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Article

# On Semi-Classical Orthogonal Polynomials Associated with a Modified Sextic Freud-Type Weight

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**Abstract:** Polynomials that are orthogonal with respect to a perturbation of the Freud weight function by some parameter, known to be modified Freudian orthogonal polynomials, are considered. In this contribution, we investigate certain properties of semi-classical modified Freud-type polynomials in which their corresponding semi-classical weight function is a more general deformation of the classical scaled sextic Freud weight $|x|^{\alpha}\exp(-cx^6)$, $c > 0$, $\alpha > -1$. Certain characterizing properties of these polynomials such as moments, recurrence coefficients, holonomic equations that they satisfy, and certain non-linear differential-recurrence equations satisfied by the recurrence coefficients, using compatibility conditions for ladder operators for these orthogonal polynomials, are investigated. Differential-difference equations were also obtained via Shohat's quasi-orthogonality approach and also second-order linear ODEs (with rational coefficients) satisfied by these polynomials. Modified Freudian polynomials can also be obtained via Chihara's symmetrization process from the generalized Airy-type polynomials. The obtained linear differential equation plays an essential role in the electrostatic interpretation for the distribution of zeros of the corresponding Freudian polynomials.

**Keywords:** semi-classical orthogonal polynomials; Freud-type weights; moments; recurrence coefficients; difference equations; differential equations; zeros

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## 1. Introduction

The theory of orthogonal polynomials on finite intervals, which was studied in the seminal works of G. Szegö [1], is essentially not the same as the theory of orthogonal polynomials on large intervals. In the late 1970s, Géza Freud has done prominent work on polynomials orthogonal with respect to exponential weights on the real line, which referred to as Freud-type orthogonal polynomials [2]. At that time, the aim of Freud's work was to extend the theory of best approximation and Jackson-Bernstein type estimates to the real line; by expecting that orthogonal expansion may serve as a near-best approximation (cf. [2–5]).

Classical orthogonal polynomials have their weight function that satisfies *Pearson's equation*

$$[\sigma(x)w(x)]' = \tau(x)w(x), \tag{1}$$

where $\sigma(x)$ is a monic polynomial of degree at most 2 and $\tau(x)$ is a polynomial with degree 1. However for *semi-classical* orthogonal polynomials, the weight function $w(x)$ satisfies the Pearson Equation (1) with either $\deg(\sigma) > 2$ or $\deg(\tau) \neq 1$ (cf. [6–10]). The information we presently know about classical orthogonal polynomials on infinite intervals is largely due to the presence of generating functions, differential equations and recursive formulas that can be exploited to discover certain properties of such polynomials. For instance, Hermite, Laguerre, Pollaczek, Sonine, Stieltjes–Wigert, Poisson–Charlier, Krawtchouk, Meixner, Lommel, Askey–Wilson, and Al-Salam–Carlitz polynomials are some of the classical examples of orthogonal polynomials with non-compact support. On the other hand, the study of semi-classical orthogonal polynomials associated with various deformed weights is still active research since some of the properties are not yet explicitly studied. For some work on semi-classical weights, we refer the reader to [11–16].

Géza Freud (1973) considered class of weight functions

$$W_{\alpha,m}(x) = K|x|^{\alpha} \exp\left(-x^{2m}\right), \quad \alpha > -1,\ m > 0,\ x \in \mathbb{R}, \tag{2}$$

where $K$ is a positive constant [5,10,16]. Certain properties of Freud-type polynomials can be found in [12,15,17–19]. The authors in [12] considered monic orthogonal polynomials with respect to the quartic Freud-type weight function; that is, ($m = 2$ in Equation (2)) upon measure deformation by $\exp\left(tx^2\right)$. Some approximation-theoretic properties of orthonormal Freud-type polynomials is given in [19]. It is noted in [20] that slight generalizations of the weights of some of the classical orthogonal polynomials present great difficulties in exploring their properties, most of which have not been conquered yet.

Following the work in [21], a new semi-classical deformation for semi-classical Freudian weight function has been essential because of their practical applications and connections to random matrix theory, integrable systems and Painlevé equations. Motivated by the above facts, we investigate certain properties of monic orthogonal polynomials associated with the modified Freud-type inner product

$$\langle p, q\rangle_W = \int_{-\infty}^{\infty} p(x)\, q(x)\, |x|^{2\lambda+1} \exp\left(-[cx^6 + t(x^4 - x^2)]\right)\, \mathrm{d}x, \tag{3}$$

with real parameters $\lambda > 0,\ c > 0,\ t \in [0, K],\ (K \in \mathbb{R}^+)$. The modified Freudian weight in (3) emerges from deformation of the scaled Freudian weight $\exp(-cx^6),\ c > 0$. (see also [3,5,12]).

The motive for the choice of the orthogonality weight in Equation (3) is as follows. First, from some of the well-known classical orthogonal polynomials (for e.g., Laguerre polynomials), a new class of semi-classical orthogonal polynomials can be obtained by making slight modifications of the orthogonality measure; for example, the semi-classical Laguerre measure, which is well-studied in [11,22–24]. Knowing the fact that such measure modification usually leads to difficulties as mentioned in the papers of P. Nevai [5,20]; and motivated by the work in [20], a slight modification of a new orthogonality measure on non-compact support always leads to a new class of orthogonal polynomials provided the corresponding moments are finite [1,10,25]. We noticed that the generic semi-classical weights in [8] are special instances of such measure modification given in [20]. Secondly, perturbation of an orthogonality measure allows one to investigate certain fresh properties such as new-type of Toda-like differential-recurrence relations, new non-linear higher-order recursions as well as differential equations, and some properties of the zeros. For more on this, one can refer to [26,27] and the references therein. The obtained non-linear differential/difference equations for such orthogonal polynomials have considerable applications in modeling non-linear phenomena, Soliton Theory, Random matrix theory, Quantum oscillators and in the crystal structure in solid-state physics (for e.g., see the works by Chen and Its [13], Clarkson and Jordaan [11], Van Assche [10], Marcellán [28] and others).

Our aim in this paper is to investigate certain results on the recurrence coefficients for the semi-classical modified Freudian polynomials and to explore some properties for these polynomials including differential, differential-recurrence and difference equations and the zeros. For the semi-classical weight under consideration, it is not easy to obtain a concise formulation of the recurrence coefficients straightforwardly (see [11,12,15,29] for some related works). Related to the weight in (3), we accomplish some new work, by investigating Toda-type relations, differential, and differential-recurrence relations for the recurrence coefficients and the polynomials themselves.

The paper is organized as follows. In Section 2, we give some background for semi-classical modified Freud-type orthogonal polynomials. Section 3 gives some results that the recurrence coefficients for the weight in (3) obey nonlinear recursion relations. Besides, nonlinear differential-recurrence relations as well as differential equations satisfied by the recurrence coefficients as well as the polynomials are also obtained. Section 4 employs both the methods of ladder operators as well as Shohat's quasi-orthogonality approach to derive the ladder relations associated with the modified weight in (3) and note that the results obtained by both methods are the same. In Section 5, we show that the modified Freudian polynomials can be obtained from the semi-classical generalized Airy type polynomials using Chihara's symmetrization process [30]. Section 6 gives an electrostatic interpretation of the zeros of the modified Freudian polynomials from the obtained differential equation. Section 7 provides the discussion of the results and the final section ends with conclusions.

## 2. The Modified Sextic Freud-Type Weight

Consider $\{\mathcal{S}_n(x;t)\}_{n\geq 0}$ be sequence of monic polynomials that are orthogonal with respect to the semi-classical weight function

$$d\mu_\lambda(x) = W_\lambda(x;t)\,dx = |x|^{2\lambda+1}\exp\left(-[cx^6 + t(x^4 - x^2)]\right)\,dx, \tag{4}$$

with real parameters $\lambda > 0$, $c > 0$, and the orthogonality condition is given by

$$\int_{\mathbb{R}} \mathcal{S}_n(x;t)\,\mathcal{S}_m(x;t)\,W_\lambda(x;t)\,\mathrm{d}x = \hat{\zeta}_n\,\delta_{mn}, \quad \hat{\zeta}_n > 0, \quad m,n \in \{0,1,2,\ldots\}, \tag{5}$$

where $\delta_{mn}$ denotes the Kronecker delta, and $\hat{\zeta}_n$ is the square of $L^2$-norm of the polynomial $\mathcal{S}_n(x,t)$. Then the three-term recurrence relation follows as

$$\mathcal{S}_{n+1}(x;t) = x\mathcal{S}_n(x;t) - \beta_n(t;\lambda)\,\mathcal{S}_{n-1}(x;t), \;\; n \geq 1, \tag{6}$$

subject to the initial conditions $\mathcal{S}_0(x;t) := 1$ and $\beta_0\mathcal{S}_{-1}(x;t) := 0$. Multiplying both sides of Equation (6) by $\mathcal{S}_{n-1}(x;t)W_\lambda(x;t)$ and integrating this with respect to $x$ on $\mathbb{R}$, which, due to the orthogonality condition (5), gives us

$$\beta_n(t;\lambda) = \frac{1}{\hat{\zeta}_{n-1}(t)}\int_{\mathbb{R}} x\mathcal{S}_n(x;t)\,\mathcal{S}_{n-1}(x;t)\,W_\lambda(x;t)\;dx = \frac{\hat{\zeta}_n(t)}{\hat{\zeta}_{n-1}(t)} > 0. \tag{7}$$

One can see that $\mathcal{S}_n(x;t)$ contains only the terms $x^{n-j}$, $j \leq n$ and is even, since the weight function $W_\lambda(x;t)$ is even on $\mathbb{R}$. This implies that

$$\mathcal{S}_n(-x;t) = (-1)^n\mathcal{S}_n(x;t) \;\; \text{and} \;\; \mathcal{S}_n(0;t)\,\mathcal{S}_{n-1}(0;t) = 0.$$

Then we note the monic polynomials $\mathcal{S}_n(x;t)$, associated with $W_\lambda(x;t)$,

$$\mathcal{S}_n(x;t) = x^n + \mathbf{L}(n;t)\,x^{n-2} + \ldots + \mathcal{S}_n(0;t), \tag{8}$$

can be given equivalently as [30],

$$\mathcal{S}_{2j}(x;t) = x^{2j} + \mathbf{L}(2j;t)\,x^{2j-2} + \cdots + \mathcal{S}_{2j}(0;t),$$

and

$$\mathcal{S}_{2j+1}(x;t) = x^{2j+1} + \mathbf{L}(2j+1;t)\,x^{2j-1} + \cdots + \mathrm{k}.x = x\left(x^{2j} + \mathbf{L}(2j+1;t)\,x^{2j-2} + \cdots + \mathrm{k}\right),$$

where k is a real constant. If we substitute Equation (8) into Equation (6), we have

$$\beta_n(t) = \mathbf{L}(n;t) - \mathbf{L}(n+1;t), \tag{9}$$

where $\mathbf{L}(0;t) := 0$ and taking a telescoping sum of Equation (9) yields

$$\sum_{j=0}^{n-1} \beta_j = -\mathbf{L}(n;t).$$

### 2.1. *Pearson's Equation and Finite Moments for the Weight* (4)

The semi-classical modified Freud-type weight in (4) obeys Pearson's differential equation in Equation (1)

$$[xW_\lambda(x;t)]' = \left(-6cx^6 - 4tx^4 + 2tx^2 + 2\lambda + 2\right) W_\lambda(x;t), \tag{10}$$

where the prime here denotes differentiation with respect to $x$. From Equations (1) and (10), one can see that $\deg(\sigma) = 1$ and $\deg(\tau) = 6$. Hence, sequence of monic polynomials $\{\mathcal{S}_n\}_{n=0}^{\infty}$ with (10) constitute a family of *semi-classical orthogonal polynomials* (cf. [6–12,15]).

The following proposition is about the finiteness of moments for the semi-classical weight (4).

**Proposition 1.** *Let $x \in \mathbb{R}$, $\lambda > 0$ and $c > 0$. The first moment $\eta_0(t;\lambda)$ for the modified Freudian weight given in* (4) *is finite.*

By using the theory of integration [31], the higher order moments $\eta_k$, $k \in \mathbb{N}$ are also finite. The even moments $\eta_{2n}(t;\lambda)$, $n \in \mathbb{N}$ for the weight (4) are

$$\eta_{2n}(t;\lambda) = \int_{-\infty}^{\infty} x^{2n}|x|^{2\lambda+1} \exp\left(-[cx^6 + t(x^4 - x^2)]\right) \mathrm{d}x = 2\int_0^{\infty} x^{2n}|x|^{2\lambda+1} \exp\left(-[cx^6 + t(x^4 - x^2)]\right) \mathrm{d}x, \tag{11}$$

while the odd moments $\eta_{2n+1}(t;\lambda)$ are

$$\eta_{2n+1}(t;\lambda) = \int_{-\infty}^{\infty} x^{2n+1}|x|^{\alpha}|x|^{2\lambda+1} \exp\left(-[cx^6 + t(x^4 - x^2)]\right) \mathrm{d}x = 0, \quad n \in \mathbb{N}. \tag{12}$$

For $t = 0$, the moments for the symmetric weight $w(x) = |x|^{2\lambda+1} \exp\left(-cx^6\right)$, $c > 0$ are given by

$$\eta_{2n}(\lambda) = \frac{1}{3}\, c^{\frac{-2n-2\lambda-2}{6}}\, \Gamma\left(\frac{2n+2\lambda+2}{6}\right); \quad \eta_{2n+1}(\lambda) = 0.$$

However, finding the moments explicitly has not been an easy task for some semi-classical weights. We refer to an interesting work in [15], which shows that the moments for the modified sextic and dodic weights are expressed in terms of generalized hypergeometric functions and for quartic Freudian weights, see [12] (see also [18]).

