

Article

Qualitative Analyses of Differential Systems with Time-Varying Delays via Lyapunov–Krasovskii Approach

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Abstract: In this paper, a class of systems of linear and non-linear delay differential equations (DDEs) of first order with time-varying delay is considered. We obtain new sufficient conditions for uniform asymptotic stability of zero solution, integrability of solutions of an unperturbed system and boundedness of solutions of a perturbed system. We construct two appropriate Lyapunov–Krasovskii functionals (LKFs) as the main tools in proofs. The technique of the proofs depends upon the Lyapunov–Krasovskii method. For illustration, two examples are provided in particular cases. An advantage of the new LKFs used here is that they allow to eliminate using Gronwall’s inequality. When we compare our results with recent results in the literature, the established conditions are more general, less restrictive and optimal for applications.



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1. Introduction

From the relevant literature, it can be observed that numerous processes, both natural and human-made, in biology, interaction of species, population dynamics, microbiology, distributed networks, learning models, mechanics, medicine, nuclear reactors, chemistry, distributed networks, epidemiology, physics, engineering, economics, physiology, viscoelasticity, as well as many others, involve time delays. Hence, many applications in sciences, engineering and so on can be modeled as differential equations with time-varying delays (see the books of Burton [1], Hale and Verduyn Lunel [2], Kolmanovskii and Nosov [3], Krasovskii [4], Kuang [5], Lakshmikantham et al. [6], Smith [7] and bibliographies therein).

The interest of applied mathematicians, engineers, etc., to investigate qualitative properties of solutions for such numerous problems with time-varying delays has increased considerably in the last decades. In particular, see the mentioned books, the papers of Arino et al. [8], Azbelev et al. [9], Berezansky and Braverman [10], Du [11], Gil [12], Graef and Tunç [13], Slyn’ko and Tunç [14], Tian and Ren [15], Tunç [16–18], Tunç and Erdur [19], Tunç and Golmankhaneh [20], Tunç and Tunç [21–23], Zevin [24] and bibliographies therein.

It is worth mentioning that especially DDEs of first and second order with constant and time-varying delays can be encountered intensively during investigations and applications. For those reasons, during the investigations and applications, it is required to get information about various qualitative behaviors of solutions of those kind of equations

such as stability, instability, convergence, etc., of solutions of DDEs. To the best of available information, it should be noted that from the theory of DDEs, we know that analytically solving DDEs with time-varying delays is a very difficult mathematical task. Therefore, over the past decades, some methods have been developed to get information about the qualitative properties of solutions of DDEs without solving them. Among the developed methods, the Lyapunov's second method, Lyapunov–Krasovskii method, Razumikhin method and fixed point method can be effectively used to investigate the stability and some other properties of solutions of ODEs, DDEs, neutral and advanced functional differential equations. In general, the Lyapunov's second method is used to discuss numerous qualitative properties of ODEs of first and higher order. Next, the Razumikhin method is only used to study qualitative properties of a few certain forms of DDEs and impulsive differential equations. As for the fixed point method, it can be used to study stability, existence of periodic solutions, etc., of various kind of those equations of first order. However, this method is rarely used in the equations of second order and those of higher order. Meanwhile, during the last 50 years, the theory of functional differential equations (FDEs) has been developed extensively. Krasovskii [4] firstly investigated the stability of equilibria and wanted to make sure that all of the results for ODE using LKFs could be carried over to DDEs. It should be noted Krasovskii [4] suggested the use of functional defined on DDEs' trajectories instead of Lyapunov functions. Later, this functional method is very effectively used to get information on the mentioned properties of solutions of DDEs without having any prior information of solutions. When Lyapunov–Krasovskii method is used during the investigations, it is needed to define or construct a suitable LKF, which is positive definite, and its time derivative along the considered DDEs is negative or negative-semi definite. From this point of view, finding a suitable Lyapunov–Krasovskii functional for a problem under study is difficult and an open problem in the literature by this time. Next, most of researches on DDEs focus on linear differential equations with constant delay and preservation of stability; however, the number of available researches on scalar nonlinear DDEs and nonlinear system of DDEs with time-varying delays are less. From this point, it deserves to investigate the properties of solutions of systems of nonlinear DDEs with time-varying delays.

The motivation of this paper was inspired by a recent work of Tian and Ren [15]. From this point of view, we mention a related work of Tian and Ren [15]. In 2020, Tian and Ren [15] considered the following system of linear DDEs with time-varying delay,

$$\dot{x}(t) = Ax(t) + Bx(t - h(t)), \quad (1)$$

where $x(t) \in \mathbb{R}^n$, $A, B \in \mathbb{R}^{n \times n}$ and $h(t) \in C^1(\mathbb{R}^+, (0, \infty))$ is the time-varying delay, and it satisfies

$$0 \leq h_1 \leq h(t) \leq h_2, h_{12} = h_2 - h_1, 0 \leq h'(t) \leq h_0 < 1. \quad (2)$$

Tian and Ren [15] defined an LKF for the system of DDEs (1). Based on that LKF, Tian and Ren [15] proved a theorem, ([15], Theorem 1), on the asymptotically stability of the system of DDEs (1).

In this paper, motivated by the system of DDEs (1), the result of Tian and Ren ([15], Theorem 1) and those in the bibliography of this paper, as an alternative to the linear system of DDEs (1), we consider a nonlinear system of DDEs with time-varying delay as follows:

$$\dot{x}(t) = A(t)x(t) + BF(x(t - h(t))) + E(t, x(t), x(t - h(t))) \quad (3)$$

with the continuous initial function

$$x(t) = \phi(t), \quad t \in [-h_2, 0],$$

where $x \in \mathbb{R}^n, t \in \mathbb{R}^+, \mathbb{R}^+ = [0, \infty), h(t) \in C^1(\mathbb{R}^+, (0, \infty))$ is the time-varying delay, which satisfies the condition (2), $A(t) \in C(\mathbb{R}^+, \mathbb{R}^{n \times n}), B \in \mathbb{R}^{n \times n}, F \in C(\mathbb{R}^n, \mathbb{R}^n), F(0) = 0$ and $E \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$.

We now summarize the aim of this paper by the following items, respectively:

- (1) We investigate the uniformly asymptotically stability of zero solution of the system of DDEs (1), see Theorem 3. To investigate this problem, we define a very different LKF from that in Tian and Ren [15].
- (2) We study the uniformly asymptotically stability of zero solution and the integrability of the norm of solutions of the following unperturbed nonlinear system of DDEs by Theorem 4 and Theorem 5, respectively:

$$\dot{x}(t) = A(t)x(t) + BF(x(t - h(t))). \quad (4)$$

- (3) We investigate the boundedness of solutions of the perturbed system of nonlinear DDEs (3), see Theorem 6.
- (4) In particular cases, two new examples and graphs of their solutions are provided to show applications of Theorems 3–6.

2. Basic Result

Consider the system of the DDEs:

$$\frac{dx}{dt} = H(t, x_t), \quad (5)$$

where $H \in C(\mathbb{R} \times C_0, \mathbb{R}^n), H(t, 0) = 0$ and takes bounded sets into bounded sets. For some $\tau > 0, C_0 = C_0([-\tau, 0], \mathbb{R}^n)$ denotes the space of continuous functions $\phi : [-\tau, 0] \rightarrow \mathbb{R}^n$. For any $a \geq 0, \forall t_0 \geq 0$ and $x \in C_0([t_0 - \tau, t_0 + a], \mathbb{R}^n)$, we have $x_t = x(t + \theta)$ for $-\tau \leq \theta \leq 0$ and $t \geq t_0$.

Let $x \in \mathbb{R}^n$. The norm $\|\cdot\|$ is defined by $\|x\| = \sum_{i=1}^n |x_i|$. Next, let $A \in \mathbb{R}^{n \times n}$. For this case, the matrix norm, $\|A\|$ is defined by $\|A\| = \max_{1 \leq j \leq n} \left(\sum_{i=1}^n |a_{ij}| \right)$.

