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Inequalities on the Generalized *ABC* Index

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Abstract: In this work, we obtained new results relating the generalized atom-bond connectivity index with the general Randić index. Some of these inequalities for ABC_α improved, when $\alpha = 1/2$, known results on the ABC index. Moreover, in order to obtain our results, we proved a kind of converse Hölder inequality, which is interesting on its own.

Keywords: *ABC* index; *generalizedABC* index; general Randić index; topological indices; converse Hölder inequality

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1. Introduction

Mathematical inequalities are at the basis of the processes of approximation, estimation, dimensioning, interpolation, monotonicity, extremes, etc. In general, inequalities appear in models for the study or approach to a certain reality (either objective or subjective). This reason makes it clear that when working with mathematical inequalities, we can essentially find relationships and approximate values of the magnitudes and variables that are associated with one or another practical problem.

In mathematical chemistry, a topological descriptor is a function that associates each molecular graph with a real value; if it correlates well with some chemical property, it is called a topological index. For additional information see [1], for application examples see [2–7].

The atom-bond connectivity index of a graph G was defined in [8] as:

$$ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{2(d_u + d_v - 2)}{d_u d_v}} = \sqrt{2} \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}},$$

where uv denotes the edge of the graph G connecting the vertices u and v and d_u is the degree of the vertex u.

The *generalized atom-bond connectivity index* was defined in [9] as:

$$ABC_\alpha(G) = \sum_{uv \in E(G)} \left(\frac{d_u + d_v - 2}{d_u d_v} \right)^\alpha.$$

for any $\alpha \in \mathbb{R} \setminus \{0\}$. Note that $ABC_{1/2} = \frac{\sqrt{2}}{2} ABC$ and ABC_{-3} is the augmented Zagreb index.

There are many papers that have studied the ABC and ABC_α indices (see, e.g., [9–15]). In this paper, we obtained new inequalities relating these indices with the general Randić index. Some of these inequalities for ABC_α improved, when $\alpha = 1/2$, known results on the ABC index. In order to obtain our results, we proved a kind of converse Hölder inequality, Theorem 3, which is interesting on its own.

Throughout this work, a path graph P_n is a tree with n vertices and maximum degree two and a star graph S_n is a tree with n vertices and maximum degree $n-1$.

2. Inequalities Involving ABC_α

In 1998, Bollobás and Erdős [16] generalized the Randić index for $\alpha \in \mathbb{R} \setminus \{0\}$,

$$R_\beta(G) = \sum_{uv \in E(G)} (d_u d_v)^\beta.$$

The general Randić index, also called the *variable Zagreb index* in 2004 by Miličević and Nikolić [17], was extensively studied in [18–20].

The next result relates the ABC_α and R_β indices.

Theorem 1. *Let G be a graph with maximum degree Δ and minimum degree δ and $\alpha > 0$, $\beta \in \mathbb{R} \setminus \{0\}$. Denote by m_2 the cardinality of the set of isolated edges in G.*

(1) If $\beta/\alpha \le -1$ and $\delta > 1$, then:

$$(2\delta-2)^\alpha \delta^{-2\alpha-2\beta} R_\beta(G) \le ABC_\alpha(G) \le (2\Delta-2)^\alpha \Delta^{-2\alpha-2\beta} R_\beta(G).$$

The equality in each bound is attained if and only if G is a regular graph.

(2) If $\beta/\alpha \le -1$ and $\delta = 1$, then:

$$2^{-\alpha-\beta}\big(R_\beta(G) - m_2\big) \le ABC_\alpha(G) \le (2\Delta-2)^\alpha \Delta^{-2\alpha-2\beta}\big(R_\beta(G) - m_2\big).$$

The equality in the lower bound is attained if and only if G is a union of path graphs P_3 and m_2 isolated edges. The equality in the upper bound is attained if and only if G is a union of a regular graph and m_2 isolated edges.

(3) If $-1 < \beta/\alpha \le -1/2$ and $\delta > 1$, then:

$$(2\delta-2)^\alpha \delta^{-2\alpha-2\beta} R_\beta(G) \le ABC_\alpha(G).$$

The equality in the bound is attained if and only if G is a regular graph.

(4) If $-1 < \beta/\alpha \le -1/2$ and $\delta = 1$, then:

$$2^{-\alpha-\beta}\big(R_\beta(G) - m_2\big) \le ABC_\alpha(G).$$

The equality in the bound is attained if and only if G is a union of path graphs P_3 and m_2 isolated edges.

(5) If $\beta > 0$ and $\delta > 1$, then:

$$(2\Delta-2)^\alpha \Delta^{-2\alpha-2\beta} R_\beta(G) \le ABC_\alpha(G) \le (2\delta-2)^\alpha \delta^{-2\alpha-2\beta} R_\beta(G).$$

The equality in each bound is attained if and only if G is a regular graph.

(6) If $\beta > 0$, $\delta = 1$ and $1 + \alpha/\beta \ge \Delta$, then:

$$(2\Delta-2)^\alpha \Delta^{-2\alpha-2\beta}\big(R_\beta(G) - m_2\big) \le ABC_\alpha(G) \le (\Delta-1)^\alpha \Delta^{-\alpha-\beta}\big(R_\beta(G) - m_2\big).$$

The equality in the lower bound is attained if and only if G is a union of a regular graph and m_2 isolated edges. The equality in the upper bound is attained if and only if G is a union of star graphs $S_{\Delta+1}$ and m_2 isolated edges.

(7) If $\beta > 0$, $\delta = 1$ and $1 + \alpha/\beta \le 2$, then:

$$(2\Delta-2)^\alpha \Delta^{-2\alpha-2\beta}\big(R_\beta(G) - m_2\big) \le ABC_\alpha(G) \le 2^{-\alpha-\beta}\big(R_\beta(G) - m_2\big).$$

The equality in the lower bound is attained if and only if G is a union of a regular graph and m_2 isolated edges. The equality in the upper bound is attained if and only if G is a union of path graphs P_3 and m_2 isolated edges.

(8) If $\beta > 0$, $\delta = 1$ and $2 < 1 + \alpha/\beta < \Delta$, then:

$$(2\Delta - 2)^{\alpha}\Delta^{-2\alpha-2\beta}\big(R_{\beta}(G) - m_2\big) \le ABC_{\alpha}(G) \le \frac{\alpha^{\alpha}\beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}}\big(R_{\beta}(G) - m_2\big).$$

The equality in the lower bound is attained if and only if G is a union of a regular graph and m_2 isolated edges. The equality in the upper bound is attained if and only if $\alpha/\beta \in \mathbb{Z}^+$ and G is a union of star graphs $S_{\alpha/\beta+2}$ and m_2 isolated edges.

Proof. First of all, note that $ABC_{\alpha}(P_2) = 0$ and $R_{\beta}(P_2) = 1$. Therefore, it suffices to prove the theorem for the case $m_2 = 0$, i.e., when G is a graph without isolated edges. Hence, $\Delta \ge 2$.

We computed the extremal values (for fixed $\lambda \in \mathbb{R}$) of the function $f : [\delta, \Delta] \times ([\delta, \Delta] \setminus [1, 2)) \longrightarrow \mathbb{R}$ given by:

$$f(x, y) = (x + y - 2)(xy)^{-\lambda-1}.$$

(1) and (2). If $\lambda \le -1$, then $-\lambda - 1 \ge 0$ and f is a strictly increasing function in each variable, and so,

$$(2\delta - 2)\delta^{-2\lambda-2} \le f(x, y) \le (2\Delta - 2)\Delta^{-2\lambda-2}.$$

The equality in the lower (respectively, upper) bound is attained if and only if $(x, y) = (\delta, \delta)$ (respectively, $(x, y) = (\Delta, \Delta)$).