The Hankel determinant of moments $\eta_k$ for the symmetric weight in (4) is defined by [11,32]

$$\mathcal{H}_n(t;\lambda) := \det\left[\int_{-\infty}^{\infty} x^{j+k}\, W_\lambda(x;t)\,\mathrm{d}x\right]_{j,k=0}^{n-1} = \det\left[\eta_{j+k}\right]_{j,k=0}^{n-1},$$

where

$$\eta_r(t;\lambda) = \int_{-\infty}^{\infty} x^r\, W_\lambda(x;t)\,\mathrm{d}x = \begin{cases} 0 & \text{if } r \text{ is odd,} \\ 2\int_0^{\infty} x^r\, W_\lambda(x;t)\,\mathrm{d}x & \text{if } r \text{ is even,} \end{cases}$$

The following factorization for the determinants holds for the given weight in (4):

$$\mathcal{H}_{2m}(t;\lambda) = \mathcal{H}_m^{(0)}(t;\lambda)\,\mathcal{H}_m^{(1)}(t;\lambda), \qquad \mathcal{H}_{2m+1}(t;\lambda) = \mathcal{H}_{m+1}^{(0)}(t;\lambda)\,\mathcal{H}_m^{(1)}(t;\lambda),$$

where $\mathcal{H}_m^{(0)}(t;\lambda)$ and $\mathcal{H}_m^{(1)}(t;\lambda)$ are the Hankel determinants of order $m$:

$$\mathcal{H}_m^{(\theta)}(t;\lambda) = \det\left[\int_0^\infty x^{2i+2j}\, W^{(\theta)}(x)\,\mathrm{d}x\right]_{i,j=0}^{m-1}, \qquad \theta = 0,1, \tag{13}$$

where

$$W^{(0)}(x,t) = 2W_\lambda(x;t), \qquad W^{(1)}(x,t) = 2x^2 W_\lambda(x;t).$$

The following lemma shows the recurrence coefficients in terms of moment determinants for symmetric weights in general. See the recent work by Clarkson and Jordaan [15] for its proof and the references therein.

**Lemma 1.** *Given $\mathcal{H}_m^{(0)}$ and $\mathcal{H}_m^{(1)}$ be the moment determinants as given in Equation (13), then the recurrence coefficient $\beta_n$ corresponding to the semi-classical weight (4) is given by*

$$\beta_{2n}(t;\lambda) = \frac{d}{dt}\ln\left(\frac{\mathcal{H}_n^{(1)}(t;\lambda)}{\mathcal{H}_n^{(0)}(t;\lambda)}\right); \qquad \beta_{2n+1}(t;\lambda) = \frac{d}{dt}\ln\left(\frac{\mathcal{H}_{n+1}^{(0)}(t;\lambda)}{\mathcal{H}_n^{(1)}(t;\lambda)}\right).$$

Since the modified Freud-type weight in (4) depends on the parameter $t$, which means that the recurrence coefficients, Hankel determinants as well as the polynomials themselves also rely on $t$. However, unless it is mandatory we do not always state the $t$-dependence. For the Hankel determinantal characterization of the recurrence coefficients in terms of tau functions, we refer to the recent work in [15] (see also [10] and the references therein).

### 2.2. Explicit Formulation of the First Few Polynomials

In view of Equation (6), the first few monic polynomials are given by

$$\begin{aligned}
\mathcal{S}_1(x;t,\lambda) &= x,\\
\mathcal{S}_2(x;t,\lambda) &= x^2 - \beta_1(t,\lambda),\\
\mathcal{S}_3(x;t,\lambda) &= x^3 - \big(\beta_1(t,\lambda) + \beta_2(t,\lambda)\big)x,\\
\mathcal{S}_4(x;t,\lambda) &= x^4 - \big(\beta_1(t,\lambda) + \beta_2(t,\lambda) + \beta_3(t,\lambda)\big)x^2 + \beta_1(t,\lambda)\beta_3(t,\lambda),\\
\mathcal{S}_5(x;t,\lambda) &= x^5 - \big(\beta_1(t,\lambda) + \beta_2(t,\lambda) + \beta_3(t,\lambda) + \beta_4(t,\lambda)\big)x^3\\
&\quad + \Big(\beta_1(t,\lambda)\beta_3(t,\lambda) + \beta_1(t,\lambda)\beta_4(t,\lambda) + \beta_2(t,\lambda)\beta_4(t,\lambda)\Big)x.
\end{aligned}$$

The following proposition, with its proof given for completeness, provides a concise formulation for modified Freudian polynomials $\mathcal{S}_n(x;t)$. For a similar result, see also [32–34].

**Proposition 2.** *For modified Freudian polynomials, we have the following concise formulation:*

$$\mathcal{S}_n(x;t) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \Psi_k(n)\, x^{n-2k}, \tag{14}$$

*where* $\Psi_0(n) = 1$ *and, for* $k \in \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$,

$$\Psi_k(n) = (-1)^k \sum_{j_1=1}^{n+1-2k} \beta_{j_1}(t;\lambda) \sum_{j_2=j_1+2}^{n+3-2k} \beta_{j_2}(t;\lambda) \sum_{j_3=j_2+2}^{n+5-2k} \beta_{j_3}(t;\lambda) \cdots \sum_{j_k=j_{k-1}+2}^{n-1} \beta_{j_k}(t;\lambda), \tag{15}$$

*with the recurrence coefficient* $\beta_j(t;\lambda)$ *given in Equation (7).*

**Proof.** From the fact that the Freudian polynomials $\mathcal{S}_n(x;t)$ are symmetric and monic of degree $n$; i.e.,

$$\mathcal{S}_n(-x;t) = (-1)^n \mathcal{S}_n(x;t),$$

we obtain, for a fixed $t \in \mathbb{R}$,

$$\mathcal{S}_{2n}(x;t) = \sum_{\ell=0}^{n} e_{2n-2\ell}\, x^{2n-2\ell}; \qquad \mathcal{S}_{2n+1}(x;t) = \sum_{\ell=0}^{n} e_{2n-2\ell+1}\, x^{2n-2\ell+1}, \tag{16}$$

where $e_{n-2k} = \Psi_k(n)$ with $\Psi_0(n) = 1$ and $\Psi_k(n) = 0$ for $k > \lfloor \frac{n}{2} \rfloor$. Substituting Equation (14) into the recursion relation Equation (6) and comparing the coefficients yields

$$\Psi_k(n+1) - \Psi_k(n) = -\beta_n(t;\lambda)\Psi_{k-1}(n-1). \tag{17}$$

By applying induction on $k$, we prove Equation (15). For $k = 1$, we see that

$$\Psi_1(n) - \Psi_1(n-1) = -\beta_{n-1}, \tag{18}$$

By using a telescopic sum on Equation (18), we obtain

$$\Psi_1(n) = -\sum_{j_1=0}^{n-1} \beta_{j_1}(t;\lambda), \text{ for every } n \geq 1.$$

Suppose that Equation (15) is true for values up to $k-1$ for every $n \in \mathbb{N}$, i.e.,

$$\Psi_{k-1}(n) = (-1)^{k-1} \sum_{j_1=1}^{n+3-2k} \beta_{j_1}(t;\lambda) \sum_{j_2=j_1+2}^{n+5-2k} \beta_{j_2}(t;\lambda) \sum_{j_3=j_2+2}^{n+7-2k} \beta_{j_3}(t;\lambda) \cdots \sum_{j_{k-1}=j_{k-2}+2}^{n-1} \beta_{j_{k-1}}(t;\lambda). \tag{19}$$

Now, iterating Equation (17), we obtain

$$\begin{aligned}
\Psi_k(n) &= \Psi_k(n-1) - \beta_{n-1}\Psi_{k-1}(n-2), \\
&= \Psi_k(n-2) - \beta_{n-2}\Psi_{k-1}(n-3) - \beta_{n-1}\Psi_{k-1}(n-2), \\
&= \Psi_k(n-3) - \beta_{n-3}\Psi_{k-1}(n-4) - \beta_{n-2}\Psi_{k-1}(n-3) - \beta_{n-1}\Psi_{k-1}(n-2), \\
&\qquad\vdots \\
&= -\beta_{2k-1}\Psi_{k-1}(2k-2) - \beta_{2k}\Psi_{k-1}(2k-1) - \cdots - \beta_{n-2}\Psi_{k-1}(n-3) - \beta_{n-1}\Psi_{k-1}(n-2).
\end{aligned} \tag{20}$$

Substituting Equation (19) into Equation (20) yields Equation (15). Thus, the result is valid for $k \in \mathbb{N}$. □

An alternate representation for Proposition 2 is given in the result below.

**Proposition 3.** *For monic modified Freudian polynomials $\mathcal{S}_n(x;t)$, we have*

$$\mathcal{S}_n(x;t) = x^n + \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^m \left( \sum_{k \in W(n,m)} \beta_{k_1}\beta_{k_2} \cdots \beta_{k_{m-1}}\beta_{k_m} \right) x^{n-2m},$$

*where*

$$W(n,m) = \{k \in \mathbb{N}^m \mid k_{j+1} \geq k_j + 2 \text{ for } 1 \leq j \leq m-1,\ 1 \leq k_1, k_m < n\},$$

*and*

$$\lfloor \frac{n}{2} \rfloor = \begin{cases} \frac{n}{2}, & n \text{ is even}, \\ \frac{n-1}{2}, & n \text{ is odd}. \end{cases}$$

The normalization constant $\hat{\zeta}_n$ in Equation (5) for the semi-classical weight (4) is given by

$$\hat{\zeta}_n = \langle \mathcal{S}_n, \mathcal{S}_n \rangle_{|x|^{2\lambda+1} \exp(-[cx^6+t(x^4-x^2)])} = \|\mathcal{S}_n\|^2_{|x|^{2\lambda+1} \exp(-[cx^6+t(x^4-x^2)])} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \Psi_k(n)\, \eta_{2n-2k}(t;\lambda), \tag{21}$$

where $\Psi_k(n)$ is given in Equation (15).

## 3. Recurrence Coefficients of the Sextic Freud-Type Polynomials

In this section, we derive a non-linear discrete equation satisfied by the recursion coefficient $\beta_n(t;\lambda)$ for the semi-classical weight (4), which is one of the main results of this paper.

**Theorem 4.** *The recurrence coefficient $\beta_n(t;\lambda)$ in Equation (6) satisfies the non-linear difference equation*

$$n + (2\lambda+1)\Omega_n = 6c\,[\beta_n\,(Q_{n-1} + Q_n + Q_{n+1}) + \beta_{n-1}\beta_n\beta_{n+1}] + 4tQ_n - 2t\beta_n \tag{22}$$

*where $\beta_0 = 0$ and $\beta_1(t;\lambda)$ is given by*

$$\beta_1(t;\lambda) = \frac{\|x^2\|_t^2}{\|1\|_t^2} = \frac{\eta_2(t;\lambda)}{\eta_0(t;\lambda)} = \frac{\displaystyle\int_{-\infty}^{\infty} x^2|x|^{2\lambda+1} \exp\left(-[cx^6 + t(x^4 - x^2)]\right) dx}{\displaystyle\int_{-\infty}^{\infty} |x|^{2\lambda+1} \exp\left(-[cx^6 + t(x^4 - x^2)]\right) dx},$$

*where the expressions $Q_n$ and $\Omega_n$ are, respectively, given by*

$$Q_n = \beta_n(t;\lambda)\,[\beta_{n-1}(t;\lambda) + \beta_n(t;\lambda) + \beta_{n+1}(t;\lambda)]\,, \tag{23}$$

*and*

$$\Omega_n = \frac{1-(-1)^n}{2} = \begin{cases} 1, & \text{for} \quad n \text{ is odd} \\ 0, & \text{for} \quad n \text{ is even}. \end{cases} \tag{24}$$

**Proof.** (i) By employing Freud's method in [16,20], we consider, for $t \in \mathbb{R}^+$,

$$\mathbb{I}_n = \frac{1}{\hat{\zeta}_n} \int_{-\infty}^{\infty} [\mathcal{S}_n(x;t)\, \mathcal{S}_{n-1}(x;t)]'\; W_\lambda(x;t)\, \mathrm{d}x, \tag{25}$$

where $\{\mathcal{S}_n(x;t)\}_{n\geq 0}$ are sequences of monic modified Freudian polynomials and $\hat{\zeta}_n$ is given in Equation (21). Then we have

$$\mathbb{I}_n = \frac{1}{\hat{\zeta}_n}\int_{-\infty}^{\infty}\left(\mathcal{S}_n'(x;t)\,\mathcal{S}_{n-1}(x;t) + \mathcal{S}_n(x;t)\,\mathcal{S}_{n-1}'(x;t)\right)\,W_\lambda(x;t)\;\mathrm{d}x$$
$$= \frac{1}{\hat{\zeta}_n}\int_{-\infty}^{\infty}(nx^{n-1} + R_{n-2})\,\mathcal{S}_{n-1}(x;t)\,W_\lambda(x;t)\,\mathrm{d}x = \frac{\hat{\zeta}_{n-1}}{\hat{\zeta}_n}n, \tag{26}$$

where $R_{n-2} \in \mathbb{P}_{n-2}$. On the other hand, by employing technique of integration on Equation (25), we obtain

$$\mathbb{I}_n\hat{\zeta}_n = \left[\mathcal{S}_n(x;t)\,\mathcal{S}_{n-1}(x;t)w_\lambda(x;t)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\mathcal{S}_n(x;t)\,\mathcal{S}_{n-1}(x;t)\,W_\lambda'(x;t)\;\mathrm{d}x$$
$$= -(2\lambda+1)\int_{-\infty}^{\infty}\frac{\mathcal{S}_n(x;t)\,\mathcal{S}_{n-1}(x;t)}{x}\,W_\lambda(x;t)\;\mathrm{d}x + 6c\int_{-\infty}^{\infty}x^5\mathcal{S}_n(x;t)\,\mathcal{S}_{n-1}(x;t)\,W_\lambda(x;t)\;\mathrm{d}x$$
$$+ 4t\int_{-\infty}^{\infty}x^3\mathcal{S}_n(x;t)\,\mathcal{S}_{n-1}(x;t)\;\;W_\lambda(x;t)\;\mathrm{d}x - 2t\int_{-\infty}^{\infty}x\mathcal{S}_n(x;t)\,\mathcal{S}_{n-1}(x;t)\;\;W_\lambda(x;t)\;\mathrm{d}x, \tag{27}$$

where $\left[\mathcal{S}_n(x;t)\,\mathcal{S}_{n-1}(x;t)W_\lambda(x;t)\right]_{-\infty}^{\infty} = 0$, since the boundary terms vanish as the expression $-cx^6 - t(x^4 - x^2)$ (for $c > 0$) in the weight $W_\lambda(x;t)$ only consists of even powers of $x$ and hence will dominate the limit as $x \to \pm\infty$.

