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Article

On a Conjecture of Alzer, Berg, and Koumandos

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Abstract: In this paper, we find a solution of an open problem posed by Alzer, Berg, and Koumandos: determine $(\alpha, m) \in \mathbb{R}^+ \times \mathbb{N}$ such that the function $x^\alpha|\psi^{(m)}(x)|$ is completely monotonic on $(0, \infty)$, where $\psi(x)$ denotes the logarithmic derivative of Euler's gamma function.

Keywords: completely monotonic functions; inequality; polygamma function; digamma function

1. Introduction

Completely monotonic functions have attracted the attention of many authors. They play an important role in the mathematical analysis, statistics, physics and so on. For example, in the book ([1], p. 275), it can be found that Hanyga [2] showed that complete monotonicity is essential to ensure the monotone decay of the energy in isolated systems (as it appears reasonable from physical considerations); thus, restricting to completely monotonic functions is essential for the physical acceptability and realizability of the dielectric models. Next, in the paper [3], it was shown that, according to consequences of complete monotonicity properties of some functions involving the gamma function, authors established various new upper and lower bounds for the gamma function and the harmonic numbers. Monotonic functions have been studied very intensively by many researchers. A detailed list of references on completely monotonic functions can be found in [1,2,4–17].

We remind some useful definitions and theorems.

It is well known that the function $\psi(x) = \Gamma'(x)/\Gamma(x)$ is called as digamma or psi function, where $\Gamma(x)$ is the classical Euler's gamma function [6]. The following useful formula

$$\psi^{(m)}(x) = (-1)^{m+1}m!\sum_{j=0}^{\infty}\frac{1}{(x+j)^{m+1}} = (-1)^{m+1}\int_0^\infty \frac{t^m}{1-e^{-t}}e^{-xt}dt$$

is valid for $x > 0$.

Definition 1 ([14])**.** *We say that a function f is a completely monotonic on the interval I, if $f(x)$ has derivatives of all orders on I and the inequality $(-1)^n f^{(n)}(x) \geq 0$ holds for $x \in I$ and $n \in \mathbb{N}_0$.*

A characterization of completely monotonic function is given by the Bernstein–Widder theorem [17,18], which reads that a function $f(x)$ on $(0, \infty)$ is completely monotonic if and only if there exists bounded and non-decreasing function $\alpha(t)$ such that the integral

$$f(x) = \int_0^\infty e^{-xt}d\alpha(t)$$

converges for $x \in (0, \infty)$.

Definition 2 ([14]). *Let $h(t)$ be a completely monotonic function on $(0, \infty)$ and let $h(\infty) = \lim_{t\to\infty} h(t) \geq 0$. If the function $t^\alpha [h(t) - h(\infty)]$ is a completely monotonic on $(0, \infty)$ when and only when $0 \leq \alpha \leq r \in \mathbb{R}$, then we say $h(t)$ is of completely monotonic degree r; if $t^\alpha [h(t) - h(\infty)]$ is a completely monotonic on $(0, \infty)$ for all $\alpha \in \mathbb{R}$, then we say that the completely monotonic degree of $h(t)$ is ∞.*

In the paper [9], Guo designed a notation $deg^t_{cm} h(t)$ for denoting the completely monotonic degree r of $h(t)$ with respect to $t \in (0, \infty)$.

Recall that in the paper [4] Alzer and all disproved the following Conjecture 1 of Clark and Ismail [6]:

Conjecture 1 ([6]). *Let $\Phi_m(x) = -x^m \psi^{(m)}(x)$, where $\psi(x)$ denotes the logarithmic derivative of Euler's gamma function. Then, the function $\Phi_m^{(m)}(x)$ is completely monotonic on $(0, \infty)$ for each $m \in \mathbb{N}$.*