In this article, without loss of generality, sometimes instead of $x(t)$, we will simply write x .

For any $\phi \in C_0$, let

$$\|\phi\|_{C_0} = \sup_{\theta \in [-r, 0]} \|\phi(\theta)\| = \|\phi(\theta)\|_{[-r, 0]}$$

and

$$C_H = \{\phi : \phi \in C_0 \text{ and } \|\phi\|_{C_0} \leq H < \infty\}.$$

We suppose that the function H satisfies the conditions of the uniqueness of solutions of the system of DDEs (5). We note that the system of DDEs (3) is a particular case of the system of DDEs (5).

Let $x(t) = x(t, t_0, \phi)$ be a solution of the system of DDEs (5) such that $x(t) = \phi(t)$ on $[t_0 - \tau, t_0]$, where $\phi \in C([t_0 - \tau, t_0], \mathbb{R}^n)$ is an initial function.

Let

$$V_1(t, \phi) : \mathbb{R}^+ \times C_H \rightarrow \mathbb{R}^+, \mathbb{R}^+ = [0, \infty),$$

be a continuous functional in t and ϕ with $V_1(t, 0) = 0$. Further, let $\frac{d}{dt} V_1(t, x)$ denote the derivative of $V_1(t, x)$ on the right through any solution $x(t)$ of the system of DDEs (5).

Theorem 1 (Burton [1], Theorem 4.2.9). *Assume that the following conditions hold:*

(A1) The function $V_1(t, x)$ satisfies the locally Lipschitz in x , i.e., for every compact $S \subset \mathbb{R}^n$ and $\gamma > t_0$, there exists a $K_{\gamma S} \in \mathbb{R}$ with $K_{\gamma S} > 0$ such that

$$|V_1(t, x) - V_1(t, y)| \leq K_{\gamma S} \|x - y\|_{[t_0-\tau, t]}$$

for all $t \in [t_0, \gamma]$ and $x, y \in C_0([t_0 - \tau, t_0], S)$.

(A2) Let $Z(t, \phi)$ be a functional such that it satisfies the one side locally Lipschitz in t :

$$Z(t_2, \phi) - Z(t_1, \phi) \leq K(t_2 - t_1), 0 < t_1 < t_2 < \infty, K > 0, K \in \mathbb{R},$$

whenever $\phi \in C_H$, where $Z : \mathbb{R}^+ \times C_H \rightarrow \mathbb{R}^+$ is continuous.

(A3) There are four strictly increasing functions $\omega, \omega_1, \omega_2, \omega_3 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with value 0 at 0 such that

$$\omega(\|\phi(0)\|) + Z(t, \phi) \leq V_1(t, \phi) \leq \omega_1(\|\phi(0)\|) + Z(t, \phi),$$

$$Z(t, \phi) \leq \omega_2(\|\phi\|_C)$$

and

$$\frac{d}{dt} V_1(t, x(\cdot)) \leq -\omega_3(\|x(t)\|)$$

whenever $t \in \mathbb{R}^+$ and $x \in C_H$. Then, the solution $x(t) = 0$ of the system of DDEs (5) is uniformly asymptotically stable.

3. Asymptotic Stability

Firstly, we introduce the main result of Tian and Ren ([15], Theorem 1).

Theorem 2 (Tian and Ren [15], Theorem 1). For given scalars h_1, h_2 , system (1) is asymptotically stable if there exist matrices $P \in S_+^{6n}, Q_1, Q_2, Q_3, Q_4 \in S_+^n, N_1, N_2 \in \mathbb{R}^{16n \times 5n}$, such that

$$\Phi(\alpha) = \begin{bmatrix} \phi_1(\alpha) + \phi_2(\alpha) & * \\ \alpha M_1^T + (1 - \alpha) M_2^T & -\Theta \end{bmatrix} < 0$$

holds for $\alpha = \{0, 1\}$, where

$$\begin{aligned} \phi_1(\alpha) &= He\left(\sum_1^T P \Sigma_2\right) + \varepsilon_1^T Q_1 \varepsilon_1 - \varepsilon_2^T Q_1 \varepsilon_2 + \varepsilon_2^T Q_2 \varepsilon_2 \\ &\quad - \varepsilon_4^T Q_2 \varepsilon_4 + h_1^2 \varepsilon_0^T Q_3 \varepsilon_0 + h_{12}^2 \varepsilon_0^T Q_4 \varepsilon_0 - \sum_{i=0}^4 (2i+1) \Sigma_{i+3}^T Q_3 \Sigma_{i+3}, \\ \phi_2(\alpha) &= -Y^T \Sigma(\alpha) Y - He\left(Y^T \begin{bmatrix} (1-\alpha) M_1^T \\ \alpha M_2^T \end{bmatrix}\right), \\ \Sigma_1 &= [\varepsilon_1^T \quad h_1 \varepsilon_5^T \quad \alpha h_{12} \varepsilon_6^T + (1-\alpha) h_{12} \varepsilon_7^T \quad h_1^2 \varepsilon_8^T \quad \frac{h_1^2}{2} h_1^3 \varepsilon_{11}^T \quad h_1^4 \varepsilon_{14}^T], \\ \Sigma_2 &= \begin{bmatrix} \varepsilon_0^T & \varepsilon_1^T - \varepsilon_2^T & \varepsilon_2^T - \varepsilon_4^T & h_1 \varepsilon_1^T - h_1 \varepsilon_5^T & \frac{h_1^2}{2} \varepsilon_1^T - h_1^2 \varepsilon_8^T & \frac{h_1^3}{6} \varepsilon_1^T - h_1^3 \varepsilon_{11}^T \end{bmatrix}, \\ \Sigma_3 &= \varepsilon_1 - \varepsilon_2, \\ \Sigma_4 &= \varepsilon_1 + \varepsilon_2 - 2\varepsilon_5, \\ \Sigma_5 &= \varepsilon_1 - \varepsilon_2 + 6\varepsilon_5 - 12\varepsilon_8, \\ \Sigma_6 &= \varepsilon_1 - \varepsilon_2 - 12\varepsilon_5 + 60\varepsilon_8 - 120\varepsilon_{11}, \\ \Sigma_7 &= \varepsilon_1 - \varepsilon_2 + 20\varepsilon_5 - 180\varepsilon_8 + 840\varepsilon_{11} - 1680\varepsilon_{14}, \\ \Sigma_8 &= \varepsilon_2 - \varepsilon_3, \\ \Sigma_9 &= \varepsilon_2 + \varepsilon_3 - 2\varepsilon_6, \end{aligned}$$

$$\begin{aligned}
\Sigma_{10} &= \varepsilon_2 - \varepsilon_3 + 6\varepsilon_6 - 12\varepsilon_9, \\
\Sigma_{11} &= \varepsilon_2 + \varepsilon_3 - 12\varepsilon_6 + 60\varepsilon_9 - 120\varepsilon_{12}, \\
\Sigma_{12} &= \varepsilon_2 - \varepsilon_3 + 20\varepsilon_6 - 180\varepsilon_9 + 840\varepsilon_{12} - 1680\varepsilon_{15}, \\
\Sigma_{13} &= \varepsilon_3 - \varepsilon_4, \\
\Sigma_{14} &= \varepsilon_3 + \varepsilon_4 - 2\varepsilon_7, \\
\Sigma_{15} &= \varepsilon_3 - \varepsilon_4 + 6\varepsilon_7 - 12\varepsilon_{10}, \\
\Sigma_{16} &= \varepsilon_3 + \varepsilon_4 - 12\varepsilon_7 + 60\varepsilon_{10} - 120\varepsilon_{13}, \\
\Sigma_{17} &= \varepsilon_3 - \varepsilon_4 + 12\varepsilon_{17} - 60\varepsilon_{10} + 120\varepsilon_{13} - 1680\varepsilon_{16}, \\
\varepsilon_0 &= A\varepsilon_1 + B\varepsilon_3, \\
\Gamma &= \begin{bmatrix} \Sigma_8^T & \Sigma_9^T & \Sigma_{10}^T & \Sigma_{11}^T & \Sigma_{12}^T & \Sigma_{13}^T & \Sigma_{14}^T & \Sigma_{15}^T & \Sigma_{16}^T & \Sigma_{17}^T \end{bmatrix}, \\
Q &= \text{diag}(Q_4, 3Q_4, 5Q_4, 7Q_4, 9Q_4),
\end{aligned}$$

and

$$\varepsilon_i \in \mathbb{R}^{n \times 16n}$$

is defined as

$$\varepsilon_i = \begin{bmatrix} 0_{n \times (i-1)n} & I_n & 0_{n \times (16-i)n} \end{bmatrix} \text{ for } i = 1, 2, \dots, 16.$$

We now give our first result.

