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On the Finite Orthogonality of q -Pseudo-Jacobi Polynomials

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Abstract: Using the Sturm–Liouville theory in q -difference spaces, we prove the finite orthogonality of q -Pseudo Jacobi polynomials. Their norm square values are then explicitly computed by means of the Favard theorem.

Keywords: q -Pseudo Jacobi Polynomials; Sturm–Liouville problems; q -difference equations; finite sequences of q -orthogonal polynomials

1. Introduction

For $\alpha, \beta > -1$, the Jacobi polynomials are defined as [1]

$$P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} \left\{ (1-x)^\alpha (1+x)^\beta (1-x^2)^n \right\}. \quad (1)$$

Another representation of Jacobi polynomials is as [2,3]

$$\begin{aligned} P_n^{(\alpha, \beta)}(x) &= \frac{(\alpha+1)_n}{n!} {}_2F_1 \left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix} \middle| \frac{1-x}{2} \right) \\ &= \frac{(\alpha+1)_n}{n!} \sum_{k=0}^n \frac{(-n)_k (\alpha+\beta+1)_k}{(\alpha+1)_k} \frac{(1-x)^k}{2^k k!}, \end{aligned} \quad (2)$$

where

$$(a)_k := \prod_{j=0}^{k-1} (a+j), \quad (a)_0 := 1, \quad (3)$$

and

$${}_rF_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| z \right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_r)_k}{(b_1)_k \dots (b_s)_k} \frac{z^k}{k!}, \quad (4)$$

in which $a_1, a_2, \dots, a_r, b_1, b_2, \dots, b_s, z \in \mathbb{C}$ and $b_1, \dots, b_s \neq 0, -1, -2, \dots, -(k-1)$.

The weight function corresponding to Jacobi polynomials is known in statistics as the shifted beta distribution

$$w(x; \alpha, \beta) = (1-x)^\alpha (1+x)^\beta, \quad x \in [-1, 1].$$

An interesting subclass of Jacobi polynomials is when $\alpha = -u + iv$ and $\beta = -u - iv$ for $i^2 = -1$ in (2), so that the real polynomials

$$J_n^{(u, v)}(x) = (-i)^n P_n^{(-u+iv, -u-iv)}(ix), \quad (5)$$

satisfy the equation

$$(1+x^2)J_n''(x) + 2((1-u)x+v)J_n'(x) - n(n-2u+1)J_n(x) = 0. \quad (6)$$

It is proved in [4] that $\{J_n^{(u,v)}(x)\}$ are finitely orthogonal with respect to the weight function

$$w(x; u, v) = (1+x^2)^{-u} \exp(2v \arctan x),$$

on $(-\infty, \infty)$ and can be explicitly represented in form of hypergeometric functions as

$$J_n^{(u,v)}(x) = \frac{(-2i)^n (1-u+iv)_n}{(n-2u+1)_n} {}_2F_1 \left(\begin{matrix} -n, n-2u+1 \\ 1-u+iv \end{matrix} \middle| \frac{1-ix}{2} \right).$$

The so-called q -polynomials have found many applications in Eulerian series and continued fractions [3], q -algebras and quantum groups [5–7], and q -oscillators [8–10]. See also [11,12] in this regard. It has been acknowledged that the theory of q -special functions is essentially based on the relation

$$\lim_{q \rightarrow 1} \frac{1-q^\alpha}{1-q} = \alpha.$$

Hence, a basic number in q -calculus is defined as

$$[\alpha]_q = \frac{1-q^\alpha}{1-q}.$$

There is a q -analogue of the Pochhammer symbol (3) (called q -shifted factorial) as

$$(a; q)_k := \prod_{j=0}^{k-1} (1-aq^j), \quad (a; q)_0 := 1.$$

Moreover we have

$$(a; q)_\infty = \prod_{k=0}^{\infty} (1-aq^k) \quad \text{for } 0 < |q| < 1,$$

and

$$(a_1, a_2, \dots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \dots (a_m; q)_\infty. \quad (7)$$

There exist several q -analogues of classical hypergeometric orthogonal polynomials that are known as basic hypergeometric orthogonal polynomials [3].

