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Some Properties Involving q-Hermite Polynomials Arising from Differential Equations and Location of Their Zeros

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Abstract: Hermite polynomials are one of the Apell polynomials and various results were found by the researchers. Using Hermit polynomials combined with q-numbers, we derive different types of differential equations and study these equations. From these equations, we investigate some identities and properties of q-Hermite polynomials. We also find the position of the roots of these polynomials under certain conditions and their stacked structures. Furthermore, we locate the roots of various forms of q-Hermite polynomials according to the conditions of q-numbers, and look for values which have approximate roots that are real numbers.

Keywords: q-Hermite polynomials; zeros of q-Hermite polynomials; differential equation

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1. Introduction

There is a special case in the Sturm–Liouville boundary value problem the called Hermite differential equation that arises when dealing with harmonic oscillator in quantum mechanics. The ordinary Hermite differential equation is defined as

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (\rho - 1)y = 0, \tag{1}$$

where ρ is a constant. When $\rho = 2n+1, n = 0, 1, 2, \ldots$, then one of the solutions of Equation (1) becomes a polynomial. These polynomial solutions are known as Hermite polynomials $H_n(x)$, which are defined by means of the generating function

$$e^{(2x-t)t} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}, \qquad |t| < \infty. \tag{2}$$

The numbers $H_n := H_n(0)$ are the Hermite numbers. Hermite polynomials, first defined by Laplace, are one of the classic orthogonal polynomials and many studies have been conducted by mathematicians. These Hermite polynomials also have many mathematical applications, such as quantum mechanics, physics, and probability theory; see [1–6].

We define the q-numbers also referred by Jackson as follows; see [7–9]

$$[x]_q = \frac{1-q^x}{1-q}, \quad 0 < q < 1, \tag{3}$$

Note that $\lim_{q\to 1}[x]_q = x$. In [8], we recall that the q-Hermite polynomials $\mathbf{H}_{n,q}(x)$ defined by

$$\sum_{n=0}^{\infty} \mathbf{H}_{n,q}(x)\frac{t^n}{n!} = e^{2[x]_q t - t^2} = \mathcal{G}(t, [x]_q), \tag{4}$$

where $0 < q < 1$. In the definition of q-Hermite polynomials, we can observe that if $q \to 1$, then $\mathbf{H}_{n,q}(x) \to H_n(x)$.

In [10], authors defined the two-variable partially degenerate Hermite polynomials $\mathbf{H}_n(x,y,\lambda)$ as

$$\sum_{n=0}^{\infty}\mathbf{H}_n(x,y,\lambda)\frac{t^n}{n!}=(1+\lambda t)^{\frac{x}{\lambda}}e^{yt^2},\quad \lambda\neq 0, \tag{5}$$

and we can see some useful properties of these polynomials. Representatively, we can confirm the following theorems in [10].

$$\begin{aligned}&\text{(i)}\quad a^m\mathbf{H}_m(bx,b^2y,\frac{\lambda}{a})=b^m\mathbf{H}_m(ax,a^2y,\frac{\lambda}{b})\\&\text{(ii)}\quad \mathbf{H}_n(x_1+x_2,y,\lambda)=\sum_{l=0}^{n}\binom{n}{l}(x_2|\lambda)\mathbf{H}_{n-l}(x_1,y,\lambda).\end{aligned} \tag{6}$$

The differential equations derived from the generating functions of special numbers and polynomials have been studied by many mathematicians; see [11–21].

Based on the results to date, in the present work, we can investigate the differential equations generated from the generating function of q-Hermite polynomials $\mathbf{H}_{n,q}(x)$. The rest of the paper is organized as follows. In Section 2, we obtain the basic properties of the q-Hermite polynomials. In Section 3, we construct the differential equations generated from the definition of q-Hermite polynomials:

$$\left(\frac{\partial}{\partial t}\right)^N\mathcal{G}(t,[x]_q)-a_0(N,[x]_q)\mathcal{G}(t,[x]_q)-\cdots-a_N(N,[x]_q)t^N\mathcal{G}(t,[x]_q)=0. \tag{7}$$

We also consider explicit identities for $\mathbf{H}_{n,q}(x)$ using the coefficients of this differential equation. In Section 4, we find the zeros of the q-Hermite polynomials using numerical methods and observe the scattering phenomenon of the zeros of these polynomials. Finally, in Section 5, conclusions and discussions on this work are provided.

2. Basic Properties for the q-Hermite Polynomials

To derive various properties of $\mathbf{H}_{n,q}(x)$, the generating function (4) is an useful function. The following basic properties of polynomials $\mathbf{H}_{n,q}(x)$ are derived from (4). Hence, we choose to omit the details involved.

Theorem 1. *Let n be any positive integer. Then, we have*

$$\begin{aligned}&(1)\quad \mathbf{H}_{n,q}(x)=\sum_{k=0}^{n}\binom{n}{k}2^{n-k}[x]_q^{n-k}H_k.\\&(2)\quad \mathbf{H}_{n,q}(x)=n!\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k2^{n-2k}[x]_q^{n-2k}}{k!(n-2k)!}.\\&(3)\quad \mathbf{H}_{n,q}(x_1+x_2)=\sum_{k=0}^{n}\binom{n}{k}\mathbf{H}_{k,q}(x_1)2^{n-k}q^{x_1(n-k)}[x_2]_q^{n-k},\end{aligned} \tag{8}$$

where $[x]$ is the greatest integer not exceeding x.