Since the Freudian measure given in (4) is symmetric, the integral expression

$$\int_{-\infty}^{\infty}\mathcal{S}_n(x;t)\,\mathcal{S}_{n-1}(x;t)\,\frac{1}{x}\,W_\lambda(x;t)\;\mathrm{d}x = 0, \tag{28a}$$

when $n$ is even. For $n$ is odd, the expression $\dfrac{\mathcal{S}_n(x;t)}{x}$ is a polynomial of degree $n-1$ and hence

$$\int_{-\infty}^{\infty}\frac{\mathcal{S}_n(x;t)}{x}\,\mathcal{S}_{n-1}(x;t)\,W_\lambda(x;t)\;\mathrm{d}x = \hat{\zeta}_{n-1}. \tag{28b}$$

Thus,

$$\int_{-\infty}^{\infty}\frac{\mathcal{S}_{n-1}(x;t)\,\mathcal{S}_n(x;t)}{x}\,W_\lambda(x;t)\;\mathrm{d}x = \Omega_n\hat{\zeta}_{n-1}, \tag{28c}$$

where $\Omega_n$ is given in Equation (24). Using the recurrence iterated relation from Equation (6), we do have

$$x^2\mathcal{S}_n(x;t) = \mathcal{S}_{n+2}(x;t) + (\beta_n + \beta_{n+1})\,\mathcal{S}_n(x;t) + \beta_n\beta_{n-1}\mathcal{S}_{n-2}(x;t), \tag{29a}$$

$$x^3\mathcal{S}_n(x;t) = \mathcal{S}_{n+3}(x;t) + (\beta_n + \beta_{n+1} + \beta_{n+2})\,\mathcal{S}_{n+1}(x;t)$$
$$+ \beta_n\,(\beta_{n-1} + \beta_n + \beta_{n+1})\,\mathcal{S}_{n-1}(x;t) + \beta_n\beta_{n-1}\,\beta_{n-2}\mathcal{S}_{n-3}(x;t), \tag{29b}$$

$$x^4\mathcal{S}_n(x;t) = \mathcal{S}_{n+4}(x;t) + (\beta_n + \beta_{n+1} + \beta_{n+2} + \beta_{n+3})\mathcal{S}_{n+2}(x;t)$$
$$+ \left[\beta_n(\beta_{n-1} + \beta_n + \beta_{n+1}) + \beta_{n+1}(\beta_n + \beta_{n+1} + \beta_{n+2})\right]\mathcal{S}_n(x;t)$$
$$+ \beta_n\beta_{n-1}(\beta_{n-2} + \beta_{n-1} + \beta_n + \beta_{n+1})\mathcal{S}_{n-2}(x;t) + (\beta_n\beta_{n-1}\beta_{n-2}\beta_{n-3})\mathcal{S}_{n-4}(x;t), \tag{29c}$$

$$\begin{aligned} x^5\mathcal{S}_n(x;t) = {} & \mathcal{S}_{n+5}(x;t) + (\beta_n+\beta_{n+1}+\beta_{n+2}+\beta_{n+3}+\beta_{n+4})\,\mathcal{S}_{n+3}(x;t) \\ & + [\beta_n\,(Q_{n-1}+Q_n+Q_{n+1}) + \beta_{n-1}\beta_n\beta_{n+1}]\,\mathcal{S}_{n+1}(x;t) \\ & + [\beta_n\beta_{n-2}Q_{n-1} + \beta_{n-2}\beta_{n-1}\beta_n\beta_{n+1} + \beta_n\beta_{n-1}\beta_{n-2}\beta_{n-3}]\,\mathcal{S}_{n-3}(x;t) \\ & + (\beta_n\beta_{n-1}\beta_{n-2}\beta_{n-3}\beta_{n-4})\,\mathcal{S}_{n-5}(x;t), \end{aligned} \tag{29d}$$

where $Q_n$ is given in Equation (23). Applying the identities in Equation (29) and the Pearson equation for the weight (4) together with Equation (28) into Equation (27), we obtain

$$\begin{aligned} n\hat{\zeta}_{n-1} = \mathbb{I}_n\hat{\zeta}_n = {} & 6c\,[(\beta_nQ_{n+1}+\beta_n\beta_{n-1}\beta_{n-2}) + (\beta_n+\beta_{n-1})\,Q_n]\,\hat{\zeta}_{n-1} \\ & + 4t\,[\beta_n\,(\beta_{n-1}+\beta_n+\beta_{n+1})]\,\hat{\zeta}_{n-1} - 2t\beta_n\hat{\zeta}_{n-1} - (2\lambda+1)\Omega_n\hat{\zeta}_{n-1}, \end{aligned} \tag{30}$$

which simplifies to

$$n + (2\lambda+1)\Omega_n = 6c\,[(\beta_n+\beta_{n-1})\,Q_n + (\beta_nQ_{n+1}+\beta_n\beta_{n-1}\beta_{n-2})] + 4t\,[\beta_n\,(\beta_{n-1}+\beta_n+\beta_{n+1})] - 2t\beta_n, \tag{31}$$

from the fact that $\hat{\zeta}_{n-1} \neq 0$ and $\Omega_n$ is given in Equation (24). Note that Equations (30) and (26) yield Equation (22).
□

**Remark 5.** *Quite similar non-linear discrete equations like Equation (31) can be given in [10,12,17,26].*

We obtain the following result for recurrence coefficients for the modified Freudian weight in (4).

**Theorem 6.** *The recurrence coefficients $\beta_n(t;\lambda)$ obey the differential-difference formula*

$$\frac{d\beta_n}{dt} = \beta_n\big[\,(\beta_{n+1}-Q_{n+1}) - (\beta_{n-1}-Q_{n-1})\,\big], \tag{32}$$

*where the expression $Q_n$ is given by Equation (23).*

**Proof.** By differentiating the normalization constant $\hat{\zeta}_n(t)$ with respect to $t$, we have

$$\begin{aligned} \frac{d\hat{\zeta}_n}{dt} &= 2\int_{\mathbb{R}} \frac{d\mathcal{S}_n(x;t)}{dt}\,\mathcal{S}_n(x;t)\,W_\lambda(x;t)\,\mathrm{d}x + \int_{\mathbb{R}}(x^2-x^4)\,\mathcal{S}_n^2(x;t)\,W_\lambda(x;t)\,\mathrm{d}x, \\ &= \int_{\mathbb{R}} x^2\,\mathcal{S}_n^2(x;t)\,W_\lambda(x;t)\,\mathrm{d}x - \int_{\mathbb{R}} x^4\,\mathcal{S}_n^2(x;t)\,W_\lambda(x;t)\,\mathrm{d}x, \end{aligned} \tag{33}$$

where the first integral vanishes by orthogonality since $\frac{d\mathcal{S}_n(x;t)}{dt}$ is a monic polynomial in $x$ of degree $n-1$. By using orthogonality and the recurrence relation in Equation (6), we rewrite Equation (33)

$$\frac{d\hat{\zeta}_n}{dt} = (\beta_n+\beta_{n+1})\,\hat{\zeta}_n - (Q_n+Q_{n+1})\,\hat{\zeta}_n = [(\beta_n-Q_n) + (\beta_{n+1}-Q_{n+1})]\,\hat{\zeta}_n, \tag{34}$$

On the other hand, applying differentiation on both sides of Equation (7) with respect to $t$ gives

$$\frac{d\beta_n}{dt} = \frac{d}{dt}\left(\frac{\hat{\zeta}_n}{\hat{\zeta}_{n-1}}\right) = \beta_n\left[\frac{d}{dt}\ln\hat{\zeta}_n - \frac{d}{dt}\ln\hat{\zeta}_{n-1}\right] = \beta_n\Big[\,(\beta_{n+1}-\beta_{n-1}) - [Q_{n+1}-Q_{n-1}]\,\Big],$$

and substituting Equation (34) into above equation, we obtain the desired result. □

**Remark 7.** *The discrete equation in Equation (6) is also known to be non-linear discrete mKdV-like equation (see [26] for similar equations).*

Our next result provides a higher order non-linear differential-recurrence relation satisfied by the recursion coefficients $\beta_n(t;\lambda)$ in reference to the semi-classical weight (4).

**Theorem 8.** *For the modified Freud-type weight in (4), the recursion coefficients $\beta_n(t;\lambda)$ satisfy the following differential-recurrence equation*

$$\begin{aligned}
\frac{d^2\beta_n}{dt^2} &= \frac{1}{6c}\left[n+(2\lambda+1)\Omega_n-\vartheta(t)\right]+\left(-\beta_{n-1}-\beta_{n+1}\right)\beta_n^4\\
&+\left(-\beta_{n-2}\beta_{n-1}-\beta_{n-1}^2-6\beta_{n-1}\beta_{n+1}-\beta_{n+1}^2-\beta_{n+1}\beta_{n+2}+2\beta_{n-1}+2\beta_{n+1}\right)\beta_n^3\\
&+\Bigg(\beta_{n-3}\beta_{n-2}\beta_{n-1}+\beta_{n-2}^2\beta_{n-1}+2\beta_{n-2}\beta_{n-1}^2-4\beta_{n-2}\beta_{n-1}\beta_{n+1}+\beta_{n-1}^3-5\beta_{n-1}^2\beta_{n+1}-4\beta_{n-1}\beta_{n+1}\beta_{n+2}\\
&\quad-5\beta_{n-1}\beta_{n+1}^2+\beta_{n+1}^3+2\beta_{n+1}^2\beta_{n+2}+\beta_{n+1}\beta_{n+2}^2+\beta_{n+1}\beta_{n+2}\beta_{n+3}+8\beta_{n-1}\beta_{n+1}-\beta_{n-1}-\beta_{n+1}\Bigg)\beta_n^2\\
&+\Bigg(\beta_{n-4}\beta_{n-3}\beta_{n-2}\beta_{n-1}+\beta_{n-3}^2\beta_{n-2}\beta_{n-1}+2\beta_{n-3}\beta_{n-2}^2\beta_{n-1}+2\beta_{n-3}\beta_{n-2}\beta_{n-1}^2+\beta_{n-2}^3\beta_{n-1}\\
&\quad+3\beta_{n-2}^2\beta_{n-1}^2+3\beta_{n-2}\beta_{n-1}^3-2\beta_{n-2}\beta_{n-1}\beta_{n+1}^2-2\beta_{n-2}\beta_{n-1}\beta_{n+1}\beta_{n+2}+\beta_{n-1}^4-2\beta_{n-1}^2\beta_{n+1}^2\\
&\quad-2\beta_{n-1}^2\beta_{n+1}\beta_{n+2}+\beta_{n+1}^4+3\beta_{n+1}^3\beta_{n+2}+3\beta_{n+1}^2\beta_{n+2}^2+2\beta_{n+1}^2\beta_{n+2}\beta_{n+3}+\beta_{n+1}\beta_{n+2}^3\\
&\quad+2\beta_{n+1}\beta_{n+2}^2\beta_{n+3}+\beta_{n+1}\beta_{n+2}\beta_{n+3}^2-2\beta_{n-2}^2\beta_{n-1}+\beta_{n+1}\beta_{n+2}\beta_{n+3}\beta_{n+4}-2\beta_{n-3}\beta_{n-2}\beta_{n-1}\\
&\quad-4\beta_{n-2}\beta_{n-1}^2+2\beta_{n-2}\beta_{n-1}\beta_{n+1}-2\beta_{n-1}^3+2\beta_{n-1}^2\beta_{n+1}+2\beta_{n-1}\beta_{n+1}^2+2\beta_{n-1}\beta_{n+1}\beta_{n+2}-2\beta_{n+1}^3\\
&\quad-4\beta_{n+1}^2\beta_{n+2}-\beta_n^2-2\beta_{n+1}\beta_{n+2}^2-2\beta_{n+1}\beta_{n+2}\beta_{n+3}-2\beta_{n-1}\beta_{n+1}-2\beta_n\beta_{n-1}-2\beta_n\beta_{n+1}-\beta_{n+1}\beta_{n-1}\Bigg)\beta_n, \qquad \text{(35a)}
\end{aligned}$$

*where $\Omega_n$ and $Q_n$ are given in Equations (24) and (23), respectively and the expression $\vartheta(t)$ is given by*

$$\vartheta(t) = 4tQ_n - 2\beta_n t = 2t\beta_n\left[2(\beta_{n-1}+\beta_n+\beta_{n+1})-1\right]. \qquad \text{(35b)}$$

**Proof.** The above result is obtained by applying differentiation of Equation (32) with respect to $t$; i.e.,

$$\frac{d^2\beta_n}{dt^2} = \frac{d}{dt}\left(\beta_n\Big[(\beta_{n+1}-Q_{n+1})-[\beta_{n-1}-Q_{n-1}]\Big]\right) = \frac{d}{dt}\left(\beta_n\Big[(\beta_{n+1}-\beta_{n-1})-[Q_{n+1}-Q_{n-1}]\Big]\right),$$

together with employing the following identity (22)

$$\beta_n\left(\beta_{n-1}^2+\beta_{n+1}^2+\beta_{n-1}\beta_{n-2}+\beta_{n+1}\beta_{n+2}\right) = \frac{1}{6c}\left[n+\gamma\Omega_n-\vartheta(t)\right]-\beta_n\left(\beta_n^2+2\beta_n\beta_{n-1}+2\beta_n\beta_{n+1}+\beta_{n+1}\beta_{n-1}\right),$$

where $\gamma := 2\lambda+1$ and $\Omega_n$ is given in (24) and the expression $\vartheta(t)$ is given in (35b). $\square$

## 4. The Differential-Difference Equation Satisfied by Modified Sextic Freud-Type Polynomials

In view of an extension of the ladder operators technique given in [35], Chen and Feigin [13] examined the weight $\widetilde{w}(x)|x-t|^K$, $x,t,K\in\mathbb{R}$, for any smooth reference weight $\widetilde{w}(x)$. They indicated that when $\widetilde{w}(x)$ is the Gaussian (Hermite) weight ($\mathrm{e}^{-x^2}, x\in\mathbb{R}$), the recurrence coefficients fulfill a particular two-parameter Painlevé IV condition. Filipuk et al. [23] found that the recurrence coefficients for the quartic Freudian weight $|x|^{2\alpha+1}\mathrm{e}^{-x^4+tx^2}$, $x,t\in\mathbb{R}$, $\alpha>-1$ are identified with solutions of the Painlevé IV and the first discrete Painlevé equation. Clarkson et al. [12,15] gave a methodical investigation on Freud weights and some generalized work for [13] (see also [18,36]).

The following result gives the differential-difference equation (i.e., lowing operator) satisfied by the modified Freud-type polynomials corresponding to the weight given in Equation (4). This result works also for semi-classical polynomials $\Phi_n(z)$ that are orthogonal with reference to a general symmetric weight in the semi-classical class, and specifically, it holds for the modified Freudian polynomials $\mathcal{S}_n(x,t)$.