Clark and Ismail [6] showed that the function $\Phi_m^{(m)}(x)$ is completely monotonic for $m = 1, ..., 16$, and they conjectured that it is true for all $m \in \mathbb{N}$. Alzer and all proved [4] that there is $m^* \in \mathbb{N}$ such that if $m > m^*$ then $\Phi_m^{(m)}(x)$ is not completely monotonic on $(0, \infty)$. The proof of Alzer, Berg and Koumandos [4] was based on properties of new function

$$s(x) = \frac{1}{2} + \frac{1}{\pi} H\left(\frac{x}{2\pi}\right)$$

where

$$H(x) = \sum_{k=1}^{\infty} \frac{1}{k} \sin\left(\frac{x}{k}\right)$$

is the Hardy–Littlewood function [7,8,10,11,15] defined for $x \in \mathbb{C}$. Authors showed that the functions $\Phi_m^{(m)}(x)$ are all completely monotonic on $(0, \infty)$, $m \in \mathbb{N}$ if and only if $s(x) \geq 0$ for $x > 0$. In their proof, it was shown that, for each $K > 0$, there is $x_K > 0$ such that $H(x_K) < -K$. It implies Conjecture 1 is not valid. In the paper [13] Matejíčka showed that the result of Alzer is valid for function $x^m \beta^{(m)}(x)$, where $\beta(x)$ is the Nielsen β function and he also showed that the functions $x^{m-1}\psi^{(m)}(x)$ and $x^{m-1}\beta^{(m)}(x)$ are completely monotonic on $(0, \infty)$ for each $m \in \mathbb{N}$, $m > 2$. In the paper [12] it was shown that the function $x^m \left|\beta^{(m)}(x)\right|$ is completely monotonic on $(0, \infty)$ for $m = 1, 2, 3$. Recall that the Nielsen β function can be defined as

$$\begin{aligned} \beta(x) &= \int_0^\infty \frac{e^{-xt}}{1+e^{-t}} dt = \int_0^1 \frac{t^{x-1}}{1+t} dt = \sum_{m=0}^{\infty} \frac{(-1)^m}{m+x} = \\ &= \frac{1}{2}\left(\psi\left(\frac{1+x}{2}\right) - \psi\left(\frac{x}{2}\right)\right) \end{aligned}$$

for $x > 0$.

We believe that the information mentioned and the ideas used in the paper can give some directions for obtaining new results for the Nielsen β function.

Definition 3 ([16]). *A function f has exponential order α if there exist constants $M > 0$ and α such that, for some $t_0 \geq 0$,*

$$|f(t)| \leq Me^{\alpha t},\ t \geq t_0.$$

Definition 4 ([13]). *We say that*

$$\alpha^* = \inf\left\{\alpha;\ \textit{there are constants } M,\ t_0 \textit{ such that } |f(t)| \leq Me^{\alpha t},\ t \geq t_0\right\}$$

is a lower exponential order of function f.

In the paper [13], the following theorem was proved.

Theorem 1 ([13]). *Let $m \in \mathbb{N}$ and φ, φ', ... $\varphi^{(m)}$ be continuous functions of lower exponential orders L_0^*, L_1^*, ... L_m^*, respectively, on $(0, \infty)$. Let*

$L = \max\left\{L_0^*, L_1^*, ..., L_m^*\right\} \geq 0$; $F(x) = \int\limits_0^\infty \varphi(t)e^{-xt}dt \geq 0$ *for* $x > L$; $\varphi^{(m)}(t) \geq 0$ *on* $(0, \infty)$. *Let*

$$p_1 : \qquad \lim_{t\to+\infty} \varphi^{(k)}(t)\frac{\partial^{m-1-k}}{\partial t^{m-1-k}}\left(t^n e^{-xt}\right) = 0$$

and

$$p_2 : \qquad \lim_{t\to 0^+} \varphi^{(k)}(t)\frac{\partial^{m-1-k}}{\partial t^{m-1-k}}\left(t^n e^{-xt}\right) = 0$$

for $x > L$, $n \in \mathbb{N}$, $k = 0, ..., m-1$. Then, $x^m F(x)$ is a completely monotonic function on $(L, +\infty)$.

Remark 1 ([13]). *We note that, if the conditions*

$$\lim_{t\to 0^+} \varphi^{(k)}(t) = 0, \qquad \lim_{t\to\infty} \varphi^{(k)}(t)e^{-xt} = 0$$

are fulfilled for $k = 0, ..., m-1$, and $x > L$ where $m \in \mathbb{N}$, then p_1, p_2 are also valid.