If $\delta = 1$, then $f(x, y) \ge f(1, 2) = 2^{-\lambda-1}$, since $x \in [1, \Delta]$ and $y \in [2, \Delta]$, and the equality in this inequality is attained if and only if $(x, y) = (1, 2)$.

If $\lambda = \beta/\alpha$, then:

$$(2\delta - 2)^{\alpha}\delta^{-2\beta-2\alpha}\big(d_u d_v\big)^{\beta} \le \Big(\frac{d_u + d_v - 2}{d_u d_v}\Big)^{\alpha} \le (2\Delta - 2)^{\alpha}\Delta^{-2\beta-2\alpha}\big(d_u d_v\big)^{\beta}$$

for every $uv \in E(G)$ and, consequently,

$$(2\delta - 2)^{\alpha}\delta^{-2\alpha-2\beta}R_{\beta}(G) \le ABC_{\alpha}(G) \le (2\Delta - 2)^{\alpha}\Delta^{-2\alpha-2\beta}R_{\beta}(G).$$

The previous argument shows that the equality in the upper bound is attained if and only if $d_u = d_v = \Delta$ for every $uv \in E(G)$, i.e., G is regular. If $\delta > 1$, then the equality in the lower bound is attained if and only if $d_u = d_v = \delta$ for every $uv \in E(G)$, i.e., G is regular.

If $\lambda = \beta/\alpha$ and $\delta = 1$, then:

$$2^{-\beta-\alpha}\big(d_u d_v\big)^{\beta} \le \Big(\frac{d_u + d_v - 2}{d_u d_v}\Big)^{\alpha}$$

for every $uv \in E(G)$ and, consequently,

$$2^{-\alpha-\beta}R_{\beta}(G) \le ABC_{\alpha}(G).$$

The equality in this bound is attained if and only if $\{d_u, d_v\} = \{1, 2\}$ for every $uv \in E(G)$, i.e., G is a union of path graphs P_3.

(3) and (4). In what follows, by symmetry, we can assume that $x \le y$. We have:

$$\begin{aligned}\frac{\partial f}{\partial y}(x, y) &= x^{-\lambda-1}\big(y^{-\lambda-1} + (x + y - 2)(-\lambda - 1)y^{-\lambda-2}\big) \\ &= x^{-\lambda-1}y^{-\lambda-2}\big(y + (x + y - 2)(-\lambda - 1)\big).\end{aligned}$$

If $-1 < \lambda \le -1/2$, then $-\lambda - 1 \ge -1/2$, and so,

$$\begin{aligned}\frac{\partial f}{\partial y}(x,y) &\ge x^{-\lambda-1}y^{-\lambda-2}\Big(y - \frac{x+y-2}{2}\Big)\\ &= x^{-\lambda-1}y^{-\lambda-2}\,\frac{y-x+2}{2} \ge x^{-\lambda-1}y^{-\lambda-2} > 0.\end{aligned}$$

Hence,

$$f(x,y) \ge f(x,x) = (2x-2)x^{-2\lambda-2} = g(x).$$

We have:

$$\begin{aligned}g'(x) &= 2x^{-2\lambda-2} + (2x-2)(-2\lambda-2)x^{-2\lambda-3}\\ &= 2x^{-2\lambda-3}\big(x + (x-1)(-2\lambda-2)\big)\\ &= 2x^{-2\lambda-3}\big((-2\lambda-1)x + 2\lambda+2\big).\end{aligned}$$

Since $2\lambda + 2 > 0$ and $-2\lambda - 1 \ge 0$, we have:

$$\begin{aligned}g'(x) &= 2x^{-2\lambda-3}\big((-2\lambda-1)x + 2\lambda+2\big)\\ &\ge 2x^{-2\lambda-3}(2\lambda+2) > 0.\end{aligned}$$

Thus, $g(x) \ge g(\delta)$ and:

$$f(x,y) \ge g(x) \ge (2\delta-2)\delta^{-2\lambda-2},$$

if $\delta \ge 2$.

If $\lambda = \beta/\alpha$ and $\delta > 1$, then:

$$(2\delta-2)^{\alpha}\delta^{-2\beta-2\alpha}\big(d_u d_v\big)^{\beta} \le \Big(\frac{d_u + d_v - 2}{d_u d_v}\Big)^{\alpha}$$

for every $uv \in E(G)$ and, consequently,

$$(2\delta-2)^{\alpha}\delta^{-2\alpha-2\beta}R_{\beta}(G) \le ABC_{\alpha}(G).$$

The previous argument shows that the equality in this bound is attained if and only if $d_u = d_v = \delta$ for every $uv \in E(G)$, i.e., G is regular.

Assume that $\delta = 1$. We proved that $f(x,y) \ge g(x) \ge g(2) = 2^{-2\lambda-1}$ for every $x, y \in [2,\Delta]$. Since $\partial f/\partial y(1,y) > 0$ for every $y \in [2,\Delta]$, we have $f(1,y) \ge f(1,2) = 2^{-\lambda-1}$ for every $y \in [2,\Delta]$. Since $\lambda < 0$, we have $2^{-2\lambda-1} > 2^{-\lambda-1}$ and $f(x,y) \ge 2^{-\lambda-1}$ for every $x \in [1,\Delta] \cap \mathbb{Z}, y \in [2,\Delta] \cap \mathbb{Z}$. Furthermore, the equality in this bound is attained if and only if $(x,y) = (1,2)$.

If $\lambda = \beta/\alpha$, then:

$$2^{-\beta-\alpha}\big(d_u d_v\big)^{\beta} \le \Big(\frac{d_u + d_v - 2}{d_u d_v}\Big)^{\alpha}$$

for every $uv \in E(G)$ and, consequently,

$$2^{-\alpha-\beta}R_{\beta}(G) \le ABC_{\alpha}(G).$$

The equality in this bound is attained if and only if $\{d_u, d_v\} = \{1,2\}$ for every $uv \in E(G)$, i.e., G is a union of path graphs P_3.

(5). Assume now that $\lambda > 0$. Thus, $-\lambda - 1 < -1$ and:

$$\begin{aligned}\frac{\partial f}{\partial y}(x,y) &= x^{-\lambda-1}y^{-\lambda-2}\big(y + (x+y-2)(-\lambda-1)\big)\\ &< x^{-\lambda-1}y^{-\lambda-2}(2-x),\end{aligned}$$

and:

$$\frac{\partial f}{\partial x}(x,y) < y^{-\lambda-1}x^{-\lambda-2}(2-y).$$

If $\delta > 1$, then f is a strictly decreasing function in each variable, and so,

$$(2\Delta-2)\Delta^{-2\lambda-2} \le f(x,y) \le (2\delta-2)\delta^{-2\lambda-2}. \tag{1}$$

The equality in the lower (respectively, upper) bound is attained if and only if $(x,y) = (\Delta,\Delta)$ (respectively, $(x,y) = (\delta,\delta)$).

If $\beta > 0$ and $\lambda = \beta/\alpha$, then:

$$(2\Delta-2)^{\alpha}\Delta^{-2\beta-2\alpha}\big(d_u d_v\big)^{\beta} \le \Big(\frac{d_u+d_v-2}{d_u d_v}\Big)^{\alpha} \le (2\delta-2)^{\alpha}\delta^{-2\beta-2\alpha}\big(d_u d_v\big)^{\beta}$$

for every $uv \in E(G)$ and, consequently,

$$(2\Delta-2)^{\alpha}\Delta^{-2\alpha-2\beta}R_{\beta}(G) \le ABC_{\alpha}(G) \le (2\delta-2)^{\alpha}\delta^{-2\alpha-2\beta}R_{\beta}(G).$$

The equality in the lower bound is attained if and only if $d_u = d_v = \Delta$ for every $uv \in E(G)$, i.e., G is regular. Furthermore, the equality in the upper bound is attained if and only if $d_u = d_v = \delta$ for every $uv \in E(G)$, i.e., G is regular.

