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Strong Convergence Theorems for Generalized Split Feasibility Problems in Banach Spaces

Xuejiao Zi ¹, Zhaoli Ma ^{2,3} and Wei-Shih Du ^{4,*}

¹ Department of General Education, The College of Arts and Sciences, Yunnan Normal University, Kunming 650222, Yunnan, China; zxj89530@aliyun.com

² Division of Research Administration, The College of Arts and Sciences, Yunnan Normal University, Kunming 650222, Yunnan, China; kmszmzl@126.com

³ College of Public Foundation, Yunnan Open University (Yunnan Technical College of National Defence Industry), Kunming 650500, China

⁴ Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 82444, Taiwan

* Correspondence: wsdu@mail.nknu.edu.tw

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Abstract: In this paper, we establish new strong convergence theorems of proposed algorithms under suitable new conditions for the generalized split feasibility problem in Banach spaces. As applications, new strong convergence theorems for equilibrium problems, fixed point problems and split common fixed point problems are also studied. Our new results are distinct from recent results on the topic in the literature.

Keywords: strong convergence; generalized split feasibility problem; fixed point problem; equilibrium problem; split common fixed point problem

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1. Introduction

Let \mathcal{C} and \mathcal{Q} be nonempty closed and convex subsets of finite-dimensional Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 , respectively. The mathematical model about the split feasibility problem (SFP, in short), originally put forward by Censor and Elfving [1], was introduced as follows:

$$(SFP) \quad \text{Find a point } b^* \in \mathcal{C} \text{ such that } \mathcal{A}b^* \in \mathcal{Q},$$

where $\mathcal{A} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a bounded linear operator. The solution set of (SFP) for \mathcal{A} is denoted by $SFP(\mathcal{A})$, i.e., $SFP(\mathcal{A}) := \{b^* \in \mathcal{C} : \mathcal{A}b^* \in \mathcal{Q}\}$.

In fact, the split feasibility problem originated from modeling and inverse problems, phase retrievals and in medical image reconstruction [2]. In the past more than two decades, the split feasibility problem has been widely studied by many authors and has been applied in different disciplines, including radiation therapy treatment planning, signal processing, image restoration, computer tomography, and so forth. For details, see, e.g., [3–5] and the reference therein. Based on the idea of split feasibility problem, split variational inclusion problem, split common null point problem, split common fixed problem, split equilibrium problem, split equality problem and so on were introduced by many authors and some iteration algorithms for the approximation of solutions of these problems were established in Banach spaces or Hilbert spaces (see, e.g., [6–15] and the reference therein).

In 2014, Takahashi, Xu, and Yao [16] investigated the following generalized split feasibility problem (GSFP, in short) in Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 :

$$(GSFP) \quad \text{Find a point } b^* \in \mathcal{H}_1, \text{ such that } 0 \in \mathcal{B}b^* \text{ and } \mathcal{A}b^* \in F(\mathcal{T}),$$

where $\mathcal{B} : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$ is a maximal monotone operator, $\mathcal{A} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a bounded linear operator and $\mathcal{T} : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ is a nonexpansive mapping. We use Ω to denote the solution set of (GSFP), i.e., $\Omega := \{b^* \in \mathcal{B}^{-1}0 : \mathcal{A}b^* \in F(\mathcal{T})\}$. The algorithm shown below was established to solve (GSFP) and a weak convergence theorem was obtained under suitable control conditions as follows: for any $b_1 \in \mathcal{H}_1$,

$$b_{n+1} = \mathcal{J}_{\lambda_n}^{\mathcal{B}}(\mathcal{I} - \gamma_n \mathcal{A}^*(\mathcal{I} - \mathcal{T})\mathcal{A})b_n \quad \text{for all } n \in \mathbb{N},$$

where $\mathcal{J}_{\lambda_n}^{\mathcal{B}}$ is the resolvent operator of \mathcal{B} , \mathcal{A}^* is the adjoint of \mathcal{A} . The research on (GSFP) has extended from Hilbert spaces to Banach spaces, see, e.g., [12,17] and the reference therein.

In reality, strong convergence results are more useful and easily applied than the weak convergence results in many practical applications. Motivated by that reason, in this paper, we establish new strong convergence theorems of proposed algorithms under suitable new conditions for (GSFP) in Banach spaces. Our results established in Section 3 can be applied to study for equilibrium problems, fixed point problems and split common fixed point problems. These new results in this paper are distinct from recent results on the topic in the literature.

2. Preliminaries

Let \mathcal{E} be a real Banach space with the dual space \mathcal{E}^* . \mathcal{E} is said to be strictly convex if $\frac{\|b+e\|}{2} \leq 1$ for all $b, e \in U := \{z \in \mathcal{E} : \|z\| = 1\}$ with $b \neq e$. The modulus of convexity of \mathcal{E} is defined as

$$\delta_{\mathcal{E}}(\epsilon) = \inf \left\{ 1 - \frac{1}{2}\|b+e\| : \|b\| \leq 1, \|e\| \leq 1, \|b-e\| \geq \epsilon \right\}$$

for all $\epsilon \in [0, 2]$. \mathcal{E} is said to be uniformly convex if $\delta_{\mathcal{E}}(0) = 0$ and $\delta_{\mathcal{E}}(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. Let p be a real number with $p \geq 2$. \mathcal{E} is called p -uniformly convex if there exists a constant $\lambda > 0$ such that $\delta_{\mathcal{E}}(\epsilon) \geq \lambda\epsilon^p$ for all $\epsilon > 0$.

The function $\rho_{\mathcal{E}} : [0, \infty) \rightarrow [0, \infty)$ is the modulus of smoothness of \mathcal{E} and is defined as

$$\rho_{\mathcal{E}}(t) = \sup \left\{ \frac{1}{2}(\|b+e\| + \|b-e\|) - 1 : b \in U, \|e\| \leq t \right\}.$$

\mathcal{E} is called to be uniformly smooth if $\frac{\rho_{\mathcal{E}}(t)}{t} \rightarrow 0$ as $t \rightarrow 0$. Let $1 < q \leq 2$. \mathcal{E} is called q -uniformly smooth if there exists a constant $c > 0$ such that $\rho_{\mathcal{E}}(t) \leq ct^q$ for all $t > 0$. It is generally known that every q -uniformly smooth Banach space is uniformly smooth.

The normalized duality mapping \mathcal{J} from \mathcal{E} to $2^{\mathcal{E}^*}$ is defined as

$$\mathcal{J}(b) = \{b^* \in \mathcal{E}^* : \langle b, b^* \rangle = \|b\|^2 = \|b^*\|^2\} \quad \forall b \in \mathcal{E},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between \mathcal{E} and \mathcal{E}^* .

As is known to all, if \mathcal{E} is uniformly smooth Banach spaces, then \mathcal{J} is uniformly norm-to-norm continuous on each bounded subset of \mathcal{E} .