In the paper ([4], p. 110), Alzer and all posed the following open problem:

Conjecture 2 ([4]). *Determine all $(\alpha, m) \in \mathbb{R}^+ \times \mathbb{N}$ such that the function $\Delta_{\alpha,m} = x^\alpha |\psi^{(m)}(x)|$ is completely monotonic on $(0, \infty)$.*

Our goal is to find the solution of the Conjecture 2.

2. Results

Lemma 1. *There is only one $m_\psi \in \mathbb{N}$, $m_\psi \geq 16$ such that $x^m \mid \psi^{(m)}(x) \mid$ is completely monotonic on $(0, \infty)$ for $m \leq m_\psi$, $m \in \mathbb{N}$, and is not completely monotonic on $(0, \infty)$ for $m \geq m_\psi + 1$, $m \in \mathbb{N}$.*

Proof. In the paper [13], it was presented that $t^m \mid \psi^{(m)}(t) \mid$ is completely monotonic for $t > 0$ if and only if $f_m(t) > 0$ for $t > 0$, where

$$f_m(t) = \frac{d^m}{dt^m}\left(\frac{t^m}{1-e^{-t}}\right).$$

Simple calculation gives

$$\begin{aligned} f_m(t) &= \frac{d^{m-1}}{dt^{m-1}}\left(\frac{t^{m-1}}{1-e^{-t}}\right)+\frac{d^{m-1}}{dt^{m-1}}\left(t\frac{d}{dt}\left(\frac{t^{m-1}}{1-e^{-t}}\right)\right) \\ &= 2\frac{d^{m-1}}{dt^{m-1}}\left(\frac{t^{m-1}}{1-e^{-t}}\right)+\frac{d^{m-2}}{dt^{m-2}}\left(t\frac{d^2}{dt^2}\left(\frac{t^{m-1}}{1-e^{-t}}\right)\right) \end{aligned}$$

Repeating the above procedure $m-1$ times, we obtain

$$f_m(t) = m\frac{d^{m-1}}{dt^{m-1}}\left(\frac{t^{m-1}}{1-e^{-t}}\right)+t\frac{d^m}{dt^m}\left(\frac{t^{m-1}}{1-e^{-t}}\right)=mf_{m-1}(t)+tf'_{m-1}(t) \tag{1}$$

for $t>0$ and $m>1$, $m\in\mathbb{N}$. Now, we show that, if $f_m(t)>0$ for $t>0$ and $m>1$, $m\in\mathbb{N}$, then $f_{m-1}(t)>0$ for $t>0$. Let $m>1$, $m\in\mathbb{N}$ and $f_m(t)>0$ for $t>0$. If $f_{m-1}(t_0)\le 0$ for some $t_0>0$, then (1) implies $f'_{m-1}(t_0)>0$, so there is $0<t_1<t_0$ such that $f_{m-1}(t_1)<0$. Put $t^*=\inf\{t_1>0;\ \textit{such that}\ f_{m-1}(t_1)<0\}$. Then, $t^*=0$ or $t^*>0$. If $t^*>0$, then we have again $f_{m-1}(t^*)\le 0$ and $f'_{m-1}(t^*)>0$. Thus, there is $0<t_2<t_1$ such that $f_{m-1}(t_2)<0$. This is a contradiction with a definition of t^*. Thus, $t^*=0$ and $f_{m-1}(t^*)\le 0$.

In the paper ([4], p. 108), the following formula

$$\frac{x^m}{1-e^{-x}}=x^{m-1}+\frac{x^m}{2}+\sum_{k=2}^{\infty}\frac{B_k}{k!}x^{k+m-1} \tag{2}$$

for $0<x<2\pi$, where B_k are Bernoulli numbers, was presented.

This implies $\lim\limits_{x\to 0^+} f_m(x)=m!/2$, so $\lim\limits_{x\to 0^+} f_{m-1}(x)=(m-1)!/2$. However, this is a contradiction with $f_{m-1}(0^+)\le 0$ if $m>1$.