(6). Note that:

$$\Big(\frac{\Delta^2}{2}\Big)^{\lambda+1} > \frac{\Delta^2}{2} \ge 2\Delta-2 \qquad \Rightarrow \qquad 2^{-\lambda-1} > (2\Delta-2)\Delta^{-2\lambda-2}. \tag{2}$$

We also have:

$$\Delta^{\lambda+1} > \Delta \ge 2 \qquad \Rightarrow \qquad (\Delta-1)\Delta^{-\lambda-1} > (2\Delta-2)\Delta^{-2\lambda-2}. \tag{3}$$

Assume that $\delta = 1$. If $2 \le x,y \le \Delta$, then $f(x,y) \le f(2,2) = 2^{-2\lambda-1}$. This inequality and the lower bound in (1) give:

$$(2\Delta-2)\Delta^{-2\lambda-2} \le f(x,y) \le 2^{-2\lambda-1}, \tag{4}$$

for every $2 \le x,y \le \Delta$.

Let us consider the function $h(y) = f(1,y) = (y-1)y^{-\lambda-1}$ with $2 \le y \le \Delta$. We have:

$$h'(y) = -\lambda y^{-\lambda-1} + (\lambda+1)y^{-\lambda-2} = y^{-\lambda-2}(-\lambda y + \lambda + 1),$$

and so, h strictly increases on $(0, 1+1/\lambda)$ and strictly decreases on $(1+1/\lambda, \infty)$.

If $1+1/\lambda \ge \Delta$, then h strictly increases on $(0,\Delta]$ and:

$$2^{-\lambda-1} = h(2) \le h(y) \le h(\Delta) = (\Delta-1)\Delta^{-\lambda-1},$$

for every $2 \le y \le \Delta$. These inequalities and Equation (4) give:

$$\min\big\{2^{-\lambda-1},\, (2\Delta-2)\Delta^{-2\lambda-2}\big\} \le f(x,y) \le \max\big\{(\Delta-1)\Delta^{-\lambda-1},\, 2^{-2\lambda-1}\big\}.$$

for every $x \in [1,\Delta] \cap \mathbb{Z}, y \in [2,\Delta] \cap \mathbb{Z}$. Since we have in this case $2^{-\lambda-1} = h(2) \le h(\Delta) = (\Delta-1)\Delta^{-\lambda-1}$, we conclude:

$$\begin{aligned}(\Delta-1)\Delta^{-\lambda-1} &\le \max\big\{(\Delta-1)\Delta^{-\lambda-1},\, 2^{-2\lambda-1}\big\}\\ &\le \max\big\{(\Delta-1)\Delta^{-\lambda-1},\, 2^{-\lambda-1}\big\} = (\Delta-1)\Delta^{-\lambda-1}.\end{aligned}$$

Equation (2) gives:

$$\min\big\{2^{-\lambda-1},\, (2\Delta-2)\Delta^{-2\lambda-2}\big\} = (2\Delta-2)\Delta^{-2\lambda-2}.$$

Hence,
$$(2\Delta-2)\Delta^{-2\lambda-2} \le f(x,y) \le (\Delta-1)\Delta^{-\lambda-1},$$
for every $x \in [1,\Delta] \cap \mathbb{Z}, y \in [2,\Delta] \cap \mathbb{Z}$. The equality in the lower (respectively, upper) bound is attained if and only if $(x,y) = (\Delta,\Delta)$ (respectively, $(x,y) = (1,\Delta)$).

If $\beta > 0$ and $\lambda = \beta/\alpha$, then we obtain:

$$(2\Delta-2)^{\alpha}\Delta^{-2\beta-2\alpha}(d_u d_v)^{\beta} \le \Big(\frac{d_u+d_v-2}{d_u d_v}\Big)^{\alpha} \le (\Delta-1)^{\alpha}\Delta^{-\beta-\alpha}(d_u d_v)^{\beta},$$
$$(2\Delta-2)^{\alpha}\Delta^{-2\alpha-2\beta}R_{\beta}(G) \le ABC_{\alpha}(G) \le (\Delta-1)^{\alpha}\Delta^{-\alpha-\beta}R_{\beta}(G).$$

The equality in the lower bound is attained if and only if $d_u = d_v = \Delta$ for every $uv \in E(G)$, i.e., G is regular. The equality in the upper bound is attained if and only if $\{d_u, d_v\} = \{1, \Delta\}$ for every $uv \in E(G)$, i.e., G is a union of star graphs $S_{\Delta+1}$.

(7). If $1 + 1/\lambda \le 2$, then h strictly decreases on $[2,\Delta]$ and:

$$(\Delta-1)\Delta^{-\lambda-1} = h(\Delta) \le h(y) \le h(2) = 2^{-\lambda-1},$$

for every $2 \le y \le \Delta$. These inequalities and Equation (4) give:

$$\min\big\{(\Delta-1)\Delta^{-\lambda-1},\, (2\Delta-2)\Delta^{-2\lambda-2}\big\} \le f(x,y) \le \max\big\{2^{-\lambda-1},\, 2^{-2\lambda-1}\big\},$$

for every $x \in [1,\Delta] \cap \mathbb{Z}, y \in [2,\Delta] \cap \mathbb{Z}$. Equation (3) gives:

$$(2\Delta-2)\Delta^{-2\lambda-2} \le f(x,y) \le 2^{-\lambda-1},$$

for every $x \in [1,\Delta] \cap \mathbb{Z}, y \in [2,\Delta] \cap \mathbb{Z}$. The equality in the lower (respectively, upper) bound is attained if and only if $(x,y) = (\Delta,\Delta)$ (respectively, $(x,y) = (1,2)$).

If $\beta > 0$ and $\lambda = \beta/\alpha$, then we obtain for every $uv \in E(G)$:

$$(2\Delta-2)^{\alpha}\Delta^{-2\beta-2\alpha}(d_u d_v)^{\beta} \le \Big(\frac{d_u+d_v-2}{d_u d_v}\Big)^{\alpha} \le 2^{-\beta-\alpha}(d_u d_v)^{\beta},$$
$$(2\Delta-2)^{\alpha}\Delta^{-2\alpha-2\beta}R_{\beta}(G) \le ABC_{\alpha}(G) \le 2^{-\alpha-\beta}R_{\beta}(G).$$

The equality in the lower bound is attained if and only if $d_u = d_v = \Delta$ for every $uv \in E(G)$, i.e., G is regular. The equality in the upper bound is attained if and only if $\{d_u, d_v\} = \{1, 2\}$ for every $uv \in E(G)$, i.e., G is a union of path graphs P_3.

(8). If $2 < 1 + 1/\lambda < \Delta$, then:

$$h(y) \ge \min\big\{h(2),\, h(\Delta)\big\} = \min\big\{2^{-\lambda-1},\, (\Delta-1)\Delta^{-\lambda-1}\big\},$$

for every $2 \le y \le \Delta$. Furthermore,

$$h(y) \le h(1+1/\lambda) = \frac{1}{\lambda}\Big(\frac{\lambda+1}{\lambda}\Big)^{-\lambda-1} = \frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}},$$

for every $2 \le y \le \Delta$. These facts and (4) give:

$$\begin{aligned}\min\big\{2^{-\lambda-1},\, (\Delta-1)\Delta^{-\lambda-1},\, (2\Delta-2)\Delta^{-2\lambda-2}\big\} &\le f(x,y)\\ &\le \max\Big\{\frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}},\, 2^{-2\lambda-1}\Big\}\end{aligned}$$

for every $x \in [1,\Delta] \cap \mathbb{Z}, y \in [2,\Delta] \cap \mathbb{Z}$.

