mathematics

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A General Family of q-Hypergeometric Polynomials and Associated Generating Functions

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Abstract: Basic (or q-) series and basic (or q-) polynomials, especially the basic (or q-) hypergeometric functions and the basic (or q-) hypergeometric polynomials are studied extensively and widely due mainly to their potential for applications in many areas of mathematical and physical sciences. Here, in this paper, we introduce a general family of q-hypergeometric polynomials and investigate several q-series identities such as an extended generating function and a Srivastava-Agarwal type bilinear generating function for this family of q-hypergeometric polynomials. We give a transformational identity involving generating functions for the generalized q-hypergeometric polynomials which we have introduced here. We also point out relevant connections of the various q-results, which we investigate here, with those in several related earlier works on this subject. We conclude this paper by remarking that it will be a rather trivial and inconsequential exercise to give the so-called (p, q)-variations of the q-results, which we have investigated here, because the additional parameter p is obviously redundant.

Keywords: basic (or q-) hypergeometric series; homogeneous q-difference operator; q-binomial theorem; cauchy polynomials; Al-Salam-Carlitz q-polynomials; rogers type formulas; Srivastava-Agarwal type generating functions

MSC: Primary 05A30; 33D15; 33D45; Secondary 05A40; 11B65

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Citation: Srivastava, H.M.; Arjika, S. A General Family of q-Hypergeometric Polynomials and Associated Generating Functions. *Mathematics* **2021**, *9*, 1161. https://doi.org/10.3390/math9111161

Academic Editor: Clemente Cesarano

Received: 17 April 2021
Accepted: 19 May 2021
Published: 21 May 2021

1. Introduction, Definitions and Preliminaries

In this paper, we adopt the commonly-used conventions and notations for the basic (or q-) series and the basic (or q-) polynomials. For the convenience of the reader, we first provide a summary of the mathematical notations, basic properties and definitions to be used in this paper. We refer to the general references (see, for example [1–4], especially for the q-hypergeometric function ${}_r\Phi_s$ in the case when $r = s + 1$) for the definitions and notations.

Throughout this paper, we assume that $|q| < 1$. For complex numbers $a \in \mathbb{C}$, the q-shifted factorials are defined by

$$(a;q)_0 := 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k), \quad (a;q)_\infty := \prod_{k=0}^{\infty}(1 - aq^k)$$

and

$$(a_1, a_2, \cdots, a_r; q)_m = (a_1;q)_m(a_2;q)_m \cdots (a_r;q)_m$$

$$(m \in \mathbb{N}_0 := \{0, 1, 2, \cdots\} = \mathbb{N} \cup \{0\}).$$

We will frequently use the following q-identity (see, for example, p. 241, Entry (II.5) in [4]):

$$(aq^{-n}; q)_n = (-a)^n \, q^{-n-\binom{n}{2}} \left(\frac{q}{a}; q\right)_n. \tag{1}$$

The generalized basic (or q-) hypergeometric function ${}_r\Phi_s$, with r numerator parameters $a_1, a_2, \cdots, a_r$ and s demominator parameters $b_1, b_2, \cdots, b_s$, is defined here as follows (see, for example, p. 347, Equation 9.4 (282) in [2]; see also [1]):

$$\begin{aligned} &{}_r\Phi_s \left[\begin{array}{c} a_1, a_2, \cdots, a_r; \\ \\ b_1, b_2, \cdots, b_s; \end{array} q;\, z \right] \\ &\quad = \sum_{n=0}^{\infty} \frac{(a_1;q)_n \, (a_2;q)_n \, \cdots \, (a_r;q)_n}{(b_1;q)_n \, (b_2;q)_n \, \cdots \, (b_s;q)_n} \, \frac{z^n}{(q;q)_n} \left[(-1)^n \, q^{\binom{n}{2}} \right]^{1+s-r} \end{aligned} \tag{2}$$

$$(r \leqq s+1),$$

provided that the series converges or terminates (see, for details [1,2]).

Our present investigation is motivated essentially by the fact that basic (or q-) series and basic (or q-) polynomials, especially the basic (or q-) hypergeometric functions and the basic (or q-) hypergeometric polynomials, are potentially useful in many areas of mathematical and physical sciences (see, for details, pp. 351–352 in [2]). Here, in this paper, we are mainly concerned with the Cauchy polynomials defined by (see [5]):

$$\begin{aligned} P_0(x,y) := 0 \quad \text{and} \quad P_n(x,y) &:= (x-y)(x-qy)\cdots(x-q^{n-1}y) \\ &= (y/x;q)_n \, x^n \quad (n \in \mathbb{N}), \end{aligned} \tag{3}$$

together with their Srivastava-Agarwal type generating function as given below:

$$\sum_{n=0}^{\infty} P_n(x,y) \frac{(\lambda;q)_n \, t^n}{(q;q)_n} = {}_2\Phi_1 \left[\begin{array}{c} \lambda, y/x; \\ \\ 0; \end{array} q;\, xt \right]. \tag{4}$$

For $\lambda = 0$, we obtain the following generating function (see [5]):

$$\sum_{n=0}^{\infty} P_n(x,y) \frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt;q)_\infty}. \tag{5}$$

The generating function (5) is also the homogeneous version of the Cauchy identity or the q-binomial theorem as given by (see [1]; see also the related recent [6] on some number-theoretic applications of the q-binomial theorem)

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = {}_1\Phi_0 \left[\begin{array}{c} a; \\ \\ -\,; \end{array} q;\, z \right] = \frac{(az;q)_\infty}{(z;q)_\infty} \qquad (|z| < 1). \tag{6}$$

Putting $a = 0$, relation (6) becomes Euler's identity (see [1]):

$$\sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty} \qquad (|z| < 1) \tag{7}$$

and its inverse relation is given below (see [1]):

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} \, z^k}{(q;q)_k} = (z;q)_\infty. \tag{8}$$

Saad and Sukhi [7] introduced the dual homogeneous q-difference operator Θ_{xy} as follows (see also [8–10]):

$$\Theta_{xy}\{f(x,y)\} := \frac{f(q^{-1}x,y)-f(x,qy)}{q^{-1}x-y}, \tag{9}$$

which, when it acts upon functions of suitably-restricted variables x and y, yields

$$\Theta_{xy}^{k}\{P_n(y,x)\} = (-1)^k \frac{(q;q)_n}{(q;q)_{n-k}} P_{n-k}(y,x)$$
$$\text{and} \qquad \Theta_{xy}^{k}\left\{\frac{(xt;q)_\infty}{(yt;q)_\infty}\right\} = (-t)^k \frac{(xt;q)_\infty}{(yt;q)_\infty}. \tag{10}$$

The Hahn polynomials (see [11–13]) or, equivalently, the Al-Salam-Carlitz q-polynomials (see [14]) are defined as follows:

$$\phi_n^{(a)}(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (a;q)_k\, x^k \quad \text{and} \quad \psi_n^{(a)}(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} (aq^{1-k};q)_k\, x^k. \tag{11}$$

These polynomials are usually called the "*Al-Salam-Carlitz polynomials*" in several recent publications. Moreover, because of their considerable role in the theories of q-series and q-orthogonal polynomials, many authors investigated various extensions of the Al-Salam-Carlitz polynomials (see, for example [15–17]). Some other related developments involving the Carlitz type and the Srivastava-Agarwal type q-generating functions, the reader may be referred to the recent works [18,19].