We remind readers that Alzer and all [4] showed that there is $m^*\in\mathbb{N}$ such that $x^m\mid\psi^{(m)}(x)\mid$ is not completely monotonic on $(0,\infty)$ for all $m\ge m^*$, so $f_m(x)>0$ is not valid for all $x>0$ and each $m\ge m^*$. Put

$$m_\psi=\inf\{m\in\mathbb{N};\ \textit{such that}\ f_k(x)\ \textit{is not positive for all}\ x>0,\ \textit{if}\ k\ge m\}-1.$$

Clark and Ismail [6] proved that $x^m\mid\psi^{(m)}(x)\mid$ is completely monotonic on $(0,\infty)$ if $m=1,\ldots,16$. Thus, $m_\psi\ge 16$. It is easy to see that, if $k>m_\psi$, then $x^k\mid\psi^{(k)}(x)\mid$ is not completely monotonic on $(0,\infty)$ and, if $k\le m_\psi$, then $x^k\mid\psi^{(k)}(x)\mid$ is completely monotonic on $(0,\infty)$ according to $x^{m_\psi}\mid\psi^{(m_\psi)}(x)\mid$ is completely monotonic on $(0,\infty)$ and so $f_{m_\psi}(x)>0$ for all $x>0$. This implies $f_j(x)>0$ on $(0,\infty)$ for all $j<m_\psi$. Thus, $x^j\mid\psi^{(j)}(x)\mid$ is completely monotonic on $(0,\infty)$ for $j\le m_\psi$, which completes our proof. □

Lemma 2. *Let $m\in\mathbb{N}$, $m>m_\psi$, $0<\alpha\le 1$. Let*

$$\varphi(t)=\int_0^t\frac{1}{u^{1-\alpha}}\frac{(t-u)^m}{1-e^{-(t-u)}}du$$

for t>0. Then,

(a) $\lim\limits_{t\to 0^+}\varphi^{(k)}(t)=0\quad\textit{for}\quad k=0,\ldots,m-1,$

(b) $\lim\limits_{t\to+\infty}\varphi^{(k)}(t)e^{-xt}=0\quad\textit{for}\quad k=0,\ldots,m-1,\ x>0.$

Proof. Case (a). Using $\lim\limits_{t\to 0^+} t^m/(1-e^{-t}) = 0$ for $m \geq 2$ and integration by parts gives

$$\varphi(t) = \int_0^t \frac{1}{u^{1-\alpha}} \frac{(t-u)^m}{1-e^{-(t-u)}} du = \frac{1}{\alpha} \int_0^t u^\alpha \frac{d}{dt} \left(\frac{(t-u)^m}{1-e^{-(t-u)}} \right) du.$$

Applying the Formula (2) leads to

$$\lim_{x\to 0^+} \left(\frac{x^m}{1-e^{-x}} \right)^{(k)} = \begin{cases} 0 & \text{if} \quad 0 \leq k \leq m-2 \\ (m-1)! & \text{if} \quad k = m-1 \\ \frac{m!}{2} & \text{if} \quad k = m. \end{cases} \tag{3}$$

This implies that, for each $m > m_\psi$ and each $\varepsilon > 0$, there is $\delta_m > 0$ such that

$$\left| \left(\frac{x^m}{1-e^{-x}} \right)^{(k)} \right| \leq \frac{m!}{2} + \varepsilon \tag{4}$$

for $0 < x < \delta_m$. Thus, for $0 < u \leq t < \delta_m$, we have

$$\left| \left(\frac{(t-u)^m}{1-e^{-(t-u)}} \right)^{(k)} \right| \leq \frac{m!}{2} + \varepsilon.$$

Using mathematical induction leads to

$$\varphi^{(k)}(t) = \frac{1}{\alpha} \int_0^t u^\alpha \frac{d^{k+1}}{dt^{k+1}} \left(\frac{(t-u)^m}{1-e^{-(t-u)}} \right) du$$

for $k = 0, ..., m-2$, $t > 0$,

$$\varphi^{(m-1)}(t) = \frac{1}{\alpha} \left(t^\alpha (m-1)! + \int_0^t u^\alpha \frac{d^m}{dt^m} \left(\frac{(t-u)^m}{1-e^{-(t-u)}} \right) du \right).$$