Theorem 2. *The q-Hermite polynomials are the solutions of equation*

$$\begin{aligned}&\left(\left(\frac{d}{d[x]_q}\right)^2-2[x]_q\left(\frac{d}{d[x]_q}\right)+2n\right)\mathbf{H}_{n,q}(x)=0,\\&\mathbf{H}_{n,q}(0)=\begin{cases}(-1)^k\dfrac{(2k)!}{k!}, & \text{if } n=2k,\\ 0, & \text{otherwise}\end{cases}\end{aligned} \tag{9}$$

Proof. From Equation (4), we can note that

$$\mathcal{G}(t,[x]_q) = e^{2[x]_q t - t^2} \tag{10}$$

which is satisfied as

$$\frac{\partial \mathcal{G}(t,[x]_q)}{\partial t} - \left(2[x]_q - 2t\right)\mathcal{G}(t,[x]_q) = 0. \tag{11}$$

By substituting the series in (11) for $\mathcal{G}(t,[x]_q)$, we find

$$\mathbf{H}_{n+1,q}(x) - 2[x]_q \mathbf{H}_{n,q}(x) + 2n\mathbf{H}_{n-1,q}(x) = 0, n = 1,2,\ldots, \tag{12}$$

which is the recurrence relation for q-Hermite polynomials. Another recurrence relation comes from

$$\left(\frac{d}{d[x]_q}\right)\mathcal{G}(t,[x]_q) - 2t\mathcal{G}(t,[x]_q) = 0. \tag{13}$$

The following equation implies

$$\left(\frac{d}{d[x]_q}\right)\mathbf{H}_{n,q}(x) - 2n\mathbf{H}_{n-1,q}(x) = 0, n = 1,2,\ldots. \tag{14}$$

Remove $\mathbf{H}_{n-1,q}(x)$ from Equations (12) and (13) to obtain

$$\mathbf{H}_{n+1,q}(x) - 2[x]_q \mathbf{H}_{n,q}(x) + \left(\frac{d}{d[x]_q}\right)\mathbf{H}_{n,q}(x) = 0. \tag{15}$$

By differentiating the following equation and using Equations (12) and (13) again, we can obtain

$$\left(\frac{d}{d[x]_q}\right)^2\mathbf{H}_{n,q}(x) - 2[x]_q\left(\frac{d}{d[x]_q}\right)\mathbf{H}_{n,q}(x) + 2n\mathbf{H}_{n,q}(x) = 0, n = 0,1,2,\ldots. \tag{16}$$

From the above equation, we complete the proof of Theorem 2. □

Theorem 3. $\mathbf{H}_{n,q}(x)$ *in the Equation (4) is the solution of equation*

$$\begin{gathered}\left(\frac{q-1}{q^x \log q}\frac{d^2}{dx^2} + \left(\frac{1-q}{q^x} - \frac{2(1-q^x)}{1-q}\right)\frac{d}{dx} + 2n\frac{\log q}{q-1}q^x\right)\mathbf{H}_{n,q}(x) = 0,\\ \mathbf{H}_{n,q}(0) = \begin{cases}(-1)^k\dfrac{(2k)!}{k!}, & \text{if } n = 2k,\\ 0, & \text{otherwise.}\end{cases}\end{gathered} \tag{17}$$

Proof. We consider another form of the differential equation for $\mathbf{H}_{n,q}(x)$. We consider

$$\mathcal{G}(t,[x]_q) = e^{2[x]_q t - t^2}, \tag{18}$$

which satisfies

$$\frac{d\mathcal{G}(t,[x]_q)}{dx} - \frac{\log q}{q-1}q^x 2t\mathcal{G}(t,[x]_q) = 0. \tag{19}$$

Substitute the series in Equation (19) for $\mathcal{G}(t,[x]_q)$, in order to find

$$\frac{d\mathbf{H}_{n,q}(x)}{dx} - \frac{2n\log q}{q-1}q^x\mathbf{H}_{n-1,q}(x) = 0, n = 1,2,\ldots. \tag{20}$$

To use Equation (15), we note

$$\frac{d}{dx}\left(\frac{q-1}{q^x\log q}\frac{d}{dx}\mathbf{H}_{n,q}(x)\right)=\frac{1-q}{q^x}\frac{d}{dx}\mathbf{H}_{n,q}(x)+\frac{q-1}{q^x\log q}\left(\frac{d}{dx}\right)^2\mathbf{H}_{n,q}(x). \tag{21}$$

By differentiating Equation (15) and using the above Equation (21), we derive

$$\begin{aligned}&2n\frac{\log q}{q-1}q^x\mathbf{H}_{n,q}(x)+\left(\frac{1-q}{q^x}-\frac{2(1-q^x)}{1-q}\right)\frac{d\mathbf{H}_{n,q}(x)}{dx}\\&+\frac{q-1}{q^x\log q}\frac{d^2\mathbf{H}_{n,q}(x)}{dx^2}=0,\end{aligned} \tag{22}$$

where the equation is obtained as the required result immediately. □

3. Differential Equations Associated with q-Hermite Polynomials

In this section, we introduce differential equations arising from the generating functions of q-Hermite polynomials. By using these differential equations, we can obtain the explicit identities for these polynomials. Many authors studied differential equations derived in the generating functions of special polynomials in order to derive explicit identities for special polynomials, see [11–20].