Let \mathcal{E} be a smooth, reflexive and strictly convex Banach space. Consider the functional ψ [18,19] defined by

$$\psi(b, e) = \|b\|^2 - 2\langle b, \mathcal{J}e \rangle + \|e\|^2 \quad \text{for all } b, e \in \mathcal{E},$$

where \mathcal{J} is a normalized duality mapping. By the definition of ψ , we know that

$$(\|b\| - \|e\|)^2 \leq \psi(b, e) \leq (\|b\| + \|e\|)^2 \quad \text{for all } b, e \in \mathcal{E}.$$

From Alber [18], the generalized projection $\Pi_{\mathcal{C}} : \mathcal{E} \rightarrow \mathcal{C}$ is defined by

$$\Pi_{\mathcal{C}}(b) = \arg \min_{e \in \mathcal{C}} \psi(e, b) \quad \text{for all } b \in \mathcal{E}.$$

That is, $\Pi_{\mathcal{C}}b = \bar{b}$, where \bar{b} is the unique solution to the minimization problem $\psi(\bar{b}, b) = \inf_{e \in \mathcal{C}} \psi(e, b)$.

The following useful existence and uniqueness results for the operator $\Pi_{\mathcal{C}}$ can follow from the properties of the functional ψ and strict monotonicity of the mapping \mathcal{J} (see, e.g., [16,18–20]).

Lemma 1 (see [19]). *Let \mathcal{E} be a smooth, strictly convex and reflexive Banach space and \mathcal{C} be a nonempty closed convex subset of \mathcal{E} . Then the following conclusions hold:*

- (A) $\psi(b, \Pi_{\mathcal{C}}e) + \psi(\Pi_{\mathcal{C}}e, e) \leq \psi(b, e)$ for all $b \in \mathcal{C}$ and $e \in \mathcal{E}$;
- (B) If $b \in \mathcal{E}$ and $v \in \mathcal{C}$, then $v = \Pi_{\mathcal{C}}b$ if and only if $\langle v - e, \mathcal{J}b - \mathcal{J}v \rangle \geq 0$ for all $e \in \mathcal{C}$;
- (C) For $b, e \in \mathcal{E}$, $\psi(b, e) = 0$ if and only if $b = e$;
- (D) For $b, e \in \mathcal{E}$, $\lambda \in [0, 1]$, $\psi(b, \mathcal{J}^{-1}(\lambda \mathcal{J}e + (1 - \lambda)\mathcal{J}v)) \leq \lambda \psi(b, e) + (1 - \lambda)\psi(b, v)$ for all $b, e \in \mathcal{E}$.

Assume \mathcal{E} be a reflexive, strictly convex and smooth Banach space. The duality mapping \mathcal{J}^* from \mathcal{E}^* onto $\mathcal{E}^{**} = \mathcal{E}$ coincides with the inverse of the duality mapping \mathcal{J} from \mathcal{E} onto \mathcal{E}^* , i.e., $\mathcal{J}^* = \mathcal{J}^{-1}$.

We will use the following mapping $V : \mathcal{E} \times \mathcal{E}^* \rightarrow \mathbb{R}$, introduced in [18], to prove our main result:

$$V(b, b^*) = \|b\|^2 - 2\langle b, b^* \rangle + \|b^*\|^2$$

for all $b \in \mathcal{E}$ and $b^* \in \mathcal{E}^*$. Obviously, $V(b, b^*) = \psi(b, \mathcal{J}^{-1}(b^*))$ for all $b \in \mathcal{E}$ and $b^* \in \mathcal{E}^*$.

Lemma 2 (see [18]). *Let \mathcal{E} be a reflexive, strictly convex and smooth Banach space. Then*

$$V(b, b^*) + 2\langle \mathcal{J}^{-1}(b^*) - b, e^* \rangle \leq V(b, b^* + e^*)$$

for all $b \in \mathcal{E}$ and $b^*, e^* \in \mathcal{E}^*$.

In what follows, the symbols \rightharpoonup and \rightarrow will symbolize weak convergence and strong convergence as usual, respectively. The symbols \mathbb{N} and \mathbb{R} are used to denote the sets of positive integers and real numbers, respectively. Let \mathcal{E} be a smooth Banach space, \mathcal{C} be a nonempty closed convex subset of \mathcal{E} , and let \mathcal{T} be a mapping from \mathcal{C} into itself. We use $F(\mathcal{T})$ to denote the set of all fixed points of the mapping \mathcal{T} . A point $p \in \mathcal{C}$ is called an asymptotically fixed point of \mathcal{T} [21] if there exists a sequence $\{b_n\} \subset \mathcal{C}$ such that $b_n \rightarrow p$ and $\|b_n - \mathcal{T}b_n\| \rightarrow 0$. We will use $\hat{F}(\mathcal{T})$ denote the set of asymptotical fixed points of \mathcal{T} .

Definition 1. *A mapping $\mathcal{T} : \mathcal{C} \rightarrow \mathcal{C}$ is called*

- (i) τ -quasi-strictly pseudocontractive, if $F(\mathcal{T}) \neq \emptyset$ and there exists a constant $\tau \in [0, 1]$, such that

$$\|\mathcal{T}b - p\|^2 \leq \|b - p\|^2 + \tau\|b - \mathcal{T}b\|^2 \quad \text{for all } b \in \mathcal{C} \text{ and } p \in F(\mathcal{T});$$

- (ii) relatively nonexpansive [22], if $F(\mathcal{T}) \neq \emptyset$, $F(\mathcal{T}) = \hat{F}(\mathcal{T})$ and $\psi(p, \mathcal{T}b) \leq \psi(p, b)$ for all $b \in \mathcal{C}$ and $p \in F(\mathcal{T})$;
- (iii) strongly relatively nonexpansive [23], if \mathcal{T} is relatively nonexpansive and $\psi(\mathcal{T}b_n, b_n) \rightarrow 0$ whenever $\{b_n\}$ is bounded sequence in \mathcal{C} with $\psi(p, b_n) - \psi(p, \mathcal{T}b_n) \rightarrow 0$ for some $p \in F(\mathcal{T})$.

Recently, the class of firmly nonexpansive type mappings have been introduced by Kohsaka and Takahashi [24] in Banach spaces. Let \mathcal{C} be a nonempty closed convex subset of a smooth Banach space \mathcal{E} , and let \mathcal{T} be a mapping from \mathcal{C} into itself. Then \mathcal{T} is said to be *firmly nonexpansive type* if

$$\psi(\mathcal{T}b, \mathcal{T}e) + \psi(\mathcal{T}e, \mathcal{T}b) + \psi(\mathcal{T}b, b) + \psi(\mathcal{T}e, e) \leq \psi(\mathcal{T}b, e) + \psi(\mathcal{T}e, b)$$

for all $b, e \in \mathcal{C}$. It is easy to see that if \mathcal{T} is firmly nonexpansive type with $\mathcal{I} - \mathcal{T}$ is demi-closed at zero, then it is strongly relatively nonexpansive.

It is well know that if \mathcal{C} is a nonempty closed convex subset of smooth, strictly convex and reflexive \mathcal{E} and $\mathcal{A} \subset \mathcal{E} \times \mathcal{E}^*$ is a monotone operator such that $D(\mathcal{A}) \subset \mathcal{C} \subset \mathcal{J}^{-1}R(\mathcal{J} + r\mathcal{A})$ for all $r > 0$, then for each $r > 0$, the resolvent $Q_r^{\mathcal{A}}$ of \mathcal{A} which is defined by $Q_r^{\mathcal{A}}b = (\mathcal{J} + r\mathcal{A})^{-1}\mathcal{J}b$ for all $b \in \mathcal{C}$ is firmly nonexpansive type mapping. In particular, if $\mathcal{A} \subset \mathcal{E} \times \mathcal{E}^*$ is maximal monotone operator, then $R(\mathcal{J} + r\mathcal{A}) = \mathcal{E}^*$ for all $r > 0$, see [25]. In this case, the resolvent $Q_r^{\mathcal{A}}$ of \mathcal{A} is a firmly nonexpansive type mapping from \mathcal{E} into itself [26,27] and $\mathcal{A}^{-1}0$ is closed and convex and $F(Q_r^{\mathcal{A}}) = \mathcal{A}^{-1}0$.