Recently, Equation (4.7) in Cao [15] introduced the following two families of generalized Al-Salam-Carlitz polynomials:

$$\phi_n^{(a,b,c)}(x,y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(a,b;q)_k}{(c;q)_k}\, x^k\, y^{n-k} \tag{12}$$

and

$$\psi_n^{(a,b,c)}(x,y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1)^k q^{\binom{k+1}{2}-nk}(a,b;q)_k}{(c;q)_k}\, x^k\, y^{n-k}, \tag{13}$$

together with their generating functions given by (see Equations (4.10) and (4.11) in [15])

$$\sum_{n=0}^{\infty} \phi_n^{(a,b,c)}(x,y|q)\, \frac{t^n}{(q;q)_n} = \frac{1}{(xt;q)_\infty}\, {}_2\Phi_1 \left[\begin{matrix} a,b; & \\ & q;yt \\ c; & \end{matrix} \right] \tag{14}$$

$$(\max\{|yt|,|xt|\} < 1)$$

and

$$\sum_{n=0}^{\infty} \psi_n^{(a,b,c)}(x,y|q)\, \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n} = (xt;q)_\infty\, {}_2\Phi_1 \left[\begin{matrix} a,b; & \\ & q;yt \\ c; & \end{matrix} \right] \tag{15}$$

$$(\max\{|yt|,|xt|\} < 1).$$

Clearly, in the special case when $b = c$ and $y = 1$, the polynomials $\phi_n^{(a,b,c)}(x, y|q)$ and $\psi_n^{(a,b,c)}(x, y|q)$, defined by (12) and (13), would correspond to the simpler polynomials $\phi_n^{(a)}(x|q)$ and $\psi_n^{(a)}(x|q)$, which are given in (11).

Motivated by the above-cited work [15], Cao et al. [16] introduced the extensions $\phi_n^{\left(\substack{a,b,c\\ d,e}\right)}(x, y|q)$ and $\psi_n^{\left(\substack{a,b,c\\ d,e}\right)}(x, y|q)$ of the Al-Salam-Carlitz polynomials, which are defined by

$$\phi_n^{\left(\substack{a,b,c\\ d,e}\right)}(x, y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(a, b, c; q)_k}{(d, e; q)_k}\, x^{n-k}\, y^k \tag{16}$$

and

$$\psi_n^{\left(\substack{a,b,c\\ d,e}\right)}(x, y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1)^k q^{k(k-n)}(a, b, c; q)_k}{(d, e; q)_k}\, x^{n-k}\, y^k, \tag{17}$$

respectively.

Remark 1. *We choose to remark that, except possibly for the Hahn polynomials in* (11)*, it is an open problem to show whether or not the polynomials defined by* (12) *to* (17) *are orthogonal.*

More recently, by using the following notation:

$$\mathbf{a} := (a_1, a_2, \cdots, a_r) \qquad \text{and} \qquad \mathbf{c} := (c_1, c_2, \cdots, c_u),$$

Srivastava and Arjika [17] introduced the following two families:

$$\phi_n^{(\mathbf{a},\mathbf{b})}(x, y|q) \qquad \text{and} \qquad \psi_n^{(\mathbf{a},\mathbf{b})}(x, y|q)$$

of the generalized Al-Salam-Carlitz q-polynomials, which are defined by

$$\phi_n^{(\mathbf{a},\mathbf{b})}(x, y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(a_1, a_2, \cdots, a_{r+1}; q)_k}{(b_1, b_2, \cdots, b_r; q)_k}\, x^k\, y^{n-k} \tag{18}$$

and

$$\psi_n^{(\mathbf{a},\mathbf{b})}(x, y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(a_1, a_2, \cdots, a_{r+1}; q)_k}{(b_1, b_2, \cdots, b_r; q)_k} q^{\binom{k+1}{2}-nk}\, x^k\, y^{n-k}, \tag{19}$$

respectively. On the other hand, Cao [15] introduced and studied the following family of q-polynomials:

$$V_n^{(\mathbf{a},\mathbf{c})}(x, y, z|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(a_1, a_2, \cdots, a_r; q)_k}{(c_1, c_2, \cdots, c_u; q)_k} P_{n-k}(x, y) z^k \tag{20}$$

$$(r \leqq u + 1),$$

together with their generating function given by

$$\sum_{n=0}^{\infty} V_n^{(\mathbf{a},\mathbf{c})}(x, y, z|q) \frac{t^n}{(q; q)_n} = \frac{(yt; q)_\infty}{(xt; q)_\infty}\, {}_r\Phi_u \left[\begin{matrix} a_1, a_2, \cdots, a_r; \\ c_1, c_2, \cdots, c_u; \end{matrix} \; q; zt \right] \tag{21}$$

$$(r \leqq u + 1).$$

Our present investigation may be looked upon essentially as a sequel to the earlier works by Srivastava and Arjika [17] and Cao [15]. Our aim here is to introduce and study some further extensions of the above-mentioned q-polynomials.

Definition 1. *In terms of q-binomial coefficient, a family of generalized q-hypergeometric polynomials $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$ is defined by*

$$\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q) := (-1)^n q^{-\binom{n}{2}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} W_k(\mathbf{a},\mathbf{b}) P_{n-k}(y,x) z^k, \tag{22}$$

where, for convenience,

$$\mathbf{a} := (a_1,a_2,\cdots,a_r), \quad \mathbf{b} := (b_1,b_2,\cdots,b_s) \quad \text{and} \quad W_k(\mathbf{a},\mathbf{b}) := \frac{(a_1,a_2,\cdots,a_r;q)_k}{(b_1,b_2,\cdots,b_s;q)_k}.$$

The above-defined q-polynomials $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$ include many one-variable q-hypergeometric ${}_r\Phi_s$ series as special or limit cases. Therefore, we choose just to call them generalized q-hypergeometric polynomials.