Theorem 3. We suppose that the following conditions (C1) and (C2) hold:

(C1) There exist constants h_0 from (2) and $a_0 > 0$, $\alpha > 0$ such that

$$a_0(1 - h_0) - \|B\| \geq \alpha.$$

(C2) There exist a constant a_0 from (C1) such that

$$a_{ii} + \sum_{j=1, j \neq i}^n |a_{ji}| \leq -a_0 \text{ for all } t \in \mathbb{R}^+.$$

Then, the zero solution of the system of DDEs (1) is uniformly asymptotically stable

Proof. Define a new LKF $V := V(t, x_t)$ by

$$V(t, x_t) := \|x(t)\| + \lambda \int_{t-h(t)}^t \|x(s)\| ds, \quad (6)$$

where λ is an arbitrary positive constant which will be chosen later in the proof. This functional, the LKF (6), can be expressed as the following:

$$V(t, x_t) := |x_1(t)| + \dots + |x_n(t)| + \lambda \int_{t-h(t)}^t |x_1(s)| ds + \dots + \lambda \int_{t-h(t)}^t |x_n(s)| ds.$$

Then, we see that the functional $V(t, x_t)$ satisfies the following relations:

$$V(t, 0) = 0, \beta_1 \|x\| \leq V(t, x_t), \beta_1 \in (0, 1), \beta_1 \in \mathbb{R}.$$

Thus, it is obvious that the LKF $V(t, x_t)$ is positive definite.

Let

$$\beta_2 \geq 1, \beta_2 \in \mathbb{R}.$$

In the light of Burton ([1], Theorem 4.2.9), we define the functional $Z(t, x)$ as follows:

$$Z(t, x) := \int_{t-h(t)}^t \|x(s)\| ds.$$

From this point of view, we have

$$\beta_1 \|x\| + \lambda Z(t, x) \leq V(t, x_t) \leq \beta_2 \|x\| + \lambda Z(t, x).$$

Next, it follows that

$$\begin{aligned} |V(t, x_t) - V(t, y_t)| &\leq |\|x(t)\| - \|y(t)\|| + \lambda \int_{t-h(t)}^t |\|x(s)\| - \|y(s)\|| ds \\ &\leq \sum_{i=1}^n |x_i(t) - y_i(t)| + \lambda \int_{t-h(t)}^t \|x(s) - y(s)\| ds \\ &\leq \|x(t) - y(t)\| + \lambda h(t) \sup_{t-h(t) \leq s \leq t} \|x(s) - y(s)\| \\ &\leq \|x(t) - y(t)\| + \lambda h_2 \sup_{t-h(t) \leq s \leq t} \|x(s) - y(s)\| \\ &\leq (1 + \lambda h_2) \sup_{t-h(t) \leq s \leq t} \|x(s) - y(s)\| \\ &= K_0 \sup_{t-h(t) \leq s \leq t} \|x(s) - y(s)\|, \end{aligned}$$

where

$$K_0 := 1 + \lambda h_2.$$

Hence, we arrive at the inequality:

$$|V(t, x_t) - V(t, y_t)| \leq K_0 \|x(s) - y(s)\|_{[t-h(t), t]}.$$

This inequality proves that the functional $V(t, x_t)$ satisfies the locally Lipschitz condition in x , i.e., the condition (A1) of Theorem 1 holds.

For the next step, in view of the definition of $Z(t, x)$, it follows that

$$Z(t, x) = \int_{t-h(t)}^t \|x(s)\| ds \leq h(t) \sup_{t-h(t) \leq s \leq t} \|x(s)\| \leq h_2 \sup_{t-h(t) \leq s \leq t} \|x(s)\|.$$

Thus,

$$Z(t, x) \leq h_2 \|x(s)\|_{[t-h(t), t]}.$$

For the next step, via some simple calculations, we get

$$\begin{aligned}
 Z(t_2, x) - Z(t_1, x) &= \int_{t_2-h(t_2)}^{t_2} \|x(s)\| ds - \int_{t_1-h(t_1)}^{t_1} \|x(s)\| ds \\
 &= \int_{t_2-h(t_2)}^{t_2} \|x(s)\| ds - \int_{t_1-h(t_1)}^{t_1} \|x(s)\| ds \\
 &\quad + \int_{t_1-h(t_1)}^{t_2-h(t_2)} \|x(s)\| ds - \int_{t_1-h(t_1)}^{t_2-h(t_2)} \|x(s)\| ds \\
 &= \int_{t_1}^{t_2} \|x(s)\| ds - \int_{t_1-h(t_1)}^{t_2-h(t_2)} \|x(s)\| ds \\
 &\leq \int_{t_1}^{t_2} \|x(s)\| ds \\
 &\leq \sup_{t_1 \leq s \leq t_2} \|x(s)\| (t_2 - t_1) = M(t_2 - t_1),
 \end{aligned}$$

where

$$M = \sup_{t_1 \leq s \leq t_2} \|x(s)\|, 0 < t_1 < t_2 < \infty.$$

This result proves that the condition (A2) of Theorem 1 holds.

The derivative of $V(t, x_t)$ in (6) with respect to the system of DDEs (1) is given by

$$\frac{d}{dt} V(t, x_t) = \sum_{i=1}^n x'_i(t) x_i(t+0) + \lambda \|x(t)\| - \lambda \|x(t-h(t))\| \times (1-h'(t)). \quad (7)$$

Using the condition (C2), we obtain

$$\begin{aligned}
 \sum_{i=1}^n x_i(t+0) x'_i(t) &\leq \sum_{i=1}^n a_{ii} |x_i(t)| + \sum_{i=1}^n \sum_{j=1, j \neq i}^n |a_{ji}| |x_i(t)| \\
 &\quad + \sum_{i=1}^n \sum_{j=1}^n |b_{ij}| |x_j(t-h(t))| \\
 &= \sum_{i=1}^n \left(a_{ii}(t) + \sum_{j=1, j \neq i}^n |a_{ji}(t)| \right) |x_i(t)| + \|B\| \|x(t-h(t))\| \\
 &\leq -a_0 \|x(t)\| + \|B\| \|x(t-h(t))\|.
 \end{aligned} \quad (8)$$

Thereby, gathering the relations (7) and (8) and using the condition $0 \leq h'(t) \leq h_0 < 1$, we find

$$\begin{aligned}
 \frac{d}{dt} V(t, x_t) &\leq -a_0 \|x(t)\| + \|B\| \|x(t-h(t))\| \\
 &\quad + \lambda \|x(t)\| - \lambda \|x(t-h(t))\| \times (1-h'(t)) \\
 &\leq -a_0 \|x(t)\| + \|B\| \|x(t-h(t))\| \\
 &\quad + \lambda \|x(t)\| - \lambda (1-h_0) \|x(t-h(t))\|.
 \end{aligned}$$

Let $\lambda = \frac{\|B\|}{1-h_0}$.