Equations (2) and (3) give:

$$\min\big\{2^{-\lambda-1},\, (\Delta-1)\Delta^{-\lambda-1},\, (2\Delta-2)\Delta^{-2\lambda-2}\big\} = (2\Delta-2)\Delta^{-2\lambda-2}.$$

Since $h(2) \le h(1+1/\lambda)$, we obtain:

$$2^{-2\lambda-1} < 2^{-\lambda-1} \le \frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}},$$

and so,

$$(2\Delta-2)\Delta^{-2\lambda-2} \le f(x,y) \le \frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}}$$

for every $x \in [1,\Delta]\cap\mathbb{Z}, y \in [2,\Delta]\cap\mathbb{Z}$. The equality in the lower (respectively, upper) bound is attained if and only if $(x,y)=(\Delta,\Delta)$ (respectively, $(x,y)=(1,1+1/\lambda)$).

If $\beta > 0$ and $\lambda = \beta/\alpha$, then we obtain:

$$\Big(\frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}}\Big)^\alpha = \frac{(\beta/\alpha)^\beta}{(\beta/\alpha+1)^{\beta+\alpha}} = \frac{\alpha^\alpha\beta^\beta}{(\alpha+\beta)^{\alpha+\beta}},$$

and we have for every $uv \in E(G)$:

$$(2\Delta-2)^\alpha\Delta^{-2\beta-2\alpha}(d_ud_v)^\beta \le \Big(\frac{d_u+d_v-2}{d_ud_v}\Big)^\alpha \le \frac{\alpha^\alpha\beta^\beta}{(\alpha+\beta)^{\alpha+\beta}}\,(d_ud_v)^\beta,$$

$$(2\Delta-2)^\alpha\Delta^{-2\alpha-2\beta}R_\beta(G) \le ABC_\alpha(G) \le \frac{\alpha^\alpha\beta^\beta}{(\alpha+\beta)^{\alpha+\beta}}\,R_\beta(G).$$

The equality in the lower bound is attained if and only if $d_u = d_v = \Delta$ for every $uv \in E(G)$, i.e., G is regular. The equality in the upper bound is attained if and only if $\alpha/\beta \in \mathbb{Z}^+$ and $\{d_u,d_v\} = \{1,1+\alpha/\beta\}$ for every $uv \in E(G)$, i.e., G is a union of star graphs $S_{\alpha/\beta+2}$. □

Note that $ABC_\alpha(G)$ is not well defined if $\alpha < 0$ and G has an isolated edge. The argument in the proof of Theorem 1 gives directly the following result for $\alpha < 0$.

Theorem 2. *Let G be a graph without isolated edges, with maximum degree Δ and minimum degree δ, and $\alpha < 0, \beta \in \mathbb{R}\setminus\{0\}$.*

(1) If $\beta/\alpha \le -1$ and $\delta > 1$, then:

$$(2\Delta-2)^\alpha\Delta^{-2\alpha-2\beta}R_\beta(G) \le ABC_\alpha(G) \le (2\delta-2)^\alpha\delta^{-2\alpha-2\beta}R_\beta(G).$$

The equality in each bound is attained if and only if G is a regular graph.

(2) If $\beta/\alpha \le -1$ and $\delta = 1$, then:

$$(2\Delta-2)^\alpha\Delta^{-2\alpha-2\beta}R_\beta(G) \le ABC_\alpha(G) \le 2^{-\alpha-\beta}R_\beta(G).$$

The equality in the lower bound is attained if and only if G is a regular graph. The equality in the upper bound is attained if and only if G is a union of path graphs P_3.

(3) If $-1 < \beta/\alpha \le -1/2$ and $\delta > 1$, then:

$$ABC_\alpha(G) \le (2\delta-2)^\alpha\delta^{-2\alpha-2\beta}R_\beta(G).$$

The equality in the bound is attained if and only if G is a regular graph.

(4) If $-1 < \beta/\alpha \le -1/2$ and $\delta = 1$, then:

$$ABC_\alpha(G) \le 2^{-\alpha-\beta}R_\beta(G).$$

The equality in the bound is attained if and only if G is a union of path graphs P_3.

(5) If $\beta < 0$ and $\delta > 1$, then:

$$(2\delta-2)^\alpha\delta^{-2\alpha-2\beta}R_\beta(G) \le ABC_\alpha(G) \le (2\Delta-2)^\alpha\Delta^{-2\alpha-2\beta}R_\beta(G).$$

The equality in each bound is attained if and only if G is a regular graph.

(6) If $\beta < 0$, $\delta = 1$ and $1 + \alpha/\beta \geq \Delta$, then:

$$(\Delta - 1)^{\alpha} \Delta^{-\alpha-\beta} R_{\beta}(G) \leq ABC_{\alpha}(G) \leq (2\Delta - 2)^{\alpha} \Delta^{-2\alpha-2\beta} R_{\beta}(G).$$

The equality in the lower bound is attained if and only if G is a union of star graphs $S_{\Delta+1}$. The equality in the upper bound is attained if and only if G is a regular graph.

(7) If $\beta < 0$, $\delta = 1$ and $1 + \alpha/\beta \leq 2$, then:

$$2^{-\alpha-\beta} R_{\beta}(G) \leq ABC_{\alpha}(G) \leq (2\Delta - 2)^{\alpha} \Delta^{-2\alpha-2\beta} R_{\beta}(G).$$

The equality in the lower bound is attained if and only if G is a union of path graphs P_3. The equality in the upper bound is attained if and only if G is a regular graph.

(8) If $\beta < 0$, $\delta = 1$ and $2 < 1 + \alpha/\beta < \Delta$, then:

$$\frac{|\alpha|^{\alpha} |\beta|^{\beta}}{|\alpha + \beta|^{\alpha+\beta}} R_{\beta}(G) \leq ABC_{\alpha}(G) \leq (2\Delta - 2)^{\alpha} \Delta^{-2\alpha-2\beta} R_{\beta}(G).$$

The equality in the lower bound is attained if and only if $\alpha/\beta \in \mathbb{Z}^+$ and G is a union of star graphs $S_{\alpha/\beta+2}$. The equality in the upper bound is attained if and only if G is a regular graph.

Note that Theorems 1 and 2 generalize the classical inequalities:

$$2\sqrt{\delta - 1}\, R(G) \leq ABC(G) \leq 2\sqrt{\Delta - 1}\, R(G). \quad (5)$$

Theorem 1 has the following consequence.

Corollary 1. *Let G be a graph with minimum degree δ and m_2 isolated edges.*

(1) If $\delta > 1$, then:

$$2\sqrt{1 - \frac{1}{\delta}}\, R_{-1/4}(G) \leq ABC(G).$$

The equality in the bound is attained if and only if G is a regular graph.

(2) If $\delta = 1$, then

$$2^{1/4}\big(R_{-1/4}(G) - m_2\big) \leq ABC(G).$$

The equality in the bound is attained if and only if G is a union of path graphs P_3 and m_2 isolated edges.

Corollary 1 improves the inequality:

$$2\left(1 - \frac{1}{\sqrt{\delta}}\right) R_{-1/4}(G) \leq ABC(G)$$

in ([21], Theorem 2.5).

In [22], Lemma 4, the following result appeared.

Lemma 1. *Let (X, μ) be a measure space and $f, g : X \to \mathbb{R}$ measurable functions. If there exist positive constants ω, Ω with $\omega|g| \leq |f| \leq \Omega|g|$ μ-a.e., then:*

$$\|f\|_2 \|g\|_2 \leq \frac{1}{2}\left(\sqrt{\frac{\Omega}{\omega}} + \sqrt{\frac{\omega}{\Omega}}\right) \|fg\|_1. \quad (6)$$

If these norms are finite, the equality in the bound is attained if and only if $\omega = \Omega$ and $|f| = \omega|g|$ μ-a.e. or $f = g = 0$ μ-a.e.