Definition 2. A mapping $\mathcal{T} : \mathcal{C} \rightarrow \mathcal{C}$ is called demiclosed at zero if for any sequence $\{b_n\} \subset \mathcal{C}$ with $b_n \rightarrow b \in \mathcal{C}$ and $\|b_n - \mathcal{T}b_n\| \rightarrow 0$ as $n \rightarrow \infty$, then $\mathcal{T}b = b$.

The following known results are very crucial in our proofs.

Lemma 3 (see [27]). Let \mathcal{E} be a uniformly convex and smooth Banach space, $\{b_n\}$ and $\{e_n\}$ be two sequences of \mathcal{E} . If $\lim_{n \rightarrow \infty} \psi(b_n, e_n) = 0$ and either $\{b_n\}$ or $\{e_n\}$ is bounded, then $\lim_{n \rightarrow \infty} \|b_n - e_n\| = 0$.

Lemma 4 (see [28]). Let \mathcal{C} be a nonempty closed convex subset of a real Banach space \mathcal{E} and let $\mathcal{T} : \mathcal{C} \rightarrow \mathcal{C}$ be a τ -quasi-strictly pseudocontractive mapping. If $F(\mathcal{T}) \neq \emptyset$, then $F(\mathcal{T})$ is closed and convex.

Lemma 5 (see [29]). If \mathcal{E} be a 2-uniformly smooth Banach space, then for each $k > 0$ and each $b, e \in \mathcal{E}$:

$$\|b + e\|^2 \leq \|b\|^2 + 2\langle e, \mathcal{T}b \rangle + 2\|ke\|^2.$$

Lemma 6 (see [30]). Let $\{l_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$l_{n+1} \leq (1 - \rho_n)l_n + \rho_n\sigma_n + \omega_n, \quad n \geq 1,$$

where (i) $\{\rho_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \rho_n = \infty$; (ii) $\limsup_{n \rightarrow \infty} \sigma_n < 0$; (iii) $\omega_n \geq 0$, $\sum_{n=1}^{\infty} \omega_n < \infty$. Then $l_n \rightarrow 0$ as $n \rightarrow \infty$.

Lemma 7 (see [31]). Let $\{l_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ satisfying $l_{n_i} < l_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \rightarrow \infty$ and the following properties are satisfied for all (sufficiently large) numbers $k \in \mathbb{N}$:

$$l_{m_k} \leq l_{m_k+1} \quad \text{and} \quad l_k \leq l_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : l_j < l_{j+1}\}$.

3. Main Results

In this section, we first establish a new strong convergence iterative algorithm for the generalized split feasibility problem.

Theorem 1. Let \mathcal{E}_1 and \mathcal{E}_2 be 2-uniformly convex and 2-uniformly smooth real Banach spaces with smoothness constant k satisfying $0 < k \leq \frac{1}{\sqrt{2}}$. Let $\mathcal{B} \subset \mathcal{E}_1 \times \mathcal{E}_1^*$ be a maximal monotone operator with $\mathcal{B}^{-1}0 \neq \emptyset$.

Let $Q_r^{\mathcal{B}} = (\mathcal{J}_1 + r\mathcal{B})^{-1}\mathcal{J}_1$ be the resolvent of \mathcal{B} . Let $\mathcal{T} : \mathcal{E}_2 \rightarrow \mathcal{E}_2$ be a τ -quasi-strict pseudocontractive mapping such that $F(\mathcal{T}) \neq \emptyset$, and \mathcal{T} be demiclosed at zero, $\mathcal{A} : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ be a bounded linear operator. Let $\{\alpha_n\}$ be a sequence in $(0,1)$. For any $b_1 = b \in \mathcal{E}_1$ and a fixed $u \in \mathcal{E}_1$, let $\{b_n\}$ be a sequence defined by

$$\begin{cases} e_n = \mathcal{J}_1^{-1}(\mathcal{J}_1 b_n - \gamma \mathcal{A}^* \mathcal{J}_2(\mathcal{I} - \mathcal{T})\mathcal{A} b_n), \\ b_{n+1} = \mathcal{J}_1^{-1}[\alpha_n \mathcal{J}_1 u + (1 - \alpha_n) \mathcal{J}_1 Q_r^{\mathcal{B}} e_n], \end{cases} \tag{1}$$

where \mathcal{J}_1 and \mathcal{J}_2 are the normalized duality mappings of \mathcal{E}_1 and \mathcal{E}_2 , respectively. Suppose that $\{\alpha_n\}$ and $\gamma \in \mathbb{R}$ satisfy the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (ii) $0 < \gamma < \frac{1-\tau}{\|\mathcal{A}\|^2}$.

If $\Omega := \{b^* \in \mathcal{B}^{-1}0 : \mathcal{A}b^* \in F(\mathcal{T})\} \neq \emptyset$, then the sequence $\{b_n\}$ converges strongly to a point $b^* \in \Omega$, where $b^* = \Pi_{\Omega} u$.

Proof. First, note that $\mathcal{B}^{-1}0$ is closed and convex and from Lemma 4, we have Ω is closed and convex. Let $p \in \Omega$. Then $Q_r^{\mathcal{B}} p = p$ and $\mathcal{T}(\mathcal{A}p) = \mathcal{A}p$. For any $n \in \mathbb{N}$, from (1) and Lemma 5, we have

$$\begin{aligned} \psi(p, e_n) &= \psi(p, \mathcal{J}_1^{-1}(\mathcal{J}_1 b_n - \gamma \mathcal{A}^* \mathcal{J}_2(\mathcal{I} - \mathcal{T})\mathcal{A} b_n)) \\ &= \|p\|^2 - 2\langle p, \mathcal{J}_1 b_n - \gamma \mathcal{A}^* \mathcal{J}_2(\mathcal{I} - \mathcal{T})\mathcal{A} b_n \rangle + \|\mathcal{J}_1 b_n - \gamma \mathcal{A}^* \mathcal{J}_2(\mathcal{I} - \mathcal{T})\mathcal{A} b_n\|^2 \\ &\leq \|p\|^2 - 2\langle p, \mathcal{J}_1 b_n \rangle + 2\gamma \langle \mathcal{A}p, \mathcal{J}_2(\mathcal{I} - \mathcal{T})\mathcal{A} b_n \rangle + \|b_n\|^2 \\ &\quad - 2\gamma \langle \mathcal{A}^* \mathcal{J}_2(\mathcal{I} - \mathcal{T})\mathcal{A} b_n, b_n \rangle + 2k^2 \gamma^2 \|\mathcal{A}\|^2 \|(\mathcal{I} - \mathcal{T})\mathcal{A} b_n\|^2 \\ &\leq \psi(p, b_n) + 2\gamma \langle \mathcal{A}p - \mathcal{A}b_n, \mathcal{J}_2(\mathcal{I} - \mathcal{T})\mathcal{A} b_n \rangle + \gamma^2 \|\mathcal{A}\|^2 \|(\mathcal{I} - \mathcal{T})\mathcal{A} b_n\|^2, \end{aligned} \tag{2}$$