**Lemma 2.** *For $\alpha \geq 1$, consider a sequence of monic polynomials $\Phi_n(z)$, which are orthogonal with respect to the semi-classical weight of the form*

$$w(z) = |z|^{\alpha}\, w_0(z), \tag{36}$$

*(where the modified weight in (4) is a special case when $w_0(z) := \exp\left(-v_0(z)\right)$ with $v_0(z) := cz^6 + t(z^4 - z^2)$, $c > 0$ on $\mathbb{R}$). Then the orthogonal polynomials $\Phi_n(z)$ associated with the weight in (36) satisfy the differential-difference relation*

$$\Phi_n'(z) = \beta_n\, A_n(z)\, \Phi_{n-1}(z) - B_n(z)\, \Phi_n(z), \tag{37a}$$

*where*

$$A_n(z) := \frac{1}{\zeta_n} \int_{-\infty}^{\infty} \frac{v_0'(z) - v_0'(y)}{z-y}\, \Phi_n^2(y)\, w(y)\, \mathrm{d}y, \tag{37b}$$

$$B_n(z) := \frac{1}{\zeta_{n-1}} \int_{-\infty}^{\infty} \frac{v_0'(z) - v_0'(y)}{z-y}\, \Phi_n(y)\, \Phi_{n-1}(y)\, w(y)\, \mathrm{d}y + \frac{\alpha\, [1-(-1)^n]}{2z}. \tag{37c}$$

**Proof.** Since the derivative of $\Phi_n(z)$ is a polynomial of degree $n-1$ in $z$, hence $\Phi_n'(z)$ can be given by

$$\Phi_n'(z) = \sum_{j=0}^{n-1} c_{n,j}\, \Phi_j(z). \tag{38}$$

From orthogonality and integrating by parts, we have

$$c_{n,j} = \frac{1}{\zeta_j} \int_{-\infty}^{\infty} \Phi_n'(y)\, \Phi_j(y)\, w(y) dy = \frac{1}{\zeta_j} \int_{-\infty}^{\infty} \Phi_n(y)\, \Phi_j(y) \left[ v_0'(y) - \frac{\alpha}{y} \right] w(y) dy, \tag{39}$$

where

$$v_0(y) := \left[ cy^6 + t(y^4 - y^2) \right].$$

By substituting Equation (39) into Equation (38) and summing over $j$ applying the Christoffel-Darboux formula [25]

$$\sum_{j=0}^{n-1} \frac{\Phi_j(z)\, \Phi_j(y)}{\zeta_j} = \frac{\Phi_n(z)\, \Phi_{n-1}(y) - \Phi_n(y) \Phi_{n-1}(z)}{(z-y)\zeta_{n-1}},$$

we have

$$\begin{aligned}
\Phi_n'(z) &= \int_{-\infty}^{\infty} \Phi_n(y) \sum_{j=0}^{n-1} \frac{\Phi_j(z)\, \Phi_j(y)}{\zeta_j} \left[ v_0'(y) - \frac{\alpha}{y} \right] w(y)\ dy, \\
&= \frac{\Phi_{n-1}(z)}{\zeta_{n-1}} \int_{-\infty}^{\infty} \Phi_n^2(y)\, \frac{v_0'(z) - v_0'(y)}{z-y}\, w(y)\ dy - \frac{\alpha}{z} \frac{\Phi_n(z)}{\zeta_{n-1}} \int_{-\infty}^{\infty} \frac{\Phi_n(y)\, \Phi_{n-1}(y)}{y} w(y)\ dy, \\
&\quad - \frac{\Phi_n(z)}{\zeta_{n-1}} \int_{-\infty}^{\infty} \Phi_n(y)\, \Phi_{n-1}(y) \frac{v_0'(z) - v_0'(y)}{z-y} w(y) dy.
\end{aligned} \tag{40}$$

Besides, by an inductive argument based on the recursion relation in Equation (6) with the initial conditions $\Phi_0(z) := 1$, and $\beta_0\, \Phi_{-1}(z) = 0$, we obtain

$$\begin{aligned}
\frac{1}{\zeta_{n-1}} \int_{-\infty}^{\infty} \frac{\Phi_n(y)\, \Phi_{n-1}(y)}{y}\, w(y)\, \mathrm{d}y &= \frac{1}{\zeta_{n-1}} \int_{-\infty}^{\infty} \frac{[y\, \Phi_{n-1}(y) - \beta_{n-1}\, \Phi_{n-2}(y)]\ \Phi_{n-1}(y)}{y} w(y) dy \\
&= \begin{cases} 1, & n = 1, \\ 1 - \dfrac{1}{\zeta_{n-2}} \displaystyle\int_{-\infty}^{\infty} \frac{\Phi_{n-1}(y)\Phi_{n-2}(y)}{y} w(y) dy, & n \geq 2, \end{cases} \\
&= \begin{cases} 0, & n = 2, \\ \dfrac{1}{\zeta_{n-3}} \displaystyle\int_{-\infty}^{\infty} \frac{\Phi_{n-2}(y)\, \Phi_{n-3}(y)}{y} w(y) \mathrm{d}y, & n \geq 3, \end{cases} \\
&\ \ \vdots \\
&= \begin{cases} 0, & n \text{ even}, \\ 1, & n \text{ odd}. \end{cases}
\end{aligned} \tag{41}$$

Therefore, (noting that $\beta_n = \zeta_n/\zeta_{n-1}$) the result follows from Equations (40) and (41) immediately. □

**Lemma 3.** *The ladder coefficients $A_n(z)$ and $B_n(z)$ defined by Lemma 2 obey the relation*

$$A_n(z) = \frac{v_0'(z)}{z} + \frac{B_n(z) + B_{n+1}(z)}{z} - \frac{\alpha}{z^2}. \tag{42}$$

**Proof.** From the definition of $A_n(z)$, we have that

$$\begin{aligned}
A_n(z) =& \frac{1}{z\zeta_n} \left\{ \int_{-\infty}^{\infty} \frac{v_0'(z) - v_0'(y)}{z - y}\, y\, \Phi_n^2(y) w(y) \mathrm{d}y + \int_{-\infty}^{\infty} \left[v_0'(z) - v_0'(y)\right]\ \Phi_n^2(y)\, w(y)\, \mathrm{d}y \right\} \\
=& \frac{1}{z\zeta_n} \left\{ \int_{-\infty}^{\infty} \frac{v_0'(z) - v_0'(y)}{z - y} \left[\Phi_{n+1}(y) + \beta_n\, \Phi_{n-1}(y)\right]\ \Phi_n(y)\, w(y)\, \mathrm{d}y + v_0'(z)\, \zeta_n \right. \\
& \left. - \int_{-\infty}^{\infty} \Phi_n^2(y) \left[\frac{\alpha}{y} w(y) - w'(y)\right] \mathrm{d}y \right\} \\
=& \frac{v_0'(z)}{z} + \frac{1}{z} \left\{ B_{n+1}(z) - \frac{\alpha}{2z} \left[1 - (-1)^{n+1}\right] + B_n(z) - \frac{\alpha}{2z} \left[1 - (-1)^n\right] \right\} \\
=& \frac{v_0'(z)}{z} + \frac{B_n(z) + B_{n+1}(z)}{z} - \frac{\alpha}{z^2},
\end{aligned}$$

which completes the proof. □

**Remark 9.** *Equation (42) is the well-known supplementary condition (S1), which given in [25,35]. The other supplementary condition [25] is given by*

$$1 + (z - \alpha_n)\, (B_{n+1}(z) - B_n(z)) = \beta_{n+1}\, A_{n+1}(z) - \beta_n\, A_{n-1}(z), \tag{43}$$

*where $v(z) = -\ln w(z)$, since $w(x)$ is non-negative in an interval $[a, b] \subseteq \mathbb{R}$.*

We next recall an important identity involving $\sum_{k=0}^{n-1} A_k(z)$ by combining Equations (42) and (43).

**Lemma 4.** *([14], Lemma 1). The functions $A_n(z)$, $B_n(z)$ and $\sum_{k=0}^{n-1} A_k(z)$ satisfy the identity*

$$B_n^2(z) + v'(z)\, B_n(z) + \sum_{k=0}^{n-1} A_k(z) = \beta_n\, A_n(z)\, A_{n-1}(z). \tag{44}$$

**Lemma 5.** *([35], Theorem 2.2). The monic polynomials $\Phi_n(z)$ that are orthogonal with respect to the semi-classical weight in Equation (36) satisfy a linear second-order differential equation*

$$\frac{d^2\Phi_n(z)}{dz^2} + \left(-v'(z) - \frac{A_n'(z)}{A_n(z)}\right)\frac{d\Phi_n(z)}{dz} + \left(B_n'(z) - B_n(z)\frac{A_n'(z)}{A_n(z)} + \sum_{j=0}^{n-1} A_j(z)\right)\Phi_n(z) = 0,$$

### *4.1. Lowering Operator for the Modified Freud-Type Weight in (4)*

For the modified Freud-type weight in (4),

$$v(x) = -\ln W_\lambda(x;t) = -(2\lambda+1)\ln|x| + cx^6 + t(x^4 - x^2), \tag{45}$$

for $x \in \mathbb{R}$, we obtain

$$v'(x) = -\frac{(2\lambda+1)}{x} + 6cx^5 + t(4x^3 - 2x),$$

and hence

$$\psi(x,y) := \frac{v'(x) - v'(y)}{x-y} = \frac{2\lambda+1}{xy} + 6c\{x^4 + x^3y + x^2y^2 + xy^3 + y^4\} + 4t(x^2 + xy + y^2) - 2t. \tag{46}$$

**Theorem 10.** *For monic modified Freudian polynomials $\mathcal{S}_n(x;t)$, we have the differential-difference relation*

$$\mathcal{S}_n'(x;t) = -B_n(x;t)\,\mathcal{S}_n(x;t) + \beta_n(t)\,A_n(x;t)\,\mathcal{S}_{n-1}(x;t),$$

*where*

$$A_n(x;t) = 6cx^4 + 6c(\beta_n + \beta_{n+1})x^2 + 6c\,(Q_{n+1} + Q_n) + 4tx^2 + 4t(\beta_n + \beta_{n+1}) - 2t, \tag{47a}$$

$$B_n(x;t) = 6c\beta_n x^3 + 6cQ_n x + 4tx\beta_n + \left(\frac{2\lambda+1}{x}\right)\Omega_n, \tag{47b}$$

*where the expressions $Q_n$ and $\Omega_n$ are given in Equations (23) and (24) respectively.*

**Proof.** From Equations (37b) and (46), we have that

$$\begin{aligned} A_n(x;t) &= \frac{1}{\hat{\zeta}_n}\int_{\mathbb{R}} \mathcal{S}_n^2(y)\,\psi(x,y)\,W_\lambda(y;t)\,dy \\ &= \frac{1}{\hat{\zeta}_n}\int_{\mathbb{R}} \mathcal{S}_n^2(y)\Big(\frac{2\lambda+1}{xy} + 6c\{x^4 + x^3y + x^2y^2 + xy^3 + y^4\} + 4t(x^2 + xy + y^2) - 2t\Big)W_\lambda(y;t)\,dy \\ &= 6cx^4 + 6c(\beta_n + \beta_{n+1})x^2 + 6c\,(Q_{n+1} + Q_n) + 4tx^2 + 4t(\beta_n + \beta_{n+1}) - 2t, \end{aligned} \tag{48}$$

and the integrand in Equation (48) is odd and hence it vanishes.

Similarly, from Equation (37c), using orthogonality and three-term recurrence relation Equation (6), we obtain

$$\begin{aligned} B_n(x;t) &= \frac{1}{\hat{\zeta}_{n-1}} \int_{\mathbb{R}} \mathcal{S}_n(y)\, \mathcal{S}_{n-1}(y) \left( \frac{2\lambda+1}{xy} + 6c\{x^4 + x^3y + x^2y^2 + xy^3 + y^4\} + 4t(x^2+xy+y^2) - 2t \right) W_\lambda(y;t)\, dy \\ &= 6c\beta_n x^3 + 6cQ_n x + 4tx\beta_n + \left( \frac{2\lambda+1}{x} \right) \Omega_n, \end{aligned} \tag{49}$$

where the expressions for $Q_n$ and $\Omega_n$ are given respectively in (23) and (24). □

*4.2. More Non-Linear Difference Equations for the Recursion Coefficients*

This subsection provides several nonlinear difference equations fullfilled by the recurrence coefficients corresponding to the weight (4).

**Theorem 11.** *For sextic Freud-type weight (4), the recurrence coefficients $\beta_n$ in Equation (6) satisfy the following system of difference equations*

$$\beta_{n+1}\Big[6c\,(Q_{n+2}+Q_{n+1}) - 2t\Big] - \beta_n\Big[6c\,(Q_n + Q_{n-1}) - 2t\Big] + 4t\,(Q_{n+1} - Q_n) = 1 + (2\lambda+1)(-1)^n, \tag{50}$$

$$n + (2\lambda+1)\Omega_n = 6c\beta_n\,(Q_{n-1} + Q_n + Q_{n+1}) + 4tQ_n + 6c\beta_{n-1}\beta_n\beta_{n+1} - 2t\beta_n, \tag{51}$$

$$\begin{aligned} &Q_n\,(6cQ_n - 2t) + \beta_n\,(2\gamma\Omega_n - 1) + \sum_{k=0}^{n-1} (\beta_k + \beta_{k+1}) \\ &\quad = \beta_n \Big[ (\beta_n + \beta_{n+1})\,[6cQ_n + 6cQ_{n-1} - 2t] + (\beta_n + \beta_{n-1})\,[6cQ_n + 6cQ_{n+1} - 2t] + 4t\,(Q_n + \beta_{n-1}\beta_{n+1}) \Big], \end{aligned} \tag{52}$$

$$\begin{aligned} &\beta_n\,(6cQ_{n-1} + 6cQ_n - 2t)\,(6cQ_{n+1} + 6cQ_n - 2t) \\ &\quad = 6c \sum_{k=0}^{n-1} (Q_k + Q_{k+1}) + (6cQ_n\gamma + 4t\beta_n\gamma - 4t\beta_n)\,(2\Omega_n - 1) - 4t\,[Q_n\,(6cQ_n - 2t)] - 2t\,(n + \gamma\Omega_n), \end{aligned} \tag{53}$$

*where $\gamma = 2\lambda + 1$, $\Omega_n$ and $Q_n$ are given in Equations (23) and (24) respectively.*

**Proof.** Substituting Equations (48) and (49) into Equation (43) and taking into account the modified Freud-type weight (4) is symmetric; i.e., $\alpha_n = 0$, and comparing the constant coefficients, we obtain Equation (50). If we also substitute Equations (48) and (49) into Equation (44) together with Equation (45), and comparing the coefficients of $x^4, x^2, x^0$, we obtain Equations (51), (52) and (53) respectively. □