We need the following converse Hölder inequality, which is interesting on its own. This result generalizes Lemma 1 and improves the inequality in [23] (Theorem 2).

Theorem 3. *Let (X, μ) be a measure space, $f, g : X \to \mathbb{R}$ measurable functions, and $1 < p, q < \infty$ with $1/p + 1/q = 1$. If there exist positive constants a, b with $a|g|^q \le |f|^p \le b|g|^q$ μ-a.e., then:*

$$\|f\|_p \|g\|_q \le K_p(a, b) \|fg\|_1, \tag{7}$$

with:

$$K_p(a,b) = \begin{cases} \dfrac{1}{p}\Big(\dfrac{a}{b}\Big)^{1/(2q)} + \dfrac{1}{q}\Big(\dfrac{b}{a}\Big)^{1/(2p)}, & \text{if } 1 < p < 2, \\[2ex] \dfrac{1}{p}\Big(\dfrac{b}{a}\Big)^{1/(2q)} + \dfrac{1}{q}\Big(\dfrac{a}{b}\Big)^{1/(2p)}, & \text{if } p \ge 2. \end{cases}$$

If these norms are finite, the equality in the bound is attained if and only if $a = b$ and $|f|^p = a|g|^q$ μ-a.e. or $f = g = 0$ μ-a.e.

Remark 1. *Since:*

$$K_2(a,b) = \frac{1}{2}\Big(\frac{b}{a}\Big)^{1/4} + \frac{1}{2}\Big(\frac{a}{b}\Big)^{1/4},$$

Theorem 3 generalizes Lemma 1 (note that $a = \omega^2$ and $b = \Omega^2$).

Proof. If $p = 2$, then Lemma 1 (with $\omega = a^{1/2}$ and $\Omega = b^{1/2}$) gives the result. Assume now $p \ne 2$, and let us define:

$$k_p(a,b) = \max\Big\{ \frac{1}{p}\Big(\frac{a}{b}\Big)^{1/(2q)} + \frac{1}{q}\Big(\frac{b}{a}\Big)^{1/(2p)}, \frac{1}{p}\Big(\frac{b}{a}\Big)^{1/(2q)} + \frac{1}{q}\Big(\frac{a}{b}\Big)^{1/(2p)} \Big\}.$$

We will check at the end of the proof that $k_p(a, b) = K_p(a, b)$.

Let us consider $t \in (0, 1)$ and define:

$$G_t(x) := t x^{1-t} + (1-t) x^{-t}$$

for $x > 0$. Since:

$$G_t'(x) = t(1-t)x^{-t} - t(1-t)x^{-t-1} = t(1-t)x^{-t-1}(x-1),$$

G_t is strictly decreasing on $(0, 1)$ and strictly increasing on $(1, \infty)$. Thus, if $0 < s \le S$ are two constants and we consider $s \le x \le S$, then:

$$G_t(x) \le \max\{G_t(s), G_t(S)\} =: A,$$

and if $G_t(x) = A$ for some $s \le x \le S$, then $x = s$ or $x = S$.

Note that if $G_t(s) \ne G_t(S)$, the following facts hold: if $G_t(s) > G_t(S)$ and $G_t(x) = A = G_t(s)$, then $x = s$; if $G_t(s) < G_t(S)$ and $G_t(x) = A = G_t(S)$, then $x = S$.

If $x_1, x_2 > 0$ and $s x_2 \le x_1 \le S x_2$, then:

$$t\Big(\frac{x_1}{x_2}\Big)^{1-t} + (1-t)\Big(\frac{x_2}{x_1}\Big)^{t} \le A,$$
$$t x_1 + (1-t) x_2 \le A x_1^t x_2^{1-t}.$$

By continuity, this last inequality holds for every $x_1, x_2 \ge 0$ with $s x_2 \le x_1 \le S x_2$. If the equality is attained for some $x_1, x_2 \ge 0$ with $s x_2 \le x_1 \le S x_2$, then $x_1 = s x_2$ or $x_1 = S x_2$ (the cases $x_1 = 0$ and $x_2 = 0$ are direct).

Choose $t = 1/p$ (thus, $1 - t = 1/q$), $x = x_1^t = x_1^{1/p}$ and $y = x_2^{1-t} = x_2^{1/q}$. Thus,

$$\frac{x^p}{p} + \frac{y^q}{q} \le Axy \tag{8}$$

for every $x, y \ge 0$ with $sy^q \le x^p \le Sy^q$. If the equality is attained for some $x, y \ge 0$ with $sy^q \le x^p \le Sy^q$, then $x^p = sy^q$ or $x^p = Sy^q$.

If $\|f\|_p = 0$ or $\|g\|_q = 0$, then $a|g|^q \le |f|^p \le b|g|^q$ μ-a.e. gives $\|f\|_p = \|g\|_q = 0$, and the equality in (7) holds. Assume now that $\|f\|_p \neq 0 \neq \|g\|_q$.

Let us define the function:

$$h := (ab)^{1/(2q)}|g|.$$

We have:

$$\sqrt{\frac{a}{b}}\, h^q = a|g|^q, \qquad \sqrt{\frac{b}{a}}\, h^q = b|g|^q, \qquad \sqrt{\frac{a}{b}}\, h^q \le |f|^p \le \sqrt{\frac{b}{a}}\, h^q.$$

If $x = |f|$, $y = h$, $s = (a/b)^{1/2}$, and $S = (b/a)^{1/2}$, then $sh^q \le |f|^p \le Sh^q$ and (8) gives:

$$\frac{1}{p}\,|f|^p + \frac{1}{q}\,h^q \le A|f|h.$$

If the equality in this inequality is attained at some point, then:

$$|f|^p = \sqrt{\frac{a}{b}}\, h^q \qquad \text{or} \qquad |f|^p = \sqrt{\frac{b}{a}}\, h^q$$

at that point.

Note that:

$$G_{1/p}(x) = \frac{1}{p}\, x^{1/q} + \frac{1}{q}\left(\frac{1}{x}\right)^{1/p}$$

and so,

$$A = \max\{G_t(s),\, G_t(S)\} = \max\left\{G_{1/p}\big((a/b)^{1/2}\big),\, G_{1/p}\big((b/a)^{1/2}\big)\right\} = k_p(a,b).$$

Hence,

$$\frac{1}{p}\,|f|^p + \frac{1}{q}\,h^q \le k_p(a,b)|f|h,$$
$$\frac{1}{p}\,\|f\|_p^p + \frac{1}{q}\,\|h\|_q^q \le k_p(a,b)\|fh\|_1.$$

Recall that these norms are well defined, although they can be infinite.