where

$$\begin{aligned} \langle \mathcal{A}p - \mathcal{A}b_n, \mathcal{J}_2(\mathcal{I} - \mathcal{T})\mathcal{A} b_n \rangle &= \langle \mathcal{A}p - \mathcal{T}\mathcal{A}b_n + \mathcal{T}\mathcal{A}b_n - \mathcal{A}b_n, \mathcal{J}_2(\mathcal{I} - \mathcal{T})\mathcal{A} b_n \rangle \\ &= -\langle \mathcal{T}\mathcal{A}b_n - \mathcal{A}p, \mathcal{J}_2(\mathcal{I} - \mathcal{T})\mathcal{A} b_n \rangle - \|(\mathcal{I} - \mathcal{T})\mathcal{A} b_n\|^2 \\ &\leq \frac{1}{2} \|(\mathcal{I} - \mathcal{T})\mathcal{A} b_n\|^2 + k^2 \|\mathcal{T}\mathcal{A}b_n - \mathcal{A}p\|^2 \\ &\quad - \frac{1}{2} \|\mathcal{A}b_n - \mathcal{A}p\|^2 - \|(\mathcal{I} - \mathcal{T})\mathcal{A} b_n\|^2 \\ &\leq -\frac{1}{2} \|(\mathcal{I} - \mathcal{T})\mathcal{A} b_n\|^2 + \frac{1}{2} \|\mathcal{T}\mathcal{A}b_n - \mathcal{A}p\|^2 - \frac{1}{2} \|\mathcal{A}b_n - \mathcal{A}p\|^2 \tag{3} \\ &\leq -\frac{1}{2} \|(\mathcal{I} - \mathcal{T})\mathcal{A} b_n\|^2 + \frac{1}{2} \|\mathcal{A}b_n - \mathcal{A}p\|^2 \\ &\quad + \frac{\tau}{2} \|(\mathcal{I} - \mathcal{T})\mathcal{A} b_n\|^2 - \frac{1}{2} \|\mathcal{A}b_n - \mathcal{A}p\|^2 \\ &= -\frac{1}{2} (1 - \tau) \|(\mathcal{I} - \mathcal{T})\mathcal{A} b_n\|^2. \end{aligned}$$

Substituting (3) into (2), and by condition (ii), we get

$$\begin{aligned} \psi(p, e_n) &\leq \psi(p, b_n) - \gamma(1 - \tau) \|(\mathcal{I} - \mathcal{T})\mathcal{A} b_n\|^2 + \gamma^2 \|\mathcal{A}\|^2 \|(\mathcal{I} - \mathcal{T})\mathcal{A} b_n\|^2 \\ &\leq \psi(p, b_n) - \gamma(1 - \tau - \gamma \|\mathcal{A}\|^2) \|(\mathcal{I} - \mathcal{T})\mathcal{A} b_n\|^2 \\ &\leq \psi(p, b_n), \end{aligned} \tag{4}$$

for all $n \in \mathbb{N}$. Furthermore, because $Q_r^{\mathcal{B}}$ is the resolvent of a maximal monotone operator, it is a strongly relative nonexpansive mapping. For any $n \in \mathbb{N}$, by taking into account (1), (4) and Lemma 1(D), we obtain

$$\begin{aligned} \psi(p, b_{n+1}) &= \psi(p, \mathcal{J}_1^{-1}[\alpha_n \mathcal{J}_1 u + (1 - \alpha_n) \mathcal{J}_1 Q_r^{\mathcal{B}} e_n]) \\ &\leq \alpha_n \psi(p, u) + (1 - \alpha_n) \psi(p, Q_r^{\mathcal{B}} e_n) \\ &\leq \alpha_n \psi(p, u) + (1 - \alpha_n) \psi(p, e_n) \\ &\leq \alpha_n \psi(p, u) + (1 - \alpha_n) \psi(p, b_n) \\ &\leq \max\{\psi(p, u), \psi(p, b_n)\} \\ &\leq \max\{\psi(p, u), \psi(p, b_1)\}. \end{aligned} \tag{5}$$

Therefore, we prove that $\{\psi(p, b_n)\}$ is bounded. Consequently, $\{b_n\}$, $\{e_n\}$ and $\{Q_r^{\mathcal{B}} e_n\}$ are also bounded. Next, according to Lemma 2, we get

$$\begin{aligned} \psi(p, e_{n+1}) &\leq \psi(p, b_{n+1}) \\ &= \psi(p, \mathcal{J}_1^{-1}[\alpha_n \mathcal{J}_1 u + (1 - \alpha_n) \mathcal{J}_1 Q_r^{\mathcal{B}} e_n]) \\ &= V(p, \alpha_n \mathcal{J}_1 u + (1 - \alpha_n) \mathcal{J}_1 Q_r^{\mathcal{B}} e_n) \\ &\leq V(p, \alpha_n \mathcal{J}_1 u + (1 - \alpha_n) \mathcal{J}_1 Q_r^{\mathcal{B}} e_n - \alpha_n(\mathcal{J}_1 u - \mathcal{J}_1 p)) \\ &\quad - 2\langle \mathcal{J}_1^{-1}(\alpha_n \mathcal{J}_1 u + (1 - \alpha_n) \mathcal{J}_1 Q_r^{\mathcal{B}} e_n) - p, -\alpha_n(\mathcal{J}_1 u - \mathcal{J}_1 p) \rangle \\ &= V(p, \alpha_n \mathcal{J}_1 u + (1 - \alpha_n) \mathcal{J}_1 Q_r^{\mathcal{B}} e_n - \alpha_n(\mathcal{J}_1 u - \mathcal{J}_1 p)) \\ &\quad + 2\alpha_n \langle b_{n+1} - p, \mathcal{J}_1 u - \mathcal{J}_1 p \rangle \\ &= \psi(p, \mathcal{J}_1^{-1}[(1 - \alpha_n) \mathcal{J}_1 Q_r^{\mathcal{B}} e_n + \alpha_n \mathcal{J}_1 p]) + 2\alpha_n \langle b_{n+1} - p, \mathcal{J}_1 u - \mathcal{J}_1 p \rangle \\ &\leq (1 - \alpha_n) \psi(p, Q_r^{\mathcal{B}} e_n) + \alpha_n \psi(p, p) + 2\alpha_n \langle b_{n+1} - p, \mathcal{J}_1 u - \mathcal{J}_1 p \rangle \\ &\leq (1 - \alpha_n) \psi(p, e_n) + 2\alpha_n \langle b_{n+1} - p, \mathcal{J}_1 u - \mathcal{J}_1 p \rangle. \end{aligned} \tag{6}$$

The rest of the proof is going to be divided into two possible cases.