Let

$$\mathcal{G}:=\mathcal{G}(t,[x]_q)=e^{2[x]_qt-t^2}=\sum_{n=0}^{\infty}H_{n,q}(x)\frac{t^n}{n!},\quad x,t\in\mathbb{R}. \tag{23}$$

Then, we obtain the following equations using mathematical induction:

$$\begin{aligned}\mathcal{G}^{(1)}&=\frac{\partial}{\partial t}\mathcal{G}(t,[x]_q)=\frac{\partial}{\partial t}\left(e^{2[x]_qt-t^2}\right)=e^{2[x]_qt-t^2}(2[x]_q-2t)\\&=(2[x]_q-2t)\mathcal{G}(t,[x]_q)\\&=(2[x]_q)\mathcal{G}(t,[x]_q)\\&\quad+(-2)t\mathcal{G}(t,[x]_q),\end{aligned} \tag{24}$$

$$\begin{aligned}\mathcal{G}^{(2)}&=\frac{\partial}{\partial t}\mathcal{G}^{(1)}(t,[x]_q)=-2\mathcal{G}(t,[x]_q)+(2x-2t)\mathcal{G}^{(1)}(t,[x]_q)\\&=(-2+4[x]_q^2)\mathcal{G}(t,[x]_q)\\&\quad+(-8[x]_q)t\mathcal{G}(t,[x]_q)\\&\quad+(-2)^2t^2\mathcal{G}(t,[x]_q),\end{aligned} \tag{25}$$

and

$$\begin{aligned}\mathcal{G}^{(3)}&=\frac{\partial}{\partial t}\mathcal{G}^{(2)}(t,[x]_q)\\&=(-8[x]_q+8t)\mathcal{G}(t,[x]_q)+(-2+4[x]_q^2-8[x]_qt+4t^2)\mathcal{G}^{(1)}(t,[x]_q)\\&=(-12[x]_q+8[x]_q^3)\mathcal{G}(t,[x]_q)\\&\quad+(12-24[x]_q^2)t\mathcal{G}(t,[x]_q)\\&\quad+(24[x]_q)t^2\mathcal{G}(t,[x]_q)\\&\quad+(-2)^3t^3\mathcal{G}(t,[x]_q).\end{aligned} \tag{26}$$

If we continue this process N-times, we can conjecture as follows.

$$\mathcal{G}^{(N)}=\left(\frac{\partial}{\partial t}\right)^N\mathcal{G}(t,[x]_q)=\sum_{i=0}^{N}a_i(N,[x]_q)t^i\mathcal{G}(t,[x]_q),(N=0,1,2,\ldots). \tag{27}$$

By differentiating $\mathcal{G}^{(N)}$ with respect to t in Equation (27), we find

$$\begin{aligned}
\mathcal{G}^{(N+1)} &= \frac{\partial \mathcal{G}^{(N)}}{\partial t} \\
&= \sum_{i=0}^{N} a_i(N,[x]_q) i t^{i-1} \mathcal{G}(t,[x]_q) + \sum_{i=0}^{N} a_i(N,[x]_q) t^i \mathcal{G}^{(1)}(t,[x]_q) \\
&= \sum_{i=0}^{N} a_i(N,[x]_q) i t^{i-1} \mathcal{G}(t,[x]_q) + \sum_{i=0}^{N} a_i(N,[x]_q) t^i (2[x]_q - 2t) \mathcal{G}(t,[x]_q) \\
&= \sum_{i=0}^{N} i a_i(N,[x]_q) t^{i-1} \mathcal{G}(t,[x]_q) + \sum_{i=0}^{N} (2[x]_q) a_i(N,[x]_q) t^i \mathcal{G}(t,[x]_q) \\
&\qquad + \sum_{i=0}^{N} (-2) a_i(N,[x]_q) t^{i+1} \mathcal{G}(t,[x]_q) \\
&= \sum_{i=0}^{N-1} (i+1) a_{i+1}(N,[x]_q) t^i \mathcal{G}(t,[x]_q) + \sum_{i=0}^{N} (2[x]_q) a_i(N,[x]_q) t^i \mathcal{G}(t,[x]_q) \\
&\qquad + \sum_{i=1}^{N+1} (-2) a_{i-1}(N,[x]_q) t^i \mathcal{G}(t,[x]_q).
\end{aligned} \tag{28}$$

Replace N by $N+1$ in (27), and we obtain

$$\mathcal{G}^{(N+1)} = \sum_{i=0}^{N+1} a_i(N+1,[x]_q) t^i \mathcal{G}(t,[x]_q). \tag{29}$$

