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Oscillation Theorems for Advanced Differential Equations with p-Laplacian Like Operators

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Abstract: The main objective of this paper is to establish new oscillation results of solutions to a class of even-order advanced differential equations with a p-Laplacian like operator. The key idea of our approach is to use the Riccati transformation and the theory of comparison with first and second-order delay equations. Some examples are provided to illustrate the main results.

Keywords: oscillation; even-order; advanced differential equations; p-Laplacian equations

1. Introduction

We provide oscillation properties of even order advanced differential equation with a p-Laplacian like operator

$$\left(a\left(v\right)\left|y^{(\kappa-1)}\left(v\right)\right|^{p-2}y^{(\kappa-1)}\left(v\right)\right)'+\sum_{i=1}^{j}q_i\left(v\right)g\left(y\left(\eta_i\left(v\right)\right)\right)=0,\ \ v\geq v_0, \tag{1}$$

where $j\geq 1$, κ is even and $p>1$ is a real number. Throughout the paper, we assume that (1) satisfy the following assumptions.

L1: $a\in C^1\left(\left[v_0,\infty\right),\mathbb{R}\right), a\left(v\right)>0, a'\left(v\right)\geq 0,$

L2: $q_i,\eta_i\in C\left(\left[v_0,\infty\right),\mathbb{R}\right), q_i\left(v\right)\geq 0, \eta_i\left(v\right)\geq v, \lim_{v\to\infty}\eta_i\left(v\right)=\infty, i=1,2,..,j,$

L3: $g\in C\left(\mathbb{R},\mathbb{R}\right)$ such that $g\left(x\right)/\left|x\right|^{p-2}x\geq k>0$, for $x\neq 0$, k is a constant, and under the condition

$$\int_{v_0}^{\infty}\frac{1}{a^{1/(p-1)}\left(s\right)}\mathrm{d}s=\infty. \tag{2}$$

Definition 1. *The function $y\in C^{\kappa-1}[v_y,\infty)$, $v_y\geq v_0$, is called a solution of (1), if $\left|y^{(\kappa-1)}\left(v\right)\right|^{p-2}y^{(\kappa-1)}\left(v\right)\in C^1[v_y,\infty)$, for $a\in C^1\left(\left[v_0,\infty\right),\mathbb{R}\right)$, $a\left(v\right)>0$ and $y\left(v\right)$ satisfies (1) on $[v_y,\infty)$.*

The p-Laplace equations have some applications in continuum mechanics, see [1–3]. Advanced differential equations naturally appear in models concerning physical, biological, and chemical phenomena, mechanical, see [4], and in the mathematical modeling of engineering problems, such as electrical power systems, see [5], materials, see [6], and energy, see [7].

During this decade, Several works have been accomplished in the development of the oscillation theory of higher order advanced equations by using the Riccati transformation and the theory of comparison between first and second-order delay equations, see [8–19]. Further, the oscillation theory of fourth and second order equations has been studied and developed by using integral averaging technique and the Riccati transformation, see [20–27].

Our aim of this paper is complement and improve the results contained in [28–30]. For this purpose we discuss first these results.

Moaaz et al. [31] considered the fourth-order differential equation

$$\left(a\left(v\right)\left(y'''\left(v\right)\right)^{\beta}\right)'+q\left(v\right)y^{\alpha}\left(\eta\left(v\right)\right)=0,$$

where β, α are quotients of odd positive integers.

Grace et al. [32] considered the equation

$$\left(a\left(v\right)\left(y''\left(v\right)\right)^{\beta}\right)''+q\left(v\right)g\left(y\left(\eta\left(v\right)\right)\right)=0, \tag{3}$$

where $\eta\left(v\right)\leq v$, β is a quotient of odd positive integers.

Zhang et al. in [3] studied the qualitative behavior of the fourth-order differential equation

$$\left(a\left(v\right)\left(w'''\left(v\right)\right)^{\beta}\right)'+q\left(v\right)w\left(\eta\left(v\right)\right)=0,$$

where $\eta\left(v\right)\leq v$, β is a quotient of odd positive integers and they used the Riccati transformation.