Case 1. Assume that there exists $n_0 \in \mathbb{N}$ such that $\{\psi(p, e_n)\}$ is monotonically decreasing as $n \geq n_0$. Obviously, $\{\psi(p, e_n)\}$ converges and

$$\psi(p, e_{n+1}) - \psi(p, e_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{7}$$

It follows (4) and (5) that

$$\begin{aligned} \psi(p, e_n) - \psi(p, Q_r^{\mathcal{B}} e_n) &= \psi(p, e_n) - \psi(p, b_{n+1}) + \psi(p, b_{n+1}) - \psi(p, Q_r^{\mathcal{B}} e_n) \\ &\leq \psi(p, e_n) - \psi(p, e_{n+1}) + \psi(p, b_{n+1}) - \psi(p, Q_r^{\mathcal{B}} e_n) \\ &\leq \psi(p, e_n) - \psi(p, e_{n+1}) + \alpha_n \psi(p, u) \\ &\quad + (1 - \alpha_n) \psi(p, Q_r^{\mathcal{B}} e_n) - \psi(p, Q_r^{\mathcal{B}} e_n) \\ &= \psi(p, e_n) - \psi(p, e_{n+1}) + \alpha_n (\psi(p, u) - \psi(p, Q_r^{\mathcal{B}} e_n)), \end{aligned} \tag{8}$$

for all $n \in \mathbb{N}$. In view of condition (i) and (7), we know

$$\lim_{n \rightarrow \infty} (\psi(p, e_n) - \psi(p, Q_r^{\mathcal{B}} e_n)) = 0. \tag{9}$$

Therefore, by the definition of strongly relatively nonexpansive mapping, we obtain

$$\lim_{n \rightarrow \infty} \psi(Q_r^{\mathcal{B}} e_n, e_n) = 0. \tag{10}$$

Furthermore, by Lemma 3, we have

$$\lim_{n \rightarrow \infty} \|Q_r^{\mathcal{B}} e_n - e_n\| = 0. \tag{11}$$

Since \mathcal{E} is reflexive and $\{e_n\}$ is bounded, there exists a subsequence $\{e_{n_j}\}$ of $\{e_n\}$ such that $\{e_{n_j}\}$ converges weakly to $\bar{b} \in \mathcal{E}$. Since $Q_r^{\mathcal{B}}$ is strongly relatively nonexpansive, from (11), we have $Q_r^{\mathcal{B}} \bar{b} = \bar{b}$, i.e., $0 \in \mathcal{B}(\bar{b})$. For any $n \in \mathbb{N}$, by taking into account (4), (6), (7) and conditions (i) and (ii), we get

$$\begin{aligned} 0 &< \gamma(1 - \tau - \gamma\|\mathcal{A}\|^2)\|(\mathcal{I} - \mathcal{T})\mathcal{A}b_n\|^2 \\ &\leq \psi(p, b_n) - \psi(p, e_n) \\ &\leq (1 - \alpha_{n-1})\psi(p, e_{n-1}) + 2\alpha_{n-1}\langle b_n - p, \mathcal{J}_1 u - \mathcal{J}_1 p \rangle - \psi(p, e_n) \\ &= \psi(p, e_{n-1}) - \psi(p, e_n) - \alpha_{n-1}\psi(p, e_{n-1}) + 2\alpha_{n-1}\langle b_n - p, \mathcal{J}_1 u - \mathcal{J}_1 p \rangle \\ &\leq \psi(p, e_{n-1}) - \psi(p, e_n) + 2\alpha_{n-1}\langle b_n - p, \mathcal{J}_1 u - \mathcal{J}_1 p \rangle \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned} \tag{12}$$

which implies

$$\lim_{n \rightarrow \infty} \|(\mathcal{I} - \mathcal{T})\mathcal{A}b_n\| = 0. \tag{13}$$

Hence, from the definition of $\{e_n\}$, we obtain

$$0 \leq \|\mathcal{J}_1 b_n - \mathcal{J}_1 e_n\| \leq \gamma\|\mathcal{A}^*\| \|(\mathcal{I} - \mathcal{T})\mathcal{A}b_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{14}$$

Because \mathcal{J}_1 is norm to norm uniformly continuous, we obtain

$$\lim_{n \rightarrow \infty} \|b_n - e_n\| = 0. \tag{15}$$

By the continuity of \mathcal{A} and (15), we obtain that $\mathcal{A}b_{n_j} \rightarrow \mathcal{A}\bar{b}$ as $j \rightarrow \infty$. Thus, by (13) and \mathcal{T} is demiclosed at zero, we get $\mathcal{T}(\mathcal{A}\bar{b}) = \mathcal{A}\bar{b}$. Therefore, $\bar{b} \in \Omega$.

Next, we show that $\{b_n\}$ converges strongly to $\Pi_{\Omega}u$. Let $b^* = \Pi_{\Omega}u$. For any $n \in \mathbb{N}$, from (6), we know that

$$\begin{aligned} \psi(b^*, b_{n+1}) &\leq (1 - \alpha_n)\psi(b^*, e_n) + 2\alpha_n\langle b_{n+1} - b^*, \mathcal{J}_1 u - \mathcal{J}_1 b^* \rangle \\ &\leq (1 - \alpha_n)\psi(b^*, b_n) + 2\alpha_n\langle b_{n+1} - b^*, \mathcal{J}_1 u - \mathcal{J}_1 b^* \rangle. \end{aligned} \tag{16}$$

Now, we observe that

$$\begin{aligned} \psi(e_n, b_{n+1}) &= \psi(e_n, \mathcal{J}_1^{-1}[\alpha_n \mathcal{J}_1 u + (1 - \alpha_n) \mathcal{J}_1 Q_r^{\mathcal{B}} e_n]) \\ &\leq \alpha_n \psi(e_n, u) + (1 - \alpha_n) \psi(e_n, Q_r^{\mathcal{B}} e_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \tag{17}$$

Hence, $\lim_{n \rightarrow \infty} \|e_n - b_{n+1}\| = 0$. Thus

$$\|b_n - b_{n+1}\| \leq \|b_n - e_n\| + \|e_n - b_{n+1}\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{18}$$

By choosing a subsequence $\{b_{n_j}\}$ of $\{b_n\}$ and from Lemma 1(B), we obtain

$$\limsup_{n \rightarrow \infty} \langle b_n - b^*, \mathcal{J}_1 u - \mathcal{J}_1 b^* \rangle = \limsup_{j \rightarrow \infty} \langle b_{n_j} - b^*, \mathcal{J}_1 u - \mathcal{J}_1 b^* \rangle = \langle \bar{b} - b^*, \mathcal{J}_1 u - \mathcal{J}_1 b^* \rangle \leq 0.$$

By (18), we have

$$\limsup_{n \rightarrow \infty} \langle b_{n+1} - b^*, \mathcal{J}_1 u - \mathcal{J}_1 b^* \rangle = \limsup_{n \rightarrow \infty} \langle b_n - b^*, \mathcal{J}_1 u - \mathcal{J}_1 b^* \rangle \leq 0. \tag{19}$$

Therefore, in view of (16), (19) and Lemma 6, we conclude that $\lim_{n \rightarrow \infty} \psi(b^*, b_n) = 0$, which means $\lim_{n \rightarrow \infty} \|b_n - b^*\| = 0$. Therefore, $\{b_n\}$ converges strongly to b^* .