Remark 2. *The generalized q-hypergeometric polynomials $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$ defined in (22) are a generalized and unified form of the Hahn polynomials and the Al-Salam-Carlitz polynomials. The functional notation, which is used in each of the following simpler cases, can easily be expressed explicitly by specializing the definition (22) as we have indicated below, together with the reference cited with the case.*

1. *Upon setting*

$$(r,x,y) = (s+1,y,x) \qquad \text{and} \qquad (b_1,b_2,\cdots,b_s) =: \mathbf{b} = \mathbf{c} := (c_1,c_2,\cdots,c_s),$$

the generalized q-hypergeometric polynomials (22) would reduce to the generalized Al-Salam-Carlitz q-polynomials $V_n^{(\mathbf{a},\mathbf{c})}(x,y|q)$ (see [20]) :

$$\Psi_n^{(\mathbf{a},\mathbf{c})}(y,x,z|q) = (-1)^n q^{-\binom{n}{2}} V_n^{(\mathbf{a},\mathbf{c})}(x,y,z|q) \qquad (r \leqq u+1). \tag{23}$$

2. *By choosing $r = s+1,\ x = 0$ and $z = x$, the generalized q-hypergeometric polynomials (22) reduce to the generalized Al-Salam-Carlitz q-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ (see [17]) :*

$$\Psi_n^{(\mathbf{a},\mathbf{b})}(0,y,x|q) = (-1)^n q^{-\binom{n}{2}} \phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q) \qquad (r \leqq s+1). \tag{24}$$

3. *Upon setting $r = s+1,\ y = 0,\ z = -x$ and $x = y$, the q-hypergeometric polynomials (22) reduce to the generalized Al-Salam-Carlitz q-polynomials $\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ (see [17]) :*

$$\Psi_n^{(\mathbf{a},\mathbf{b})}(y,0,-x|q) = \psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q). \tag{25}$$

4. *For $r = s = 1, \mathbf{a} = a$ and $\mathbf{b} = 0$, the q-hypergeometric polynomials $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$ are the generalized trivariate Hahn polynomials $\Psi_n^{(a)}(x,y,z|q)$ (see [21]) :*

$$\Psi_n^{(a,0)}(x,y,z|q) = \Psi_n^{(a)}(x,y,z|q). \tag{26}$$

5. *For $r = 3,\ s = 2,\ \mathbf{a} = (a,0,0),\ \mathbf{b} = (0,0)$ and $z = b$, the q-hypergeometric polynomials (22) reduce to the generalized Hahn polynomials $h_n(x,y,a,b|q)$ (see [22]) :*

$$\Psi_n^{(\mathbf{a},\mathbf{c})}(y,x,z|q) = (-1)^n q^{-\binom{n}{2}} h_n(x,y,a,b|q). \tag{27}$$

6. *For $r = s$ and $\mathbf{a} = \mathbf{b} = \mathbf{0}$, the q-hypergeometric polynomials $\Psi_n^{(\mathbf{a},\mathbf{b})}(x, y, z|q)$ are the well known trivariate q-polynomials $F_n(x, y, z; q)$ (see [23]) :*

$$\Psi_n^{(\mathbf{0},\mathbf{0})}(x, y, z|q) = F_n(x, y, z; q). \tag{28}$$

7. *For $r = 3$, $s = 2$, $\mathbf{a} = (a, b, c)$, $\mathbf{b} = (d, e)$ and $(x, y, z) = (0, x, y)$, the generalized q-hypergeometric polynomials $\Psi_n^{(\mathbf{a},\mathbf{b})}(y, x, z|q)$ reduce to $\phi_n^{\binom{a,b,c}{d,e}}(x, y, z|q)$ (see [16]) :*

$$\Psi_n^{(\mathbf{a},\mathbf{b})}(y, x, z|q) = (-1)^n q^{-\binom{n}{2}} \phi_n^{\binom{a,b,c}{d,e}}(x, y, z|q). \tag{29}$$

8. *For $r = s = 2$, $\mathbf{a} = (a, b, c)$, $\mathbf{b} = (d, e)$ and $(x, y, z) = (0, x, y)$, the q-hypergeometric polynomials (22) reduce to the polynomials $\psi_n^{\binom{a,b,c}{d,e}}(x, y, z|q)$ (see [16]) :*

$$\Psi_n^{(\mathbf{a},\mathbf{b})}(y, x, z|q) = (-1)^n q^{-\binom{n}{2}} \psi_n^{\binom{a,b,c}{d,e}}(x, y, z|q). \tag{30}$$

9. *If we let $r = 2$, $s = 1$, $\mathbf{a} = (a, 0)$, $\mathbf{b} = (0)$, $x = 0$ and $z = x$, the polynomials $\Psi_n^{(\mathbf{a},\mathbf{b})}(x, y, z|q)$ reduce to the second Hahn polynomials $\phi_n^{(a)}(x, y|q)$ (see [24]) :*

$$\Psi_n^{(\mathbf{0},\mathbf{0})}(x, ax, y|q) = (-1)^n q^{-\binom{n}{2}} \phi_n^{(a)}(x, y|q). \tag{31}$$

10. *If we let $r = 2$, $s = 1$, $(\mathbf{a}, \mathbf{b}) = (\mathbf{0}, \mathbf{0})$, $y = ax$ and $z = y$, the polynomials $\Psi_n^{(\mathbf{a},\mathbf{b})}(x, y, z|q)$ reduce to the second Hahn polynomials $\psi_n^{(a)}(x, y|q)$ (see [24]) :*

$$\Psi_n^{(\mathbf{0},\mathbf{0})}(x, ax, y|q) = \psi_n^{(a)}(x, y|q). \tag{32}$$

11. *For $r = 2$, $s = 1$, $(\mathbf{a}, \mathbf{b}) = (\mathbf{0}, \mathbf{0})$, $x = y = 0$ and $z = x$, the polynomials $\Psi_n^{(\mathbf{a},\mathbf{b})}(x, y, z|q)$ reduce to the generalized Hahn polynomials $\phi_n^{(a)}(x|q)$ (see [11–13]) or, equivalently, the Al-Salam-Carlitz q-polynomials (see [14]) :*

$$\Psi_n^{(\mathbf{0},\mathbf{0})}(0, 0, x|q) = (-1)^n q^{-\binom{n}{2}} \phi_n^{(a)}(x|q). \tag{33}$$

12. *Upon putting $r = s$, $(\mathbf{a}, \mathbf{b}) = (\mathbf{0}, \mathbf{0})$, $y = ax$ and $z = 1$, the generalized q-hypergeometric polynomials $\Psi_n^{(\mathbf{a},\mathbf{b})}(x, y, z|q)$ reduce to the Hahn polynomials $\psi_n^{(a)}(x|q)$ (see [11–13]) or, equivalently, the Al-Salam-Carlitz q-polynomials (see [14]):*

$$\Psi_n^{(\mathbf{0},\mathbf{0})}(x, ax, 1|q) = \psi_n^{(a)}(x|q). \tag{34}$$

This paper is organized as follows. In Section 2, we introduce the following homogeneous q-difference operator:

$${}_r\Phi_s \left[\begin{matrix} a_1, a_2, \cdots, a_r; \\ \\ b_1, b_2, \cdots, b_s; \end{matrix} \; q; -z\Theta_{xy} \right]$$

and apply it to investigate several q-series properties. In addition, we derive several extended generating functions for the generalized q-hypergeometric polynomials. In Section 3, we first state and prove the Rogers type formulas. The Srivastava-Agarwal type bilinear generating functions involving the generalized q-hypergeometric polynomials are derived in Section 4. In Section 5, we give a transformational identity involving generating functions for generalized q-hypergeometric polynomials. Finally, in our concluding Section 6, we present several remarks and observations. We also reiterate the well-documented fact that it will be a rather trivial exercise to give the so-called (p, q)-variation of the q-results, which we have investigated here, because the additional parameter p is obviously redundant.