Agarwal and Grace [28] considered the equation

$$\left(\left(y^{(\kappa-1)}\left(v\right)\right)^{\beta}\right)'+q\left(v\right)y^{\beta}\left(\eta\left(v\right)\right)=0, \tag{4}$$

where κ is an even and they established some new oscillation criteria by using the comparison technique. Among others, they proved it oscillatory if

$$\liminf_{v\to\infty}\int_{v}^{\eta(v)}\left(\eta\left(s\right)-s\right)^{\kappa-2}\left(\int_{\eta(v)}^{\infty}q\left(v\right)dv\right)^{1/\beta}ds>\frac{\left(\kappa-2\right)!}{e}. \tag{5}$$

Agarwal et al. in [29] extended the Riccati transformation to obtain new oscillatory criteria for ODE (4) under the condition

$$\limsup_{v\to\infty}v^{\beta(\kappa-1)}\int_{v}^{\infty}q\left(s\right)ds>\left(\left(\kappa-1\right)!\right)^{\beta}. \tag{6}$$

The authors in [30] studied oscillatory behavior of (4) where $\beta=1$ and if there exists a function $\tau\in C^{1}\left(\left[v_{0},\infty\right),\left(0,\infty\right)\right)$, also, they proved it oscillatory by using the Riccati transformation if

$$\int_{v}^{\infty}\left(\tau\left(s\right)q\left(s\right)-\frac{\left(\kappa-2\right)!\left(\tau'\left(s\right)\right)^{2}}{2^{3-2\kappa}s^{\kappa-2}\tau\left(s\right)}\right)ds=\infty. \tag{7}$$

The main aim of this paper is to establish new oscillation results of solutions to a class of even-order differential equations and they essentially complement and improve the results contained in [28–30].

The rest of the paper is organized as follows. In Section 2, three lemmas are given to prove the main results. In Section 3, we establish new oscillation results for (1), comparisons are carried out with oscillations of first and second-order delay differential equations and two examples are presented to illustrate the main results. Some conclusions are discussed in the last Section 4.

2. Some Auxiliary Lemmas

We shall employ the following lemmas which are found in the references [8,15,20,27] respectively:

Lemma 1. *If* $y^{(i)}(v) > 0$, $i = 0, 1, ..., \kappa$, *and* $y^{(\kappa+1)}(v) < 0$, *then*

$$\frac{y(v)}{v^{\kappa}/\kappa!} \geq \frac{y'(v)}{v^{\kappa-1}/(\kappa-1)!}.$$

Lemma 2. *Suppose that* $y \in C^{\kappa}([v_0, \infty), (0, \infty))$, $y^{(\kappa)}$ *is of a fixed sign on* $[v_0, \infty)$, $y^{(\kappa)}$ *not identically zero and there exists a* $v_1 \geq v_0$ *such that*

$$y^{(\kappa-1)}(v) y^{(\kappa)}(v) \leq 0,$$

for all $v \geq v_1$. *If we have* $\lim_{v\to\infty} y(v) \neq 0$, *then there exists* $v_\theta \geq v_1$ *such that*

$$y(v) \geq \frac{\theta}{(\kappa-1)!} v^{\kappa-1} \left| y^{(\kappa-1)}(v) \right|,$$

for every $\theta \in (0, 1)$ *and* $v \geq v_\theta$.

Lemma 3. *Suppose that* y *is an eventually positive solution of (1). Then, there exist two possible cases:*

$$\begin{array}{ll} (\mathbf{S}_1) & y(v) > 0,\ y'(v) > 0, y''(v) > 0,\ y^{(\kappa-1)}(v) > 0,\ y^{(\kappa)}(v) < 0, \\ (\mathbf{S}_2) & y(v) > 0, y^{(r)}(v) > 0, y^{(r+1)}(v) < 0 \text{ for all odd integer} \\ & r \in \{1, 3, ..., \kappa-3\}, y^{(\kappa-1)}(v) > 0, y^{(\kappa)}(v) < 0, \end{array}$$

for $v \geq v_1$, *where* $v_1 \geq v_0$ *is sufficiently large.*

Remark 1. *Let the differential equation*

$$\left[a(v)\left(y'(v)\right)^{\beta}\right]' + q(v) y^{\beta}(g(v)) = 0, \quad v \geq v_0, \tag{8}$$

where $\beta > 0$ *is the ratio of odd positive integers,* a , $q \in C([v_0, \infty), \mathbb{R}^+)$, *is nonoscillatory if and only if there exist a number* $v \geq v_0$, *and a function* $\varsigma \in C^1([v, \infty), \mathbb{R})$, *satisfying the inequality*

$$\varsigma'(v) + \gamma a^{-1/\beta}(v) (\varsigma(v))^{(1+\beta)/\beta} + q(v) \leq 0, \quad on\ [v, \infty).$$