Case 2. Let $\{n_i\}$ be a subsequence of $\{n\}$ such that $\psi(p, e_{n_i}) < \psi(p, e_{n_i+1})$ for all $i \in \mathbb{N}$. Therefore, from Lemma 7, there exists a nondecreasing sequence $\{m_k\} \subseteq \mathbb{N}$ such that $m_k \rightarrow \infty$,

$$\psi(p, e_{m_k}) \leq \psi(p, e_{m_k+1}),$$

and

$$\psi(p, e_k) \leq \psi(p, e_{m_k+1}).$$

Using the same lines of arguments as in (7)–(10), we can show that

$$\lim_{k \rightarrow \infty} \|Q_r^{\mathcal{B}} e_{m_k} - e_{m_k}\| = 0.$$

Similarly as in the proof of case 1, we get

$$\limsup_{n \rightarrow \infty} \langle b_{m_k+1} - b^*, \mathcal{J}_1 u - \mathcal{J}_1 b^* \rangle \leq 0.$$

By (6), we have

$$\psi(b^*, e_{m_k+1}) \leq (1 - \alpha_{m_k})\psi(b^*, e_{m_k}) + 2\alpha_{m_k} \langle b_{m_k+1} - b^*, \mathcal{J}_1 u - \mathcal{J}_1 b^* \rangle, \tag{20}$$

which deduces

$$\alpha_{m_k} \psi(b^*, e_{m_k}) \leq \psi(b^*, e_{m_k}) - \psi(b^*, e_{m_k+1}) + 2\alpha_{m_k} \langle b_{m_k+1} - b^*, \mathcal{J}_1 u - \mathcal{J}_1 b^* \rangle.$$

for all $k \in \mathbb{N}$. Due to $\psi(b^*, e_{m_k}) \leq \psi(b^*, e_{m_k+1})$, we obtain

$$\psi(b^*, e_{m_k}) \leq 2 \langle b_{m_k+1} - b^*, \mathcal{J}_1 u - \mathcal{J}_1 b^* \rangle \quad \text{for all } k \in \mathbb{N}.$$

Therefore $\lim_{k \rightarrow \infty} \psi(b^*, e_{m_k}) = 0$. Furthermore, it follows from (20) that $\lim_{k \rightarrow \infty} \psi(b^*, e_{m_k+1}) = 0$. Since $\psi(b^*, e_k) \leq \psi(b^*, e_{m_k+1})$ for all $k \in \mathbb{N}$, we conclude that $e_k \rightarrow b^*$ as $k \rightarrow \infty$. On the other hand, since $\|e_n - b_n\| \rightarrow 0$ as $n \rightarrow \infty$, we obtain $b_k \rightarrow b^*$ as $k \rightarrow \infty$. The proof is completed. \square

Remark 1.

- (a) All results established in [16] were considered in the setting of Hilbert spaces. It is worth noting that Theorem 1 is a strong convergence theorem for the generalized split feasibility problem in the setting of Banach spaces, so it is different from any result in [16];
- (b) Recently, Ansari and Rehan [17] studied (GSFP) and established weak convergence theorems of the iterative algorithm shown below in the setting of two Banach spaces:

$$b_{n+1} = \mathcal{J}_\lambda^{\mathcal{B}} (\mathcal{J}_{\mathcal{E}_1}^{-1} (\mathcal{J}_{\mathcal{E}_1} (b_n) - \gamma \mathcal{A}^* \mathcal{J}_{\mathcal{E}_2} (\mathcal{I} - \mathcal{T}) \mathcal{A} b_n)) \quad \text{for all } n \in \mathbb{N},$$

where \mathcal{E}_1 and \mathcal{E}_2 are uniformly convex and 2-uniformly smooth real Banach spaces, $\mathcal{B} : \mathcal{E}_1 \rightarrow \mathcal{E}_1^*$ be a maximal monotone set-valued mapping such that $\mathcal{B}^{-1}0 \neq \emptyset$, $\mathcal{T} : \mathcal{E}_2 \rightarrow \mathcal{E}_2$ be a quasi-nonexpansive mapping and $\mathcal{A} : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ be a bounded linear operator whose adjoint is denoted by \mathcal{A}^* . $\mathcal{J}_\lambda^{\mathcal{B}}$ be the resolvent operator of \mathcal{B} for $\lambda > 0$, $\mathcal{J}_{\mathcal{E}_1}$ and $\mathcal{J}_{\mathcal{E}_2}$ be the normalized duality mappings on \mathcal{E}_1 and \mathcal{E}_2 , respectively. It is worth noting that Theorem 1 is distinct from any result in [17].

Let \mathcal{C} be a smooth strictly convex and reflexive Banach space and let \mathcal{C} be a nonempty closed convex subset of \mathcal{E} . Let $i_{\mathcal{C}}$ be the indicator function of $\mathcal{C} \subseteq \mathcal{E}$, i.e., $i_{\mathcal{C}}(b) = 0$ if $b \in \mathcal{C}$ and ∞

otherwise. Then $i_{\mathcal{C}} : \mathcal{E} \rightarrow (-\infty, \infty]$ is a proper lower semicontinuous convex function. Rockafellar’s maximal monotonicity theorem [32] guarantees that the subdifferential $\partial_{i_{\mathcal{C}}} \subset \mathcal{B} \times \mathcal{B}^*$ of $i_{\mathcal{C}}$ is maximal monotone. In this case, it is known that $\partial_{i_{\mathcal{C}}}$ is reduced to the normality operator $N_{\mathcal{C}}$ for \mathcal{C} , i.e.,

$$N_{\mathcal{C}}(b) = \{b^* \in \mathcal{E}^* : \langle e - b, b^* \rangle \text{ for all } e \in \mathcal{C}\}.$$

Indeed, for any $b \in \mathcal{C}$,

$$\begin{aligned} \partial_{i_{\mathcal{C}}}(b) &= \{b^* \in \mathcal{E}^* : i_{\mathcal{C}}(b) + \langle e - b, b^* \rangle \leq i_{\mathcal{C}}(e) \text{ for all } e \in \mathcal{E}\} \\ &= \{b^* \in \mathcal{E}^* : \langle e - b, b^* \rangle \leq 0 \text{ for all } e \in \mathcal{C}\} = N_{\mathcal{C}}(b). \end{aligned}$$

We also know that $\Pi_{\mathcal{C}}$ is the resolvent of $N_{\mathcal{C}}$. In fact, $\Pi_{\mathcal{C}} = (\mathcal{J} + 2^{-1}N_{\mathcal{C}})^{-1}\mathcal{J}$ (see, e.g., [24] for more details).

Let \mathcal{C} and \mathcal{Q} be a nonempty closed convex subsets of \mathcal{E}_1 and \mathcal{E}_2 , respectively. Consider $\mathcal{K} = \partial_{i_{\mathcal{C}}}$ and $\mathcal{T} = P_{\mathcal{Q}}$, where $P_{\mathcal{Q}}$ is the metric projection from \mathcal{E}_2 onto \mathcal{Q} . Therefore, we have $Q_{\mathcal{K}}^{\mathcal{K}} = \Pi_{\mathcal{C}}$ and $Fix(\mathcal{T}) = \mathcal{Q}$. By virtue of Theorem 1, we can establish the following strong convergence algorithm of the split feasibility problem for metric projections in Banach spaces.