If these norms are finite and the equality in the last inequality is attained, then:

$$|f|^p = \sqrt{\frac{a}{b}}\, h^q \qquad \text{or} \qquad |f|^p = \sqrt{\frac{b}{a}}\, h^q$$

μ-a.e. Young's inequality states that:

$$xy \le \frac{x^p}{p} + \frac{y^q}{q}$$

for every $x, y \ge 0$, and the equality holds if and only if $x^p = y^q$. Thus,

$$\|f\|_p\|h\|_q \le \frac{1}{p}\,\|f\|_p^p + \frac{1}{q}\,\|h\|_q^q \le k_p(a,b)\|fh\|_1.$$

Therefore, by homogeneity, we conclude:

$$\|f\|_p \|g\|_q \le k_p(a,b) \|fg\|_1.$$

Let us prove now that $k_p(a,b) = K_p(a,b)$. Consider the function $H_t(x) := G_t(x) - G_t(1/x)$ for $t \in (0,1)$ and $x \in (0,1]$. We have:

$$\begin{aligned}
H_t'(x) &= G_t'(x) + \frac{1}{x^2}\, G_t'\Big(\frac{1}{x}\Big) \\
&= t(1-t)\, x^{-t-1}(x-1) + t(1-t)\frac{1}{x^2}\, x^{t+1}\Big(\frac{1}{x} - 1\Big) \\
&= t(1-t)\, x^{-t-1}(x-1) + t(1-t)\, x^{t-2}(1-x) \\
&= t(1-t)(1-x)\, x^{-t-1}(x^{2t-1} - 1).
\end{aligned}$$

If $t \in (0,1/2)$, then $2t-1 < 0$ and $H_t'(x) > 0$ for every $x \in (0,1)$, and so, $H_t(x) < H_t(1) = 0$ for every $x \in (0,1)$. Hence, $G_t(x) < G_t(1/x)$ for every $x \in (0,1)$. If $p > 2$ and $a < b$, then $G_{1/p}\big((a/b)^{1/2}\big) < G_{1/p}\big((b/a)^{1/2}\big)$, and:

$$k_p(a,b) = \frac{1}{p}\Big(\frac{b}{a}\Big)^{1/(2q)} + \frac{1}{q}\Big(\frac{a}{b}\Big)^{1/(2p)}.$$

If $t \in (1/2,1)$, then $2t-1 > 0$ and $H_t'(x) < 0$ for every $x \in (0,1)$, and so, $H_t(x) > H_t(1) = 0$ for every $x \in (0,1)$. Hence, $G_t(x) > G_t(1/x)$ for every $x \in (0,1)$. If $1 < p < 2$ and $a < b$, then $G_{1/p}\big((a/b)^{1/2}\big) > G_{1/p}\big((b/a)^{1/2}\big)$, and:

$$k_p(a,b) = \frac{1}{p}\Big(\frac{a}{b}\Big)^{1/(2q)} + \frac{1}{q}\Big(\frac{b}{a}\Big)^{1/(2p)}.$$

Therefore, $k_p(a,b) = K_p(a,b)$.

If $a = b$ and $|f|^p = a|g|^q$ μ-a.e. or $f = g = 0$ μ-a.e., then a computation gives that the equality in (7) is attained.

Finally, assume that the equality in (7) is attained. Seeking for a contradiction, assume that $a \ne b$. The previous argument gives that:

$$|f|^p = \sqrt{\frac{a}{b}}\, h^q \qquad \text{or} \qquad |f|^p = \sqrt{\frac{b}{a}}\, h^q$$

μ-a.e. Since we proved $G_{1/p}\big((a/b)^{1/2}\big) \ne G_{1/p}\big((b/a)^{1/2}\big)$ (recall that $p \ne 2$ and $a < b$), we can conclude that:

$$|f|^p = \sqrt{\frac{a}{b}}\, h^q \;\; \mu\text{-a.e.} \qquad \text{or} \qquad |f|^p = \sqrt{\frac{b}{a}}\, h^q \;\; \mu\text{-a.e.}$$

Hence,

$$\|f\|_p^p = \sqrt{\frac{a}{b}}\, \|h\|_q^q \qquad \text{or} \qquad \|f\|_p^p = \sqrt{\frac{b}{a}}\, \|h\|_q^q.$$

Since the equality in Young's inequality gives $\|f\|_p^p = \|h\|_q^q$, we obtain $a = b$, a contradiction. Therefore, $a = b$ and $|f|^p = h^q$ μ-a.e. Hence, $|f|^p = a\,|g|^q$ μ-a.e. □

Theorem 3 has the following consequence.

Corollary 2. *If $1 < p, q < \infty$ with $1/p + 1/q = 1$, $x_j, y_j \ge 0$ and $ay_j^q \le x_j^p \le by_j^q$ for $1 \le j \le k$ and some positive constants a, b, then:*

$$\Big(\sum_{j=1}^{k} x_j^p \Big)^{1/p} \Big(\sum_{j=1}^{k} y_j^q \Big)^{1/q} \le K_p(a,b) \sum_{j=1}^{k} x_j y_j,$$

where $K_p(a,b)$ is the constant in Theorem 3. If $x_j > 0$ for some $1 \le j \le k$, then the equality in the bound is attained if and only if $a = b$ and $x_j^p = ay_j^q$ for every $1 \le j \le k$.

The *Platt number* is defined (see, e.g., [24]) as:

$$F(G) = \sum_{uv \in E(G)} \big(d_u + d_v - 2\big).$$

Theorem 4. *Let G be a graph with m_2 isolated edges and $0 < \alpha < 1$.*

(1) Then:

$$ABC_\alpha(G) \le F(G)^\alpha \big(R_{-\alpha/(1-\alpha)}(G) - m_2\big)^{1-\alpha}.$$

The equality in this bound is attained for the union of any regular or biregular graph and m_2 isolated edges; if G is the union of a connected graph and m_2 isolated edges, then the equality in this bound is attained if and only if G is the union of any regular or biregular connected graph and m_2 isolated edges.

(2) If $\delta > 1$, then:

$$ABC_\alpha(G) \ge \frac{(\Delta-1)^{\alpha/2}\Delta^{\alpha^2/(1-\alpha)}(\delta-1)^{(1-\alpha)/2}\delta^\alpha F(G)^\alpha R_{-\alpha/(1-\alpha)}(G)^{1-\alpha}}{\alpha(\Delta-1)^{1/2}\Delta^{\alpha/(1-\alpha)} + (1-\alpha)(\delta-1)^{1/2}\delta^{\alpha/(1-\alpha)}},$$

if $\alpha \in (0, 1/2]$, and:

$$ABC_\alpha(G) \ge \frac{(\delta-1)^{\alpha/2}\delta^{\alpha^2/(1-\alpha)}(\Delta-1)^{(1-\alpha)/2}\Delta^\alpha F(G)^\alpha R_{-\alpha/(1-\alpha)}(G)^{1-\alpha}}{\alpha(\delta-1)^{1/2}\delta^{\alpha/(1-\alpha)} + (1-\alpha)(\Delta-1)^{1/2}\Delta^{\alpha/(1-\alpha)}},$$

if $\alpha \in (1/2, 1)$. The equality in these bounds is attained if and only if G is regular.

(3) If $\delta = 1$, then:

$$ABC_\alpha(G) \ge \frac{2^\alpha(\Delta-1)^{\alpha/2}\Delta^{\alpha^2/(1-\alpha)} F(G)^\alpha \big(R_{-\alpha/(1-\alpha)}(G) - m_2\big)^{1-\alpha}}{\alpha(2\Delta-2)^{1/2}\Delta^{\alpha/(1-\alpha)} + (1-\alpha)2^{\alpha/(2-2\alpha)}},$$

if $\alpha \in (0, 1/2]$, and:

$$ABC_\alpha(G) \ge \frac{2^{\alpha^2/(2-2\alpha)}\Delta^\alpha(2\Delta-2)^{(1-\alpha)/2} F(G)^\alpha \big(R_{-\alpha/(1-\alpha)}(G) - m_2\big)^{1-\alpha}}{\alpha 2^{\alpha/(2-2\alpha)} + (1-\alpha)(2\Delta-2)^{1/2}\Delta^{\alpha/(1-\alpha)}},$$

if $\alpha \in (1/2, 1)$.

Proof. Since $ABC_\alpha(P_2) = 0$ and $R_\beta(P_2) = 1$, it suffices to prove the theorem for the case $m_2 = 0$, i.e., when G is a graph without isolated edges. Hence, $\Delta \ge 2$.