If $0 < t < \delta_m$, then

$$\left| \varphi^{(k)}(t) \right| \leq \frac{1}{\alpha} t^{1+\alpha} \left(\frac{m!}{2} + \varepsilon \right)$$

for $k = 0, ..., m-2$ and

$$\left| \varphi^{(m-1)}(t) \right| \leq \frac{1}{\alpha} t^\alpha \left((m-1)! + t \left(\frac{m!}{2} + \varepsilon \right) \right).$$

Thus, $\lim\limits_{t\to 0^+} \varphi^{(k)}(t) = 0$ for $k = 0, ..., m-1$, $0 < \alpha \leq 1$.

Case (b). Denote again

$$f_m(s) = \frac{d^m}{ds^m} \left(\frac{s^m}{1-e^s} \right)$$

for $s > 0$ and $m \in \mathbb{N}$, $m > 2$. It is obvious that

$$\varphi^{(m-1)}(t) = \frac{1}{\alpha}\left(t^{\alpha}(m-1)! + \int_{0}^{t} u^{\alpha} f_m(t-u)du\right) = \frac{1}{\alpha}\left(t^{\alpha}(m-1)! + \int_{0}^{t}(t-s)^{\alpha} f_m(s)ds\right).$$

Considering that $f_m(0) = m!/2$, we obtain, for each $\varepsilon > 0$, the inequality

$$f_m(s) \le \frac{m!}{2} + \varepsilon$$

is valid for $0 < s < \delta_m$. Thus,

$$\varphi^{(m-1)}(t) = \frac{1}{\alpha}\left(t^{\alpha}(m-1)! + \int_{0}^{\delta_m}(t-s)^{\alpha} f_m(s)ds + \int_{\delta_m}^{t}(t-s)^{\alpha} f_m(s)ds\right)$$

if $t > \delta_m$. Alzer and all presented ([4], p. 112) that

$$|f_m(x)| < m!\frac{e^{-\frac{x}{2}}}{1-e^{-\frac{x}{2}}} \tag{5}$$

for $x > 0$. It is easy to show that $g(t) = e^{-t}/(1-e^{-t})$ is a decreasing function on $(0,\infty)$. Thus,

$$|f_m(s)| < m!\frac{e^{-\frac{\delta_m}{2}}}{1-e^{-\frac{\delta_m}{2}}}$$

for $s > \delta_m$. This implies

$$\left|\varphi^{(m-1)}(t)\right| \le \frac{1}{\alpha}\left(t^{\alpha}(m-1)! + \frac{1}{1+\alpha}\left(\frac{m!}{2}+\varepsilon\right)\left(t^{\alpha+1} - (t-\delta_m)^{1+\alpha}\right) + \frac{1}{1+\alpha}(t-\delta_m)^{1+\alpha}\frac{m!e^{-\frac{\delta_m}{2}}}{1-e^{-\frac{\delta_m}{2}}}\right).$$

for $t > \delta_m$. Thus, we observe that $\lim\limits_{t\to+\infty} \varphi^{(m-1)}(t)e^{-xt} = 0$ for $x > 0$. Next, it is evident that

$$\varphi^{(k)}(t) = \frac{1}{\alpha}\int_{0}^{t} u^{\alpha}\frac{d^{k+1}}{dt^{k+1}}\left(\frac{(t-u)^m}{1-e^{-(t-u)}}\right)du$$

for $k = 0, ..., m-2$ and $t > 0$. Putting $s = t-u$ yields

$$\varphi^{(k)}(t) = \frac{1}{\alpha}\int_{0}^{t}(t-s)^{\alpha}\frac{d^{k+1}}{ds^{k+1}}\left(\frac{s^m}{1-e^{-s}}\right)ds.$$

By mathematical induction, it is easy to show that

$$\varphi^{(m-k)}(t) = \frac{1}{\alpha(\alpha+1)...(\alpha+k-1)} \left\{ (m-1)! t^{\alpha+k-1} + \int_0^t (t-s)^{\alpha+k-1} f_m(s) ds \right\}$$

for $k = 2, ..., m$ and $t > 0$. Using the Formula (3), the inequality (5) and the similar way of estimation for $\varphi^{(m-1)}(t)$ yields to $\lim\limits_{t\to+\infty} \varphi^{(k)}(t) e^{-xt} = 0$ for $x > 0$, $k = 0, ..., m-2$. The proof is complete. □