Theorem 4. *For $N = 0, 1, 2, \ldots$, the differential equation*

$$\mathcal{G}^{(N)} = \left(\frac{\partial}{\partial t}\right)^N \mathcal{G}(t,[x]_q) = \left(\sum_{i=0}^{N} a_i(N,[x]_q) t^i\right) \mathcal{G}(t,[x]_q) \tag{30}$$

has a solution

$$\mathcal{G} = \mathcal{G}(t,[x]_q) = e^{2[x]_q t - t^2}, \tag{31}$$

where

$$\begin{aligned}
& a_0(N,[x]_q) = \sum_{k=0}^{N-1} [x]_q^i a_1(N-1-k,[x]_q) + (2[x]_q)^N, \\
& a_{N-1}(N,[x]_q) = (-2)^{N-1} N (2[x]_q), \\
& a_N(N,[x]_q) = (-2)^N, \\
& a_i(N+1,[x]_q) \\
& = (i+1) \sum_{k=0}^{N} 2^k [x]_q^k a_{i+1}(N-k,[x]_q) + (-2) \sum_{k=0}^{N} 2^k [x]_q^k a_{i-1}(N-k,[x]_q), \\
& (1 \le i \le N-2).
\end{aligned} \tag{32}$$

Proof. Comparing the coefficients on both sides of (28) and (29), we obtain

$$\begin{aligned}
& a_0(N+1,[x]_q) = a_1(N,[x]_q) + (2[x]_q) a_0(N,[x]_q), \\
& a_N(N+1,[x]_q) = (2[x]_q) a_N(N,[x]_q) + (-2) a_{N-1}(N,[x]_q), \\
& a_{N+1}(N+1,[x]_q) = (-2) a_N(N,[x]_q),
\end{aligned} \tag{33}$$

and

$$\begin{aligned}
a_i(N+1,[x]_q) &= (i+1) a_{i+1}(N,[x]_q) \\
&\quad + (2[x]_q) a_i(N,[x]_q) + (-2) a_{i-1}(N,[x]_q), (1 \le i \le N-1).
\end{aligned} \tag{34}$$

In addition, from Equation (27), we get

$$\mathcal{G}(t,[x]_q) = \mathcal{G}^{(0)}(t,[x]_q) = a_0(0,[x]_q)\mathcal{G}(t,[x]_q), \tag{35}$$

which gives

$$a_0(0,[x]_q) = 1. \tag{36}$$

It is not difficult to show that

$$\begin{aligned}
&(2[x]_q)\mathcal{G}(t,[x]_q) + (-2)t\mathcal{G}(t,[x]_q) \\
&= \mathcal{G}^{(1)}(t,[x]_q) \\
&= \sum_{i=0}^{1} a_i(1,[x]_q)\mathcal{G}(t,[x]_q) \\
&= a_0(1,[x]_q)\mathcal{G}(t,[x]_q) + a_1(1,[x]_q)t\mathcal{G}(t,[x]_q).
\end{aligned} \tag{37}$$

By using Equation (29), we can present the following as

$$a_0(1,[x]_q) = 2[x]_q, \quad a_1(1,[x]_q) = -2. \tag{38}$$

From the Equation (33), we express

$$\begin{aligned}
&a_0(N+1,[x]_q) = a_1(N,[x]_q) + (2[x]_q)a_0(N,[x]_q), \\
&a_0(N,[x]_q) = a_1(N-1,[x]_q) + (2x)a_0(N-1,[x]_q), \ldots \\
&a_0(N+1,[x]_q) = \sum_{i=0}^{N} (2[x]_q)^i a_1(N-i,[x]_q) + (2[x]_q)^{N+1},
\end{aligned} \tag{39}$$

$$\begin{aligned}
&a_N(N+1,[x]_q) = (2[x]_q)a_N(N,[x]_q) + (-2)a_{N-1}(N,[x]_q), \\
&a_{N-1}(N,[x]_q) = (2[x]_q)a_{N-1}(N-1,[x]_q) + (-2)a_{N-2}(N-1,[x]_q), \ldots \\
&a_N(N+1,[x]_q) = (-2)^N(N+1)(2[x]_q),
\end{aligned} \tag{40}$$

and

$$\begin{aligned}
&a_{N+1}(N+1,[x]_q) = (-2)a_N(N,[x]_q), \\
&a_N(N,[x]_q) = (-2)a_{N-1}(N-1,[x]_q), \ldots \\
&a_{N+1}(N+1,[x]_q) = (-2)^{N+1}.
\end{aligned} \tag{41}$$

Choose $i = 1$ in (34). Then, we can find

$$a_1(N+1,[x]_q) = 2\sum_{k=0}^{N} (2[x]_q)^k a_2(N-k,[x]_q) + (-2)\sum_{k=0}^{N} (2[x]_q)^k a_0(N-k,[x]_q). \tag{42}$$