2. Generalized q-Hypergeometric Polynomials

In this section, we begin by introducing a homogeneous q-difference hypergeometric operator as follows.

Definition 2. *The homogeneous q-difference hypergeometric operator is defined by*

$$ {}_r\Phi_s\left[\begin{matrix} a_1, a_2, \cdots, a_r; \\ \\ b_1, b_2, \cdots, b_s; \end{matrix} q; -z\Theta_{xy}\right] = \sum_{k=0}^{\infty} W_k(\mathbf{a}, \mathbf{b}) \frac{(-z\Theta_{xy})^k}{(q;q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}. \tag{35} $$

We now derive the q-series identities (36) to (38) below, which will be used later in order to derive the extended generating functions, the Rogers type formulas and the Srivastava-Agarwal type bilinear generating functions involving the generalized q-hypergeometric polynomials.

Lemma 1. *Each of the following operational formulas holds true for the homogeneous q-difference operator defined by* (35):

$$ {}_r\Phi_s\left[\begin{matrix} a_1, a_2, \cdots, a_r; \\ \\ b_1, b_2, \cdots, b_s; \end{matrix} q; -z\Theta_{xy}\right] \left\{(-1)^n q^{-\binom{n}{2}} P_n(y, x)\right\} = \Psi_n^{(\mathbf{a},\mathbf{b})}(x, y, z|q), \tag{36} $$

$$ \begin{aligned} &{}_r\Phi_s\left[\begin{matrix} a_1, a_2, \cdots, a_r; \\ \\ b_1, b_2, \cdots, b_s; \end{matrix} q; -z\Theta_{xy}\right] \left\{\frac{(xt;q)_\infty}{(yt;q)_\infty}\right\} \\ &\qquad = \frac{(xt;q)_\infty}{(yt;q)_\infty}\, {}_r\Phi_s\left[\begin{matrix} a_1, a_2, \cdots, a_r; \\ \\ b_1, b_2, \cdots, b_s; \end{matrix} q; zt\right] \end{aligned} \tag{37} $$

and

$$ \begin{aligned} &{}_r\Phi_s\left[\begin{matrix} a_1, a_2, \cdots, a_r; \\ \\ b_1, b_2, \cdots, b_s; \end{matrix} q; -z\Theta_{xy}\right] \left\{\frac{P_k(y, x)\,(xt;q)_\infty}{(xt;q)_k (yt;q)_\infty}\right\} \\ &\qquad = t^{-k} \frac{(xt;q)_\infty}{(yt;q)_\infty} \sum_{j=0}^{k} \frac{(q^{-k}, yt;q)_j\, q^j}{(xt, q;q)_j}\, {}_r\Phi_s\left[\begin{matrix} a_1, a_2, \cdots, a_r; \\ \\ b_1, b_2, \cdots, b_s; \end{matrix} q; ztq^j\right] \\ &\qquad\qquad (|yt| < 1). \end{aligned} \tag{38} $$

Proof. Firstly, by applying (35) and (10), we obtain (36). Secondly, in light of (10), we obtain the desired identity (37). Thirdly, by making use of the q-Chu-Vandermonde formula (II.6) in [1]

$$ {}_2\Phi_1\left[\begin{matrix} q^{-n}, a; \\ \\ c; \end{matrix} q; q\right] = \frac{(c/a;q)_n}{(c;q)_n}\, a^n, \tag{39} $$

we find that

$$
\begin{aligned}
\frac{P_k(y,x)\,(xt;q)_\infty}{(xt;q)_k(yt;q)_\infty} &= t^{-k}\,\frac{(xt;q)_\infty}{(yt;q)_\infty}\,{}_2\Phi_1\left[\begin{matrix} q^{-k}, yt; \\ \\ xt; \end{matrix}\; q;q\right] \\
&= t^{-k}\sum_{j=0}^{\infty}\frac{(q^{-k};q)_j\,q^j}{(q;q)_j}\,\frac{(xtq^j;q)_\infty}{(ytq^j;q)_\infty}.
\end{aligned}
\tag{40}
$$

Therefore, by using (40) and (37) successively, we obtain

$$
\begin{aligned}
&{}_r\Phi_s\left[\begin{matrix} a_1,a_2,\cdots,a_r; \\ \\ b_1,b_2,\cdots,b_s; \end{matrix}\; q;-z\Theta_{xy}\right]\left\{\frac{P_k(y,x)\,(xt;q)_\infty}{(xt;q)_k(yt;q)_\infty}\right\} \\
&\quad = t^{-k}\sum_{j=0}^{k}\frac{(q^{-k};q)_j\,q^j}{(q;q)_j}\,{}_r\Phi_s\left[\begin{matrix} a_1,a_2,\cdots,a_r; \\ \\ b_1,b_2,\cdots,b_s; \end{matrix}\; q;-z\Theta_{xy}\right]\left\{\frac{(xtq^j;q)_\infty}{(ytq^j;q)_\infty}\right\} \\
&\quad = t^{-k}\sum_{j=0}^{k}\frac{(q^{-k};q)_j\,q^j}{(q;q)_j}\,\frac{(xtq^j;q)_\infty}{(ytq^j;q)_\infty}\,{}_r\Phi_s\left[\begin{matrix} a_1,a_2,\cdots,a_r; \\ \\ b_1,b_2,\cdots,b_s; \end{matrix}\; q;ztq^j\right],
\end{aligned}
$$

which is asserted by Lemma 1. □

We now derive an extended generating function for the generalized q-hypergeometric polynomials $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$ by using the operator representation (36).