Corollary 1. *Let \mathcal{E}_1 and \mathcal{E}_2 be 2-uniformly convex and 2-uniformly smooth real Banach space with smoothness constant k satisfying $0 < k \leq \frac{1}{\sqrt{2}}$, \mathcal{C} and \mathcal{Q} be nonempty closed convex subsets of \mathcal{E}_1 and \mathcal{E}_2 , respectively. Let $P_{\mathcal{Q}}$ be the metric projection from \mathcal{E}_2 onto \mathcal{Q} and $\mathcal{A} : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ be a bounded linear operator. Let $\{\alpha_n\}$ be a sequence in $(0,1)$. For any $b_1 = b \in \mathcal{E}_1$ and a fixed $u \in \mathcal{E}_1$, suppose that $\{b_n\}$ is a sequence defined by*

$$\begin{cases} e_n = \mathcal{J}_1^{-1}(\mathcal{J}_1 b_n - \gamma \mathcal{A}^* \mathcal{J}_2 (\mathcal{I} - P_{\mathcal{Q}}) \mathcal{A} b_n), \\ b_{n+1} = \mathcal{J}_1^{-1}[\alpha_n \mathcal{J}_1 u + (1 - \alpha_n) \mathcal{J}_1 \Pi_{\mathcal{C}} e_n], \end{cases} \tag{21}$$

where \mathcal{J}_1 and \mathcal{J}_2 are the normalized duality mappings of \mathcal{E}_1 and \mathcal{E}_2 , respectively. Suppose that $\{\alpha_n\}$ and $\gamma \in \mathbb{R}$ satisfy the following conditions: (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$; (ii) $0 < \gamma < \frac{1}{\|\mathcal{A}\|^2}$. If $\Omega_1 := \{b^* \in \mathcal{C} : \mathcal{A} b^* \in \mathcal{Q}\} \neq \emptyset$, then the sequence $\{b_n\}$ converges strongly to a point $b^* \in \Omega_1$, where $b^* = \Pi_{\Omega_1} u$.

4. Some Applications

In this section, we will show some applications of the generalized split feasibility problem and Theorem 1.

(I) Equilibrium problem and fixed point problem

Let $\mathcal{F} : \mathcal{C} \times \mathcal{C} \rightarrow \mathbb{R}$ be a bi-function. Recall that the classical equilibrium problem (EP, in short) is defined as follows.

$$(EP) \quad \text{Find } p \in \mathcal{C} \text{ such that } \mathcal{F}(p, y) \geq 0, \quad \forall y \in \mathcal{C}.$$

The symbol $EP(\mathcal{F})$ is used to denote the set of all solutions of the problem (EP) for \mathcal{F} , i.e.,

$$EP(\mathcal{F}) = \{u \in \mathcal{C} : \mathcal{F}(u, v) \geq 0, \quad \forall v \in \mathcal{C}\}.$$

Let us consider the following hybrid problem for equilibrium problem and fixed point problem (HEFP, in short):

$$(HEFP) \quad \text{Find } b^* \in \mathcal{C}, \text{ such that } \mathcal{F}(b^*, z) \geq 0 \text{ and } \mathcal{A} b^* \in F(\mathcal{T}) \text{ for all } z \in \mathcal{C},$$

where \mathcal{C} is a nonempty closed and convex subset of \mathcal{E}_1 , \mathcal{E}_1 and \mathcal{E}_2 are 2-uniformly convex and 2-uniformly smooth real Banach spaces with smoothness constant k satisfying $0 < k \leq \frac{1}{\sqrt{2}}$, $\mathcal{A} : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ is a bounded linear operator, $\mathcal{T} : \mathcal{E}_2 \rightarrow \mathcal{E}_2$ is a τ -quasi-strict pseudocontractive mapping such that $F(\mathcal{T}) \neq \emptyset$.

Let $\mathcal{F} : \mathcal{C} \times \mathcal{C} \rightarrow \mathbb{R}$ be a bi-function satisfying the following conditions (C1)–(C4):

- (C1) $\mathcal{F}(b, b) = 0, \forall b \in \mathcal{C}$;
- (C2) \mathcal{F} is monotone, i.e., $\mathcal{F}(b, e) + \mathcal{F}(e, b) \leq 0, \forall b, e \in \mathcal{C}$;
- (C3) For all $b, e, z \in \mathcal{C}, \lim_{t \downarrow 0} \mathcal{F}(tz + (1 - t)b, e) \leq \mathcal{F}(b, e)$;
- (C4) For each $b \in \mathcal{C}$, the function $e \mapsto \mathcal{F}(b, e)$ is convex and lower semi-continuous .

The resolvent mapping $\mathcal{T}_\mu^{\mathcal{F}}$ of \mathcal{F} is defined as

$$\mathcal{T}_\mu^{\mathcal{F}}(b) = \{z \in \mathcal{C} : \mathcal{F}(z, e) + \frac{1}{\mu} \langle e - z, \mathcal{J}_{\mathcal{E}_1}^p z - \mathcal{J}_{\mathcal{E}_1}^p b \rangle \geq 0, \forall e \in \mathcal{C}\}, \quad \mu > 0.$$

It is known that the following assertions hold (see [33]):

- (1) $\mathcal{T}_\mu^{\mathcal{F}}$ is single-valued ;
- (2) $\mathcal{T}_\mu^{\mathcal{F}}$ is a firmly nonexpansive-type mapping;
- (3) $Fix(\mathcal{T}_\mu^{\mathcal{F}}) = EP(\mathcal{F})$;
- (4) $EP(\mathcal{F})$ is closed and convex.

The following result is a special case of the result by Aoyama et al. [34].

Lemma 8. Let $\mathcal{F} : \mathcal{C} \times \mathcal{C} \rightarrow \mathbb{R}$ be bi-functions satisfying (C1)-(C4) and let $\mathcal{B}_{\mathcal{F}} : \mathcal{E}_1 \rightarrow \mathcal{E}_1^*$ be a set-valued mapping defined as follows:

- For any $b \in \mathcal{C}, \mathcal{B}_{\mathcal{F}}(b) := \{b^* \in \mathcal{E}_1^* : \mathcal{F}(b, z) \geq \langle z - b, b^* \rangle \text{ for all } z \in \mathcal{C}\}$;
- For any $b \notin \mathcal{C}, \mathcal{B}_{\mathcal{F}}(b) := \emptyset$.

Then, $\mathcal{B}_{\mathcal{F}}$ is a maximal monotone operator with $D(\mathcal{B}_{\mathcal{F}}) \subseteq \mathcal{C}$ and $EP(\mathcal{F}) = \mathcal{B}_{\mathcal{F}}^{-1}0$. Furthermore, for $\mu > 0$, the resolvent $\mathcal{T}_\mu^{\mathcal{F}}$ of \mathcal{F} coincides with the resolvent $(\mathcal{J} + \mu\mathcal{B}_{\mathcal{F}})^{-1}\mathcal{J}$ of $\mathcal{B}_{\mathcal{F}}$, i.e.,

$$\mathcal{T}_\mu^{\mathcal{F}}(b) = (\mathcal{J} + \mu\mathcal{B}_{\mathcal{F}})^{-1}\mathcal{J}(b).$$

As a consequence of Theorem 1, we can get the following result for finding a solution of (HEFP).