3. Comparison Theorems with Second/First-Order Equations

Theorem 1. *Assume that (2) holds. If the differential equations*

$$\left(\frac{(\kappa-2)! a^{\frac{1}{(p-1)}}(v)}{(\theta v^{\kappa-2})^{p-1}} \left(y'(v)\right)^{p-1} \right)' + k \sum_{i=1}^{j} q_i(v) y^{p-1}(v) = 0 \tag{9}$$

for every $\theta \in (0, 1)$, *and*

$$y''(v) + y(v) \frac{1}{(\kappa-4)!} \int_v^{\infty} (\varsigma - v)^{\kappa-4} \left(\frac{1}{a(\varsigma)} \int_{\varsigma}^{\infty} \sum_{i=1}^{j} q_i(s)\, \mathrm{d}s \right)^{1/(p-1)} \mathrm{d}\varsigma = 0 \tag{10}$$

are oscillatory. Then, every solution of (1) is oscillatory.

Proof. On the contrary, assume that y is a positive solution of (1). Then, we can suppose that $y(v)$ and $y(\eta_i(v))$ are positive for all $v \geq v_1$ sufficiently large. From Lemma 3, we have two possible cases $(\mathbf{S}_1)$ and $(\mathbf{S}_2)$.

Let case $(\mathbf{S}_1)$ holds. Using Lemma 2, we find

$$y'(v) \geq \frac{\theta}{2} v^{\kappa-2} y^{(\kappa-1)}(v), \tag{11}$$

for every $\theta \in (0,1)$ and for all large v.

Define

$$\varphi(v) := \tau(v) \left(\frac{a(v) \left(y^{(\kappa-1)}(v) \right)^{p-1}}{y^{p-1}(v)} \right), \tag{12}$$

we see that $\varphi(v) > 0$ for $v \geq v_1$, where $\tau \in C^1([v_0, \infty), (0, \infty))$ and

$$\begin{aligned} \varphi'(v) &= \tau'(v) \frac{a(v) \left(y^{(\kappa-1)}(v) \right)^{p-1}}{y^{p-1}(v)} + \tau(v) \frac{\left(a \left(y^{(\kappa-1)} \right)^{p-1} \right)'(v)}{y^{p-1}(v)} \\ &\quad -\beta\tau(v) \frac{y^{p-2}(v) y'(v) a(v) \left(y^{(\kappa-1)}(v) \right)^{p-1}}{y^{2(p-1)}(v)}. \end{aligned}$$

Using (11) and (12), we obtain

$$\begin{aligned} \varphi'(v) &\leq \frac{\tau'_+(v)}{\tau(v)} \varphi(v) + \tau(v) \frac{\left(a(v) \left(y^{(\kappa-1)}(v) \right)^{p-1} \right)'}{y^{p-1}(v)} \\ &\quad - (p-1)\tau(v) \frac{\theta}{(\kappa-2)!} v^{\kappa-2} \frac{a(v) \left(y^{(\kappa-1)}(v) \right)^{p}}{y^{p}(v)} \\ &\leq \frac{\tau'(v)}{\tau(v)} \varphi(v) + \tau(v) \frac{\left(a(v) \left(y^{(\kappa-1)}(v) \right)^{p-1} \right)'}{y^{p-1}(v)} \\ &\quad - \frac{(p-1)\theta v^{\kappa-2}}{(\kappa-2)! \left(\tau(v) a(v) \right)^{\frac{1}{p-1}}} \varphi(v)^{\frac{p}{p-1}}. \end{aligned} \tag{13}$$

From (1) and (13), we obtain

$$\varphi'(v) \leq \frac{\tau'(v)}{\tau(v)} \varphi(v) - k\tau(v) \frac{\sum_{i=1}^{j} q_i(v) y^{p-1}(\eta_i(v))}{y^{p-1}(v)} - \frac{(p-1)\theta v^{\kappa-2}}{(\kappa-2)! \left(\tau(v) a(v) \right)^{\frac{1}{p-1}}} \varphi(v)^{\frac{p}{p-1}}.$$

Note that $y'(v) > 0$ and $\eta_i(v) \geq v$, thus, we find

$$\varphi'(v) \leq \frac{\tau'(v)}{\tau(v)} \varphi(v) - k\tau(v) \sum_{i=1}^{j} q_i(v) - \frac{(p-1)\theta v^{\kappa-2}}{(\kappa-2)! \left(\tau(v) a(v) \right)^{\frac{1}{p-1}}} \varphi(v)^{\frac{p}{p-1}}. \tag{14}$$

If we set $\tau(v) = k = 1$ in (14), then we find

$$\varphi'(v) + \frac{(p-1)\theta v^{\kappa-2}}{(\kappa-2)! \left(\tau(v) a(v) \right)^{\frac{1}{p-1}}} \varphi(v)^{\frac{p}{p-1}} + \sum_{i=1}^{j} q_i(v) \leq 0.$$

Therefore, we note that the Equation (9) is nonoscillatory, which is a contradiction.