Theorem 1 (Extended generating function)**.** *For $k \in \mathbb{N}$ and $\max\{|yt|,|zt|\} < 1$, it is asserted that*

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\Psi_{n+k}^{(\mathbf{a},\mathbf{b})}(x,y,z|q)(-1)^{n+k}q^{\binom{n+k}{2}}\,\frac{t^n}{(q;q)_n} \\
&\quad = \frac{(xt;q)_\infty}{t^k(yt;q)_\infty}\sum_{j=0}^{k}\frac{(q^{-k},yt;q)_j\,q^j}{(q,xt;q)_j}\,{}_r\Phi_s\left[\begin{matrix} a_1,a_2,\cdots,a_r; \\ \\ b_1,b_2,\cdots,b_s; \end{matrix}\; q;ztq^j\right] \\
&\qquad\qquad (|yt|<1).
\end{aligned}
\tag{41}
$$

Proof. In light of (36) and (5), we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\Psi_{n}^{(\mathbf{a},\mathbf{b})}(x,y,z|q)(-1)^{n+k}q^{\binom{n+k}{2}}\,\frac{t^n}{(q;q)_n} \\
&\quad = \sum_{n=0}^{\infty}{}_r\Phi_s\left[\begin{matrix} a_1,a_2,\cdots,a_r; \\ \\ b_1,b_2,\cdots,b_s; \end{matrix}\; q;-z\Theta_{xy}\right]\left\{(-1)^{n+k}q^{\binom{n+k}{2}}P_{n+k}(y,x)\right\} \\
&\qquad\qquad \cdot\frac{(-1)^{n+k}q^{\binom{n+k}{2}}\,t^n}{(q;q)_n} \\
&\quad = {}_r\Phi_s\left[\begin{matrix} a_1,a_2,\cdots,a_r; \\ \\ b_1,b_2,\cdots,b_s; \end{matrix}\; q;-z\Theta_{xy}\right]\left\{P_k(y,x)\sum_{n=0}^{\infty}P_n(y,xq^k)\,\frac{t^n}{(q;q)_n}\right\} \\
&\quad = {}_r\Phi_s\left[\begin{matrix} a_1,a_2,\cdots,a_r; \\ \\ b_1,b_2,\cdots,b_s; \end{matrix}\; q;-z\Theta_{xy}\right]\left\{\frac{P_k(y,x)\,(xt;q)_\infty}{(xt;q)_k(yt;q)_\infty}\right\}.
\end{aligned}
$$

The proof of Theorem 1 is completed by using (38). □

Remark 3. *Setting $k = 0$ in Theorem 1, we obtain the following generating function for the generalized q-hypergeometric polynomials*:

$$\sum_{n=0}^{\infty} \Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)(-1)^n q^{\binom{n}{2}} \frac{t^n}{(q;q)_n} = \frac{(xt;q)_\infty}{(yt;q)_\infty} {}_r\Phi_s \left[\begin{matrix} a_1, a_2, \cdots, a_r; \\ b_1, b_2, \cdots, b_s; \end{matrix} \; q; zt \right]. \tag{42}$$

3. The Rogers Formula

In this section, we use the assertion (38) of Lemma 1 in order to derive several q-identities such as the Rogers type formula for the generalized q-hypergeometric polynomials $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$.

Theorem 2. (The Rogers formula for $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$) *For* $\max\{|\frac{t}{\omega}|, |y\omega|\} < 1$, *the following Rogers type formula holds true*:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \Psi_{n+k}^{(\mathbf{a},\mathbf{b})}(x,y,z|q)(-1)^{n+k} q^{\binom{n+k}{2}} \frac{t^n}{(q;q)_n} \frac{\omega^k}{(q;q)_k}$$
$$= \frac{(x\omega;q)_\infty}{(t/\omega, y\omega;q)_\infty} \sum_{k=0}^{\infty} \frac{(y\omega;q)_k\, q^k}{(q\omega/t, x\omega, q;q)_k} {}_r\Phi_s \left[\begin{matrix} a_1, a_2, \cdots, a_r; \\ b_1, b_2, \cdots, b_s; \end{matrix} \; q; z\omega q^k \right]. \tag{43}$$

Proof. In light of (36), we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \Psi_{n+k}^{(\mathbf{a},\mathbf{b})}(x,y,z|q)(-1)^{n+k} q^{\binom{n+k}{2}} \frac{t^n}{(q;q)_n} \frac{\omega^k}{(q;q)_k}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} {}_r\Phi_s \left[\begin{matrix} a_1, a_2, \cdots, a_r; \\ b_1, b_2, \cdots, b_s; \end{matrix} \; q; -z\Theta_{xy} \right] \{P_{n+k}(y,x)\} \frac{t^n}{(q;q)_n} \frac{\omega^k}{(q;q)_k}$$
$$= {}_r\Phi_s \left[\begin{matrix} a_1, a_2, \cdots, a_r; \\ b_1, b_2, \cdots, b_s; \end{matrix} \; q; -z\Theta_{xy} \right] \left\{ \sum_{n=0}^{\infty} P_n(y,x) \frac{t^n}{(q;q)_n} \sum_{k=0}^{\infty} P_k(y,q^n x) \frac{\omega^k}{(q;q)_k} \right\}$$
$$= {}_r\Phi_s \left[\begin{matrix} a_1, a_2, \cdots, a_r; \\ b_1, b_2, \cdots, b_s; \end{matrix} \; q; -z\Theta_{xy} \right] \left\{ \sum_{n=0}^{\infty} P_n(y,x) \frac{t^n}{(q;q)_n} \frac{(x\omega q^n;q)_\infty}{(y\omega;q)_\infty} \right\}$$
$$= \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} {}_r\Phi_s \left[\begin{matrix} a_1, a_2, \cdots, a_r; \\ b_1, b_2, \cdots, b_s; \end{matrix} \; q; -z\Theta_{xy} \right] \left\{ \frac{P_n(y,x)\,(x\omega;q)_\infty}{(x\omega;q)_n (y\omega;q)_\infty} \right\}. \tag{44}$$

Now, by using (38) as well as (1), this last relation takes the following form:

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\Psi_{n+k}^{(\mathbf{a},\mathbf{b})}(x,y,z|q)(-1)^{n+k}\,q^{\binom{n+k}{2}}\,\frac{t^n}{(q;q)_n}\,\frac{\omega^k}{(q;q)_k}\\
&\quad=\frac{(x\omega;q)_\infty}{(y\omega;q)_\infty}\sum_{n=0}^{\infty}\frac{(t/\omega)^n}{(q;q)_n}\sum_{k=0}^{n}\frac{(q^{-n},y\omega;q)_k\,q^k}{(x\omega,q;q)_k}\\
&\qquad\cdot{}_r\Phi_s\left[\begin{matrix}a_1,a_2,\cdots,a_r;\\ \\ b_1,b_2,\cdots,b_s;\end{matrix}\;q;z\omega q^k\right]\\
&\quad=\frac{(x\omega;q)_\infty}{(y\omega;q)_\infty}\sum_{k=0}^{\infty}\frac{(y\omega;q)_k\,q^k}{(x\omega,q;q)_k}\\
&\qquad\cdot{}_r\Phi_s\left[\begin{matrix}a_1,a_2,\cdots,a_r;\\ \\ b_1,b_2,\cdots,b_s;\end{matrix}\;q;z\omega q^k\right]\sum_{n=k}^{\infty}\frac{(t/\omega)^n(-1)^k q^{\binom{k}{2}-nk}}{(q;q)_{n-k}}\\
&\quad=\frac{(x\omega;q)_\infty}{(y\omega;q)_\infty}\sum_{k=0}^{\infty}\frac{(y\omega;q)_k\,(-t/\omega)^k q^{-\binom{k}{2}}}{(x\omega,q;q)_k}\,{}_r\Phi_s\left[\begin{matrix}a_1,a_2,\cdots,a_r;\\ \\ b_1,b_2,\cdots,b_s;\end{matrix}\;q;z\omega q^k\right]\\
&\qquad\cdot\sum_{n=0}^{\infty}\frac{(t/\omega)^n\,q^{-nk}}{(q;q)_n}\\
&\quad=\frac{(x\omega;q)_\infty}{(y\omega;q)_\infty}\sum_{k=0}^{\infty}\frac{(y\omega;q)_k\,(-t/\omega)^k\,q^{-\binom{k}{2}}}{(x\omega,q;q)_k}\,{}_r\Phi_s\left[\begin{matrix}a_1,a_2,\cdots,a_r;\\ \\ b_1,b_2,\cdots,b_;\end{matrix}\;q;z\omega q^k\right]\\
&\qquad\cdot\frac{1}{(tq^{-k}/\omega;q)_\infty}\\
&\quad=\frac{(x\omega;q)_\infty}{(t/\omega,y\omega;q)_\infty}\sum_{k=0}^{\infty}\frac{(y\omega;q)_k\,(-t/\omega)^k q^{-\binom{k}{2}}}{(x\omega,tq^{-k}/\omega,q;q)_k}\,{}_r\Phi_s\left[\begin{matrix}a_1,a_2,\cdots,a_r;\\ \\ b_1,b_2,\cdots,b_s;\end{matrix}\;q;z\omega q^k\right]\\
&\quad=\frac{(x\omega;q)_\infty}{(t/\omega,y\omega;q)_\infty}\sum_{k=0}^{\infty}\frac{(y\omega;q)_k\,q^k}{(q\omega/t,x\omega,q;q)_k}\,{}_r\Phi_s\left[\begin{matrix}a_1,a_2,\cdots,a_r;\\ \\ b_1,b_2,\cdots,b_s;\end{matrix}\;q;z\omega q^k\right],
\end{aligned}
\tag{45}
$$

which evidently completes the proof of Theorem 2. □

4. The Srivastava-Agarwal Type Bilinear Generating Functions for the Generalized q-Hypergeometric Polynomials $\Psi_{\mathbf{n}}^{(\mathbf{a},\mathbf{b})}(\mathbf{x},\mathbf{y},\mathbf{z}|\mathbf{q})$

In this section, by applying the following homogeneous q-difference hypergeometric operator:

$$
{}_r\Phi_s\left[\begin{matrix}a_1,a_2,\cdots,a_r;\\ \\ b_1,b_2,\cdots,b_s;\end{matrix}\;q;-z\Theta_{xy}\right],
$$

which is given by (35) in Definition 2, we derive the Srivastava-Agarwal type generating functions for the generalized q-hypergeometric polynomials $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$ defined by (22). We also deduce a bilinear generating function for the Al-Salam-Carlitz polynomials $\psi_n^{(\alpha)}(x|q)$ as an application of the Srivastava-Agarwal type generating functions.

Lemma 2. (see Equation (3.20) in [25] and Equation (5.4) in [26]). *Each of the following generating relations holds true*:

$$\sum_{n=0}^{\infty} \phi_n^{(\alpha)}(x|q)(\lambda;q)_n \frac{t^n}{(q;q)_n} = \frac{(\lambda t;q)_\infty}{(t;q)_\infty} \, {}_2\Phi_1 \left[\begin{matrix} \lambda, \alpha; \\ \\ \lambda t; \end{matrix} \; q; xt \right] \tag{46}$$

$$\big(\max\{|t|, |xt|\} < 1\big)$$

and

$$\sum_{n=0}^{\infty} \psi_n^{(\alpha)}(x|q)(1/\lambda;q)_n \frac{(\lambda tq)^n}{(q;q)_n} = \frac{(xtq;q)_\infty}{(\lambda xtq;q)_\infty} \, {}_2\Phi_1 \left[\begin{matrix} 1/\lambda, 1/(\alpha x); \\ \\ 1/(\lambda xt); \end{matrix} \; q; \alpha q \right] \tag{47}$$

$$\big(\max\{|\lambda xtq|, |\alpha q|\} < 1\big).$$

For more information about the Srivastava-Agarwal type generating functions for the Al-Salam-Carlitz polynomials, one may refer to [24,25].

We now state and prove the Srivastava-Agarwal type bilinear generating functions asserted by Theorem 3 below.

Theorem 3. *Suppose that* $\max\{|\alpha q|, |vxtq|, |xztq|\} < 1$*. Then*

$$\begin{aligned} &\sum_{n=0}^{\infty} \psi_n^{(\alpha)}(x|q) \Psi_n^{(\mathbf{a},\mathbf{b})}(u,v,z|q) \frac{(-1)^n q^{\binom{n+1}{2}} t^n}{(q;q)_n} \\ &\quad = \frac{(q/x, uxtq;q)_\infty}{(\alpha q, vxtq;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (1/(\alpha x), 1/(uxt);q)_n}{(q/x, 1/(vxt), q;q)_n} \left(\frac{\alpha u q}{v}\right)^n \\ &\qquad \cdot {}_r\Phi_s \left[\begin{matrix} a_1, a_2, \cdots, a_r; \\ \\ b_1, b_2, \cdots, b_s; \end{matrix} \; q; xztq^{1-n} \right]. \end{aligned} \tag{48}$$

Proof. In order to prove Theorem 3, we need the q-Chu-Vandermonde summation theorem given by (see Equation (II.7) [1])

$${}_2\Phi_1 \left[\begin{matrix} q^{-n}, a; \\ \\ c; \end{matrix} \; q; \frac{cq^n}{a} \right] = \frac{(c/a;q)_n}{(c;q)_n} \qquad (n \in \mathbb{N}_0). \tag{49}$$