**Remark 12.** *It is important to acknowledge the work of Clarkson and Jordaan [15] on differential-difference and differential equations as we have similar results (see also [12] for the quartic case). In this contribution, our choice of the modified Freud-type weight (4) is a more general measure modification by $d\mu(x;t) = e^{t(x^4-x^2)}d\mu(x;0)$, (a similar one is given in [21]); while the one in [15] is measure deformation by $d\mu(x;t) = e^{tx^2}d\mu(x;0)$. This choice led us to a different version of results of new type of Toda-like relations, new non-linear recursion relations, differential equations and some properties of the zeros for the polynomials. For the semi-classical weight under consideration, we have obtained differential-recurrence relations using two different methods; the method ladder operators and Shohat's quasi-orthogonality method [37]. In the following subsection, we show that both methods provide us same results.*

### 4.3. The Method of Quasi-Orthogonality Due to J. A. Shohat

Shohat [37] gave a strategy using quasi-orthogonality to determine Equation (37a) for semi-classical weight functions $w(x;t)$ with the fact that $w'(x;t)/w(x;t)$ is a rational function that we apply to the modified Freudian weight in (4) ([12], Section 4.5). In recent years, this method has gained the attention of some experts including Bonan, Freud, Mhaskar and Nevai to list, but a few. See [5] for more details. The concept of quasi-orthogonality is studied in [9] (see also [37,38]). By using the ideas given in [5,37], one can see that the monic polynomials $\mathcal{S}_n(x;t)$ that are orthogonal with respect to the modified Freudian weight (4) are quasi-orthogonal of order $m = 7$ and henceforth we can write

$$x\frac{d\mathcal{S}_n}{dx}(x;t) = \sum_{j=n-6}^{n} \mathfrak{f}_{n,j}\, \mathcal{S}_j(x;t), \tag{54}$$

where the expression $\mathfrak{f}_{n,j}$ is obtained by

$$\mathfrak{f}_{n,j} = \frac{1}{\zeta_j}\int_{-\infty}^{\infty} x\,\left(\frac{d}{dx}\mathcal{S}_n(x;t)\right)\,\mathcal{S}_j(x;t)\,W_\lambda(x;t)\,\mathrm{d}x, \tag{55}$$

with $n-6 \le j \le n$ and $\zeta_j \ne 0$. By using the technique of integration, which gives for $n-6 \le j \le n-1$,

$$\begin{aligned}
\zeta_j\,\mathfrak{f}_{n,j} &= \Big[x\,\mathcal{S}_j(x;t)\,\mathcal{S}_n(x;t)\,W_\lambda(x;t)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{d}{dx}\left[x\mathcal{S}_j(x;t)W_\lambda(x;t)\right]\mathcal{S}_n(x;t)\,\mathrm{d}x\\
&= -\int_{-\infty}^{\infty}\left[\mathcal{S}_n(x;t)\,\mathcal{S}_j(x;t) + x\,\mathcal{S}_n(x;t)\,\frac{d\mathcal{S}_j}{dx}(x;t)\right]\,W_\lambda(x;t)\,\mathrm{d}x - \int_{-\infty}^{\infty} x\mathcal{S}_n(x;t)\,\mathcal{S}_j(x;t)\,\frac{dW_\lambda}{dx}(x;t)\,\mathrm{d}x,\\
&= -\int_{-\infty}^{\infty} x\mathcal{S}_n(x;t)\,\mathcal{S}_j(x;t)\,\frac{dW_\lambda}{dx}(x;t)\,\mathrm{d}x\\
&= -\int_{-\infty}^{\infty}\mathcal{S}_n(x;t)\,\mathcal{S}_j(x;t)\,\Big(-6cx^6 - 4tx^4 + 2tx^2 + 2\lambda + 1\Big)W_\lambda(x;t)\,\mathrm{d}x\\
&= \int_{-\infty}^{\infty}\left(6cx^6 + 4tx^4 - 2tx^2 - (2\lambda+1)\right)\,\mathcal{S}_n(x;t)\,\mathcal{S}_j(x;t)\,W_\lambda(x;t)\,\mathrm{d}x,
\end{aligned} \tag{56}$$

since

$$x\,\frac{dW_\lambda}{dx}(x;t) = \big[-6cx^6 - 4tx^4 + 2tx^2 + 2\lambda + 1\big]W_\lambda(x;t).$$

The following relations follow from iterating the three-term recurrence relation Equation (6):

$$x^2\mathcal{S}_n(x;t) = \mathcal{S}_{n+2}(x;t) + (\beta_n + \beta_{n+1})\,\mathcal{S}_n(x;t) + \beta_n\beta_{n-1}\mathcal{S}_{n-2}(x;t), \tag{57a}$$

$$\begin{aligned}
x^4\mathcal{S}_n(x;t) &= \mathcal{S}_{n+4}(x;t) + (\beta_n + \beta_{n+1} + \beta_{n+2} + \beta_{n+3})\,\mathcal{S}_{n+2}(x;t)\\
&\quad + \left[\beta_n(\beta_{n-1} + \beta_n + \beta_{n+1}) + \beta_{n+1}(\beta_n + \beta_{n+1} + \beta_{n+2})\right]\mathcal{S}_n(x;t)\\
&\quad + \beta_n\beta_{n-1}(\beta_{n-2} + \beta_{n-1} + \beta_n + \beta_{n+1})\mathcal{S}_{n-2}(x;t) + (\beta_n\beta_{n-1}\beta_{n-2}\beta_{n-3})\mathcal{S}_{n-4}(x;t).
\end{aligned} \tag{57b}$$

$$\begin{aligned}
x^6\mathcal{S}_n(x;t) &= \mathcal{S}_{n+6}(x;t) + (\beta_n + \beta_{n+1} + \beta_{n+2} + \beta_{n+3} + \beta_{n+4} + \beta_{n+5})\,\mathcal{S}_{n+4}(x;t)\\
&\quad + \Big[\beta_{n+3}\,(\beta_n + \beta_{n+1} + \beta_{n+2} + \beta_{n+3} + \beta_{n+4}) + \beta_{n+2}\,(\beta_n + \beta_{n+1} + \beta_{n+2} + \beta_{n+3}) + Q_n + Q_{n+1}\Big]\mathcal{S}_{n+2}(x;t)\\
&\quad + \Big[\beta_{n+1}\beta_{n+2}\,[\beta_n + \beta_{n+1} + \beta_{n+2} + \beta_{n+3}] + [(\beta_n + \beta_{n+1})(Q_n + Q_{n+1})]\\
&\qquad\quad + \beta_n\beta_{n-1}\,(\beta_{n-2} + \beta_{n-1} + \beta_n + \beta_{n+1})\Big]\mathcal{S}_n(x;t)\\
&\quad + \beta_n\beta_{n-1}\Big[Q_{n-1} + Q_n + Q_{n+1} + \beta_{n-1}\beta_{n+1} + \beta_{n-2}\,(\beta_{n-3} + \beta_{n-2} + \beta_{n-1} + \beta_n + \beta_{n+1})\Big]\mathcal{S}_{n-2}(x;t)\\
&\quad + \beta_n\beta_{n-1}\beta_{n-2}\beta_{n-3}\Big[\beta_{n-4} + \beta_{n-3} + \beta_{n-2} + \beta_{n-1} + \beta_n + \beta_{n+1}\Big]\mathcal{S}_{n-4}(x;t)\\
&\quad + (\beta_n\beta_{n-1}\beta_{n-2}\beta_{n-3}\beta_{n-4}\beta_{n-5})\,\mathcal{S}_{n-6}(x;t),
\end{aligned} \tag{57c}$$

where the expression $Q_n$ is as given in Equation (23). Substituting Equation (57) into Equation (56) yields the coefficients $\{\mathfrak{f}_{n,j}\}_{j=n-4}^{n-1}$ in Equation (54) as:

$$\mathfrak{f}_{n,n-6} = 6c\left(\prod_{j=0}^{5}\beta_{n-j}\right) = 6c\left[\beta_n\beta_{n-1}\beta_{n-2}\beta_{n-3}\beta_{n-4}\beta_{n-5}\right], \tag{58a}$$

$$\mathfrak{f}_{n,n-5} = 0, \tag{58b}$$

$$\mathfrak{f}_{n,n-4} = 6c\left(\prod_{j=0}^{3}\beta_{n-j}\right)\left[\beta_{n-4}+\beta_{n-3}+\beta_{n-2}+\beta_{n-1}+\beta_n+\beta_{n+1}\right], \tag{58c}$$

$$\mathfrak{f}_{n,n-3} = 0, \tag{58d}$$

$$\begin{aligned}\mathfrak{f}_{n,n-2} = \beta_n\beta_{n-1}\Big[&6c\{Q_{n-2}+Q_{n-1}+Q_n+Q_{n+1}+\beta_{n-1}\beta_{n-2}+\beta_{n+1}(\beta_{n-2}+\beta_{n-1})\}\\ &+4t(\beta_{n-2}+\beta_{n-1}+\beta_n+\beta_{n+1})-2t\Big],\end{aligned} \tag{58e}$$

$$\mathfrak{f}_{n,n-1} = 0. \tag{58f}$$

Finally we take the case into consideration for $j = n$. Using the technique of integration in Equation (55), we get

$$\begin{aligned}\zeta_n\mathfrak{f}_{n,n} &= \int_{-\infty}^{\infty} x\frac{d\mathcal{S}_n}{dx}(x;t)\mathcal{S}_n(x;t)W_\lambda(x;t)\,\mathrm{d}x = -\tfrac{1}{2}\int_{-\infty}^{\infty}\mathcal{S}_n^2(x;t)\left[W_\lambda(x;t)+x\frac{dW_\lambda}{dx}(x;t)\right]dx\\ &= -\tfrac{1}{2}\zeta_n + \int_{-\infty}^{\infty}\mathcal{S}_n^2(x;t)\left(3cx^6-2tx^4+tx^2-\lambda-\tfrac{1}{2}\right)W_\lambda(x;t)\,\mathrm{d}x\\ &= 3c\int_{-\infty}^{\infty}x^6\mathcal{S}_n^2(x;t)\,W_\lambda(x;t)\,\mathrm{d}x - 2t\int_{-\infty}^{\infty}x^4\mathcal{S}_n^2(x;t)\,W_\lambda(x;t)\,\mathrm{d}x\\ &\quad + t\int_{-\infty}^{\infty}x^2\mathcal{S}_n^2(x;t)\,W_\lambda(x;t)\,\mathrm{d}x - (\lambda+1)\zeta_n.\end{aligned} \tag{59}$$

Again employing the recursion relation in Equation (6) for Equation (59), we obtain

$$x^2\mathcal{S}_n^2 = (\mathcal{S}_{n+1}+\beta_n\mathcal{S}_{n-1})^2 = \mathcal{S}_{n+1}^2 + 2\beta_n\mathcal{S}_{n+1}\mathcal{S}_{n-1} + \beta_n^2\mathcal{S}_{n-1}^2, \tag{60}$$

$$\begin{aligned}x^4\mathcal{S}_n^2 &= x^2\left(\mathcal{S}_{n+1}^2+2\beta_n\mathcal{S}_{n+1}\mathcal{S}_{n-1}+\beta_n^2\mathcal{S}_{n-1}^2\right) = x^2\mathcal{S}_{n+1}^2 + 2\beta_n(x\mathcal{S}_{n+1})(x\mathcal{S}_{n-1}) + \beta_n^2x^2\mathcal{S}_{n-1}^2\\ &= \left(\mathcal{S}_{n+2}+\beta_{n+1}\mathcal{S}_n\right)^2 + 2\beta_n\left(\mathcal{S}_{n+2}+\beta_{n+1}\mathcal{S}_n\right)\left(\mathcal{S}_n+\beta_{n-1}\mathcal{S}_{n-2}\right) + \beta_n^2\left(\mathcal{S}_n+\beta_{n-1}\mathcal{S}_{n-2}\right)^2\\ &= \mathcal{S}_{n+2}^2 + 2(\beta_{n+1}+\beta_n)\mathcal{S}_{n+2}\mathcal{S}_n + (\beta_{n+1}+\beta_n)^2\mathcal{S}_n^2 + 2\beta_n\beta_{n-1}\mathcal{S}_{n+2}\mathcal{S}_{n-2}\\ &\quad + 2\beta_n\beta_{n-1}(\beta_n+\beta_{n+1})\mathcal{S}_n\mathcal{S}_{n-2} + \beta_n^2\beta_{n-1}^2\mathcal{S}_{n-2}^2,\end{aligned} \tag{61}$$

$$
\begin{aligned}
x^6\mathcal{S}_n^2 &= x^2\Big[\mathcal{S}_{n+2}^2 + 2(\beta_{n+1}+\beta_n)\mathcal{S}_{n+2}\mathcal{S}_n + (\beta_{n+1}+\beta_n)^2\mathcal{S}_n^2 + 2\beta_n\beta_{n-1}\mathcal{S}_{n+2}\mathcal{S}_{n-2}\\
&\qquad + 2\beta_n\beta_{n-1}(\beta_n+\beta_{n+1})\mathcal{S}_n\mathcal{S}_{n-2} + \beta_n^2\beta_{n-1}^2\mathcal{S}_{n-2}^2\Big]\\
&= (\mathcal{S}_{n+3}^2 + 2\beta_{n+2}\mathcal{S}_{n+3}\mathcal{S}_{n+1} + \beta_{n+2}^2\mathcal{S}_{n+1}^2) + (\beta_{n+1}+\beta_n)^2(\mathcal{S}_{n+1}^2 + 2\beta_n\mathcal{S}_{n+1}\mathcal{S}_{n-1} + \beta_n^2\mathcal{S}_{n-1}^2)\\
&\qquad + 2(\beta_{n+1}+\beta_n)\left[\mathcal{S}_{n+4} + (\beta_{n+3}+\beta_{n+2})\mathcal{S}_{n+2} + \beta_{n+2}\beta_{n+1}\mathcal{S}_n\right]\mathcal{S}_n\\
&\qquad + 2\beta_n\beta_{n-1}\left[\mathcal{S}_{n+4} + (\beta_{n+2}+\beta_{n+3})\mathcal{S}_{n+2} + \beta_{n+2}\beta_{n+1}\mathcal{S}_n\right]\mathcal{S}_{n-2}\\
&\qquad + 2\beta_n\beta_{n-1}(\beta_n+\beta_{n+1})\left[\mathcal{S}_{n+2} + (\beta_n+\beta_{n+1})\mathcal{S}_n + \beta_n\beta_{n-1}\mathcal{S}_{n-2}\right]\mathcal{S}_{n-2}\\
&\qquad + \beta_n^2\beta_{n-1}^2(\mathcal{S}_{n-1}^2 + 2\beta_{n-2}\mathcal{S}_{n-1}\mathcal{S}_{n-3} + \beta_{n-2}^2\mathcal{S}_{n-3}^2).
\end{aligned}
\tag{62}
$$