In the present work, using the Sturm–Liouville theory in q -difference spaces, we prove that a special case of big q -Jacobi polynomials is finitely orthogonal on $(-\infty, \infty)$. The big q -Jacobi polynomials are defined as

$$P_n(x; a, b, c; q) = {}_3\phi_2 \left(\begin{matrix} q^{-n}, a b q^{n+1}, x \\ a q, c q \end{matrix} \middle| q; q \right), \quad (8)$$

where

$${}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| q; z \right) := \sum_{k=0}^{\infty} \frac{(a_1; q)_k \dots (a_r; q)_k}{(b_1; q)_k \dots (b_s; q)_k} \frac{z^k}{(q; q)_k} \left((-1)^k q^{\frac{k(k-1)}{2}} \right)^{1+s-r}, \quad (9)$$

is known as the basic hypergeometric series.

Again, $a_1, a_2, \dots, a_r, b_1, b_2, \dots, b_s, z \in \mathbb{C}$ and $b_1, b_2, \dots, b_s \neq 1, q^{-1}, q^{-2}, \dots, q^{1-k}$.

Notice that [3] (p. 15)

$$\lim_{q \rightarrow 1} {}_r\phi_s \left(\begin{matrix} q^{a_1}, \dots, q^{a_r} \\ q^{b_1}, \dots, q^{b_s} \end{matrix} \middle| q; (q-1)^{1+s-r} z \right) = {}_rF_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| z \right). \quad (10)$$

On the other side, if we set $c = 0$, $a = q^\alpha$ and $b = q^\beta$ in (8) and then let $q \rightarrow 1$, we find the Jacobi polynomials (2) as

$$\lim_{q \rightarrow 1} P_n(x; q^\alpha, q^\beta, 0; q) = \frac{P_n^{(\alpha, \beta)}(2x-1)}{P_n^{(\alpha, \beta)}(1)}.$$

Moreover, by referring to (8), one can define another family of big q -Jacobi polynomials [13] with four free parameters as

$$P_n^*(x; a, b, c, d; q) = P_n(qac^{-1}x; a, b, -ac^{-1}d; q) = {}_3\phi_2 \left(\begin{matrix} q^{-n}, abq^{n+1}, qac^{-1}x \\ aq, -qac^{-1}d \end{matrix} \middle| q; q \right),$$

which yields

$$P_n(x; a, b, c; q) = P_n^*(-q^{-1}c^{-1}x; a, b, -ac^{-1}, 1; q).$$

Because a particular case of Jacobi polynomials (5) are called the pseudo Jacobi polynomials, it is reasonable to similarly consider a special case of big q -Jacobi polynomials preserving the limit relation as $q \rightarrow 1$. This means that the q -pseudo Jacobi polynomials will be derived by substituting

$$a = iq^{\frac{1}{2}(u-iv)}, \quad b = -iq^{\frac{1}{2}(u+iv)}, \quad c = iq^{\frac{1}{2}(-u+iv)} \quad \text{and} \quad d = -iq^{\frac{1}{2}(-u-iv)}$$

in a special case of the polynomials (8) as

$$P_n(cx; c/b, d/a, c/a; q) \quad \text{where} \quad a, b, c, d \in \mathbb{C} \quad \text{and} \quad (ab)/(qcd) > 0,$$

so that

$$\lim_{q \rightarrow 1} P_n(iq^{\frac{1}{2}(-u+iv)}x; -q^{-u}, -q^{-u}, q^{-u+iv}; q) = \frac{J_n^{(u,v)}(x)}{J_n^{(u,v)}(i)}.$$

Therefore, the q -pseudo Jacobi polynomials are defined as

$$J_n^{(u,v)}(x; q) = P_n(iq^{\frac{1}{2}(-u+iv)}x; -q^{-u}, -q^{-u}, q^{-u+iv}; q) = {}_3\phi_2 \left(\begin{matrix} q^{-n}, q^{u+n+1}, -q^{1+u-iv}x \\ -q^{1+\frac{1}{2}(u-iv)}, iq^{1+\frac{1}{2}(u-3iv)} \end{matrix} \middle| q; q \right). \quad (11)$$

The main aim of this paper is to apply a q -Sturm–Liouville theorem in order to obtain a finite orthogonality for the real polynomials (11) on $(-\infty, \infty)$, which is a new contribution in the literature.