Theorem 2. *Let $m \in \mathbb{N}$ such that $m > m_\psi$. Let*

$$f_m(t) = \left(\frac{t^m}{1-e^{-t}} \right)^{(m)}$$

for $t > 0$. Let

$$c_m = \inf \left\{ 0 < c \leq 1; \frac{(m-1)!}{t^{1-c}} + \int_0^t \frac{1}{u^{1-c}} f_m(t-u) du > 0 \quad for \quad t > 0 \right\}.$$

Then, the function $\Delta_{\alpha,m}(x) = x^\alpha \mid \psi^{(m)}(x) \mid$ is completely monotonic on $(0, \infty)$ if and only if $0 < \alpha \leq m - c_m$.

Proof. Using the well known formulas

$$\frac{1}{x^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-xt} dt = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{1}{t^{1-\alpha}} e^{-xt} dt$$

for $0 < \alpha \leq 1$, $x > 0$ and

$$\left| \psi^{(m)}(x) \right| = \int_0^\infty \frac{t^m}{1-e^{-t}} e^{-xt} dt$$

for $x > 0$ reveals

$$x^{-\alpha} \left| \psi^{(m)}(x) \right| = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{1}{t^{1-\alpha}} e^{-xt} dt \int_0^\infty \frac{t^m}{1-e^{-t}} e^{-xt} dt.$$

Applying the convolution theorem leads to

$$x^{-\alpha} \left| \psi^{(m)}(x) \right| = \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(\int_0^t \frac{1}{(t-u)^{1-\alpha}} \frac{u^m}{1-e^{-u}} du \right) e^{-xt} dt.$$

Thus,

$$x^{m-\alpha} \left| \psi^{(m)}(x) \right| = x^m \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(\int_0^t \frac{1}{(t-u)^{1-\alpha}} \frac{u^m}{1-e^{-u}} du \right) e^{-xt} dt = \frac{1}{\Gamma(\alpha)} x^m \int_0^t \varphi(t) e^{-xt} dt,$$

where

$$\varphi(t) = \int_0^t \frac{1}{(t-u)^{1-\alpha}} \frac{u^m}{1-e^{-u}} du.$$

If the function $\varphi(t)$ fulfills the conditions p_1, p_2 of the Theorem 1, then the function $x^{m-\alpha} \left|\psi^{(m)}(x)\right|$ is completely monotonic if and only if $\varphi^{(m)}(t) > 0$ for $t > 0$. The conditions p_1, p_2 are fulfilled according to Lemmas 1 and 2. The inequality $\varphi^{(m)}(t) > 0$ for $t > 0$ is equivalent to

$$\gamma(t) = \frac{d^m}{dt^m} \int_0^t \frac{1}{u^{1-\alpha}} \frac{(t-u)^m}{1-e^{-(t-u)}} du > 0.$$

Using the Formula (2) gives

$$\begin{aligned}
\gamma(t) &= \frac{d}{dt} \int_0^t \frac{1}{u^{1-\alpha}} \frac{d^{m-1}}{dt^{m-1}} \left(\frac{(t-u)^m}{1-e^{-(t-u)}} \right) du \\
&= \frac{(m-1)!}{t^{1-\alpha}} + \int_0^t \frac{1}{u^{1-\alpha}} \frac{d^m}{dt^m} \left(\frac{(t-u)^m}{1-e^{-(t-u)}} \right) du \\
&= \frac{(m-1)!}{t^{1-\alpha}} + \int_0^t \frac{1}{u^{1-\alpha}} f_m(t-u) du > 0.
\end{aligned}$$

This completes the proof. □

3. Materials and Methods

In this paper, we used methods of mathematical analysis.

4. Conclusions

In this paper, we found conditions for $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}^+$ such that the function $\Upsilon_m(x) = x^\alpha \mid \beta^{(m)}(x) \mid$ is completely monotonic on $(0, \infty)$.

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