Then, we have

$$\frac{d}{dt}V(t, x_t) \leq -\left[a_0 - \frac{\|B\|}{1-h_0}\right]\|x(t)\| = -\frac{1}{1-h_0}[a_0(1-h_0) - \|B\|]\|x(t)\|.$$

Using the condition (C1), we have

$$\frac{d}{dt}V(t, x_t) \leq -\frac{\alpha}{1-h_0}\|x(t)\| = -K\|x(t)\| \leq 0, \quad (9)$$

where, $K = \frac{\alpha}{1-h_0} > 0$. \square

Thus, we discover that the time derivative of the LKF $V(t, x_t)$ is negative definite. This is a desirable and necessary result for the investigation of uniform asymptotic stability. From the inequality (9), it follows that the condition (A3) of Theorem 1 is satisfied. From the whole discussion, we see that the conditions of (A1)–(A3) of Theorem 1 hold (see Burton ([1], Theorem 4.2.9). Therefore, the zero solution of the nonlinear system of DDEs (4) is uniformly asymptotically stable.

4. Uniformly Asymptotic Stability and Integrability

In the nonlinear system of DDEs (3), we take $E(t, x(t), x(t-h(t))) = 0$. Then, we consider the unperturbed nonlinear system of DDEs (4). We now generalize and optimize the main result of Tian and Ren ([15], Theorem 1) under less restrictive conditions and also give one more result for the non-linear system of DDEs (4). These results are proved by the Lyapunov–Krasovskii functional approach.

Theorem 4. We suppose that the following conditions (H1) and (H2) hold:

(H1) There exist a constant a_0 such that

$$a_{ii}(t) + \sum_{j=1, j \neq i}^n |a_{ji}(t)| \leq -a_0 \text{ for all } t \in \mathbb{R}^+.$$

(H2) There exist constants h_0 and a_0 from (2) and (H1), respectively, and $f_0 > 0, \alpha_1 > 0$, such that

$$F(0) = 0, \|F(u) - F(v)\| \leq f_0\|u - v\| \text{ for all } u, v \in \mathbb{R}^n,$$

and

$$a_0(1-h_0) - f_0\|B\| \geq \alpha_1.$$

Then, the zero solution of the system of DDEs (4) is uniformly asymptotically stable.

Proof. Define a new LKF $V_1 := V_1(t, x_t)$ by

$$V_1(t, x_t) := \|x(t)\| + \mu \int_{t-h(t)}^t \|F(x(s))\| ds, \quad (10)$$

where μ is an arbitrary positive constant, which will be chosen in the proof.

This functional can be expressed as the following:

$$V_1(t, x_t) := |x_1(t)| + \dots + |x_n(t)| + \mu \int_{t-h(t)}^t |f_1(x(s))| ds + \dots + \mu \int_{t-h(t)}^t |f_n(x(s))| ds.$$

Then, we see that the functional $V_1(t, x_t)$ satisfies the following relations:

$$V_1(t, 0) = 0, \beta_1\|x\| \leq V_1(t, x_t), \beta_1 \in (0, 1), \beta_1 \in \mathbb{R}.$$

Let

$$\beta_2 \geq 1, \beta_2 \in \mathbb{R}$$

and define

$$Z_1(t, x) := \int_{t-h(t)}^t \|F(x(s))\| ds.$$

Hence, it follows that

$$\beta_1 \|x\| + \mu Z_1(t, x) \leq V_1(t, x_t) \leq \beta_2 \|x\| + \mu Z_1(t, x).$$

Next, using the condition (H2) and following some simple calculations, we get

$$\begin{aligned} |V_1(t, x_t) - V_1(t, y_t)| &\leq |\|x(t)\| - \|y(t)\|| + \mu \int_{t-h(t)}^t |\|F(x(s))\| - \|F(y(s))\|| ds \\ &\leq \sum_{i=1}^n |x_i(t) - y_i(t)| + \mu \int_{t-h(t)}^t \|F(x(s)) - F(y(s))\| ds \\ &\leq \|x(t) - y(t)\| + \mu f_0 h_2 \sup_{t-h(t) \leq s \leq t} \|x(s) - y(s)\| \\ &\leq (1 + \mu f_0 h_2) \sup_{t-h(t) \leq s \leq t} \|x(s) - y(s)\| \\ &= L_0 \sup_{t-h(t) \leq s \leq t} \|x(s) - y(s)\|, \end{aligned}$$

where

$$L_0 := 1 + \mu f_0 h_2.$$

Hence, we conclude the inequality

$$|V_1(t, x_t) - V_1(t, y_t)| \leq L_0 \|x(s) - y(s)\|_{[t-h(t), t]}.$$

This inequality proves that the functional $V_1(t, x_t)$ satisfies the local Lipschitz condition in x . Thus, the condition (A1) of Theorem 1 holds.

For the next step, from the definition of $Z_1(t, x)$ and the condition (H2), it follows that

$$Z_1(t, x) = \int_{t-h(t)}^t \|F(x(s))\| ds \leq f_0 h(t) \sup_{t-h(t) \leq s \leq t} \|x(s)\| \leq f_0 h_2 \sup_{t-h(t) \leq s \leq t} \|x(s)\|.$$

Thus, it follows that

$$Z_1(t, x) \leq f_0 h_2 \|x(s)\|_{[t-h(t), t]}.$$

As the following step, using some simple calculations and the condition (H2), we have

$$\begin{aligned}
 Z_1(t_2, x) - Z_1(t_1, x) &= \int_{t_2-h(t_2)}^{t_2} \|F(x(s))\| ds - \int_{t_1-h(t_1)}^{t_1} \|F(x(s))\| ds \\
 &= \int_{t_2-h(t_2)}^{t_2} \|F(x(s))\| ds - \int_{t_1-h(t_1)}^{t_1} \|F(x(s))\| ds \\
 &\quad + \int_{t_1-h(t_1)}^{t_2-h(t_2)} \|F(x(s))\| ds - \int_{t_1-h(t_1)}^{t_2-h(t_2)} \|F(x(s))\| ds \\
 &= \int_{t_1}^{t_2} \|F(x(s))\| ds - \int_{t_1-h(t_1)}^{t_2-h(t_2)} \|F(x(s))\| ds \\
 &\leq f_0 \int_{t_1}^{t_2} \|x(s)\| ds \\
 &\leq f_0 \sup_{t_1 \leq s \leq t_2} \|x(s)\| (t_2 - t_1) = M(t_2 - t_1),
 \end{aligned}$$

where

$$M_1 = f_0 \sup_{t_1 \leq s \leq t_2} \|x(s)\|, 0 < t_1 < t_2 < \infty.$$

The last inequality shows that the condition (A2) of Theorem 1 holds.

The derivative of $V_1(t, x_t)$ in (10) along the system of DDEs (4) is given by

$$\begin{aligned}
 \frac{d}{dt} V_1(t, x_t) &= \sum_{i=1}^n x'_i(t) x_i(t+0) + \mu \|F(x(t))\| \\
 &\quad - \mu \|F(x(t-h(t)))\| \times (1-h'(t)).
 \end{aligned} \tag{11}$$

Consider the first term on the right hand side of the equality (11). Using the condition (H1), we obtain

$$\begin{aligned}
 \sum_{i=1}^n x_i(t+0) x'_i(t) &\leq \sum_{i=1}^n \left(a_{ii}(t) + \sum_{j=1, j \neq i}^n |a_{ji}(t)| \right) |x_i(t)| \\
 &\quad + \|B\| \|F(x(t-h(t)))\| \\
 &\leq -a_0 \|x(t)\| + \|B\| \|F(x(t-h(t)))\|.
 \end{aligned} \tag{12}$$