A regular Sturm–Liouville problem of continuous type is a boundary value problem of the form

$$\frac{d}{dx} \left(K(x) \frac{dy_n(x)}{dx} \right) + \lambda_n w(x) y_n(x) = 0, \quad (K(x) > 0, w(x) > 0), \quad (12)$$

which is defined on an open interval, say (γ_1, γ_2) with the boundary conditions

$$\alpha_1 y(\gamma_1) + \beta_1 y'(\gamma_1) = 0 \quad \text{and} \quad \alpha_2 y(\gamma_2) + \beta_2 y'(\gamma_2) = 0, \quad (13)$$

where α_1, α_2 and β_1, β_2 are constant numbers and $K(x)$, and $w(x)$ in (12) are to be assumed continuous functions for $x \in [\gamma_1, \gamma_2]$. The function $w(x)$ is called the weight or density function.

Let y_n and y_m be two eigenfunctions of Equation (12). According to the Sturm–Liouville theory [14], they have an orthogonality property with respect to the weight function $w(x)$ under the given condition (13), so that we have

$$\int_{\gamma_1}^{\gamma_2} w(x) y_n(x) y_m(x) dx = \left(\int_{\gamma_1}^{\gamma_2} w(x) y_n^2(x) dx \right) \delta_{m,n}, \quad (14)$$

in which

$$\delta_{m,n} = \begin{cases} 0 & (n \neq m), \\ 1 & (n = m). \end{cases}$$

There are generally two types of orthogonality for relation (14), i.e. infinitely orthogonality and finitely orthogonality. In the finite case, one has to impose some constraints on n , while in the infinite case, n is free up to infinity [4].

By referring to the differential Equation (6), it is proved in [4] that

$$\int_{-\infty}^{\infty} (1+x^2)^{-u} \exp(2v \arctan x) J_n^{(u,v)}(x) J_m^{(u,v)}(x) dx = \frac{2\pi n! 2^{2n+1-2u} \Gamma(2u-n)}{(2u-2n-1)\Gamma(u-n+iv)\Gamma(u-n-iv)} \delta_{m,n}$$

$$\Leftrightarrow m, n = 0, 1, 2, \dots, \quad N = \max\{m, n\} < u - \frac{1}{2} \quad \text{and} \quad v \in \mathbb{R},$$

where $\Gamma(\cdot)$ is the well-known gamma function.

Similarly, q -orthogonal functions can be solutions of a q -Sturm-Liouville problem in the form [15]

$$D_q(K(x;q)D_q y_n(x;q)) + \lambda_{n,q} w(x;q) y_n(x;q) = 0, \quad (K(x;q) > 0, w(x;q) > 0), \quad (15)$$

where

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x} \quad (x \neq 0, q \neq 1),$$

and (15) satisfies a set of boundary conditions like (13). This means that if $y_n(x;q)$ and $y_m(x;q)$ are two eigenfunctions of the q -difference Equation (15), they are orthogonal with respect to a weight function $w(x;q)$ on a discrete set [16].

Let $\varphi(x)$ and $\psi(x)$ be two polynomials of degree at most 2 and 1, respectively, as

$$\varphi(x) = ax^2 + bx + c \quad \text{and} \quad \psi(x) = dx + e \quad (a, b, c, d, e \in \mathbb{C}, d \neq 0).$$

If $\{y_n(x;q)\}_n$ is a sequence of polynomials that satisfies the q -difference equation [3]

$$\varphi(x) D_q^2 y_n(x;q) + \psi(x) D_q y_n(x;q) + \lambda_{n,q} y_n(qx;q) = 0, \quad (16)$$

where

$$D_q^2(f(x)) = \frac{f(q^2x) - (1+q)f(qx) + qf(x)}{q(q-1)^2x^2},$$

$\lambda_{n,q} \in \mathbb{C}$ and $q \in \mathbb{R} \setminus \{-1, 0, 1\}$, then the following orthogonality relation holds