For $1 \le i \le N-1$, by containing this process, we can deduce

$$\begin{aligned}
a_i(N+1,[x]_q) = (i+1)&\sum_{k=0}^{N} (2[x]_q)^k a_{i+1}(N-k,[x]_q) \\
&+ (-2)\sum_{k=0}^{N} (2[x]_q)^k a_{i-1}(N-k,[x]_q).
\end{aligned} \tag{43}$$

Here, notice that the matrix $a_i(j,[x]_q)_{0\le i,j\le N+1}$ is given by

$$\begin{pmatrix} 1 & 2[x]_q & -2+4[x]_q^2 & -12[x]_q+8[x]_q^3 & \cdots & \cdot \\ 0 & (-2) & (-2)2(2[x]_q) & 12-24[x]_q^2 & \cdots & \cdot \\ 0 & 0 & (-2)^2 & (-2)^2 3(2[x]_q) & \cdots & \cdot \\ 0 & 0 & 0 & (-2)^3 & \ddots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots & (-2)^N(N+1)(2[x]_q) \\ 0 & 0 & 0 & 0 & \cdots & (-2)^{N+1} \end{pmatrix} \tag{44}$$

From (33) to (43), we investigate the desired result immediately. □

Theorem 5. *For $N=0,1,2,\ldots$, we have*

$$\mathbf{H}_{m+N,q}(x)=\sum_{i=0}^{m}\frac{\mathbf{H}_{m-i,q}(x)a_i(N,[x]_q)m!}{(m-i)!}, \tag{45}$$

where

$$\begin{aligned} &a_0(N,[x]_q)=\sum_{k=0}^{N-1}2^k[x]_q^k a_1(N-1-k,[x]_q)+(2[x]_q)^N,\\ &a_{N-1}(N,[x]_q)=(-2)^{N-1}N(2[x]_q),\\ &a_N(N,[x]_q)=(-2)^N,\\ &a_i(N+1,[x]_q)\\ &=(i+1)\sum_{k=0}^{N}2^k[x]_q^k a_{i+1}(N-k,[x]_q)+(-2)\sum_{k=0}^{N}2^k[x]_q^k a_{i-1}(N-k,[x]_q),\\ &(1\le i\le N-2). \end{aligned} \tag{46}$$

Proof. By making the N-times derivative for (4) with respect to t, we get

$$\left(\frac{\partial}{\partial t}\right)^N\mathcal{G}(t,[x]_q)=\left(\frac{\partial}{\partial t}\right)^N e^{2[x]_q t-t^2}=\sum_{m=0}^{\infty}\mathbf{H}_{m+N,q}(x)\frac{t^m}{m!}. \tag{47}$$

From (46) and (47), we obtain

$$a_0(N,[x]_q)\mathcal{G}(t,[x]_q)+\cdots+a_1(N,[x]_q)t^N\mathcal{G}(t,[x]_q)=\sum_{m=0}^{\infty}\mathbf{H}_{m+N,q}(x)\frac{t^m}{m!}, \tag{48}$$

which makes the required result. □

Corollary 1. *For $N=0,1,2,\ldots$, if we take $m=0$ in (45), then, the following holds*

$$\mathbf{H}_{N,q}(x)=a_0(N,[x]_q),$$

where,

$$
\begin{aligned}
& a_0(N,[x]_q) = \sum_{k=0}^{N-1} 2^k [x]_q^k a_1(N-1-k,[x]_q) + (2[x]_q)^N, \\
& a_1(N,[x]_q) \\
& = 2\sum_{k=0}^{N-1}(2[x]_q)^k a_2(N-k-1,[x]_q) + (-2)\sum_{k=0}^{N-1}(2[x]_q)^k a_0(N-k-1,[x]_q).
\end{aligned} \tag{49}
$$

For $N = 0, 1, 2, \ldots$, the differential equation

$$
\mathcal{G}^{(N)} = \left(\frac{\partial}{\partial t}\right)^N \mathcal{G}(t,[x]_q) = \left(\sum_{i=0}^{N} a_i(N,[x]_q) t^i\right) \mathcal{G}(t,[x]_q) \tag{50}
$$

has a solution

$$
\mathcal{G} = \mathcal{G}(t,[x]_q) = e^{2[x]_q t - t^2}. \tag{51}
$$

The following Figure 1 is the graph representation for this solution by using MATHEMATICA.

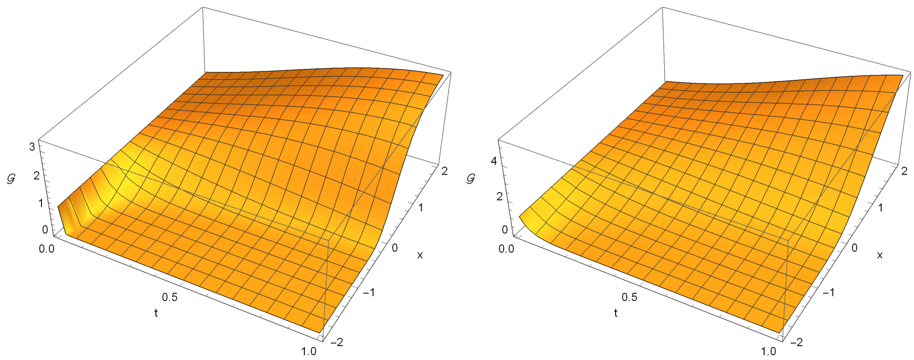

Figure 1. The surface for the solution $\mathcal{G}(t,[x]_q)$.