Thereby, gathering the inequalities (11), (12) and using the condition $0 \leq h'(t) \leq h_0 < 1$, we have

$$\begin{aligned}
 \frac{d}{dt} V_1(t, x_t) &\leq -a_0 \|x(t)\| + \|B\| \|F(x(t-h(t)))\| \\
 &\quad + \mu \|F(x(t))\| - \mu \|F(x(t-h(t)))\| \times (1-h'(t)) \\
 &\leq -a_0 \|x(t)\| + \|B\| \|F(x(t-h(t)))\| \\
 &\quad + \mu f_0 \|x(t)\| - \mu (1-h_0) \|F(x(t-h(t)))\|.
 \end{aligned}$$

Choosing μ as $\mu = \frac{\|B\|}{1-h_0}$, we have

$$\frac{d}{dt} V_1(t, x_t) \leq - \left[a_0 - \frac{f_0 \|B\|}{1-h_0} \right] \|x(t)\| = - \frac{1}{1-h_0} [a_0(1-h_0) - f_0 \|B\|] \|x(t)\|.$$

Using the condition (H2), we conclude that

$$\frac{d}{dt} V_1(t, x_t) \leq \frac{-\alpha_1}{1-h_0} \|x(t)\| = -K_1 \|x(t)\| \leq 0, \quad (13)$$

where $K_1 = \frac{\alpha_1}{1-h_0}$. \square

As in the proof of Theorem 3, we find that the time derivative of the LKF $V_1(t, x_t)$ is negative definite. From the inequality (13), it follows that the condition (A3) of Theorem 1 is satisfied. From the whole discussion of this proof, it can be followed that the conditions of (A1)–(A3) of Theorem 1 hold (see Burton ([1], Theorem 4.2.9)). Therefore, the zero solution of the nonlinear system of DDEs (4) is uniformly asymptotically stable.

Theorem 5. *If the conditions (H1) and (H2) of Theorem 4 hold, then the norm of solutions of the system of DDEs (4) is integrable in the sense of Lebesgue on $\mathbb{R}^+ = [0, \infty)$.*

Proof. As in the proof of Theorem 4, the main tool in this proof is the LKF $V_1(t, x_t)$. It is clear that the conditions (H1) and (H2) yield that

$$\frac{d}{dt} V_1(t, x_t) \leq -K_1 \|x(t)\|.$$

This result verifies that the LKF $V_1(t, x_t)$ is decreasing. That is, the LKF $V_1(t, x_t)$ satisfies that

$$V_1(t, x_t) \leq V(t_0, \phi(t_0)) \text{ for all } t \geq t_0.$$

Then, integrating this inequality from t_0 to t , we obtain

$$K_1 \int_{t_0}^t \|x(s)\| ds \leq V_1(t_0, \phi(t_0)) - V_1(t, x_t) \leq V_1(t_0, \phi(t_0)) \equiv \text{a positive constant, say } D_0,$$

for all $t \geq t_0$.

Then, we find that

$$\int_{t_0}^t \|x(s)\| ds \leq K_1^{-1} V(t_0, \phi(t_0)) \equiv K_1^{-1} D_0.$$

If $t \rightarrow +\infty$, then the last inequality clearly implies that

$$\int_{t_0}^{\infty} \|x(s)\| ds \leq K_1^{-1} D_0 < \infty.$$

Therefore, we can conclude that the norm of solutions of the system of DDEs (4) is integrable in the sense of Lebesgue on $\mathbb{R}^+ = [0, \infty)$. Thus, the proof of Theorem 5 is completed. \square

Example 1. Consider the following two dimensional system of non-linear DDEs:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -19 - \frac{1}{1+\exp(t)} & \frac{1}{1+\exp(t)} \\ \frac{1}{1+\exp(t)} & -19 - \frac{1}{1+\exp(t)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \sin x_1(t - \frac{1}{2}|\arctan(t)|) \\ \sin x_2(t - \frac{1}{2}|\arctan(t)|) \end{pmatrix}, \quad (14)$$

where $h(t) = \frac{1}{2}|\arctan(t)|$ is time-varying delay, $t \geq 1$.

Then, comparing both the systems of DDEs (14) and DDEs (4), it follows that

$$A(t) = \begin{pmatrix} -19 - \frac{1}{1+\exp(t)} & \frac{1}{1+\exp(t)} \\ \frac{1}{1+\exp(t)} & -19 - \frac{1}{1+\exp(t)} \end{pmatrix},$$

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

$$F(x(t - \frac{1}{2}|\arctan(t)|)) = \begin{pmatrix} \sin x_1(t - \frac{1}{2}|\arctan(t)|) \\ \sin x_2(t - \frac{1}{2}|\arctan(t)|) \end{pmatrix}, F(0) = 0, x \in \mathbb{R}^2,$$

$$h(t) = \frac{1}{2}|\arctan(t)|.$$

Let

$$u = x(t - \frac{1}{2}|\arctan(t)|), u_1 = x_1(t - \frac{1}{2}|\arctan(t)|), u_2 = x_2(t - \frac{1}{2}|\arctan(t)|)$$

and

$$v = y(t - \frac{1}{2}|\arctan(t)|), v_1 = y_1(t - \frac{1}{2}|\arctan(t)|), v_2 = y_2(t - \frac{1}{2}|\arctan(t)|).$$

In view of the matrix $A(t)$, we have

$$a_{ii}(t) + \sum_{j=1, j \neq i}^n |a_{ji}(t)| = -19 < -18 = -a_0$$

since

$$a_{11}(t) + |a_{21}(t)| = -19 - \frac{1}{1+\exp(t)} + \frac{1}{1+\exp(t)} = -19 < -18 = -a_0,$$

$$a_{22}(t) + |a_{12}(t)| = -\frac{1}{1+\exp(t)} - 19 + \frac{1}{1+\exp(t)} - 19 < -18 = -a_0.$$

Hence, we derive

$$a_{ii}(t) + \sum_{j=1, j \neq i}^2 |a_{ji}(t)| < -a_0 = -18 \text{ for all } t \in \mathbb{R}^+.$$

Next, some simple calculations give

$$\|B\| = 3$$

and

$$\begin{aligned}
 \|F(u) - F(v)\| &= \left\| \begin{pmatrix} \sin u_1 - \sin v_1 \\ \sin u_2 - \sin v_2 \end{pmatrix} \right\| \\
 &= |\sin u_1 - \sin v_1| + |\sin u_2 - \sin v_2| \\
 &= 2 \left| \cos\left(\frac{u_1 + v_1}{2}\right) \sin\left(\frac{u_1 - v_1}{2}\right) \right| \\
 &\quad + 2 \left| \cos\left(\frac{u_2 + v_2}{2}\right) \sin\left(\frac{u_2 - v_2}{2}\right) \right| \\
 &\leq 2 \left| \sin\left(\frac{u_1 - v_1}{2}\right) \right| + 2 \left| \sin\left(\frac{u_2 - v_2}{2}\right) \right| \\
 &\leq 2 \left| \frac{u_1 - v_1}{2} \right| + 2 \left| \frac{u_2 - v_2}{2} \right| \\
 &= \|u - v\|, f_0 = 1. \\
 h(t) &= \frac{1}{2} |\arctan(t)|, 0 < 0.001 = h_1 = \frac{1}{2} |\arctan(t)| \leq \frac{\pi}{4} = h_2, \\
 h_{12} &= h_2 - h_1 = \frac{\pi}{4} - 0.001, \\
 h'(t) &= \frac{1}{2 + 2t^2}, 0 \leq h'(t) \leq \frac{1}{2} = h_0 < 1, \\
 a_0(1 - h_0) - f_0\|B\| &= 18 \left(1 - \frac{1}{2}\right) - 3 = 6 \geq \alpha.
 \end{aligned}$$

It follows that the conditions (C1), (C2) of Theorem 3 and (H1) and (H2) of Theorems 4 and 5 hold. So, the solution $(x_1(t), x_2(t)) = (0, 0)$ of the system of DDEs (14) is uniformly asymptotic stable. Furthermore, the norm of solutions of the system of DDEs (14) is integrable.