$$\int_{\rho_1}^{\rho_2} w(x;q) y_n(x;q) y_m(x;q) d_q x = \left(\int_{\rho_1}^{\rho_2} w(x;q) y_n^2(x;q) d_q x \right) \delta_{n,m},$$

in which

$$\int_{\rho_1}^{\rho_2} f(t) d_q t = (1-q) \sum_{j=0}^{\infty} q^j \left(\rho_2 f(q^j \rho_2) - \rho_1 f(q^j \rho_1) \right),$$

and $w(x;q)$ is a solution of the Pearson q -difference equation

$$D_q(w(x;q)\varphi(q^{-1}x)) = w(qx;q)\psi(x). \quad (17)$$

Note that $w(x;q)$ is assumed to be positive and $w(q^{-1}x;q)\varphi(q^{-2}x)x^k$ for $k \in \mathbb{N}$ must vanish at $x = \rho_1, \rho_2$.

If $\bar{P}_n(x) = x^n + \dots$ is a monic solution of Equation (16), the eigenvalue $\lambda_{n,q}$ is explicitly derived as

$$\lambda_{n,q} = -\frac{[n]_q}{q^n} (a[n-1]_q + d).$$

The q -integral as an inverse of the q -difference operator [3,17,18] is defined as

$$\int_0^x f(t) d_q t = (1-q)x \sum_{j=0}^{\infty} q^j f(q^j x) \quad (x \in \mathbb{R})$$

provided that the series converges absolutely. Furthermore, we have

$$\int_0^\infty f(t) d_q t = (1-q) \sum_{n=-\infty}^\infty q^n f(q^n),$$

and

$$\int_{-\infty}^\infty f(t) d_q t = (1-q) \sum_{n=-\infty}^\infty q^n (f(q^n) + f(-q^n)).$$

2. Finite Orthogonality of q -Pseudo Jacobi Polynomials

Let us consider the following q -difference equation

$$\begin{aligned} & (q^{2-u}x^2 + 2\sin\left(\frac{v}{2}\ln q\right)x + 1) D_q^2 y_n(x; q) \\ & + \left(\frac{q^u - q^{2-u}}{1-q} x - 2\sin\left(\frac{v}{2}\ln q\right)(q^{1-\frac{u}{2}} - q^{\frac{u}{2}}) \right) D_q y_n(x; q) + \lambda_{n,q}^* y_n(qx; q) = 0, \end{aligned} \quad (18)$$

with

$$\lambda_{n,q}^* = -\frac{[n]_q}{q^n} \left(q^{2-u}[n-1]_q + \frac{q^u - q^{2-u}}{1-q} \right),$$

for $n = 0, 1, 2, \dots$ and $q \in \mathbb{R} \setminus \{-1, 0, 1\}$.

It is clear that

$$\lim_{q \rightarrow 1} \lambda_{n,q}^* = -n(n-2u+1),$$

gives the same eigenvalues as in the continuous case (6).

Theorem 1. Let $\{J_n^{(u,v)}(x; q)\}_n$ defined in (11) be a sequence of polynomials that satisfies the q -difference Equation (18). Subsequently, we have

$$\int_{-\infty}^\infty w^{(u,v)}(x; q) J_n^{(u,v)}(x; q) J_m^{(u,v)}(x; q) d_q x = \left(\int_{-\infty}^\infty w^{(u,v)}(x; q) \left(J_n^{(u,v)}(x; q) \right)^2 d_q x \right) \delta_{n,m},$$

where $N < u - \frac{1}{2}$ for $N = \max\{m, n\}$ and the positive function $w^{(u,v)}(x; q)$ is a solution of the Pearson-type q -difference equation

$$\begin{aligned} & D_q \left(w^{(u,v)}(x; q) (q^{2-u}x^2 + 2\sin\left(\frac{v}{2}\ln q\right)x + 1) \right) \\ & = \left(\frac{q^u - q^{2-u}}{1-q} x - 2\sin\left(\frac{v}{2}\ln q\right)(q^{1-\frac{u}{2}} - q^{\frac{u}{2}}) \right) w^{(u,v)}(qx; q), \end{aligned}$$

which is equivalent to

$$\frac{w^{(u,v)}(x; q)}{w^{(u,v)}(qx; q)} = \frac{q^u x^2 - 2q^{\frac{u}{2}} \sin\left(\frac{v}{2}\ln q\right)x + 1}{q^{-u}x^2 + 2q^{-\frac{u}{2}} \sin\left(\frac{v}{2}\ln q\right)x + 1}. \quad (19)$$