Upon letting $(\lambda, t) = (v/u, tu)$ in the Equation (47), we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty} \psi_n^{(\alpha)}(x|q) p_n(v,u) \frac{(qt)^n}{(q;q)_n} \\ &= \frac{(uxtq;q)_\infty}{(vxtq;q)_\infty} \, {}_2\Phi_1 \left[\begin{matrix} 1/(\alpha x), u/v; \\ \\ 1/(vxt); \end{matrix} \; q; \alpha q \right] \\ &= \frac{(uxtq;q)_\infty}{(vxtq;q)_\infty} \sum_{k=0}^{\infty} \frac{(1/(\alpha x);q)_k (\alpha q)^k}{(q;q)_k} \frac{(u/v;q)_k}{(1/(vxt);q)_k} \\ &= \frac{(uxtq;q)_\infty}{(vxtq;q)_\infty} \sum_{k=0}^{\infty} \frac{(1/(\alpha x);q)_k (\alpha q)^k}{(q;q)_k} \, {}_2\Phi_1 \left[\begin{matrix} q^{-k}, 1/(uxt); \\ \\ 1/(vxt); \end{matrix} \; q; \frac{uq^k}{v} \right] \\ &= \sum_{k=0}^{\infty} \frac{(1/(\alpha x);q)_k (\alpha q)^k}{(q;q)_k} \sum_{n=0}^{\infty} \frac{(q^{-k};q)_n \, q^{nk}}{(q;q)_n} \frac{(1/(uxt);q)_n (uxtq;q)_\infty}{(1/(vxt);q)_n (vxtq;q)_\infty} \left(\frac{u}{v}\right)^n \end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{(1/(\alpha x);q)_k(\alpha q)^k}{(q;q)_k} \sum_{n=0}^{\infty} \frac{(q^{-k};q)_n\, q^{nk}}{(q;q)_n} \frac{(uxtq^{1-n};q)_\infty}{(vxtq^{1-n};q)_\infty}, \tag{50}$$

where we have made use of (49).

By appealing appropriately to (50), the left-hand side of (48) becomes

$$\begin{aligned}
&\sum_{n=0}^{\infty} \psi_n^{(\alpha)}(x|q)\, {}_r\Phi_s\left[\begin{matrix} a_1,a_2,\cdots,a_r; \\ & q;-z\Theta_{uv} \\ b_1,b_2,\cdots,b_s; \end{matrix}\right]\{p_n(v,u)\}\frac{(qt)^n}{(q;q)_n} \\
&= {}_r\Phi_s\left[\begin{matrix} a_1,a_2,\cdots,a_r; \\ & q;-z\Theta_{uv} \\ b_1,b_2,\cdots,b_s; \end{matrix}\right]\left\{\sum_{n=0}^{\infty} \psi_n^{(\alpha)}(x|q)p_n(v,u)\frac{(qt)^n}{(q;q)_n}\right\} \\
&= {}_r\Phi_s\left[\begin{matrix} a_1,a_2,\cdots,a_r; \\ & q;-z\Theta_{uv} \\ b_1,b_2,\cdots,b_s; \end{matrix}\right] \\
&\qquad\cdot\left\{\sum_{k=0}^{\infty} \frac{(1/(\alpha x);q)_k(\alpha q)^k}{(q;q)_k} \sum_{n=0}^{\infty} \frac{(q^{-k};q)_n\, q^{nk}}{(q;q)_n} \frac{(uxtq^{1-n};q)_\infty}{(vxtq^{1-n};q)_\infty}\right\} \\
&= \sum_{k=0}^{\infty} \frac{(1/(\alpha x);q)_k(\alpha q)^k}{(q;q)_k} \sum_{n=0}^{\infty} \frac{(q^{-k};q)_n\, q^{nk}}{(q;q)_n}\, {}_r\Phi_s\left[\begin{matrix} a_1,a_2,\cdots,a_r; \\ & q;-z\Theta_{uv} \\ b_1,b_2,\cdots,b_s; \end{matrix}\right] \\
&\qquad\cdot\left\{\frac{(uxtq^{1-n};q)_\infty}{(vxtq^{1-n};q)_\infty}\right\} \\
&= \sum_{k=0}^{\infty} \frac{(1/(\alpha x);q)_k(\alpha q)^k}{(q;q)_k} \sum_{n=0}^{\infty} \frac{(q^{-k};q)_n\, q^{nk}}{(q;q)_n} \frac{(uxtq^{1-n};q)_\infty}{(vxtq^{1-n};q)_\infty} \\
&\qquad\cdot {}_r\Phi_s\left[\begin{matrix} a_1,a_2,\cdots,a_r; \\ & q;xztq^{1-n} \\ b_1,b_2,\cdots,b_s; \end{matrix}\right] \\
&= \frac{(uxtq;q)_\infty}{(vxtq;q)_\infty} \sum_{k=0}^{\infty} \frac{(1/(\alpha x);q)_k(\alpha q)^k}{(q;q)_k} \sum_{n=0}^{\infty} \frac{(q^{-k},uxtq^{1-n},q)_n\, q^{nk}}{(vxtq^{1-n},q;q)_n} \\
&\qquad\cdot {}_r\Phi_s\left[\begin{matrix} a_1,a_2,\cdots,a_r; \\ & q;xztq^{1-n} \\ b_1,b_2,\cdots,b_s; \end{matrix}\right] \\
&= \frac{(uxtq;q)_\infty}{(vxtq;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}(1/(uxt);q)_n}{(1/(vxt),q;q)_n}\left(\frac{u}{v}\right)^n \\
&\qquad\cdot {}_r\Phi_s\left[\begin{matrix} a_1,a_2,\cdots,a_r; \\ & q;xztq^{1-n} \\ b_1,b_2,\cdots,b_s; \end{matrix}\right] \sum_{k=n}^{\infty} \frac{(1/(\alpha x);q)_k(\alpha q)^k}{(q;q)_{k-n}} \\
&= \frac{(uxtq;q)_\infty}{(vxtq;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}(1/(\alpha x),1/(uxt);q)_n}{(1/(vxt),q;q)_n}\left(\frac{\alpha uq}{v}\right)^n \\
&\qquad\cdot {}_r\Phi_s\left[\begin{matrix} a_1,a_2,\cdots,a_r; \\ & q;xztq^{1-n} \\ b_1,b_2,\cdots,b_s; \end{matrix}\right] \sum_{k=0}^{\infty} \frac{(q^n/(\alpha x);q)_k(\alpha q)^k}{(q;q)_k} \\
&= \frac{(uxtq;q)_\infty}{(vxtq;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}(1/(\alpha x),1/(uxt);q)_n}{(1/(vxt),q;q)_n}\left(\frac{\alpha uq}{v}\right)^n \\
&\qquad\cdot {}_r\Phi_s\left[\begin{matrix} a_1,a_2,\cdots,a_r; \\ & q;xztq^{1-n} \\ b_1,b_2,\cdots,b_s; \end{matrix}\right] \frac{(q^{1+n}/x;q)_\infty}{(\alpha q;q)_\infty} \\
&= \frac{(q/x,uxtq;q)_\infty}{(\alpha q,vxtq;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}(1/(\alpha x),1/(uxt);q)_n}{(q/x,1/(vxt),q;q)_n}\left(\frac{\alpha uq}{v}\right)^n \\
&\qquad\cdot {}_r\Phi_s\left[\begin{matrix} a_1,a_2,\cdots,a_r; \\ & q;xztq^{1-n} \\ b_1,b_2,\cdots,b_s; \end{matrix}\right],
\end{aligned}$$

which is precisely the right-hand side of the assertion (48) of Theorem 3. The proof of Theorem 3 is thus completed. □