In Figures 1 and 2, the two dimensional system of non-linear DDEs (14) was solved by MATLAB software.

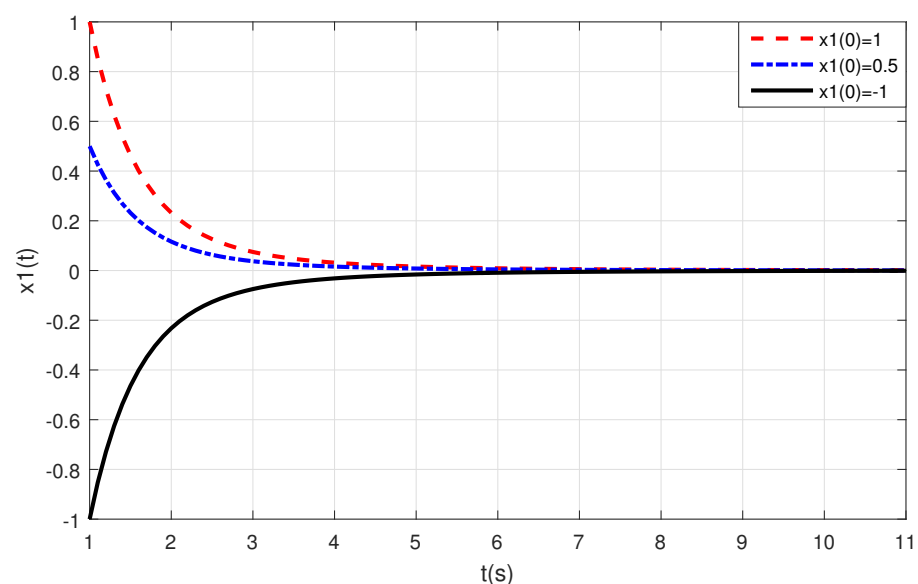


Figure 1. This figure shows that the solution $x_1(t)$ of the system of DDEs (14) is uniformly asymptotically stable and the norm of this solution is integrable for $h(t) = \frac{1}{2} |\arctan(t)|$, $t \geq 1$ and different initial values.

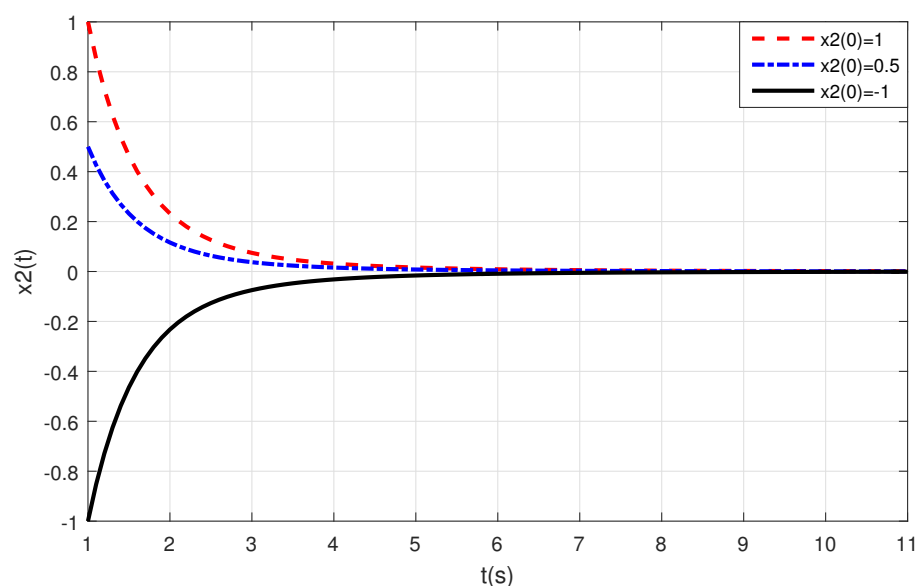


Figure 2. This figure shows that the solution $x_2(t)$ of the system of DDEs (14) is uniformly asymptotically stable and the norm of this solution is integrable for $h(t) = \frac{1}{2}|\arctan(t)|$, $t \geq 1$ and different initial values.

5. Boundedness

For the boundedness of the solutions of the system of nonlinear DDEs (3), we need the following condition in addition to some of those above, (H1):

(H3) There exist positive constants h_0 , a_0 , f_0 from (H2) and a function $e_0 \in C(\mathbb{R}, \mathbb{R})$ such that

$$\begin{aligned} F(0) &= 0, \|F(u) - F(v)\| \leq f_0 \|u - v\| \text{ for all } u, v \in \mathbb{R}^n, \\ \|E(t, x(t), x(t - h(t)))\| &\leq |e_0(t)| \|x(t)\| \text{ for all } t \in \mathbb{R}^+, x, x(t - h(t)) \in \mathbb{R}^n, \\ [a_0(1 - h_0) - f_0 \|B\| - (1 - h_0)|e_0(t)|] &\geq 0. \end{aligned}$$

Theorem 6. If the conditions (H1) and (H3) hold, then the solutions of the system of DDEs (3) are bounded as $t \rightarrow +\infty$.

Proof. As in the proofs of the former theorems, the main tool in this proof is the LKF $V_1(t, x_t)$. From the conditions (H2) and (H3), we can arrive at

$$\begin{aligned} \frac{d}{dt} V_1(t, x_t) &\leq -\frac{1}{1 - h_0} [a_0(1 - h_0) - f_0 \|B\|] \|x(t)\| + \|E(t, x(t), x(t - h(t)))\| \\ &\leq -\frac{1}{1 - h_0} [a_0(1 - h_0) - f_0 \|B\| - (1 - h_0)|e_0(t)|] \|x(t)\|. \end{aligned}$$

Hence, using condition (H3), we derived that

$$\frac{d}{dt} V_1(t, x_t) \leq 0.$$

Integrating this inequality, we obtain

$$V_1(t, x_t) \leq V_1(t_0, \phi(t_0)) \equiv D_0 > 0. \quad (15)$$

Using (15) and the definition the definition of the LKF $V_1(t, x_t)$, we derive that

$$\|x(t)\| \leq \|x(t)\| + \mu \int_{t-h(t)}^t \|F(x(s))\| ds = V_1(t, x_t) \leq V_1(t_0, \phi(t_0)) \equiv D_0 > 0. \quad (16)$$

From the first and last terms of (16), we derive

$$\|x(t)\| \leq D_0.$$

By the calculating the limit as $t \rightarrow +\infty$, it is derived from the last inequality that

$$\lim_{t \rightarrow +\infty} \|x(t)\| \leq \lim_{t \rightarrow +\infty} D_0 = D_0.$$

Then, we conclude that the solutions of the system of nonlinear DDEs (3) are bounded as $t \rightarrow +\infty$. The proof of Theorem 6 is now completed. \square

Example 2. Consider the following perturbed system of DDEs:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -19 - \frac{1}{1+\exp(t)} & \frac{1}{1+\exp(t)} \\ \frac{1}{1+\exp(t)} & -19 - \frac{1}{1+\exp(t)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ + \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \sin x_1(t - \frac{1}{2}|\arctan(t)|) \\ \sin x_2(t - \frac{1}{2}|\arctan(t)|) \end{pmatrix} \\ + \begin{pmatrix} \frac{x_1}{\exp(t) + x_1^2(t - \frac{1}{2}|\arctan(t)|)} \\ \frac{x_2}{\exp(t) + x_2^2(t - \frac{1}{2}|\arctan(t)|)} \end{pmatrix}, \quad (17)$$

where $h(t) = \frac{1}{2}|\arctan t|$ is time-varying delay, $t \geq 1$.