We can find the left surface of Figure 1 when we choose $-1 \le x \le 1, q = 1/10$, and $0 \le t \le 1$. Additionally, we can see the right surface of Figure 1 when we choose a condition such as $-1 \le x \le 1, q = 1/3$, and $0 \le t \le 1$. It particularly shows a higher-resolution density of the plots in the right surface of Figure 1.

4. Distribution and Pattern of Zeros of q-Hermite Polynomials

In this section, we examine the distribution and pattern of zeros of q-Hermite polynomials $\mathbf{H}_{n,q}(x)$ according to the change in degree n. Based on these results, we present a problem that needs to be approached theoretically. Many mathematicians now explore concepts more easily than in the past by using software. These experiments allow them to quickly create and visualize new ideas, review properties of various figures, as well as find and guess patterns. This numerical survey is particularly interesting since it helps them understand the basic concepts and solve numerous problems. Here, we use MATHEMATICA to find Figures 2–4 and approximate roots for q-Hermite polynomials.

The q-Hermite polynomials $\mathbf{H}_{n,q}(x)$ can be explicitly determined; see [21,22]. First, several examples are given, as follows.

$$\begin{aligned}
&\mathbf{H}_{0,q}(x) = 1, \\
&\mathbf{H}_{1,q}(x) = -\frac{2}{-1+q} + \frac{2q^x}{-1+q}, \\
&\mathbf{H}_{2,q}(x) = -2 + \frac{4}{(-1+q)^2} - \frac{8q^x}{(-1+q)^2} + \frac{4q^{2x}}{(-1+q)^2}, \\
&\mathbf{H}_{3,q}(x) = \frac{4}{(-1+q)^3} - \frac{24q}{(-1+q)^3} + \frac{12q^2}{(-1+q)^3} + \frac{12q^x}{(-1+q)^3} - \frac{24q^{2x}}{(-1+q)^3} \\
&\qquad + \frac{8q^{3x}}{(-1+q)^3} + \frac{24q^{1+x}}{(-1+q)^3} - \frac{12q^{2+x}}{(-1+q)^3}.
\end{aligned} \tag{52}$$

We observe the distribution of zeros of the q-Hermite polynomials $\mathbf{H}_{n,q}(x) = 0$. In Figure 2, plots for the zeros of the q-Hermite polynomials $\mathbf{H}_{n,q}(x)$ for $n = 20$ and $x \in \mathbb{R}$ are as follows.

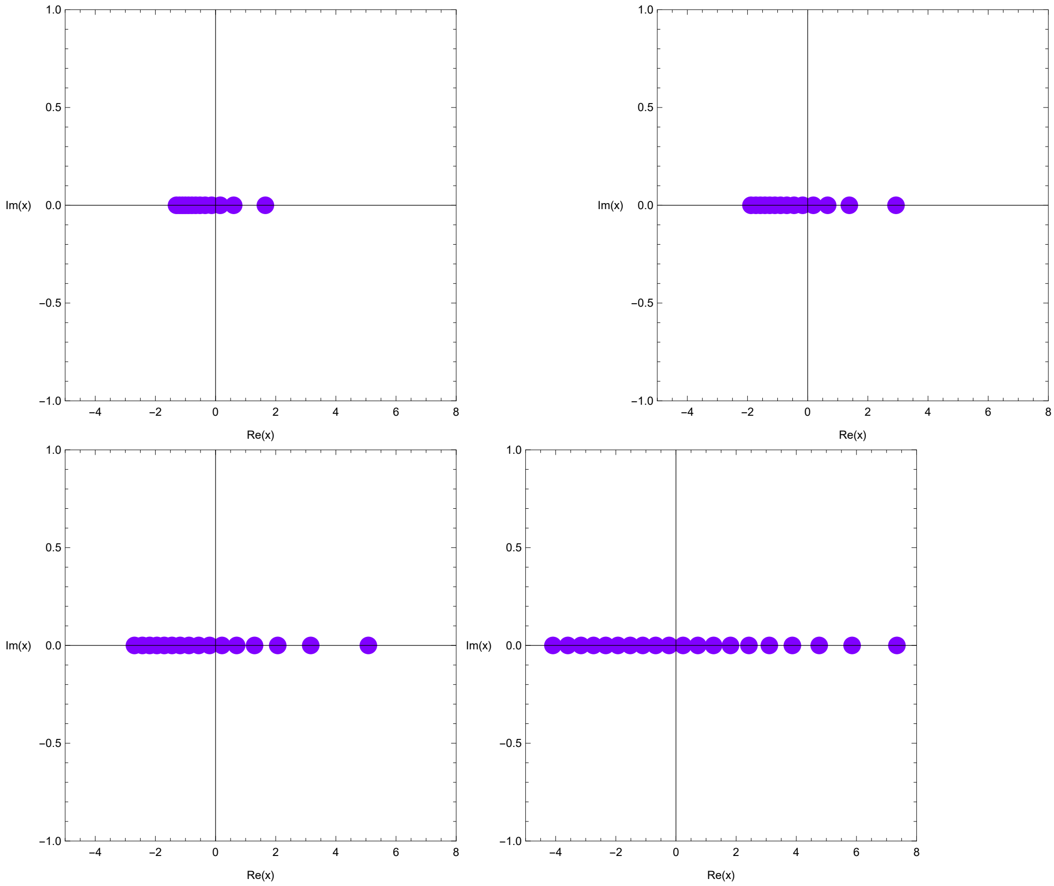

Figure 2. Zeros of $\mathbf{H}_{n,q}(x)$.