Let case $(\mathbf{S}_2)$ holds. Define

$$\psi(v) := \vartheta(v) \frac{y'(v)}{y(v)},$$

we see that $\psi(v) > 0$ for $v \geq v_1$, where $\vartheta \in C^1([v_0, \infty), (0, \infty))$. By differentiating of $\psi(v)$, we find

$$\psi'(v) = \frac{\vartheta'(v)}{\vartheta(v)}\psi(v) + \vartheta(v)\frac{y''(v)}{y(v)} - \frac{1}{\vartheta(v)}\psi(v)^2. \tag{15}$$

Now, we integrating (1) from v to m and using $y'(v) > 0$, we find

$$a(m)\left(y^{(\kappa-1)}(m)\right)^{p-1} - a(v)\left(y^{(\kappa-1)}(v)\right)^{p-1} = -\int_v^m \sum_{i=1}^{j} q_i(s)\, g(y(\eta_i(s)))\, ds.$$

By virtue of $y'(v) > 0$ and $\eta_i(v) \geq v$, we get

$$a(m)\left(y^{(\kappa-1)}(m)\right)^{p-1} - a(v)\left(y^{(\kappa-1)}(v)\right)^{p-1} \leq -ky^{p-1}(v)\int_v^u \sum_{i=1}^{j} q_i(s)\, ds.$$

Letting $m \to \infty$, we see that

$$a(v)\left(y^{(\kappa-1)}(v)\right)^{p-1} \geq ky^{p-1}(v)\int_v^\infty \sum_{i=1}^{j} q_i(s)\, \mathrm{d}s$$

and so

$$y^{(\kappa-1)}(v) \geq y(v)\left(\frac{k}{a(v)}\int_v^\infty \sum_{i=1}^{j} q_i(s)\, \mathrm{d}s\right)^{1/(p-1)}.$$

Integrating again from v to ∞ for a total of $(\kappa - 4)$ times, we get

$$y''(v) + \frac{y(v)}{(\kappa-4)!}\int_v^\infty (\varsigma - v)^{\kappa-4}\left(\frac{k}{a(\varsigma)}\int_\varsigma^\infty \sum_{i=1}^{j} q_i(s)\, \mathrm{d}s\right)^{1/(p-1)} \mathrm{d}\varsigma \leq 0. \tag{16}$$

From (15) and (16), we obtain

$$\psi'(v) \leq \frac{\vartheta'(v)}{\vartheta(v)}\psi(v) - \frac{\vartheta(v)}{(\kappa-4)!}\int_v^\infty (\varsigma - v)^{\kappa-4}\left(\frac{k}{a(\varsigma)}\int_\varsigma^\infty \sum_{i=1}^{j} q_i(s)\, \mathrm{d}s\right)^{1/(p-1)} \mathrm{d}\varsigma - \frac{1}{\vartheta(v)}\psi(v)^2. \tag{17}$$

If we now set $\vartheta(v) = k = 1$ in (17), then we obtain

$$\psi'(v) + \psi^2(v) + \frac{1}{(\kappa-4)!}\int_v^\infty (\varsigma - v)^{\kappa-4}\left(\frac{1}{a(\varsigma)}\int_\varsigma^\infty \sum_{i=1}^{j} q_i(s)\, \mathrm{d}s\right)^{1/(p-1)} \mathrm{d}\varsigma \leq 0.$$

Hence, we see that the Equation (10) is nonoscillatory, which is a contradiction. Theorem 1 is proved. □

Remark 2. *It is well known (see [23]) that if*

$$\int_{v_0}^\infty \frac{1}{a(v)}\mathrm{d}v = \infty, \text{ and } \liminf_{v\to\infty}\left(\int_{v_0}^v \frac{1}{a(s)}\mathrm{d}s\right)\int_v^\infty q(s)\, \mathrm{d}s > \frac{1}{4},$$

then Equation (8) with $p = 2$ is oscillatory.

Based on the above results and Theorem 1, we can easily obtain the following Hille and Nehari type oscillation criteria for (1) with $\beta = 1$.

