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Gröbner–Shirshov Bases Theory for Trialgebras

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Abstract: We establish a method of Gröbner–Shirshov bases for trialgebras and show that there is a unique reduced Gröbner–Shirshov basis for every ideal of a free trialgebra. As applications, we give a method for the construction of normal forms of elements of an arbitrary trisemigroup, in particular, A.V. Zhuchok's (2019) normal forms of the free commutative trisemigroups are rediscovered and some normal forms of the free abelian trisemigroups are first constructed. Moreover, the Gelfand–Kirillov dimension of finitely generated free commutative trialgebra and free abelian trialgebra are calculated, respectively.

Keywords: Gröbner–Shirshov basis; normal form; Gelfand–Kirillov dimension; trialgebra; trisemigroup

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1. Introduction

The notion of a trialgebra (trioid also known as trisemigroup) was introduced by Loday [1] and investigated in many papers (see, for example, [1–5]). There are several motivations for the study of trialgebras. First, trialgebras and trioids are closely related to Leibniz 3-Algebras [6], and Rota–Baxter operators [7]. Secondly, many results obtained for trialgebras can be applied to trioids. In addition, lastly, if the operation $\perp$ coincides with $\dashv$ or $\vdash$, then we obtain a dialgebra (dimonoid) [8]. If all operations of a trialgebra (trioid) coincide, we obtain an associative algebra (semigroup). Thus, trialgebras are generalizations of dialgebras and associative algebras. The classes of dialgebras and dimonoids were studied by various authors (see, for instance, [9–13]). Loday [1] constructed a free one generator trialgebra and a free trioid of rank 1. A.V. Zhuchok [2,3] constructed the free trioids of an arbitrary rank and the free commutative trioids.

Gröbner or Gröbner–Shirshov bases theory was first introduced by Buchberger [14] for commutative algebras and independently by Shirshov [15] for non-associative algebras [16,17] and Lie algebras [16]. Then, it was developed for various kind of algebras and widely used in different branches of mathematics. Gröbner bases and Gröbner–Shirshov bases theories have become an effective computational tool for solving the following classical problems about: rewriting system; normal form; word problem; growth function; conjugacy problem; embedding theorem; PBW-type theorem; extension, etc. See, for example, the books [18–22] and the survey [23–27].

The key in establishing Gröbner–Shirshov bases theory for certain algebras is to establish the "Composition-Diamond lemma (CD lemma)" for such algebras. The name "CD lemma" combines the Neuman Diamond Lemma [28], the Shirshov Composition Lemma [17], and the Bergman Diamond Lemma [29].

Trialgebras are generalizations of dialgebras and associative algebras, so it is natural to ask what kind of properties of associative algebras and dialgebras remain valid for trialgebras. For instance, CD lemma for dialgebras has been established by Bokut, Chen, and Liu in 2010 [12] and by Zhang and Chen in 2017 [13]. Thus, we shall establish the CD lemma for trialgebras and thus offer a way of constructing normal forms of elements of an arbitrary trisemigroup. Moreover, we prove that every ideal of a free trialgebra has a unique reduced Gröbner–Shirshov basis. The method we used is similar to what was done for dialgebras in [12,13]. However, the extension is not obvious because more operations are involved and the difficulty increases. First, we must ensure that a well ordering on monomials is more or less compatible with trialgebraic operations. Second, a trialgebra has one more operation than dialgebra, so difficulty in the proof of some critical lemmas increases naturally. These reasons make us encounter more difficulties in the process of proving CD lemma for the trialgebra case.

The paper is organized as follows: in Section 2, we first recall the linear basis constructed by Loday and Ronco [1] of the free trialgebra. In Section 3, we elaborate the method of Gröbner–Shirshov bases for trialgebras. We show that, for an arbitrary monomial-centers ordering on the linear basis, there is a unique reduced Gröbner–Shirshov basis for every ideal of free trialgebra. In Section 4, we give a detailed method to construct a set of normal form for an arbitrary trisemigroup; in particular, we give another approach to normal forms of elements of a free commutative trisemigroup that is constructed by [2]. Moreover, we apply the method of Gröbner–Shirshov bases for certain trialgebras and trisemigroups to obtain normal forms and their Gelfand–Kirillov dimensions.