and so by orthogonality, we have that

$$
\begin{aligned}
\int_{-\infty}^{\infty} x^6\mathcal{S}_n^2(x;t)W_\lambda(x;t)\,\mathrm{d}x &= (\zeta_{n+3} + \beta_{n+2}^2\zeta_{n+1}) + 2(\beta_{n+1}+\beta_n)\beta_{n+1}\beta_{n+2}\zeta_n + (\beta_{n+1}+\beta_n)^2(\zeta_{n+1}+\beta_n^2\zeta_{n-1})\\
&\qquad + 2\beta_n^2\beta_{n-1}^2(\beta_{n+1}+\beta_n)\zeta_{n-2} + \beta_n^2\beta_{n-1}^2(\zeta_{n-2}+\beta_{n-2}^2\zeta_{n-3})\\
&= (\beta_n\beta_{n+1}\beta_{n+2})\zeta_n + Q_{n+2}\beta_{n+1}\zeta_n + \beta_{n+1}(Q_n+Q_{n+1})\zeta_n + \beta_n(Q_n+Q_{n+1})\zeta_n\\
&\qquad + \beta_{n-1}\beta_n\beta_{n+1}\zeta_n + \beta_n Q_{n-1}\zeta_n,
\end{aligned}
\tag{63}
$$

$$
\int_{-\infty}^{\infty} x^2\,\mathcal{S}_n^2(x;t)\,W_\lambda(x;t)\,\mathrm{d}x = \zeta_{n+1} + \beta_n^2\,\zeta_{n-1} = (\beta_{n+1}+\beta_n)\,\zeta_n,
\tag{64}
$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} x^4\mathcal{S}_n^2(x;t)\,W_\lambda(x;t)\,\mathrm{d}x &= \zeta_{n+2} + (\beta_{n+1}+\beta_n)^2\zeta_n + \beta_n^2\beta_{n-1}^2\,\zeta_{n-2}\\
&= \big[(\beta_{n+1}+\beta_n+\beta_{n-1})\beta_n + (\beta_{n+2}+\beta_{n+1}+\beta_n)\beta_{n+1}\big]\zeta_n\\
&= (Q_n+Q_{n+1})\,\zeta_n,
\end{aligned}
\tag{65}
$$

using $\zeta_{n+1} = \beta_{n+1}\zeta_n$, the difference equation in Equation (22) and the expression $Q_n$ is given by Equation (23).

By rewriting Equation (22) and employing $n \to n-1$ in Equation (22), we have

$$
2tQ_n - t\beta_n = \frac{n + (2\lambda+1)\Omega_n}{2} - 3c\left[\beta_n\left(Q_{n-1}+Q_n+Q_{n+1}\right) + \beta_{n-1}\beta_n\beta_{n+1}\right],
\tag{66a}
$$

$$
2tQ_{n+1} - t\beta_{n+1} = \frac{n+1+(2\lambda+1)\Omega_{n+1}}{2} - 3c\left[\beta_{n+1}\left(Q_n+Q_{n+1}+Q_{n+2}\right) + \beta_n\beta_{n+1}\beta_{n+2}\right].
\tag{66b}
$$

By using Equations (66a) and (66b), we obtain

$$
\begin{aligned}
&-2t\int_{-\infty}^{\infty} x^4\mathcal{S}_n^2(x;t)\,W_\lambda(x;t)\,\mathrm{d}x + t\int_{-\infty}^{\infty} x^2\mathcal{S}_n^2(x;t)\,W_\lambda(x;t)\,\mathrm{d}x\\
&\quad = -\left(t\beta_n - 2tQ_n\right) - \left(t\beta_{n+1} - 2tQ_{n+1}\right)\\
&\quad = -3c\Big[\beta_n\left(Q_{n-1}+Q_n+Q_{n+1}\right) + \beta_{n-1}\beta_n\beta_{n+1} + \beta_{n+1}\left(Q_n+Q_{n+1}+Q_{n+2}\right) + \beta_n\beta_{n+1}\beta_{n+2}\Big]\\
&\qquad + \frac{2n+1+(2\lambda+1)\left(\Omega_n+\Omega_{n+1}\right)}{2}\\
&\quad = -3c\Big[\beta_n\left(Q_{n-1}+Q_n+Q_{n+1}\right) + \beta_{n-1}\beta_n\beta_{n+1} + \beta_{n+1}\left(Q_n+Q_{n+1}+Q_{n+2}\right) + \beta_n\beta_{n+1}\beta_{n+2}\Big]\\
&\qquad + n + (\lambda+1),
\end{aligned}
\tag{67}
$$

where $\Omega_n$ is as given in Equation (24). Hence from Equations (63) and (67), Equation (59) becomes

$$\begin{aligned}
\mathfrak{f}_{n,n} &= \frac{1}{\zeta_n}\left\{3c\int_{-\infty}^{\infty} x^6\mathcal{S}_n^2(x;t)\,W_\lambda(x;t)\,\mathrm{d}x - (\lambda+1)\zeta_n - 2t\int_{-\infty}^{\infty} x^4\mathcal{S}_n^2(x;t)\,W_\lambda(x;t)\,\mathrm{d}x + t\int_{-\infty}^{\infty} x^2\mathcal{S}_n^2(x;t)\,W_\lambda(x;t)\,\mathrm{d}x\right\}\\
&= 3c\Big[(\beta_n\beta_{n+1}\beta_{n+2}) + Q_{n+2}\beta_{n+1} + \beta_{n+1}(Q_n+Q_{n+1})\zeta_n + \beta_n(Q_n+Q_{n+1}) + \beta_{n-1}\beta_n\beta_{n+1}\zeta_n + \beta_n Q_{n-1}\Big]\\
&\quad - (\lambda+1) - 3c\Big[\beta_n\,(Q_{n-1}+Q_n+Q_{n+1}) + \beta_{n-1}\beta_n\beta_{n+1} + \beta_{n+1}\,(Q_n+Q_{n+1}+Q_{n+2}) + \beta_n\beta_{n+1}\beta_{n+2}\Big]\\
&\qquad + n + (\lambda+1)\\
&= n. \qquad (68)
\end{aligned}$$

By combining Equation (58) with Equation (54), we have that

$$x\frac{d\mathcal{S}_n}{dx}(x;t) = \mathfrak{f}_{n,n-6}\,\mathcal{S}_{n-6}(x;t) + \mathfrak{f}_{n,n-4}\,\mathcal{S}_{n-4}(x;t) + \mathfrak{f}_{n,n-2}\,\mathcal{S}_{n-2}(x;t) + \mathfrak{f}_{n,n}\,\mathcal{S}_n(x;t). \qquad (69)$$

Rewriting $\mathcal{S}_{n-4}$ and $\mathcal{S}_{n-2}$ in Equation (69) in terms of $\mathcal{S}_n$ and $\mathcal{S}_{n-1}$ using Equation (6), we obtain

$$\mathcal{S}_{n-2}(x;t) = \frac{x\mathcal{S}_{n-1}(x;t) - \mathcal{S}_n(x;t)}{\beta_{n-1}}, \qquad (70)$$

$$\mathcal{S}_{n-3}(x;t) = \frac{x\mathcal{S}_{n-2}(x;t) - \mathcal{S}_{n-1}(x;t)}{\beta_{n-2}} = \frac{x^2-\beta_{n-1}}{\beta_{n-1}\beta_{n-2}}\,\mathcal{S}_{n-1}(x;t) - \frac{x}{\beta_{n-1}\beta_{n-2}}\,\mathcal{S}_n(x;t), \qquad (71)$$

$$\mathcal{S}_{n-4}(x;t) = \frac{x\mathcal{S}_{n-3}(x;t) - \mathcal{S}_{n-2}(x;t)}{\beta_{n-3}} = \frac{x^3-(\beta_{n-1}+\beta_{n-2})x}{\beta_{n-1}\beta_{n-2}\beta_{n-3}}\,\mathcal{S}_{n-1}(x;t) - \frac{x^2-\beta_{n-2}}{\beta_{n-1}\beta_{n-2}\beta_{n-3}}\,\mathcal{S}_n(x;t), \qquad (72)$$

$$\begin{aligned}
\mathcal{S}_{n-6}(x;t) = &\left\{\frac{x^5-(\beta_{n-1}+\beta_{n-2}+\beta_{n-3}+\beta_{n-4})x + (\beta_{n-1}\beta_{n-3}+\beta_{n-1}\beta_{n-4}+\beta_{n-2}\beta_{n-4})}{\beta_{n-1}\beta_{n-2}\beta_{n-3}\beta_{n-4}\beta_{n-5}}\right\}\mathcal{S}_{n-1}(x;t)\\
&- \left\{\frac{x^4-(\beta_{n-2}+\beta_{n-3}+\beta_{n-4})x+\beta_{n-2}\beta_{n-4}}{\beta_{n-1}\beta_{n-2}\beta_{n-3}\beta_{n-4}\beta_{n-5}}\right\}\mathcal{S}_n(x;t). \qquad (73)
\end{aligned}$$

Substituting Equations (58), (68), (70), (72) and (73) into Equation (69) yields

$$x\frac{d\mathcal{S}_n}{dx}(x;t) = A_n(x;t)\,\mathcal{S}_{n-1}(x;t) - B_n(x;t)\,\mathcal{S}_n(x;t), \qquad (74)$$

where the ladder coefficients $A_n(x;t)$ and $B_n(x;t)$ are defined in Equation (47).

### *4.4. The Differential Equation Satisfied by Modified Sextic Freud-Type Polynomials*

**Theorem 13.** *For the modified Freudian weight (4), the monic orthogonal polynomials $\mathcal{S}_n(x;t)$ satisfy the linear second-order ODE (with rational coefficients):*

$$\frac{d^2}{dx^2}\mathcal{S}_n(x;t) + \tilde{R}_n(x;t)\,\frac{d}{dx}\mathcal{S}_n(x;t) + \tilde{T}_n(x;t)\,\mathcal{S}_n(x;t) = 0, \qquad (75)$$

*where*

$$\begin{aligned}
\tilde{R}_n(x;t) &= -6cx^5 - t(4x^3-2x) + \frac{(2\lambda+1)}{x} - \left[\frac{24cx^3 + 2\left[6c(\beta_n+\beta_{n+1})+4t\right]x}{6cx^4+6c(\beta_n+\beta_{n+1})x^2+6c\,(Q_{n+1}+Q_n)-2t+4t\,(x^2+\beta_n+\beta_{n+1})}\right]\\
&\equiv -v'(x) - \frac{d}{dx}\ln\left(A_n(x;t)\right), \qquad (76a)
\end{aligned}$$

*and*

$$\begin{aligned}
\tilde{T}_n(x;t) &= 18c\beta_n x^2 + 6cQ_n - \frac{(2\lambda+1)\Omega_n}{x^2} + 4t\beta_n \\
&\quad + \beta_n \left( 6cx^4 + 6c(\beta_n+\beta_{n+1})x^2 + 6c\left(Q_{n+1}+Q_n\right) - 2t + 4t\left(x^2+\beta_n+\beta_{n+1}\right)\right) \\
&\qquad \times \left( 6cx^4 + 6c(\beta_n+\beta_{n-1})x^2 + 6c\left(Q_{n-1}+Q_n\right) - 2t + 4t\left(x^2+\beta_n+\beta_{n-1}\right)\right) \\
&\quad - \left[ \left( 6cx^5 + (6c\beta_n+4t)x^3 - \frac{2\lambda+1}{x} + (6cQ_n+4t\beta_n-2t)\,x + \frac{(2\lambda+1)\Omega_n}{x} \right.\right. \\
&\qquad \left. + \frac{24cx^3 + 2\left[6c(\beta_n+\beta_{n+1})+4t\right]x}{6cx^4 + 6c(\beta_n+\beta_{n+1})x^2 + 6c\left(Q_{n+1}+Q_n\right) - 2t + 4t\left(x^2+\beta_n+\beta_{n+1}\right)} \right) \\
&\quad \left. \times \left( 6c\beta_n x^3 + (6cQ_n+4t\beta_n)\,x + \frac{(2\lambda+1)\Omega_n}{x} \right) \right] \\
&\equiv B_n'(x;t) - B_n(x;t)\left[ v'(x) + B_n(x;t) + \frac{A_n'(x;t)}{A_n(x;t)} \right] + \beta_n A_n(x;t)A_{n-1}(x;t),
\end{aligned} \tag{76b}$$

*where $Q_n$ and $\Omega_n$ are given as in (23) and (24) respectively.*

**Proof.** The result can be proved using Equations (45), (48) and (49) where

$$v'(x) = -\frac{(2\lambda+1)}{x} + 6cx^5 + t(4x^3 - 2x), \tag{77}$$

$$A_n(x;t) = 6cx^4 + \left[6c(\beta_n+\beta_{n+1}) + 4t\right]x^2 + 6c\left(Q_{n+1}+Q_n\right) + 4t(\beta_n+\beta_{n+1}) - 2t, \tag{78}$$

$$B_n(x;t) = 6c\beta_n x^3 + 6cQ_n x + 4tx\beta_n + \left(\frac{2\lambda+1}{x}\right)\Omega_n, \tag{79}$$

and later substituting Equations (77), (78) and (79) into Lemma 5. (Note that the expressions $Q_n$ and $\Omega_n$ are given as in (23) and (24) respectively). □

One can see that a more expanded expression of the coefficients of the differential equation in Equation (76) can be obtained using the symbolic package in Maple but will be very cumbersome.