Theorem 2. *Let $p = 2$, $k = 1$. Assume that (2) holds. If*

$$\int_{v_0}^{\infty} \frac{\theta v^{\kappa-2}}{(\kappa-2)!a(v)} \mathrm{d}v = \infty$$

and

$$\liminf_{v\to\infty} \left(\int_{v_0}^{v} \frac{\theta s^{\kappa-2}}{(\kappa-2)!a(s)} \mathrm{d}s \right) \int_{v}^{\infty} \sum_{i=1}^{j} q_i(s)\, \mathrm{d}s > \frac{1}{4}, \tag{18}$$

also, if

$$\liminf_{v\to\infty} v \int_{v_0}^{v} \frac{1}{(\kappa-4)!} \int_{v}^{\infty} (\varsigma - v)^{\kappa-4} \left(\frac{1}{a(\varsigma)} \int_{\varsigma}^{\infty} \sum_{i=1}^{j} q_i(s)\, \mathrm{d}s \right) \mathrm{d}\varsigma \mathrm{d}v > \frac{1}{4}, \tag{19}$$

for all constant $\theta \in (0,1)$. Then all solutions of (1) are oscillatory.

In the theorem, we compare the oscillatory behavior of (1) with the first-order differential equations:

Theorem 3. *Assume that (2) holds. If the differential equations*

$$x'(v) + k \sum_{i=1}^{j} q_i(v) \left(\frac{\theta v^{\kappa-2}}{(\kappa-2)!a^{1/\beta}(v)} \right)^{p-1} x(\eta(v)) = 0 \tag{20}$$

and

$$z'(v) + z(v) \frac{v}{(\kappa-4)!} \int_{v}^{\infty} (\varsigma - v)^{\kappa-4} \left(\frac{k}{a(\varsigma)} \int_{\varsigma}^{\infty} \sum_{i=1}^{j} q_i(s)\, \mathrm{d}s \right)^{1/(p-1)} \mathrm{d}\varsigma = 0 \tag{21}$$

are oscillatory for all constant $\theta \in (0,1)$. Then all solutions of (1) are oscillatory.

Proof. On the contrary, assume that y is a positive solution of (1). Then, we can suppose that $y(v)$ and $y(\eta_i(v))$ are positive for all $v \geq v_1$ sufficiently large. From Lemma 3, we have two possible cases $(\mathbf{S}_1)$ and $(\mathbf{S}_2)$.

In the case where $(\mathbf{S}_1)$ holds, from Lemma 2, we see

$$y(v) \geq \frac{\theta v^{\kappa-2}}{(\kappa-2)!a^{1/(p-1)}(v)} \left(a^{1/(p-1)}(v)\, y^{(\kappa-1)}(v) \right),$$

for every $\theta \in (0,1)$ and for all large v. Thus, if we set

$$x(v) = a(v) \left(y^{(\kappa-1)}(v) \right)^{p-1} > 0,$$

then we see that ψ is a positive solution of the inequality

$$x'(v) + k \sum_{i=1}^{j} q_i(v) \left(\frac{\theta v^{\kappa-2}}{(\kappa-2)!a^{1/(p-1)}(v)} \right)^{p-1} x(\eta(v)) \leq 0. \tag{22}$$

From [24] (Theorem 1), we see that the Equation (20) also has a positive solution and it is a contradiction.

In the case where $(\mathbf{S}_2)$ holds. From Lemma 1, we get

$$y(v) \geq v y'(v), \tag{23}$$

From (16) and (23), we get

$$y''(v)+y'(v)\frac{v}{(\kappa-4)!}\int_v^\infty(\varsigma-v)^{\kappa-4}\left(\frac{k}{a(\varsigma)}\int_\varsigma^\infty\sum_{i=1}^{j}q_i(s)\,\mathrm{d}s\right)^{1/(p-1)}\mathrm{d}\varsigma\leq 0.$$

Now, we set

$$z(v)=y'(v).$$

Thus, we find ψ is a positive solution of the inequality

$$z'(v)+z(v)\frac{v}{(\kappa-4)!}\int_v^\infty(\varsigma-v)^{\kappa-4}\left(\frac{k}{a(\varsigma)}\int_\varsigma^\infty\sum_{i=1}^{j}q_i(s)\,\mathrm{d}s\right)^{1/(p-1)}\mathrm{d}\varsigma\leq 0. \tag{24}$$

It is well known (see [24] (Theorem 1)) that the Equation (21) also has a positive solution and it is a contradiction.