When the non-linear systems of DDEs (17) and DDEs (3) are compared here, we do not need to show the satisfaction of the conditions related to the matrix $A(t)$, the function $F(x(t - \frac{1}{2}|\arctan(t)|))$ and the time-varying delay function $h(t) = \frac{1}{2}|\arctan(t)|$, which have been shown in Example 1. In this case, we first consider the function

$$E(t, x(t), x(t - \frac{1}{2}|\arctan(t)|)) = \begin{pmatrix} \frac{x_1}{\exp(t) + x_1^2(t - \frac{1}{2}|\arctan(t)|)} \\ \frac{x_2}{\exp(t) + x_2^2(t - \frac{1}{2}|\arctan(t)|)} \end{pmatrix}.$$

For the next step, for all $t \geq 1$, we have:

$$\begin{aligned} \|E(t, x(t), x(t - \frac{1}{2}|\arctan(t)|))\| &= \left\| \begin{pmatrix} \frac{x_1}{\exp(t) + x_1^2(t - \frac{1}{2}|\arctan(t)|)} \\ \frac{x_2}{\exp(t) + x_2^2(t - \frac{1}{2}|\arctan(t)|)} \end{pmatrix} \right\| \\ &= \frac{|x_1|}{\exp(t) + x_1^2(t - \frac{1}{2}|\arctan(t)|)} \\ &\quad + \frac{|x_2|}{\exp(t) + x_2^2(t - \frac{1}{2}|\arctan(t)|)} \\ &\leq \frac{1}{\exp(t)} [|x_1| + |x_2|] = |e_0(t)| \|x\|, \end{aligned}$$

where

$$|e_0(t)| = \frac{1}{\exp(t)}, \|x\| = |x_1| + |x_2|.$$

Next, we have

$$[a_0(1 - h_0) - f_0\|B\| - (1 - h_0)|e_0(t)|]$$

$$\begin{aligned}
 &= 18 \left(1 - \frac{1}{2} \right) - 3 - \left(1 - \frac{1}{2} \right) \frac{1}{\exp(t)} \\
 &= 9 - 3 - \frac{1}{2 \exp(t)} \geq \frac{11}{2}.
 \end{aligned}$$

Thus, all the conditions of Theorem 6 hold. In view of the above discussion, we can conclude that all the solutions of the system of DDEs (17) are bounded as $t \rightarrow \infty$.

In Figures 3 and 4, the two dimensional system of non-linear DDEs (17) was solved by MATLAB software.

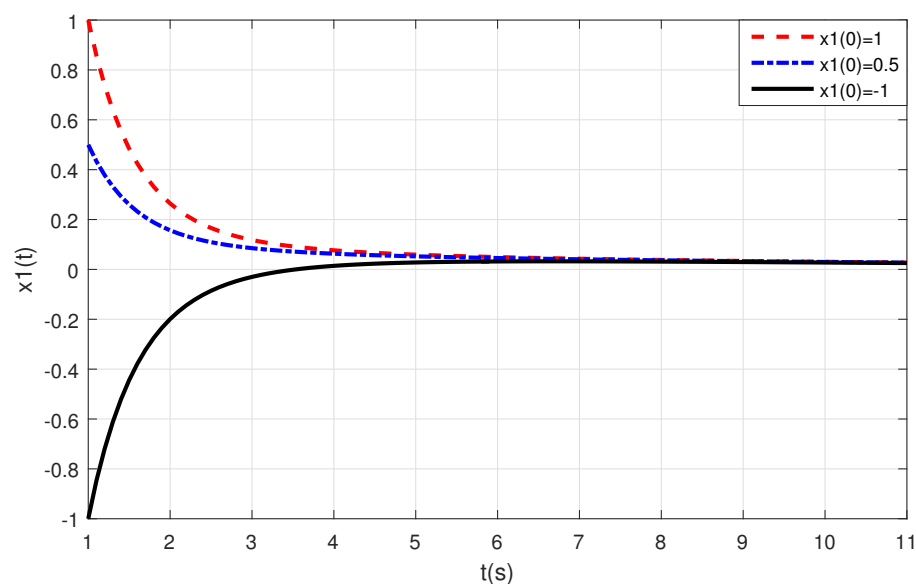


Figure 3. This figure shows that the solution $x_1(t)$ of the system of DDEs (17) is bounded for $h(t) = \frac{1}{2} |\arctan(t)|$, $t \geq 1$ and different initial values.

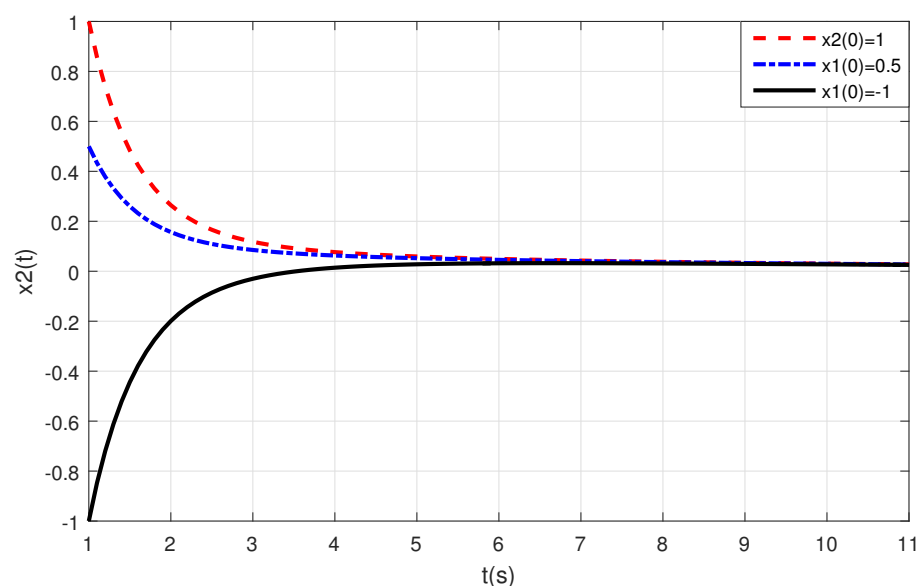


Figure 4. This figure shows that the solution $x_2(t)$ of the system of DDEs (17) is bounded for $h(t) = \frac{1}{2} |\arctan(t)|$, $t \geq 1$ and different initial values.

6. Discussion and Contribution

We now outline the contributions of Theorems 3–6 to the topic of the paper and the available related literature.

- (1) It follows that the systems of DDEs (3) and DDEs (4) extend and improve the system of DDEs (1) of Tian and Ren ([15], Theorem 1) from linear case to the non-linear case.
- (2) Tian and Ren ([15], Theorem 1) defined the following LKF:

$$\begin{aligned}
 V(x_t) = & \eta^T(t)P\eta(t) + \int_{t-h_1}^t x^T(s)Q_1x(s)ds + \int_{t-h_2}^{t-h_1} x^T(s)Q_2x(s)ds \\
 & + h_1 \int_{t-h_1}^t \int_u^t \dot{x}^T(s)Q_3\dot{x}(s)dsdu + h_{12} \int_{t-h_2}^{t-h_1} \int_u^t \dot{x}^T(s)Q_4\dot{x}(s)dsdu, \quad (18)
 \end{aligned}$$

where

$$\begin{aligned}
 \eta(t) &= \begin{bmatrix} x^T(t) & \int_{t-h_1}^t x^T(s)ds & \int_{t-h_2}^{t-h_1} x^T(s)ds & v_1^T(t) & v_2^T(t) & v_3^T(t) \end{bmatrix}^T, \\
 v_1(t) &= \int_{t-h_1}^t \int_u^t x(s)dsdu, \\
 v_2(t) &= \int_{t-h}^t \int_{u_1}^t \int_{u_2}^t y^T(s)dsdu_2du_1, \\
 v_3(t) &= \int_{t-h}^t \int_{u_1}^t \int_{u_2}^t \int_{u_3}^t y^T(s)dsdu_3du_2du_1.
 \end{aligned}$$