Proof. First, according to [3] and referring to (7) it is not difficult to verify that

$$\begin{aligned} w^{(u,v)}(x; q) &= \frac{(iq^{(u-iv)/2}x, -iq^{(u+iv)/2}x; q)_\infty}{(iq^{(-u+iv)/2}x, -iq^{(-u-iv)/2}x; q)_\infty} \\ &= x^{-2u} \frac{(-iq^{(-u+iv)/2}x^{-1}, iq^{(-u-iv)/2}x^{-1}; q^{-1})_\infty}{(-iq^{(u-iv)/2}x^{-1}, iq^{(u+iv)/2}x^{-1}; q^{-1})_\infty}, \end{aligned} \quad (20)$$

is a solution of Equation (19).

Now, if Equation (18) is written in the self-adjoint form

$$D_q \left(w^{(u,v)}(x; q) (q^{2-u}x^2 + 2\sin\left(\frac{v}{2}\ln q\right)x + 1) D_q J_n^{(u,v)}(x; q) \right) + \lambda_{n,q}^* w^{(u,v)}(qx; q) J_n^{(u,v)}(qx; q) = 0, \quad (21)$$

and for m as

$$D_q \left(w^{(u,v)}(x; q) \left(q^{2-u} x^2 + 2 \sin\left(\frac{v}{2} \ln q\right) x + 1 \right) D_q J_m^{(u,v)}(x; q) \right) + \lambda_{m,q}^* w^{(u,v)}(qx; q) J_m^{(u,v)}(qx; q) = 0, \quad (22)$$

by multiplying (21) by $J_m^{(u,v)}(qx; q)$ and (22) by $J_n^{(u,v)}(qx; q)$ and subtracting each other we get

$$\begin{aligned} & (\lambda_{m,q}^* - \lambda_{n,q}^*) w^{(u,v)}(x; q) J_m^{(u,v)}(x; q) J_n^{(u,v)}(x; q) \\ &= q^2 D_q \left(w^{(u,v)}(q^{-1}x; q) \left(q^{2-u} x^2 + 2 \sin\left(\frac{v}{2} \ln q\right) x + 1 \right) D_q J_n^{(u,v)}(q^{-1}x; q) \right) J_m^{(u,v)}(x; q) \\ &- q^2 D_q \left(w^{(u,v)}(q^{-1}x; q) \left(q^{2-u} x^2 + 2 \sin\left(\frac{v}{2} \ln q\right) x + 1 \right) D_q J_m^{(u,v)}(q^{-1}x; q) \right) J_n^{(u,v)}(x; q). \end{aligned} \quad (23)$$

Hence, q -integration by parts on both sides of (23) over $(-\infty, \infty)$ yields

$$\begin{aligned} & (\lambda_{m,q}^* - \lambda_{n,q}^*) \int_{-\infty}^{\infty} w^{(u,v)}(x; q) J_m^{(u,v)}(x; q) J_n^{(u,v)}(x; q) d_q x \\ &= q^2 \int_{-\infty}^{\infty} \left\{ D_q \left(w^{(u,v)}(q^{-1}x; q) \left(q^{2-u} x^2 + 2 \sin\left(\frac{v}{2} \ln q\right) x + 1 \right) D_q J_n^{(u,v)}(q^{-1}x; q) \right) J_m^{(u,v)}(x; q) \right. \\ &- D_q \left(w^{(u,v)}(q^{-1}x; q) \left(q^{2-u} x^2 + 2 \sin\left(\frac{v}{2} \ln q\right) x + 1 \right) D_q J_m^{(u,v)}(q^{-1}x; q) \right) J_n^{(u,v)}(x; q) \left. \right\} d_q x \\ &= q^2 \left[w^{(u,v)}(q^{-1}x; q) \left(q^{2-u} x^2 + 2 \sin\left(\frac{v}{2} \ln q\right) x + 1 \right) \right. \\ &\quad \times \left. \left(D_q J_n^{(u,v)}(q^{-1}x; q) J_m^{(u,v)}(x; q) - D_q J_m^{(u,v)}(q^{-1}x; q) J_n^{(u,v)}(x; q) \right) \right]_{-\infty}^{\infty}. \end{aligned} \quad (24)$$