Remark 4. *In view of the special case* (34) *of* $\Psi_n^{(\mathbf{a},\mathbf{b})}(u,v,z|q)$, *we deduce the bilinear generating function for the Hahn polynomials* $\psi_n^{(\alpha)}(x|q)$ *as asserted by the following Corollary.*

Corollary 1. *Let* $\max\{|\alpha q|, |axytq|\} < 1$. *Then*

$$\sum_{n=0}^{\infty} \psi_n^{(\alpha)}(x|q)\psi_n^{(a)}(y|q)\frac{(-1)^n q^{\binom{n+1}{2}} t^n}{(q;q)_n}$$
$$= \frac{(q/x, xytq, xtq;q)_\infty}{(\alpha q, axytq;q)_\infty}\, {}_3\Phi_2\left[\begin{array}{c} 1/(\alpha x), 1/(xyt), 1/(xt); \\ q/x, 1/(axyt); \end{array} q; \frac{\alpha xtq}{a}\right]. \tag{51}$$

Remark 5. *For* $r = s$, $\mathbf{a} = \mathbf{b} = \mathbf{0}$, $x = y$, $y = ay$ *and* $z = 1$, *the assertion* (48) *reduces to the above Corollary 1.*

5. A Transformational Identity Involving Generating Functions for the Generalized q-Hypergeometric Polynomials

In this section, we derive the following transformational identity involving generating functions for the generalized q-hypergeometric polynomials. Once again, in our derivation, we apply the homogeneous q-difference operator (35).

Theorem 4. *Let the coefficients* $A(n)$ *and* $B(n)$ *satisfy the following relationship*:

$$\sum_{n=0}^{\infty} A(n)P_n(v,u) = \sum_{n=0}^{\infty} B(n)\frac{(xutq^{1-n};q)_\infty}{(xvtq^{1-n};q)_\infty}. \tag{52}$$

Then

$$\sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} A(n)\Psi_n^{(\mathbf{a},\mathbf{b})}(u,v,z|q)$$
$$= \sum_{n=0}^{\infty} B(n)\frac{(xutq^{1-n};q)_\infty}{(xvtq^{1-n};q)_\infty}\, {}_r\Phi_s\left[\begin{array}{c} a_1, a_2, \cdots, a_r; \\ b_1, b_2, \cdots, b_s; \end{array} q; xztq^{1-n}\right], \tag{53}$$

provided that each of the series in (52) *and* (53) *are absolutely convergent.*

Proof. If we denote by $f(x,u,v,z)$ the right-hand side of (53), we find by using (52) and (36) that

$$
\begin{aligned}
f(x,u,v,z) &= \sum_{n=0}^{\infty} B(n)\, {}_r\Phi_s \left[\begin{matrix} a_1, a_2, \cdots, a_r; \\ \\ b_1, b_2, \cdots, b_s; \end{matrix} \quad q; -z\Theta_{uv} \right] \left\{ \frac{(xutq^{1-n};q)_\infty}{(xvtq^{1-n};q)_\infty} \right\} \\
&= {}_r\Phi_s \left[\begin{matrix} a_1, a_2, \cdots, a_r; \\ \\ b_1, b_2, \cdots, b_s; \end{matrix} \quad q; -z\Theta_{uv} \right] \left\{ \sum_{n=0}^{\infty} B(n) \frac{(xutq^{1-n};q)_\infty}{(xvtq^{1-n};q)_\infty} \right\} \\
&= {}_r\Phi_s \left[\begin{matrix} a_1, a_2, \cdots, a_r; \\ \\ b_1, b_2, \cdots, b_s; \end{matrix} \quad q; -z\Theta_{uv} \right] \left\{ \sum_{n=0}^{\infty} A(n) P_n(v,u) \right\} \\
&= \sum_{n=0}^{\infty} A(n) {}_r\Phi_s \left[\begin{matrix} a_1, a_2, \cdots, a_r; \\ \\ b_1, b_2, \cdots, b_s; \end{matrix} \quad q; -z\Theta_{uv} \right] \{P_n(v,u)\} \\
&= \sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} A(n) \Psi_n^{(\mathbf{a},\mathbf{b})}(u,v,z|q),
\end{aligned}
$$

which is precisely the left-hand side of (53). Our demonstration of Theorem 4 is thus completed. □

Remark 6. *Equation (52) is valid for the case when $u = 1$ and $v = \lambda$, so that*

$$
A(n) = \psi_n^{(\alpha)}(x|q) \frac{(qt)^n}{(q;q)_n} \qquad \text{and} \qquad B(n) = \sum_{k=0}^{\infty} \frac{(1/(\alpha x);q)_k (\alpha q)^k}{(q;q)_k} \frac{(q^{-k};q)_n\, q^{nk}}{(q;q)_n}. \tag{54}
$$

We are thus led to (47). Moreover, if we specialize the coefficients $A(n)$ and $B(n)$, which are involved in (53), just as we have done in the last Equation (54), we will obtain the assertion (3) of Theorem 3. Thus, clearly, Theorem 3 may be viewed as an interesting corollary of Theorem 4.

6. Concluding Remarks and Observations

In our present investigation, we have introduced a general family of q-hypergeometric polynomials and we have derived several q-series identities such as an extended generating function and Srivastava-Agarwal type bilinear generating functions for this family of q-hypergeometric polynomials. We have presented a transformational identity involving generating functions for the generalized q-hypergeometric polynomials which we have introduced here. We have also pointed out relevant connections of the various q-results, which we have investigated in this paper, with those in several related earlier works on this subject.

We conclude this paper by remarking that, in the recently-published survey-cum-expository review article by Srivastava [27], the so-called (p,q)-calculus was exposed to be a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter p being redundant or superfluous (see, for details, p. 340 in [27]). This observation by Srivastava [27] will indeed apply to any attempt to produce the rather straightforward (p,q)-variations of the results which we have presented in this paper.

Author Contributions: Conceptualization, S.A. and H.M.S.; methodology, S.A.; software, S.A.; validation, H.M.S. and S.A.; formal analysis, S.A. and H.M.S.; investigation, S.A.; resources, H.M.S. and S.A.; writing—original draft preparation, S.A.; writing—review and editing, H.M.S.; visualization, S.A.; supervision, H.M.S.; project administration, S.A.; funding acquisition, H.M.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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