x_3^*) + \bar{\gamma} \exp(x_3^* - x_4^*) \right) + \bar{\chi}\bar{\alpha} \exp(2x_2^* - x_3^* - x_4^*) \right] \right\}, \end{aligned}$$

and by Definition 3, we have that

$$\deg(JQN(\mathbf{x}^T, \Omega \cap \text{Ker } L, \mathbf{0}^T)) = \deg(\Phi(\mathbf{x}, 1), \Omega \cap \text{Ker } L, \mathbf{0}^T) = \text{sgn} \left| J_{\Phi}(\mathbf{x}^{*T}) \right| = -1.$$

Hence, we get that $\deg(JQN, \Omega \cap \text{Ker } L, 0) \neq 0$ and prove that (C3) is valid.

Therefore, the assertions on items (C₁)-(C₃) of the Theorem 5 are valid for the given operators. $\square$

4.2. Proof of Theorem 3

By Lemmata 7 and 8, we notice that the assumptions of the Theorem 5 are satisfied. Thus, there exist at least one solution of operator equation in Equation (12) belong $\text{Dom } L \cap \bar{\Omega} \subset X$, which implies the existence of at least one ω -periodic solution of the system in Equation (4).

5. Proof of Theorem 4

The proof of Theorem 4 is a consequence of Theorems 3 and 1. Indeed, from Theorem 3 we deduce that there exists at least one ω -periodic solution of Equation (4). Then, we get the proof of Theorem 4 by application of Theorem 1.

6. An Example

Let us consider that

$$\begin{aligned} b(t) &= 100 + \cos(\pi t), & p(t) &= \frac{2}{3} \left(1 + \frac{1}{2} \sin(\pi t) \right), \\ q(t) &= 1 - p(t), & \alpha(t) &= 1.1960e - 90 \sin^2 \left(\frac{\pi}{2} t \right), \\ \beta(t) &= \cos^2 \left(\frac{\pi}{2} t \right), & \phi(t) &= 1.1 + 1.0e - 10 \sin(\pi t), \\ \mu(t) &= 1.0e - 15(1 + \sin(\pi t)), & \gamma(t) &= \frac{1}{2} (1.1 + 1.0e - 10 \sin(\pi t)), \\ \chi(t) &= \frac{1}{4} (1.1 + 1.0e - 10 \sin(\pi t)), & \delta(t) &= \frac{3}{4} (1.1 + 1.0e - 10 \sin(\pi t)), \\ \epsilon(t) &= \frac{1.0e - 15}{4} (1 + \sin(\pi t)), & & \end{aligned} \quad (57)$$

which are 2-periodic functions. We notice that

$$(pb)^{\perp} - \phi^{\top} (\phi^{\perp})^{-1} \bar{b} \exp(2\omega\bar{\mu}) \approx 0.15,$$

$$1 - \frac{\varepsilon^{\top} \alpha^{\top} \bar{b}}{\mu^{\top} (\gamma + \varepsilon + \mu)^{\perp} (\alpha + \chi + \mu)^{\perp}} \exp\left(\omega\left[(\gamma + \varepsilon + \mu)^{\top} + \phi^{\top} (\phi^{\perp})^{-1} \bar{b} + \mu^{\top}\right]\right) \approx 0.857,$$

and we have that the hypothesis in Equation (15) is satisfied by selecting $\kappa_1 \in]0, 0.15[$ and $\kappa_2 \in]0, 0.857[$. Thus, by application of Theorem 4, we deduce that the system in Equation (2) with coefficients defined by Equation (57) has at least one positive 2-periodic solution.

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