Because

$$\max \deg \{ D_q J_n^{(u,v)}(q^{-1}x; q) J_m^{(u,v)}(x; q) - D_q J_m^{(u,v)}(q^{-1}x; q) J_n^{(u,v)}(x; q) \} = m + n - 1,$$

the left-hand side of (24) is zero if

$$\lim_{x \rightarrow \infty} w^{(u,v)}(q^{-1}x; q) \left(q^{2-u} x^2 + 2 \sin\left(\frac{v}{2} \ln q\right) x + 1 \right) x^{m+n-1} = 0. \quad (25)$$

By taking $\max\{m, n\} = N$, relation (25) would be equivalent to

$$\lim_{x \rightarrow \infty} \frac{(-iq^{(-u+iv)/2} x^{-1}, iq^{(-u-iv)/2} x^{-1}; q^{-1})_{\infty}}{(-iq^{(u-iv)/2} x^{-1}, iq^{(u+iv)/2} x^{-1}; q^{-1})_{\infty}} x^{2N-2u+1} = 0. \quad (26)$$

Note that (26) is valid if and only if

$$2N + 1 - 2u < 0 \quad \text{or} \quad N < u - \frac{1}{2}.$$

Therefore, the right-hand side of (24) tends to zero and

$$\int_{-\infty}^{\infty} w^{(u,v)}(x; q) J_m^{(u,v)}(x; q) J_n^{(u,v)}(x; q) d_q x = 0,$$

if and only if $m \neq n$ and $N < u - \frac{1}{2}$ for $N = \max\{m, n\}$. $\square$

Corollary 1. The finite polynomial set $\{J_n^{(u,v)}(x; q)\}_{n=0}^{N < u - \frac{1}{2}}$ is orthogonal with respect to the weight function (20) on $(-\infty, \infty)$.

2.1. Computing the Norm Square Value

According to (17), because $J_n^{(u,v)}(x; q)$ is a particular case of the big q -Jacobi polynomials, it satisfies the recurrence relation [3]

$$\bar{f}_{n+1}^{(u,v)}(x; q) = (x - c_n(u, v; q)) \bar{f}_n^{(u,v)}(x; q) - d_n(u, v; q) \bar{f}_{n-1}^{(u,v)}(x; q),$$

with the initial terms

$$\bar{f}_0^{(u,v)}(x; q) = 1, \quad \bar{f}_1^{(u,v)}(x; q) = x + \frac{2 \sin(\frac{v}{2} \ln q)(1 - q)(q^{2-u/2} + q^{1+u/2})}{(q^u - q^{2-u})},$$

where

$$c_n(u, v; q) = \frac{2 \sin(\frac{v}{2} \ln q) q^n}{(q^u - q^{2n-2})(q^u - q^{2n})} \\ \times \{ (q^u - q^{n-1}) (q^{-u/2} [n]_q (1 + q) + (q^{2-u/2} + q^{1+u/2})) - q^{n+1-u} (1 - q^{n+1}) (q^{1-u/2} + q^{u/2}) \},$$

and

$$d_n(u, v; q) = \frac{(q^{n+1} - q^{2n+1})(q^u - q^{n-u})}{(1 - q)^2 (q^u - q^{2n-u-1})(q^u - q^{2n-u})^2 (q^u - q^{2n-u+1})} \\ \times \{ 4 \sin^2(\frac{v}{2} \ln q) q^{n-1-u/2} (1 - q) (1 + q - q^2 + q^u - q^{1+u} - q^{n-1}) (1 - q^{n-u+1} (1 + q - q^2) - q^{n+1} (1 - q)) \\ - (q^{4n-2u} + 2q^{2n} + q^{2u}) \}.$$