In the top-left picture of Figure 2, we choose $n = 20$ and $q = 3/10$. In the top-right picture of Figure 2, we consider conditions which are $n = 20$ and $q = 5/10$. We can find the bottom-left picture of Figure 2, when we consider $n = 20$ and $q = 7/10$. If we consider $n = 20$ and $q = 9/10$, then we can observe the bottom-right picture of Figure 2.

Stacks of zeros of the q-Hermite polynomials, $\mathbf{H}_{n,q}(x)$, for $1 \leq n \leq 20$ from a 3-D structure are presented as Figure 3.

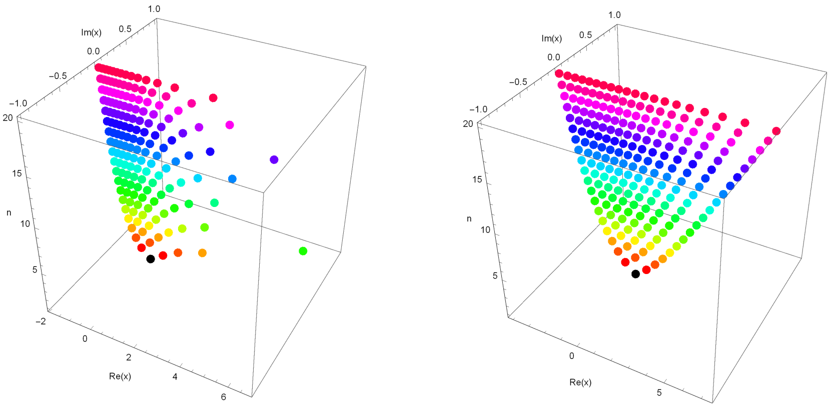

Figure 3. Stacks of zeros of $\mathbf{H}_{n,q}(x), 1 \leq n \leq 20$.

It is the left picture of Figure 3, when we consider $q = 1/2$. Additionally, if we consider $q = 9/10$, we can obtains the right picture of Figure 3.

Our numerical results for the approximate solutions of real zeros of the q-Hermite polynomials, $\mathbf{H}_{n,q}(x)$, with $q = 1/2$ and $x \in \mathbb{R}$ are displayed in Tables 1 and 2.

Table 1. Numbers of real and complex zeros of $\mathbf{H}_{n,\frac{1}{2}}(x)$.

Degree n	**Real Zeros**
1	1
2	2
3	3
4	4
5	4
6	5
7	6
8	7
9	7
10	8
11	8
12	9
13	10
14	10

The plot structures of real zeros of the q-Hermite polynomials, $\mathbf{H}_{n,q}(x)$, for $1 \leq n \leq 20$ are presented in Figure 4.

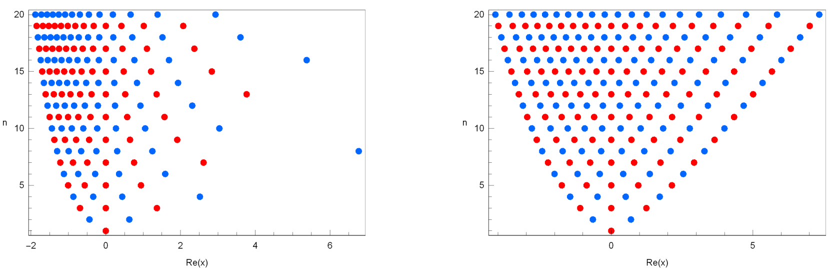

Figure 4. Stacks of zeros of $\mathbf{H}_{n,q}(x), 1 \leq n \leq 20$.

In the left picture of Figure 4, we choose $q = 5/10$. For $q = 9/10$, the right side of Figure 4 is presented. Next, we calculated an approximate solution that satisfies $H_{n,q}(x) = 0, x \in \mathbb{R}$. The results are shown in Table 2.

Table 2. Approximate solutions of $\mathbf{H}_{n,q}(x) = 0, x \in \mathbb{R}$.

Degree n	x
1	0
2	-0.436752, 0.629397
3	-0.689185, 0, 1.36726
4	-0.868165, -0.336082, 0.43894, 2.51738
5	-1.12122, -0.738054, -0.28456, 0.354831, 1.59042
6	-1.12122, -0.738054, -0.28456, 0.354831, 1.59042
7	-1.21784, -0.877176, -0.493795, 0, 0.756682, 2.61507
8	-1.30177, -0.993369, -0.658643, -0.251681, 0.305064, 1.24673, 6.76861

5. Conclusions and Discussion

In this paper, we derive a few solutions of special forms containing q-Hermit polynomials and find several properties of differential equations for these polynomials. Moreover, we find approximate values of real zeros for q-Hermit polynomials and analyze the structure of roots for these polynomials in a special condition from 3D.