Hölder's inequality gives:

$$\begin{aligned}
ABC_\alpha(G) &= \sum_{uv\in E(G)} \Big(\frac{d_u+d_v-2}{d_ud_v} \Big)^\alpha \\
&\le \Big(\sum_{uv\in E(G)} \big((d_u+d_v-2)^\alpha \big)^{1/\alpha} \Big)^\alpha \Big(\sum_{uv\in E(G)} \Big(\frac{1}{(d_ud_v)^\alpha} \Big)^{1/(1-\alpha)} \Big)^{1-\alpha} \\
&= \Big(\sum_{uv\in E(G)} (d_u+d_v-2) \Big)^\alpha \Big(\sum_{uv\in E(G)} (d_ud_v)^{-\alpha/(1-\alpha)} \Big)^{1-\alpha} \\
&= F(G)^\alpha R_{-\alpha/(1-\alpha)}(G)^{1-\alpha}.
\end{aligned}$$

If G is a regular or biregular graph with m edges, then:

$$\begin{aligned}
F(G)^\alpha R_{-\alpha/(1-\alpha)}(G)^{1-\alpha} &= \big((\Delta+\delta-2)m \big)^\alpha \big((\Delta\delta)^{-\alpha/(1-\alpha)} m \big)^{1-\alpha} \\
&= \frac{(\Delta+\delta-2)^\alpha}{(\Delta\delta)^\alpha}\, m = ABC_\alpha(G).
\end{aligned}$$

Assume that G is connected and that the equality in the first inequality is attained. Hölder's inequality gives that there exists a constant c with:

$$d_u + d_v - 2 = c(d_ud_v)^{-\alpha/(1-\alpha)}$$

for every $uv \in E(G)$. Note that the function $H : [1,\infty) \times [1,\infty) \to [0,\infty)$ given by $H(x,y) = (x+y-2)(xy)^{\alpha/(1-\alpha)}$ is increasing in each variable. If $uv, uw \in E(G)$, then:

$$c = (d_u+d_v-2)(d_ud_v)^{\alpha/(1-\alpha)} = (d_u+d_w-2)(d_ud_w)^{\alpha/(1-\alpha)}$$

implies $d_w = d_v$. Thus, for each vertex $u \in V(G)$, every neighbor of u has the same degree. Since G is a connected graph, this holds if and only if G is regular or biregular.

Assume now that $\delta > 1$. If $\alpha \in (0, 1/2]$, then:

$$\begin{aligned}
&K_{1/\alpha}\big((2\delta-2)\delta^{2\alpha/(1-\alpha)}, (2\Delta-2)\Delta^{2\alpha/(1-\alpha)} \big) \\
&= \alpha \Big(\frac{\Delta-1}{\delta-1} \Big)^{(1-\alpha)/2} \Big(\frac{\Delta}{\delta} \Big)^\alpha + (1-\alpha) \Big(\frac{\delta-1}{\Delta-1} \Big)^{\alpha/2} \Big(\frac{\delta}{\Delta} \Big)^{\alpha^2/(1-\alpha)} \\
&= \frac{\alpha(\Delta-1)^{(1-\alpha)/2}\Delta^\alpha(\Delta-1)^{\alpha/2}\Delta^{\alpha^2/(1-\alpha)} + (1-\alpha)(\delta-1)^{\alpha/2}\delta^{\alpha^2/(1-\alpha)}(\delta-1)^{(1-\alpha)/2}\delta^\alpha}{(\Delta-1)^{\alpha/2}\Delta^{\alpha^2/(1-\alpha)}(\delta-1)^{(1-\alpha)/2}\delta^\alpha} \\
&= \frac{\alpha(\Delta-1)^{1/2}\Delta^{\alpha/(1-\alpha)} + (1-\alpha)(\delta-1)^{1/2}\delta^{\alpha/(1-\alpha)}}{(\Delta-1)^{\alpha/2}\Delta^{\alpha^2/(1-\alpha)}(\delta-1)^{(1-\alpha)/2}\delta^\alpha}\,.
\end{aligned}$$

If $\alpha \in (1/2, 1)$, then a similar computation gives:

$$\begin{aligned}
&K_{1/\alpha}\big((2\delta-2)\delta^{2\alpha/(1-\alpha)}, (2\Delta-2)\Delta^{2\alpha/(1-\alpha)} \big) \\
&\qquad = \frac{\alpha(\delta-1)^{1/2}\delta^{\alpha/(1-\alpha)} + (1-\alpha)(\Delta-1)^{1/2}\Delta^{\alpha/(1-\alpha)}}{(\delta-1)^{\alpha/2}\delta^{\alpha^2/(1-\alpha)}(\Delta-1)^{(1-\alpha)/2}\Delta^\alpha}\,.
\end{aligned}$$

Since:

$$\begin{aligned}
(2\delta-2)\delta^{2\alpha/(1-\alpha)} &\le (d_u+d_v-2)(d_ud_v)^{\alpha/(1-\alpha)} = \frac{d_u+d_v-2}{(d_ud_v)^{-\alpha/(1-\alpha)}} \\
&\le (2\Delta-2)\Delta^{2\alpha/(1-\alpha)},
\end{aligned}$$

Corollary 2 gives:

$$
\begin{aligned}
ABC_\alpha(G) &= \sum_{uv\in E(G)} \Big(\frac{d_u+d_v-2}{d_u d_v} \Big)^\alpha \\
&\geq \frac{\Big(\sum_{uv\in E(G)} \big(d_u+d_v-2\big)\Big)^\alpha \Big(\sum_{uv\in E(G)} \big(d_u d_v\big)^{-\alpha/(1-\alpha)}\Big)^{1-\alpha}}{K_{1/\alpha}\big((2\delta-2)\delta^{2\alpha/(1-\alpha)}, (2\Delta-2)\Delta^{2\alpha/(1-\alpha)}\big)} \\
&= \frac{F(G)^\alpha R_{-\alpha/(1-\alpha)}(G)^{1-\alpha}}{K_{1/\alpha}\big((2\delta-2)\delta^{2\alpha/(1-\alpha)}, (2\Delta-2)\Delta^{2\alpha/(1-\alpha)}\big)}\,.
\end{aligned}
$$

This gives the second and third inequalities.
If the graph is regular, then:

$$
\begin{aligned}
&\frac{F(G)^\alpha R_{-\alpha/(1-\alpha)}(G)^{1-\alpha}}{K_{1/\alpha}\big((2\delta-2)\delta^{2\alpha/(1-\alpha)}, (2\Delta-2)\Delta^{2\alpha/(1-\alpha)}\big)} \\
&\qquad= \frac{\big((2\delta-2)m\big)^\alpha \big(\delta^{-2\alpha/(1-\alpha)}m\big)^{1-\alpha}}{K_{1/\alpha}\big((2\delta-2)\delta^{2\alpha/(1-\alpha)}, (2\delta-2)\delta^{2\alpha/(1-\alpha)}\big)} \\
&\qquad= \frac{(2\delta-2)^\alpha}{\delta^{2\alpha}}\, m = ABC_\alpha(G).
\end{aligned}
$$

If we have the equality in the second or third inequality, then Corollary 2 gives $(2\delta-2)\delta^{2\alpha/(1-\alpha)} = (2\Delta-2)\Delta^{2\alpha/(1-\alpha)}$. Since the function $h(t) = (2t-2)t^{2\alpha/(1-\alpha)}$ is strictly increasing on $[1,\infty)$, we conclude that $\delta = \Delta$ and G is regular.