**Remark 14.** *(i) For the semi-classical weight in (4), the lowering operator (cf. Theorem 10) is rewritten as*

$$\mathcal{S}_n'(x,t) = \tilde{A}(x,n)\mathcal{S}_{n-1}(x,t) - \tilde{B}(x,n)\mathcal{S}_n(x,t), \tag{80}$$

*where the coefficients $\tilde{A}(x,n)$ and $\tilde{B}(x,n)$ are given by*

$$\begin{aligned}
\tilde{A}(x,n) &= 6c\beta_n x^4 + 6c(\beta_n+\beta_{n+1})\beta_n x^2 + 6c\beta_n\left(Q_{n+1}+Q_n\right) \\
&\qquad + 4t\beta_n x^2 + 4t\beta_n(\beta_n+\beta_{n+1}) - 2t\beta_n \equiv \beta_n\, A_n(x),
\end{aligned} \tag{81}$$

$$\tilde{B}(x,n) = B_n(x) = 6c\beta_n x^3 + 6cQ_n x + 4tx\beta_n + \left(\frac{2\lambda+1}{x}\right)\Omega_n, \tag{82}$$

*where the coefficients $A_n(x)$ and $B_n(x)$ are given in Equation (47).*

(ii) *The holonomic representation [15,25] of the differential Equation (75) satisfied by semi-classical modified Freudian polynomials can be given by*

$$\tilde{M}(x,n,t)\frac{d^2\mathcal{S}_n(x;t)}{dx^2} + \tilde{N}(x,n,t)\frac{d\mathcal{S}_n(x;t)}{dx} + \tilde{R}(x,n,t)\mathcal{S}_n(x;t) = 0, \quad n \geq 0, \tag{83}$$

*where*

$$\begin{aligned}
\tilde{M}(x,n,t) &= x^2\left[6cx^4 + 6c(\beta_n + \beta_{n+1})x^2 + 6c\left(Q_{n+1} + Q_n\right) + 4tx^2 + 4t(\beta_n + \beta_{n+1}) - 2t\right],\\
&= 6cx^6 + 6c(\beta_n + \beta_{n+1})x^4 + 6c\left(Q_{n+1} + Q_n\right)x^2 + 4tx^4 + 4t(\beta_n + \beta_{n+1})x^2 - 2tx^2,\\
&\equiv x^2 A_n(x)\tilde{R}_n(x;t),\\
\tilde{N}(x,n,t) &= x^2\left[6cx^4 + 6c(\beta_n + \beta_{n+1})x^2 + 6c\left(Q_{n+1} + Q_n\right) + 4tx^2 + 4t(\beta_n + \beta_{n+1}) - 2t\right]\tilde{R}_n(x;t),\\
&\equiv x^2 A_n(x)\tilde{R}_n(x;t),\\
\tilde{U}(x,n,t) &= x^2\left[6cx^4 + 6c(\beta_n + \beta_{n+1})x^2 + 6c\left(Q_{n+1} + Q_n\right) + 4tx^2 + 4t(\beta_n + \beta_{n+1}) - 2t\right]\tilde{T}_n(x;t),\\
&\equiv x^2 A_n(x)\tilde{T}_n(x;t),
\end{aligned}$$

*where $\tilde{R}_n(x;t)$ and $\tilde{T}_n(x;t)$ are given in Equation (76).*

## 5. Symmetrizing Semi-Classical Modified Airy-Type Weight

Orthogonal polynomials associated with a symmetric measure can be obtained from classical orthogonal polynomials by means of quadratic transformation (cf. [12,30]) and for general case, we refer to [39]. For instance, a class of generalized Hermite polynomials is generated from Laguerre polynomials while a class of generalized Ultraspherical polynomials is obtained from Jacobi polynomials [30].

We will, next, show that symmetrizing the semi-classical generalized Airy-type weight function (cf. [34])

$$w_\lambda(x;t) = x^\lambda \exp\left(-[cx^3 + t(x^2 - x)]\right), \ \lambda > 0, \tag{84}$$

gives rise to the modified sextic Freud-type weight function (4).

Suppose $\{\mathcal{P}_n^{(\lambda)}(x;t)\}_{n=0}^{\infty}$ be a sequence of monic semi-classical generalized Airy-type polynomials, orthogonal with reference to the semi-classical weight in (84).

Let's define the monic polynomials

$$\mathcal{S}_{2n}(x;t) = \mathcal{P}_n^{(\lambda)}(x^2;t); \quad \mathcal{S}_{2n+1}(x;t) = x\mathcal{Q}_n^{(\lambda)}(x^2;t),$$

where the polynomial

$$\mathcal{Q}_n^{(\lambda)}(x;t) = \frac{1}{x}\left[\mathcal{P}_{n+1}^{(\lambda)}(x;t) - \frac{\mathcal{P}_{n+1}^{(\lambda)}(0;t)}{\mathcal{P}_n^{(\lambda)}(0;t)}\,\mathcal{P}_n^{(\lambda)}(x;t)\right]$$

is of degree $n$ and monic in nature. Since $x\mathcal{P}_n^{(\lambda+1)} \in \mathbb{P}_{n+1}$, we now write $x\mathcal{P}_n^{(\lambda+1)}$ in terms of the semi-classical generalized Airy-type basis $\{\mathcal{P}_k^{(\lambda)}\}_{k=0}^{n+1}$ as

$$x\mathcal{P}_n^{(\lambda+1)}(x;t) = \sum_{k=0}^{n+1} a_{n+1,k}(t)\, \mathcal{P}_k^{(\lambda)}(x;t),$$

where the expression $a_{n+1,k}(t)$, with fixed $t \in \mathbb{R}$, is given by

$$\begin{aligned} a_{n+1,k}(t)\, \langle \mathcal{P}_k^{(\lambda)}(x;t), \mathcal{P}_k^{(\lambda)}(x;t)\rangle &= \int_0^\infty x\mathcal{P}_k^{(\lambda)}(x;t)\, \mathcal{P}_n^{(\lambda+1)}(x;t)\, x^\lambda \,\exp\left(-[cx^3 + t(x^2-x)]\right) \mathrm{d}x \\ &= \int_0^\infty \mathcal{P}_k^{(\lambda)}(x;t)\, \mathcal{P}_n^{(\lambda+1)}(x;t)\, x^{\lambda+1} \,\exp\left(-[cx^3 + t(x^2-x)]\right) \mathrm{d}x \\ &= 0, \quad \text{for } k < n. \end{aligned} \tag{85}$$

Using Equation (85), we have that

$$x\, \mathcal{P}_n^{(\lambda+1)}(x;t) = a_{n+1,n+1}(t)\, \mathcal{P}_{n+1}^{(\lambda)}(x;t) + a_{n+1,n}(t)\, \mathcal{P}_n^{(\lambda)}(x;t). \tag{86}$$

We see that $a_{n+1,n+1}(t) = 1$ since $\mathcal{P}_n^{(\lambda+1)}(x;t)$ is monic, and Equation (86) becomes

$$x\mathcal{P}_n^{(\lambda+1)}(x;t) = \mathcal{P}_{n+1}^{(\lambda)}(x;t) \;+\; a_{n+1,n}(t)\, \mathcal{P}_n^{(\lambda)}(x;t). \tag{87}$$

Computing Equation (87) at $x = 0$, we obtain $a_{n+1,n}(t) = -\dfrac{\mathcal{P}_{n+1}^{(\lambda)}(0;t)}{\mathcal{P}_n^{(\lambda)}(0;t)}$ and hence

$$x\mathcal{P}_n^{(\lambda+1)}(x;t) = \mathcal{P}_{n+1}^{(\lambda)}(x;t) - \frac{\mathcal{P}_{n+1}^{(\lambda)}(0;t)}{\mathcal{P}_n^{(\lambda)}(0;t)}\, \mathcal{P}_n^{(\lambda)}(x;t) := x\, \mathcal{Q}_n^{\lambda}(x;t).$$

Now,

$$\begin{aligned} &\int_0^\infty \mathcal{P}_m^{(\lambda)}(x;t)\, \mathcal{P}_n^{(\lambda)}(x;t)\, x^\lambda \,\exp\left(-[cx^3 + t(x^2-x)]\right) \mathrm{d}x \\ &\qquad = 2\int_0^\infty \mathcal{P}_m^{(\lambda)}(x^2;t)\, \mathcal{P}_n^{(\lambda)}(x^2;t)\, |x|^{2\lambda+1} \exp\left(-[cx^6 + t(x^4-x^2)]\right) \mathrm{d}x \\ &\qquad = \int_{-\infty}^\infty \mathcal{P}_m^{(\lambda)}(x^2;t)\, \mathcal{P}_n^{(\lambda)}(x^2;t)\, |x|^{2\lambda+1} \exp\left(-[cx^6 + t(x^4-x^2)]\right) \mathrm{d}x \\ &\qquad = \int_{-\infty}^\infty \mathcal{S}_{2m}(x;t)\, \mathcal{S}_{2n}(x;t)\, |x|^{2\lambda+1} \exp\left(-[cx^6 + t(x^4-x^2)]\right) \mathrm{d}x \\ &\qquad = K_n(t)\, \delta_{mn}. \end{aligned}$$

Hence, the polynomial sequence $\{\mathcal{S}_{2m}(x;t)\}_{m=0}^\infty$ is orthogonal with respect to the symmetric weight $W_\lambda(x;t) = |x|^{2\lambda+1} \exp\left(-[cx^6 + t(x^4-x^2)]\right)$ on $\mathbb{R}$. It is proved in [30] that the kernel polynomials $\mathcal{Q}_m^{(\lambda)}(x;t)$ are orthogonal with respect to the weight $xW_\lambda(x;t) = x^{2\lambda+2} \exp\left(-[cx^6 + t(x^4-x^2)]\right)$. Hence

$$\begin{aligned} K_n(t)\, \delta_{mn} &= \int_0^\infty \mathcal{Q}_m^{(\lambda)}(x;t)\, \mathcal{Q}_n^{(\lambda)}(x;t)\, x^{\lambda+1} \exp\left(-[cx^3 + t(x^2-x)]\right) \mathrm{d}x \\ &= 2\int_0^\infty \mathcal{Q}_m^{(\lambda)}(x^2;t)\, \mathcal{Q}_n^{(\lambda)}(x^2;t)\, x^{2\lambda+3} \exp\left(-[cx^6 + t(x^4-x^2)]\right) \mathrm{d}x \\ &= \int_{-\infty}^\infty \mathcal{S}_{2m+1}(x;t)\, \mathcal{S}_{2n+1}(x;t)\, |x|^{2\lambda+1} \exp\left(-[cx^6 + t(x^4-x^2)]\right) \mathrm{d}x. \end{aligned}$$

Finally, we obtain

$$\int_{-\infty}^{\infty} \mathcal{S}_{2m+1}(x;t)\, \mathcal{S}_{2n}(x;t)\, W_\lambda(x;t)\,\mathrm{d}x = \int_{-\infty}^{\infty} \mathcal{S}_{2m}(x;t)\, \mathcal{S}_{2n+1}(x;t) W_\lambda(x;t)\,\mathrm{d}x = 0,$$

since the integrand is odd. Thus, the polynomial sequence $\{\mathcal{S}_n(x;t)\}_{n=0}^{\infty}$ is orthogonal with reference to the modified weight (4). We note that $\widetilde{W_\lambda}(x,t) = |x|^{-1} W_\lambda(x^2;t) = |x|^{2\lambda-1} \exp\left(-[cx^6 + t(x^4 - x^2)]\right)$ is another symmetric dual weight for the semi-classical generalized Airy-type weight function (cf. [32]).

**Remark 15.** *By employing quadratic transformation to the non-symmetric measure (84), we can see that the symmetrization process due to Chihara [30] preserves orthogonality and hence the modified sextic Freud-type polynomials follows from generalized Airy-type polynomials [34].*

## 6. Electrostatic Interpretation of the Zeros

In this section, let's first consider a sequence of orthonormal polynomials $\{\Psi_n(x)\}_{n\geq 0}$ with respect to a weight function $w(x) = \exp(-v(x))$ satisfying a three-term recurrence relation [40]

$$x\Psi_n(x) = a_{n+1}\Psi_{n+1}(x) + b_n\Psi_n(x) + a_n\Psi_{n-1}(x), \quad n \geq 0, \quad a_n > 0,$$

with $\Psi_{-1}(x) \equiv 0$ and $\Psi_0(x) \equiv 1$. It is known in [25,41] that under certain assumptions on $w$ the polynomials $\Psi_n$ satisfy differential-difference relation

$$\Psi_n'(x) = \mathcal{A}_n(x)\Psi_{n-1}(x) - \mathcal{B}_n(x)\Psi_n(x)\,,$$

where the coefficients $\mathcal{A}_n$ and $\mathcal{B}_n$ are explicitly given in terms of $w$, the recurrence coefficients $a_n$, and the values of $\Psi_n$ at the endpoints $c$ and $d$. Consequently, $\Psi_n$ obeys a linear differential equation [42]

$$\Psi_n''(x) - 2\mathcal{R}_n(x)\Psi_n'(x) + \mathcal{Q}_n(x)\Psi_n(x) = 0\,, \tag{88}$$

with

$$\begin{aligned}
\mathcal{R}_n(x) &= \frac{v'(x)}{2} + \frac{\mathcal{A}_n'(x)}{2A_n(x)}\,,\\
\mathcal{Q}_n(x) &= \mathcal{B}_n'(x) - \mathcal{B}_n(x)\frac{\mathcal{A}_n'(x)}{\mathcal{A}_n(x)} - \mathcal{B}_n(x)[v'(x) + \mathcal{B}_n(x)] + \frac{a_n}{a_{n-1}}\mathcal{A}_n(x)\mathcal{A}_{n-1}(x)\,.
\end{aligned}$$

(see Lemma 5 for its equivalent monic form). Specifically, if $X^* \subset (\alpha,\beta)^n$ is the ordered set of zeros of $\Psi_n$, then

$$\Psi_n''(x) - 2\mathcal{R}_n(x)\Psi_n'(x) = 0 \quad \text{for} \quad x \in X^*\,. \tag{89}$$

By considering an electrostatic model for $X^*$ with logarithmic interaction between particles and an external field

$$\varphi(x) = \frac{v(x)}{2} + \frac{\ln\left(k_n\mathcal{A}_n(x)\right)}{2} = \varphi_{\text{long}}(x) + \varphi_{\text{short}}(x) \tag{90}$$

($k_n$ is any appropriate standardization constant, taken in [43] equal to $a_n^{-1}$), one can see that this external field has two components: the first term in the right hand side of Equation (90) has its origin in the orthogonality weight $w(x) = \exp(-v(x))$, and Ismail [25] named it the *long range potential*. The second term, known to be the *short range potential*, is slightly troublesome, but informs many features of the classical models and at the same time, allows to provide a generalization of the electrostatic interpretation (see [42–44] for more details). M. E. H. Ismail in [43] states that, assuming $w(x) > 0$ on $(a,b)$, both $v$ and $\ln(\mathcal{A}_n)$ in $C^2(a,b)$, and the external field (90) convex, the total energy has a unique point of global minimum, which is precisely $X^*$ (that is, the zeros of the orthonormal polynomial $\Psi_n$).

Clearly, what is used here is in fact the relation (89) equivalent to the fact that $X^*$ is a critical point for $E(X)$, and convexity assures that this is the unique point of minimum (see also [25,43,45]).

*Properties of Electrostatic Properties of the Zeros for the Modified Freud-Type Weight*

For electrostatic interpretation of the zeros for classical orthogonal polynomials, we refer to [1]. In this discussion, an electrostatic representation of distribution of the zeros for modified Freudian polynomials is provided.