Theorem 2. Let \mathcal{E}_1 and \mathcal{E}_2 be 2-uniformly convex and 2-uniformly smooth real Banach space with smoothness constant k satisfying $0 < k \leq \frac{1}{\sqrt{2}}$. Let \mathcal{C} and \mathcal{Q} be nonempty closed and convex subsets of \mathcal{E}_1 and \mathcal{E}_2 , respectively. Let $\mathcal{A} : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ is bounded linear operators. Let $\mathcal{F} : \mathcal{C} \times \mathcal{C} \rightarrow \mathbb{R}$ be bi-function satisfying the condition (C1)-(C4) and $\mathcal{T}_\mu^{\mathcal{F}}$ be the resolvent mapping of $\mathcal{B}_{\mathcal{F}}$ defined in Lemma 8. Let $\mathcal{T} : \mathcal{E}_2 \rightarrow \mathcal{E}_2$ be a τ -quasi-strict pseudocontractive mapping with $F(\mathcal{T}) \neq \emptyset$, and \mathcal{T} be demiclosed at zero. Let $\{\alpha_n\}$ be a sequence in $(0,1)$. For any $b_1 = b \in \mathcal{E}_1$ and a fixed $u \in \mathcal{E}_1$, let $\{b_n\}$ be a sequence defined by

$$\begin{cases} e_n = \mathcal{J}_1^{-1}(\mathcal{J}_1 b_n - \gamma \mathcal{A}^* \mathcal{J}_2 (\mathcal{I} - \mathcal{T}) \mathcal{A} b_n), \\ b_{n+1} = \mathcal{J}_1^{-1}[\alpha_n \mathcal{J}_1 u + (1 - \alpha_n) \mathcal{J}_1 \mathcal{T}_\mu^{\mathcal{F}} e_n], \end{cases} \tag{22}$$

where \mathcal{J}_1 and \mathcal{J}_2 are the normalized duality mappings of \mathcal{E}_1 and \mathcal{E}_2 , respectively. Suppose that $\{\alpha_n\}$ and $\gamma \in \mathbb{R}$ satisfy the following conditions: (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$; (ii) $0 < \gamma < \frac{1-\tau}{\|\mathcal{A}\|^2}$. If $\Omega_2 := \{b^* \in EP(\mathcal{F}) : \mathcal{A} b^* \in F(\mathcal{T})\} \neq \emptyset$, then the sequence $\{b_n\}$ converges strongly to a point $b^* \in \Omega_2$, where $b^* = \Pi_{\Omega_2} u$.

(II) Split common fixed point problem

Since $Q_r^{\mathcal{B}}$ is the resolvent of a maximal monotone operator, we know that it is a strongly relative nonexpansive mapping. Therefore the following result of split common fixed point problem for τ -quasi-strict pseudocontractive mappings and strongly relatively nonexpansive mappings can be established from Theorem 1 immediately.

Theorem 3. Let \mathcal{E}_1 and \mathcal{E}_2 be 2-uniformly convex and 2-uniformly smooth real Banach spaces with smoothness constant k satisfying $0 < k \leq \frac{1}{\sqrt{2}}$. Let $\mathcal{S} : \mathcal{E}_1 \rightarrow \mathcal{E}_1$ be a strongly relatively nonexpansive mapping with $F(\mathcal{S}) \neq \emptyset$. Let $\mathcal{T} : \mathcal{E}_2 \rightarrow \mathcal{E}_2$ be a τ -quasi-strict pseudocontractive mapping such that $F(\mathcal{T}) \neq \emptyset$, and \mathcal{T} be demiclosed at zero, $\mathcal{A} : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ be a bounded linear operator. Let $\{\alpha_n\}$ be a sequence in $(0,1)$. For any $b_1 = b \in \mathcal{E}_1$ and a fixed $u \in \mathcal{E}_1$, let $\{b_n\}$ be a sequence defined by

$$\begin{cases} e_n = \mathcal{J}_1^{-1}(\mathcal{J}_1 b_n - \gamma \mathcal{A}^* \mathcal{J}_2(\mathcal{I} - \mathcal{T})\mathcal{A} b_n), \\ b_{n+1} = \mathcal{J}_1^{-1}[\alpha_n \mathcal{J}_1 u + (1 - \alpha_n) \mathcal{J}_1 \mathcal{S} e_n], \end{cases} \tag{23}$$

where \mathcal{J}_1 and \mathcal{J}_2 are the normalized duality mappings of \mathcal{E}_1 and \mathcal{E}_2 , respectively. Suppose that $\{\alpha_n\}$ and $\gamma \in \mathbb{R}$ satisfy the following conditions: (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$; (ii) $0 < \gamma < \frac{1-\tau}{\|\mathcal{A}\|^2}$. If $\Omega_3 := \{b^* \in F(\mathcal{S}) : \mathcal{A} b^* \in F(\mathcal{T})\} \neq \emptyset$, then the sequence $\{b_n\}$ converges strongly to a point $b^* \in \Omega_3$, where $b^* = \Pi_{\Omega_3} u$.

The following conclusion is an immediate consequence of Theorem 3 due to the fact that $P_{\mathcal{Q}}$ is a special τ -quasi-strict pseudocontractive mapping.

Corollary 2. Let \mathcal{E}_1 and \mathcal{E}_2 be 2-uniformly convex and 2-uniformly smooth real Banach spaces with smoothness constant k satisfying $0 < k \leq \frac{1}{\sqrt{2}}$. Let \mathcal{C} and \mathcal{Q} be nonempty, closed and convex subsets of \mathcal{E}_1 and \mathcal{E}_2 respectively. Let $P_{\mathcal{Q}}$ be the metric projection from \mathcal{E}_2 onto \mathcal{Q} and $\mathcal{S} : \mathcal{C} \rightarrow \mathcal{C}$ be a strongly relatively nonexpansive mapping with $F(\mathcal{S}) \neq \emptyset$. $\mathcal{A} : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ be a bounded linear operator. Let $\{\alpha_n\}$ be a sequence in $(0,1)$. For any $b_1 = b \in \mathcal{E}_1$ and a fixed $u \in \mathcal{E}_1$, let $\{b_n\}$ be a sequence defined by

$$\begin{cases} e_n = \Pi_{\mathcal{C}} \mathcal{J}_1^{-1}(\mathcal{J}_1 b_n - \gamma \mathcal{A}^* \mathcal{J}_2(\mathcal{I} - P_{\mathcal{Q}})\mathcal{A} b_n), \\ b_{n+1} = \Pi_{\mathcal{C}} \mathcal{J}_1^{-1}[\alpha_n \mathcal{J}_1 u + (1 - \alpha_n) \mathcal{J}_1 \mathcal{S} e_n], \end{cases} \tag{24}$$

where \mathcal{J}_1 and \mathcal{J}_2 are the normalized duality mappings of \mathcal{E}_1 and \mathcal{E}_2 , respectively. Suppose that $\{\alpha_n\}$ and $\gamma \in \mathbb{R}$ satisfy the following conditions: (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$; (ii) $0 < \gamma < \frac{1}{\|\mathcal{A}\|^2}$. If $\Omega_4 := \{b^* \in F(\mathcal{S}) : \mathcal{A} b^* \in \mathcal{Q}\} \neq \emptyset$, then the sequence $\{b_n\}$ converges strongly to a point $b^* \in \Omega_4$, where $b^* = \Pi_{\Omega_4} u$.

5. Conclusions

New strong convergence theorems of proposed algorithms under suitable new conditions for the generalized split feasibility problem in Banach spaces are established in this paper. As applications, we study new strong convergence theorems for equilibrium problems, fixed point problems and split common fixed point problems. Our new results are distinct from recent results on the topic in the literature.

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