Finally, assume that $\delta = 1$. If $\alpha \in (0,1/2]$, then:

$$
\begin{aligned}
&K_{1/\alpha}\big(2^{\alpha/(1-\alpha)}, (2\Delta-2)\Delta^{2\alpha/(1-\alpha)}\big) \\
&= \alpha\,(2\Delta-2)^{(1-\alpha)/2}\Big(\frac{\Delta}{2^{1/2}}\Big)^\alpha + (1-\alpha)\Big(\frac{1}{2\Delta-2}\Big)^{\alpha/2}\Big(\frac{2^{1/2}}{\Delta}\Big)^{\alpha^2/(1-\alpha)} \\
&= \frac{\alpha(2\Delta-2)^{(1-\alpha)/2}\Delta^\alpha(2\Delta-2)^{\alpha/2}\Delta^{\alpha^2/(1-\alpha)} + (1-\alpha)2^{\alpha^2/(2-2\alpha)}2^{\alpha/2}}{(2\Delta-2)^{\alpha/2}\Delta^{\alpha^2/(1-\alpha)}2^{\alpha/2}} \\
&= \frac{\alpha(2\Delta-2)^{1/2}\Delta^{\alpha/(1-\alpha)} + (1-\alpha)2^{\alpha/(2-2\alpha)}}{2^\alpha(\Delta-1)^{\alpha/2}\Delta^{\alpha^2/(1-\alpha)}}\,.
\end{aligned}
$$

If $\alpha \in (1/2,1)$, then a similar computation gives:

$$
\begin{aligned}
&K_{1/\alpha}\big(2^{\alpha/(1-\alpha)}, (2\Delta-2)\Delta^{2\alpha/(1-\alpha)}\big) \\
&\qquad= \frac{\alpha 2^{\alpha/(2-2\alpha)} + (1-\alpha)(2\Delta-2)^{1/2}\Delta^{\alpha/(1-\alpha)}}{2^{\alpha^2/(2-2\alpha)}\Delta^\alpha(2\Delta-2)^{(1-\alpha)/2}}\,.
\end{aligned}
$$

Since:

$$
\begin{aligned}
2^{\alpha/(1-\alpha)} &\leq (d_u+d_v-2)(d_u d_v)^{\alpha/(1-\alpha)} = \frac{d_u+d_v-2}{(d_u d_v)^{-\alpha/(1-\alpha)}} \\
&\leq (2\Delta-2)\Delta^{2\alpha/(1-\alpha)},
\end{aligned}
$$

Corollary 2 gives:

$$ABC_\alpha(G) = \sum_{uv\in E(G)} \Big(\frac{d_u + d_v - 2}{d_u d_v} \Big)^\alpha$$
$$\geq \frac{\Big(\sum_{uv\in E(G)} (d_u + d_v - 2) \Big)^\alpha \Big(\sum_{uv\in E(G)} (d_u d_v)^{-\alpha/(1-\alpha)} \Big)^{1-\alpha}}{K_{1/\alpha}\big(2^{\alpha/(1-\alpha)}, (2\Delta-2)\Delta^{2\alpha/(1-\alpha)}\big)}$$
$$= \frac{F(G)^\alpha R_{-\alpha/(1-\alpha)}(G)^{1-\alpha}}{K_{1/\alpha}\big(2^{\alpha/(1-\alpha)}, (2\Delta-2)\Delta^{2\alpha/(1-\alpha)}\big)}.$$

This gives the fourth and fifth inequalities. □

Theorem 4 has the following consequence.

Corollary 3. *Let G be a graph with m_2 isolated edges.*

(1) Then:

$$ABC(G) \leq \sqrt{2F(G)\big(R_{-1}(G) - m_2\big)}.$$

The equality in this bound is attained for the union of any regular or biregular graph and m_2 isolated edges; if G is the union of a connected graph and m_2 isolated edges, then the equality in this bound is attained if and only if G is the union of any regular or biregular connected graph and m_2 isolated edges.

(2) If $\delta > 1$, then:

$$ABC(G) \geq \frac{2\sqrt{2\Delta\delta}\,(\Delta-1)^{1/4}(\delta-1)^{1/4}F(G)^{1/2}R_{-1}(G)^{1/2}}{\Delta\sqrt{\Delta-1}+\delta\sqrt{\delta-1}}.$$

The equality in this bound is attained if and only if G is regular.

(3) If $\delta = 1$, then:

$$ABC(G) \geq \frac{2\sqrt{2\Delta}\,(\Delta-1)^{1/4}F(G)^{1/2}\big(R_{-1}(G)-m_2\big)^{1/2}}{\Delta\sqrt{\Delta-1}+1}.$$

Theorem 5. *If G is a graph with m edges and m_2 isolated edges and $\alpha \in \mathbb{R}$, then:*

$$ABC_\alpha(G) \leq (m - m_2 - 1)^\alpha \big(R_{-\alpha}(G) - m_2\big), \quad \text{if } \alpha > 0,$$
$$ABC_\alpha(G) \geq (m-1)^\alpha R_{-\alpha}(G), \quad \text{if } \alpha < 0 \text{ and } m_2 = 0.$$

The equality in the first bound is attained if and only if G is the union of a star graph and m_2 isolated edges. The equality in the second bound is attained if and only if G is a star graph.

Proof. Since $ABC_\alpha(P_2) = 0$ and $R_\beta(P_2) = 1$, it suffices to prove the theorem for the case $m_2 = 0$, i.e., when G is a graph without isolated edges.

In any graph, the inequality $d_u + d_v \leq m + 1$ holds for every $uv \in E(G)$. If $\alpha > 0$, then:

$$\frac{\Big(\frac{d_u+d_v-2}{d_u d_v}\Big)^\alpha}{\Big(\frac{1}{d_u d_v}\Big)^\alpha} = (d_u + d_v - 2)^\alpha \leq (m-1)^\alpha,$$
$$\Big(\frac{d_u+d_v-2}{d_u d_v}\Big)^\alpha \leq (m-1)^\alpha (d_u d_v)^{-\alpha},$$
$$ABC_\alpha(G) \leq (m-1)^\alpha R_{-\alpha}(G).$$

If $\alpha < 0$, then we obtain the converse inequality.

If G is a star graph, then $d_u + d_v = m + 1$ for every $uv \in E(G)$, and the equality is attained for every α.

If the equality is attained in some inequality, then the previous argument gives that $d_u + d_v = m + 1$ for every $uv \in E(G)$. In particular, G is a connected graph. If $m = 2$, then $\{d_u, d_v\} = \{1, 2\}$ for every $uv \in E(G)$, and so, $G = P_3 = S_3$. Assume now $m \geq 3$. Seeking for a contradiction, assume that $\{d_u, d_v\} \neq \{m, 1\}$ for some $uv \in E(G)$. Since $d_u + d_v = m + 1$, we have $2 \leq d_u, d_v \leq m - 1$, and so, there exist two different vertices $u', v' \in V(G) \setminus \{u, v\}$ with $uu', vv' \in E(G)$. Since vv' is not incident on u and u', we have $d_u + d_{u'} < m + 1$, a contradiction. Hence, $\{d_u, d_v\} = \{m, 1\}$ for every $uv \in E(G)$, and so, G is a star graph. □

Corollary 4. *If G is a graph with m edges and m_2 isolated edges, then:*

$$ABC(G) \leq \sqrt{2(m - m_2 - 1)}\,\big(R(G) - m_2\big),$$

and the equality is attained if and only if G is the union of a star graph and m_2 isolated edges.

Note that Theorem 5 (and Corollary 4) improves Items (1) and (2) in Theorems 1 and 2 for many graphs (when $m < 2\Delta - 1$).

3. Conclusions

Topological indices have become a useful tool for the study of theoretical and practical problems in different areas of science. An important line of research associated with topological indices is to find optimal bounds and relations between known topological indices, in particular to obtain bounds for the topological indices associated with invariant parameters of a graph (see [1]).

From the theoretical point of view in this research, a new type of Hölder converse inequality was proposed (Theorem 3 and Corollary 2). From the practical point of view, this inequality was successfully applied to establish new relationships of the generalizations of the indexes ABC and R; in particular, it was applied to prove Theorem 4 and Corollary 3. In addition, other new relationships were obtained between these indices (Theorems 1, 2, and 5) that generalized and improved already known results.

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