This LKF was used as a main tool to prove Theorem 2 in Section 3 by the authors. At the next step, Tian and Ren ([15], Theorem1) calculated the derivative of this LKF along the system of DDEs (1) and obtained the following relations:

$$\begin{aligned}
 \dot{V}(x_t) = & 2\eta^T(t)P\dot{\eta}(t) + x^T(t)Q_1x(t) - x^T(t-h_1)Q_1x(t-h_1) \\
 & + x^T(t-h_1)Q_2x(t-h_1) - x^T(t-h_2)Q_2x(t-h_2) \\
 & + h_1^2\dot{x}^T(t)Q_3\dot{x}(t) + h_{12}^2\dot{x}^T(t)Q_4\dot{x}(t) \\
 & - h_1 \int_{t-h_1}^t \dot{x}^T(s)Q_3\dot{x}(s)ds - h_{12} \int_{t-h_1}^{t-h_2} \dot{x}^T(s)Q_4\dot{x}(s)ds \\
 = & \xi^T(t) \left[He\left(\sum_1^T P \sum_2\right) + \varepsilon_1^T Q_1 \varepsilon_1 - \varepsilon_2^T Q_1 \varepsilon_2 + \varepsilon_2^T Q_2 \varepsilon_2 \right. \\
 & \left. - \varepsilon_4^T Q_2 \varepsilon_4 + h_1^2 \varepsilon_0^T Q_3 \varepsilon_0 + h_{12}^2 \varepsilon_0^T Q_4 \varepsilon_0 \right] \xi(t) \\
 & - h_1 \int_{t-h_1}^t \dot{x}^T(s)Q_3\dot{x}(s)ds - \int_{t-h_2}^{t-h_1} \dot{x}^T(s)Q_4\dot{x}(s)ds, \quad (19)
 \end{aligned}$$

where

$$\begin{aligned}
 \xi(t) &= [x^T(t) \ x^T(t-h_1) \ x^T(t-h(t)) \ x^T(t-h_2) \ \varphi_1^T(t) \ \varphi_2^T(t) \ \varphi_3^T(t) \ \varphi_4^T(t)]^T, \\
 \rho_1(t) &= h(t) - h_1, \\
 \rho_2(t) &= h_2 - h(t), \\
 \varphi_1(t) &= \left[\frac{1}{h_1} \int_{t-h_1}^t x^T(s)ds \ \frac{1}{\rho_1(t)} \int_{t-h(t)}^{t-h_1} x^T(s)ds \ \frac{1}{\rho_2(t)} \int_{t-h_2}^{t-h(t)} x^T(s)ds \right]^T,
 \end{aligned}$$

$$\begin{aligned}\varphi_2(t) &= \left[\frac{1}{h_1^2} \int_{t-h_1}^t \int_u^t x^T(s) ds du \frac{1}{\rho_1^2(t)} \int_{t-h(t)}^{t-h_1} \int_u^{t-h_1} x^T(s) ds du \frac{1}{\rho_2^2(t)} \int_{t-h_2}^{t-h(t)} \int_u^{t-h(t)} x^T(s) ds du \right]^T, \\ \varphi_3(t) &= \left[\frac{1}{h_1^3} \int_{t-h_1}^t \int_u^t \int_v^t x^T(s) ds dv du \frac{1}{\rho_1^3(t)} \int_{t-h(t)}^{t-h_1} \int_u^{t-h_1} \int_v^{t-h_1} x^T(s) ds dv du \right. \\ &\quad \left. \frac{1}{\rho_2^3(t)} \int_{t-h_2}^{t-h(t)} \int_u^{t-h(t)} \int_v^{t-h(t)} x^T(s) ds dv du \right]^T, \\ \varphi_4(t) &= \left[\frac{1}{h_1^4} \int_{t-h_1}^t \int_u^t \int_v^t \int_w^t y^T(s) ds dw dv du \frac{1}{\rho_1^4(t)} \int_{t-h(t)}^{t-h_1} \int_u^{t-h_1} \int_v^{t-h_1} \int_w^{t-h_1} x^T(s) ds dw dv du \right. \\ &\quad \left. \frac{1}{\rho_2^4(t)} \int_{t-h_2}^{t-h(t)} \int_u^{t-h(t)} \int_v^{t-h(t)} \int_w^{t-h(t)} x^T(s) ds dw dv du \right]^T.\end{aligned}$$

Based upon the results of Lemmas 1–4 (see [15]), Tian and Ren (cite15, Theorem 1) proved a result on the asymptotic stability of the linear system of DDEs (1) utilizing the LKF (18) and its time derivative in (19). In fact, the LKF (18) and its time derivative (19) satisfy the conditions of Lyapunov-Krasovskii's asymptotic stability theorem (see [1,2]). From this result, i.e., Theorem 2 of Section 3, a new and interesting delay-dependent stability criterion is derived in terms of the LMIs.

In this paper, we define two more convenient LKFs, the first one is given by (6) such that

$$V(t, x_t) := \|x(t)\| + \lambda \int_{t-h(t)}^t \|x(s)\| ds.$$

In view of the first LKF and its time derivative, applying the famous result of Burton, (ref. [1], Theorem 4.2.9), we improve the result of Tian and Ren ([15], Theorem 1) under weaker conditions. In spite of Tian and Ren ([15], Theorem 1) investigating the asymptotic stability of the linear system of DDEs (1), we discuss the uniformly asymptotically stability of (1), such that the uniformly asymptotic stability implies asymptotic stability, but its converse is not true.

Next, it is worth mentioning that the main result of Tian and Ren ([15], Theorem 1) is very interesting and has a good scientific novelty. However, the weaker conditions of Theorem 3 can be clearly observed and checked if we compare the conditions of Tian and Ren ([15], Theorem 1) with those of Theorem 3, such that taking into account Lemmas 1–4 of Tian and Ren [15], the LKF (18) and its time derivative (19) and our LKF and its time derivative, i.e., (5) and (6), respectively. Here, indeed, when we compare the conditions of Theorem 3 with those of Tian and Ren ([15], Theorem 1), we see that the conditions of Theorem 3 are very convenient and much optimal, easier to verify and apply as seen in Example 1.

As the next step, we define the following LKF:

$$V(t, x_t) := \|x(t)\| + \mu \int_{t-h(t)}^t \|F(x(s))\| ds.$$

Then, we extend and improve the main result of Tian and Ren ([15], Theorem 1) for the uniformly asymptotically stability of the zero solution, the integrability of the norm of the solutions of the system of DDE (4) as well as for the boundedness of solutions of the system of DDEs (3) using this LKF. For the sake of the brevity, we will not give more details about proper discussions of Theorems 4–6. These are the novelty, originality and contributions of this paper. Next, the mentioned observations are desirable facts for proper works to be done in the literature on the topic.

- (3) In this paper, we give two examples. These examples satisfy the conditions of Theorems 3–6 and verify the applications of the results of this paper.
- (4) An advantage of the new LKFs used here is that they eliminate using Gronwall's inequality for the boundedness of solutions at infinity. Compared to related results in the literature, the conditions here are more general, simple, and convenient for application.

7. Conclusions

In this paper, a class of systems of DDEs with time-varying delay is considered. Four new results, which are given by Theorems 3–6, are proved on the uniformly asymptotically stability of zero solution and the integrability of solutions of two non-perturbed systems of DDEs as well as the boundedness of solutions of a perturbed system. The technique used in the proofs of Theorems 3–6 depends upon definitions of two new Lyapunov–Krasovskii functionals. An advantage of the new LKFs used here is that they can lead to more optimal, general and less restrictive results for the given results, and also eliminate the need to use Gronwall's inequality for the boundedness of solutions. Since Gronwall's inequality is not used, the conditions for the boundedness of solutions are also more general, simple, and convenient to apply. Our results improve and extend the result of Tian and Ren ([15], Theorem 1), add three more new results on the qualitative properties of solutions. We give two examples to provide and to illustrate the applications of the new results of this paper.

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