The proof is complete. □

Corollary 1. *Let (2) holds. If*

$$\liminf_{v\to\infty}\int_{\eta_i(v)}^{v}\sum_{i=1}^{j}q_i(s)\left(\frac{\theta v^{\kappa-2}}{(\kappa-2)!a^{1/(p-1)}(v)}\right)^{p-1}\mathrm{d}s>\frac{((\kappa-1)!)^{p-1}}{\mathrm{e}} \tag{25}$$

and

$$\liminf_{v\to\infty}\int_{\eta_i(v)}^{v}\frac{s}{(\kappa-4)!}\int_v^\infty(\varsigma-v)^{\kappa-4}\left(\frac{k}{a(\varsigma)}\int_\varsigma^\infty\sum_{i=1}^{j}q_i(s)\,\mathrm{d}s\right)^{1/(p-1)}\mathrm{d}\varsigma\mathrm{d}s>\frac{1}{\mathrm{e}}, \tag{26}$$

for all constant $\theta\in(0,1)$. Then all solutions of (1) are oscillatory.

Let us consider the differential equation

Example 1. *Let the equation*

$$y^{(4)}(v)+\frac{q_0}{v^4}y(3v)=0,\ v\geq 1, \tag{27}$$

where $q_0>0$ is a constant. Note that $p=2$, $\kappa=4$, $a(v)=1$, $q(v)=q_0/v^4$ and $\eta(v)=3v$. If we set $k=1$, then condition (18) becomes

$$\begin{aligned}\liminf_{v\to\infty}\left(\int_{v_0}^{v}\frac{\theta s^{\kappa-2}}{(\kappa-2)!a(s)}\mathrm{d}s\right)\int_v^\infty\sum_{i=1}^{j}q_i(s)\,\mathrm{d}s &= \liminf_{v\to\infty}\left(\frac{v^3}{3}\right)\int_v^\infty\frac{q_0}{s^4}ds\\ &= \frac{q_0}{9}>\frac{1}{4}\end{aligned}$$

and condition (19) becomes

$$\begin{aligned}\liminf_{v\to\infty}v\int_{v_0}^{v}\frac{1}{(\kappa-4)!}\int_v^\infty(\varsigma-v)^{\kappa-4}\left(\frac{1}{a(\varsigma)}\int_\varsigma^\infty\sum_{i=1}^{j}q_i(s)\,\mathrm{d}s\right)^{1/(p-1)}\mathrm{d}\varsigma\mathrm{d}v &= \liminf_{v\to\infty}v\left(\frac{q_0}{6v}\right)\\ &= \frac{q_0}{6}>\frac{1}{4}.\end{aligned}$$

Therefore, from Theorem 2, all the solutions of the equation (27) is oscillatory if $q_0>2.25$.

Remark 3. *We compare our result with the known related criteria for oscillation of this equation as follows:*

The condition	*(5)*	*(6)*	*(7)*	*our condition*
The criterion	$q_0 > 13.6$	$q_0 > 18$	$q_0 > 576$	$q_0 > 2.25$

Therefore, our result improve the results contained in [28–30].

Example 2. *Consider the differential equation*

$$y^{(4)}(v) + \frac{q_0}{v^4} y(2v) = 0, \ v \geq 1, \tag{28}$$

where $q_0 > 0$ is a constant. Note that $p = 2$, $\kappa = 4$, $a(v) = 1$, $q(v) = q_0/v^4$ and $\eta(v) = 2v$. If we set $k = 1$, then condition (18) becomes

$$\frac{q_0}{9} > \frac{1}{4}.$$

Therefore, from Theorem 2, all solution equation (28) is oscillatory if $q_0 > 2.25$.

Remark 4. *We compare our result with the known related criteria for oscillation of this equation as follows:*

The condition	*(5)*	*(6)*	*(7)*	*our condition*
The criterion	$q_0 > 25.5$	$q_0 > 18$	$q_0 > 1728$	$q_0 > 2.25$

Therefore, our result improve the results contained in [28–30].

4. Conclusions

In this article, we study the oscillatory behavior of a class of non-linear even-order differential equations with a p-Laplacian like operator and establish sufficient conditions for oscillation of a even-order differential equation by using the theory of comparison with first and second-order delay equations and Riccati substitution technique.

For researchers interested in this field, and as part of our future research, there is a nice open problem which is finding new results in the following case:

$$\int_{v_0}^{\infty} \frac{1}{a^{1/(p-1)}(s)} \mathrm{d}s < \infty. \tag{29}$$

For all this there is some research in progress.

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