We also identified the structure of q-Hermit polynomials under special several conditions. These conditions change the structure of the roots and the form of polynomials, and further research needs to be done on finding various properties. In addition, by simulating the structure of roots for Hermit polynomials through various methods using the results of this paper and multiple software, it is also thought that the characteristics of the roots' structure for higher-order equations will evolve into one area.

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References

1. Andrews, L.C. *Special Functions for Engineers and Mathematicians*; Macmillan. Co.: New York, NY, USA, 1985.
2. Appell, P.; Hermitt, J. *Fonctions Hypergéométriques et Hypersphériques: Polynomes d Hermite*; Gauthier-Villars: Paris, France, 1926.
3. Erdelyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F.G. *Higher Transcendental Functions*; Krieger: New York, NY, USA, 1981; Volume 3.
4. Hermite, C. Sur un nouveau développement en série des fonctions. *Comptes Rendus l'Académie Sci.* **1864**, *58*, 93–100.
5. Laplace, P.S. Mémoire sur les intégrales définies, et leur application aux probabilités, et spécialment à la recherche du milieu qu'il faut choisir entre les résultats des observations. In *Mémoires de la Classe des Sciences Mathematiques et Physiques de l'Institut National de France*; Mémoires de l'Académie des sciences: Ière série, France, 1810; Volume 9, pp. 279–347.
6. Temme, N. *Special Functions: An Introduction to the Classical Functions of Mathematical Physics*; Wiley: New York, NY, USA, 1996.
7. Kim, M.S.; Hu, S. On p-adic Hurwitz-type Euler Zeta functions. *J. Number Theory* **2012**, *132*, 2977–3015. [CrossRef]
8. Kim, T.; Choi, J.; Kim, Y.H.; Ryoo, C.S. On q-Bernstein and q-Hermite polynomials. *Proc. Jangjeon Math. Soc.* **2011**, *14*, 215–221.
9. Robert, A.M. *A Course in p-adic Analysis, Graduate Text in Mathematics*; Springer: Berlin/Heidelberg, Germany, 2000; Volume 198.
10. Hwang, K.Y.; Ryoo, C.S. Some Identities Involving Two-Variable Partially Degenerate Hermite Polynomials Induced from Differential Equations and Structure of Their Roots. *Mathematics* **2020**, *8*, 632. [CrossRef]

11. Agarwal, R.P.; Ryoo, C.S. Differential equations associated with generalized Truesdell polynomials and distribution of their zeros. *J. Appl. Pure Math.* **2019**, *1*, 11–24.
12. Hwang, K.W.; Ryoo, C.S. Differential equations associated with two variable degenerate Hermite polynomials. *Mathematics* **2020**, *8*, 228. [CrossRef]
13. Kim, T.; Kim, D.S. Identities involving degenerate Euler numbers and polynomials arising from non-linear differential equations. *J. Nonlinear Sci. Appl.* **2016**, *9*, 2086–2098. [CrossRef]
14. Ryoo, C.S. Differential equations associated with generalized Bell polynomials and their zeros. *Open Math.* **2016**, *14*, 807–815. [CrossRef]
15. Ryoo, C.S. Differential equations associated with the generalized Euler polynomials of the second kind. *J. Comput. Appl. Math.* **2018**, *24*, 711–716.
16. Ryoo, C.S. Differential equations associated with tangent numbers. *J. Appl. Math. Inform.* **2016**, *34*, 487–494. [CrossRef]
17. Ryoo, C.S. Some identities involving Hermitt Kamp*é* de Fériet polynomials arising from differential equations and location of their zeros. *Mathematics* **2019**, *7*, 23. [CrossRef]
18. Ryoo, C.S. Differential equations associated with degenerate tangent polynomials and computation of their zeros. *Dyn. Syst. Appl.* **2019**, *28*, 153–165.
19. Ryoo, C.S.; Agarwal, R.P.; Kang, J.Y. Differential equations associated with Bell-Carlitz polynomials and their zeros. *Neural Parallel Sci. Comput.* **2016**, *24*, 453–462.
20. Ryoo, C.S.; Agarwal, R.P.; Kang, J.Y. Some properties involving 2-variable modified partially degenerate Hermite polynomials derived from differential equations and distribution of their zeros. *Dyn. Syst. Appl.* **2020**, *29*, 248–269. [CrossRef]
21. Agarwal, R.P.; O'Regan, D. *Ordinary and Partial Differential Equations: With Special Unctions, Fourier Series and Boundary Value Problems*; Springer Science & Business Media: New York, NY, USA, 2009.
22. Ruiz, L.M.S.; Blanes, S. *Complementos de Ecuaciones Diferenciales: Resolucion Analitica, EDPs y Mathematica*; Academica Editorial UPV: Editorial Universitat Politècnica de València: València, Spain, 2012.