In [40], the authors considered a perturbation of the quartic Freud weight ($w(x) = \exp(-x^4)$) by the addition of a fixed charged point of mass $\lambda$ at the origin. (See also [43] and the recent work in [28] for the Freudian-Sobolev case). It was shown in [40] that the semi-classical quartic Freud polynomials obey a differential equation of the form Equation (88), and the electrostatic model was studied. Recall from Equation (45) that the potential $v$ for the modified Freudian weight (4) is given by

$$v(x) = -\ln W_\lambda(x;t) = -(2\lambda+1)\ln|x| + cx^6 + t(x^4 - x^2), \quad x \in \mathbb{R}.$$

The following result immediately follows from Remark 14.

**Proposition 16.** *For the semi-classical weight (4), we have the following equation that corresponds to Equation (80):*

$$x^2 \mathcal{S}_n'(x,t) = -\bar{B}(x,n)\,\mathcal{S}_n(x,t) + \bar{A}(x,n)\,\mathcal{S}_{n-1}(x,t), \tag{91}$$

*where the ladder coefficients $\bar{A}(x,n)$ and $\bar{B}(x,n)$ are given by*

$$\begin{aligned} \bar{A}(x,n) &= 6c\beta_n\, x^6 + [6c(\beta_n+\beta_{n+1}) + 4t]\ \beta_n\, x^4 + [6c\,(Q_{n+1}+Q_n) + 4t(\beta_n+\beta_{n+1}) - 2t]\ \beta_n\, x^2 \\ &\equiv x^2\,\beta_n\, A_n(x), \end{aligned} \tag{92}$$

$$\bar{B}(x,n) = 6c\beta_n\, x^5 + (6cQ_n + 4t\beta_n)\, x^3 + \left(\frac{2\lambda+1}{x}\right)\Omega_n\, x^2 \equiv x^2\, B(x,n), \tag{93}$$

*where the coefficients $A_n(x)$ and $B_n(x)$ are given in Equation (47) and the expression for $Q_n$ and $\Omega_n$ are also given in Equations (23) and (24) respectively.*

The electrostatic interpretation of the zeros corresponding to the semi-classical Freudian weight (4) is attained by finding the zeros of the coefficient $\hat{A}(x,n)$, which provide us the location of some fixed charges.

**Corollary 1.** *For $n \geq 1$, the coefficient $\bar{A}(x,n)$ in Equation (92) has the following six roots $\{r_j\}_{j=1}^6$ are given by*

$$r_{1,2}(n) = 0, \quad r_{3,4}^2(n) = \frac{-B + \sqrt{B^2 - 4AE}}{2A}, \quad r_{5,6}^2(n) = \frac{-B - \sqrt{B^2 - 4AE}}{2A},$$

*where the values of $A$, $B$ and $E$ are respectively*

$$\left.\begin{aligned} A &:= 6c\beta_n \\ B &:= [6c(\beta_n+\beta_{n+1}) + 4t]\,\beta_n \\ E &:= [6c\,(Q_{n+1}+Q_n) + 4t(\beta_n+\beta_{n+1}) - 2t]\,\beta_n \end{aligned}\right\}, \tag{94}$$

*where the parameter $t$ is assumed to be a positive constant, $c > 0$ and $Q_n$ are given in (23) respectively.*

**Remark 17.**

(i) Generally, one can see that the coefficient $\bar{A}(x,n)$ in Corollary 1 has one double root 0, and the other four roots in the complex plane. We also see from Equation (94) that $A > 0$, $B > 0$ and

$Q_n > 0$ since the recurrence coefficient $\beta_n$ is always positive. For the particular cases, we have the following:

(ii) When $B^2 - 4AE > 0$, the coefficient $\bar{A}(x,n)$ in Corollary 1 has a root $r_{1,2}(n) = 0$ of multiplicity 2, two real roots $r_{3,4}(n)$ and the remaining two roots $r_{5,6}$ which are complex for $\sqrt{B^2 - 4AE} > B$.

(iii) When $B^2 - 4AE < 0$, the coefficient $\bar{A}(x,n)$ in Corollary 1 has a root $r_{1,2}(n) = 0$ of multiplicity 2, the remaining other roots $r_{1,2,3,4}(n)$ are entirely complex.

We now propose an electrostatic model in the presence of a varying external potential from the second-order linear differential equation obtained in Section 4.4. In other words, the electrostatic interpretation of the zeros for polynomials orthogonal with respect to (4) using Theorem 13 and Remarks 14 is studied.

We begin by denoting $\{x_{n,k}\}_{k=1}^{n}$ for the zeros of $\mathcal{S}_n(x)$ associated with the semi-classical weight (4). Evaluating the second-order differential equation at $x_{n,k}$, we have

$$M(x,n,t)\frac{d^2\mathcal{S}_n(x)}{dx^2} + N(x,n,t)\frac{d\mathcal{S}_n(x)}{dx} = 0, \quad 1 \le k \le n. \tag{95}$$

Then

$$\frac{\mathcal{S}_n''(x_{n,k})}{\mathcal{S}_n'(x_{n,k})} = -\frac{N(x_{n,k},n,t)}{M(x_{n,k},n,t)} = -\frac{\lambda}{x_{n,k}} - \frac{1}{2x_{n,k}} + 3cx_{n,k}^5 + t(2x_{n,k}^3 - x_{n,k}) + \frac{A'(x_{n,k},n)}{2A_n(x_{n,k},n)}, \quad 1 \le k \le n. \tag{96}$$

Applying the following property (cf. [43,46])

$$\frac{\mathcal{S}_n''(x_{n,k})}{\mathcal{S}_n'(x_{n,k})} = -2\sum_{j=1, j\neq k}^{n} \frac{1}{x_{n,j} - x_{n,k}},$$

Equation (96) becomes

$$\sum_{j=1,j\neq k}^{n} \frac{1}{x_{n,j} - x_{n,k}} + \frac{A_n'(x_{n,k},n)}{4A_n(x_{n,k},n)} - \frac{\lambda}{2x_{n,k}} - \frac{1}{2}tx_{n,k} - \frac{1}{4x_{n,k}} + tx_{n,k}^3 + \frac{3}{2}cx_{n,k}^5 = 0, \tag{97}$$

for $1 \le k \le n$, and where $t$ is assumed to be a fixed positive real constant.

For the semi-classical weight in (4), the total external potential $V(x)$ is the sum of an external field,

$$v(x) = -\ln W_\lambda(x;t) = -(2\lambda+1)\ln|x| + cx^6 + t(x^4 - x^2),$$

and a varying external potential $\frac{1}{2}\ln|A(x,n)| - \ln|x|$, called by Ismail [43] long range field and short range field respectively; i.e.,

$$\varphi(x) = \frac{v(x)}{2} + \frac{\ln(k_n A_n(x))}{2} = \varphi_{\text{long}}(x) + \varphi_{\text{short}}(x),$$

which turns out to be

$$
\begin{aligned}
V(x) &= \varphi_{\text{long}}(x) + \varphi_{\text{short}}(x),\\
&= \frac{-(2\lambda+1)\ln|x| + cx^6 + t(x^4 - x^2)}{2} + \frac{1}{2}\ln\left|\frac{\hat{A}(x,n)}{x^2}\right|,\\
&= \frac{-(2\lambda+1)\ln|x| + cx^6 + t(x^4 - x^2)}{2} + \frac{1}{2}\ln|\hat{A}(x,n)| - \ln|x|,\\
&= \frac{-(2\lambda+1)\ln|x| + cx^6 + t(x^4 - x^2)}{2}\\
&\qquad + \frac{1}{2}\ln|(x-r_1(n))(x-r_2(n))(x-r_3(n))(x-r_4(n))(x-r_5(n))(x-r_6(n))| - \ln|x|,\\
&= \frac{-(2\lambda+1)\ln|x| + cx^6 + t(x^4 - x^2)}{2} + \frac{1}{2}\ln|(x-r_1(n))| + \frac{1}{2}\ln|(x-r_2(n))| + \frac{1}{2}\ln|(x-r_3(n))|\\
&\qquad + \frac{1}{2}\ln|(x-r_4(n))| + \frac{1}{2}\ln|(x-r_5(n))| + \frac{1}{2}\ln|(x-r_6(n))| - \ln|x|, \quad x \in \mathbb{R}\backslash\{0\},
\end{aligned}
\tag{98}
$$

We consider the potential energy at $x$ of a point charge $q$ located at $s$ is $-q\ln|x-s|$. We notice from Equation (98) that the zeros of the coefficient $\bar{A}(x,n)$ give us the position relying on $n$ of six fixed charges.Then the external field is generated by a fixed charge $+1$ at the origin—due to a perturbation of the weight function- plus six fixed charges of magnitude $-1/2$; two of them located at the real positions $r_3(n), r_4(n)$, and the remaining ones at the complex positions $r_{1,2}(n), r_{5,6}(n)$ (see Remark 17).

Let us introduce the following electrostatic model corresponding to the weight (4):

*Consider the system of $n$ movable unit positive charges in $n$ distinct points $\{x_{n,i}\}_{i=1}^{n}$ of the real line in the presence of the total external potential $V(x)$.*

By finding the zeros of $\bar{A}(x,n)$, we obtain the position relying on $n$ of six fixed charges distributed on the real line. By denoting the position vector as

$$\mathbf{x} = (x_{n,1}, x_{n,2}, \ldots, x_{n,n})$$

where $x_{n,j} < x_{n,k}$ if $j < k$ and considering the total energy of the system as

$$E(\mathbf{x}) = \sum_{k=1}^{n} V(x_{n,k}) - \sum_{1\le j<k\le n} \ln|x_{n,j} - x_{n,k}|, \tag{99}$$

we attain Equation (97), which is the derivative of the energy function. This means the zeros of the modified Freud-type orthogonal polynomials are critical points of the energy function. From the physical point of view, these kind of possibilities may want to correspond to the steady or unstable equilibrium situations. The equilibrium is understood as the zero gradient of the complete energy of the system. Stable equilibrium means the existence of a global minimum of the total energy. However, studying the stability for the equilibrium configuration (global minimum) seeks a thorough discussion, which we will pass over here. Nevertheless, we partly resolve this quest by targeting in the study of the local minima of the energy function. For this purpose, the Hessian matrix is considered as

$$\mathsf{H} = \left(h_{i,j}\right), \quad h_{i,j} = \frac{\partial^2 E}{\partial x_i \partial x_j},$$

we will infer when $E(x)$ attains local minima at the zeros of the monic polynomials $\mathcal{S}_n(x)$ with reference to the weight (4). Indeed, taking into account

$$
h_{k,\ell} = \begin{cases}
\dfrac{1}{4}\dfrac{\partial}{\partial x_{n,k}}\left(\dfrac{\bar{A}'(x_{n,k},n)}{\bar{A}(x_{n,k},n)}\right) - \frac{2\lambda+1}{8x_{n,k}^2} - \frac{1}{2}t + 3tx_{n,k}^2 + \dfrac{15}{2}cx_{n,k}^4 + \displaystyle\sum_{j=1, j\neq k}^{n} \dfrac{1}{(x_{n,\ell} - x_{n,k})^2}, & \text{if} \quad k = \ell,\\
-\dfrac{1}{(x_{n,\ell} - x_{n,k})^2}, & \text{if} \quad k \neq \ell,
\end{cases}
$$

and the matrix H is a symmetric real matrix. It is noted in [47] that the Hessian matrix H is positive definite when the matrix H is strictly diagonally dominant and all its diagonal terms are positive. In this case, by setting conditions, which guarantee the equilibrium position of the proposed system will be reached at the zeros of the monic Freud-type polynomial [40] and coming along this way, one can find the electrostatic equilibrium position in the presence of the external field at the zeros $\{x_{n,k}\}_{k=1}^{n}$ for the semi-classical modified Freud-type polynomials.

## 7. Discussion

In this work, we have investigated certain properties of monic orthogonal polynomials associated with a scaled Freud-type weight upon deformation of the Freudian measure $|x|^{\alpha}\exp\left(-Cx^6\right)$, $C > 0$ by $\exp\left(t(x^4 - x^2)\right)$. This slightly different Freudian measure could be considered as a generalization in some sense for the weights in (2). For this case, we have investigated certain properties of orthogonal polynomials corresponding to the semiclassical weight given in (4). Of these properties, we have studied the finite moments, nonlinear recursion relations and differential-recurrence relations for the recurrence coefficients as well as a linear differential equation for the polynomials themselves. This work has certain similarity with the work in [12]. However, the motive of the current work is different as it deals with the modified Freud-type weight function, which already involves a higher-order of the polynomial in the exponential factor in the semi-classical weight (4). A similar work for semi-classical Laguerre weight has been given in [11] and for the Freud-type weight in [18,36].

Besides, this work mainly shows the relation between both the semi-classical polynomials that are orthogonal with respect to the modified weight in (4) as well as the expression of the recurrence coefficients corresponding to these polynomials. Since the recurrence coefficients for sextic Freudian measure (4) are not explicitly found, it is important to obtain several nonlinear recursion relations as well as differential-recurrence relations associated with the weight function (4). Following the method given by J. A. Shohat [12,37] using quasi-orthogonality, we applied this technique as a powerful tool to obtain the ladder relations associated with the weight under investigation. The differential-difference equations obtained are good references to characterize the corresponding semi-classical modified Freudian polynomials. The second-order linear differential equation also follows by combining the ladder relations with the three-term recurrence relation in a similar way to the work in [12]. As an application of the results, an electrostatic interpretation of the zeros from the obtained linear differential equation with rational coefficients is also explored.

## 8. Conclusions

In this paper, we have investigated certain properties of modified sextic Freudian polynomials. Knowing the fact that the recurrence coefficients associated with modified Freudian measure are not straightforwardly formulated; we have derived certain properties such as several non-linear recursion equations, Toda-like equation, differential and difference equations satisfied by the recurrence coefficients as well as the polynomials that are orthogonal with respect to the considered weight function.

Special attention, using the method of ladder operators, is given for the recurrence coefficients associated with the modified Freudian weight given in (4) as this weight function is derived from the generalized Airy-type weight (84) using Chihara's symmetrization process [30]. A differential-difference equation as well as linear second-order differential equation associated with the semi-classical weight (4) were obtained, both via methods of ladder operators as well as Shohat's method of quasi-orthogonality [37]. As an application of the properties, we have investigated an electrostatic interpretation of the zeros from the obtained linear second differential equation with rational coefficients. Following the work and methods used in [15], some asymptotic properties of the recurrence coefficients in terms of Hankel determinants, asymptotics of the differential equation satisfied by the studied orthogonal polynomials will be addressed in a future contribution. We also hope our results motivate further investigation of the recurrence coefficients via Hankel moments of semi-classical orthogonal polynomials in relation to certain (discrete) integrable systems and identifying these connections would also be an interesting continuation of this work.

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