Now, by applying the Favard theorem [19] for the monic type of polynomials (11), we get

$$\int_{-\infty}^{\infty} w^{(u,v)}(x; q) \bar{f}_m^{(u,v)}(x; q) \bar{f}_n^{(u,v)}(x; q) d_q x = \left(\mu_0 \prod_{k=1}^n d_k(u, v; q) \right) \delta_{n,m},$$

where

$$\mu_0 = \int_{-\infty}^{\infty} \frac{(iq^{(u-iv)/2} x, -iq^{(u+iv)/2} x; q)_{\infty}}{(iq^{(-u+iv)/2} x, -iq^{(-u-iv)/2} x; q)_{\infty}} d_q x.$$

Hence, it remains to explicitly compute the above μ_0 . For this purpose, we can refer to the general formula ([13] Formula 128)

$$\int_{z_- q^{\mathbb{Z}} \cup z_+ q^{\mathbb{Z}}} \frac{(ax, bx; q)_{\infty}}{(cx, dx; q)_{\infty}} d_q x = \frac{(q, a/c, a/d, b/c, b/d; q)_{\infty}}{(a b / (q c d); q)_{\infty}} \frac{\theta(z_- / z_+; q) \theta(c d z_- z_+; q)}{\theta(c z_-; q) \theta(d z_-; q) \theta(c z_+; q) \theta(d z_+; q)}, \quad (27)$$

in which

$$\theta(x; q) = (x, q/x; q)_{\infty}.$$

Therefore, it is enough to replace $z_- = -1, z_+ = 1$ in (27) to finally obtain

$$\mu_0 = \frac{(q, q^{u-iv}, -q^u, -q^u, q^{u+iv}; q)_{\infty}}{(q^{2u-1}; q)_{\infty}} \times \\ \frac{(-1, -q, -q^u, -q^{u+1}; q)_{\infty}}{(-iq^{\frac{-u+iv}{2}}, iq^{\frac{-u+iv}{2}}, -iq^{\frac{-u-iv}{2}}, iq^{\frac{-u-iv}{2}}, -iq^{1-\frac{-u+iv}{2}}, iq^{1-\frac{-u+iv}{2}}, -iq^{1-\frac{-u-iv}{2}}, iq^{1-\frac{-u-iv}{2}}; q)_{\infty}}.$$

For example, the set $\{J_n^{(21,1)}(x; q)\}_{n=0}^{20}$ is a finite sequence of q -orthogonal polynomials that satisfies the orthogonality relation

$$\int_{-\infty}^{\infty} \frac{(iq^{(21-i)/2} x, -iq^{(21+i)/2} x; q)_{\infty}}{(iq^{-(21-i)/2} x, -iq^{-(21+i)/2} x; q)_{\infty}} \bar{f}_m^{(21,1)}(x; q) \bar{f}_n^{(21,1)}(x; q) d_q x = \\ \left(\frac{(q, q^{21-i}, -q^{21}, -q^{21}, q^{21+i}, -1, -q, -q^{21}, -q^{22}; q)_{\infty}}{(q^{41}, -iq^{\frac{-21+i}{2}}, iq^{\frac{-21+i}{2}}, -iq^{\frac{-21-i}{2}}, iq^{\frac{-21-i}{2}}, -iq^{\frac{23-i}{2}}, iq^{\frac{23-i}{2}}, -iq^{\frac{23+i}{2}}, iq^{\frac{23+i}{2}}; q)_{\infty}} \prod_{k=1}^n d_k(21, 1; q) \right) \delta_{m,n} \\ \iff m, n < 20,$$

where

$$d_k(21, 1; q) = \frac{(q^{k+1} - q^{2k+1})(q^{21} - q^{k-21})}{(1-q)^2(q^{21} - q^{2k-22})(q^{21} - q^{2k-21})^2(q^{21} - q^{2k-20})} \\ \times \{4 \sin^2(\frac{1}{2} \ln q) q^{k-23/2} (1-q) (1+q-q^2+q^{21}-q^{22}-q^{k-1}) (1-q^{k-20}(1+q-q^2)-q^{k+1}(1-q)) \\ - (q^{4k-42} + 2q^{2k} + q^{42})\}. \quad (28)$$

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