2. Preliminaries

Throughout the paper, we fix a field $\mathbf{k}$. For a nonempty set X, we denote by X^+ the free semigroup generated by X, which consists of all associative words on X. Then, we denote by $X^* = X^+ \cup \{\varepsilon\}$ the free monoid generated by X, where ε is the empty word. For every $u = x_1x_2...x_n \in X^+$, where $x_1, ..., x_n \in X$, we define the *length* $\ell(u)$ of u to be n. For convenience, we define $\ell(\varepsilon) = 0$.

Definition 1 ([1])**.** *An* associative trialgebra *(resp.* trisemigroup*), trialgebra for short, is a* $\mathbf{k}$*-module T (resp. a set T) equipped with three binary associative operations: $\dashv$ called* left*, $\vdash$ called* right*, and $\perp$ called* middle*, satisfying the following eight identities:*

$$\begin{cases} a \dashv (b \vdash c) = a \dashv (b \dashv c), \\ (a \dashv b) \vdash c = (a \vdash b) \vdash c, \\ a \vdash (b \dashv c) = (a \vdash b) \dashv c, \\ a \dashv (b \perp c) = a \dashv (b \dashv c), \\ (a \perp b) \vdash c = (a \vdash b) \vdash c, \\ a \vdash (b \perp c) = (a \vdash b) \perp c, \\ a \perp (b \dashv c) = (a \perp b) \dashv c, \\ a \perp (b \vdash c) = (a \dashv b) \perp c \end{cases} \tag{1}$$

for all $a,\ b,\ c \in T$.

Note that, in [1–3], the authors call trisemigroups trioids, and, in [30], they are called trisemigroups. Here, we follow the terminology of [30].

Definition 2. *For an arbitrary set X, the* triwords *over X are defined inductively as follows:*

(i) For every $x \in X$, the expression (x) is a triword over X of length 1;
(ii) For all triwords (v) and (w) of lengths n and m, respectively, all monomials $((v) \dashv (w))$, $((v) \vdash (w))$ and $((v) \perp (w))$ are triwords over X of length $n + m$.

Recall that, for every trialgebra T, for all $b_1, ..., b_m \in T$, every parenthesizing of

$$(b_1 \vdash \cdots \vdash b_{m_1-1}) \vdash (b_{m_1} \dashv \cdots \dashv b_{m_2-1}) \perp (b_{m_2} \dashv \cdots \dashv b_{m_3-1}) \perp \cdots \perp (b_{m_r} \dashv \cdots \dashv b_m)$$

gives the same element in T [1], and we denote such an element by $[b_1...b_m]_U$, where U is defined to be the set $\{m_i \mid 1 \leq i \leq r\}$. In particular, assume that T is the free trialgebra generated by X. Then, the triword (with an arbitrary bracketing way)

$$(x_{i_1} \vdash \cdots \vdash x_{i_{m_1-1}}) \vdash (x_{i_{m_1}} \dashv \cdots \dashv x_{i_{m_2-1}}) \perp \cdots \perp (x_{i_{m_r}} \dashv \cdots \dashv x_{i_{m_r+t_r}})$$

over X can be determined by the sequence $u := x_{i_1} \dots x_{i_{m_r+t_r}}$ and the set of index

$$U := \{m_i \mid 1 \leq i \leq r\}.$$

Therefore, we call such a triword a *normal triword* over X and denote it by $[u]_U$, and call $x_{m_1}, x_{m_2}, ..., x_{m_r}$ the middle entries of $[u]_U$. In case we would like to emphasize the middle entries, we also denote

$$[u]_U := x_{i_1}...x_{i_{m_1-1}}\dot{x}_{i_{m_1}}...x_{i_{m_2-1}}...\dot{x}_{i_{m_r}}...x_{i_{m_r+t_r}}.$$

We call u the *associative word* of the triword $[u]_U$. Let $\mathcal{P}(\mathbb{N})$ be the power set of the positive integers $\mathbb{N}$. We define

$$[X^+]_{\mathcal{P}(\mathbb{N})} := \{[u]_U \mid u \in X^+,\, \emptyset \neq U \subseteq \{1, ..., \ell(u)\}\}$$

to be the set of all normal triwords on X.

In [1], Loday and Ronco constructed a linear basis for a one-generated free trialgebra, which can be easily generalized for the construction of a linear basis for an arbitrary free trialgebra, see also [3].

Proposition 1 ([1]). *The set $[X^+]_{\mathcal{P}(\mathbb{N})}$ of all normal triwords over X forms a linear basis of the free trialgebra generated by X.*

For every integer $k \in \mathbb{Z}$ and $\emptyset \neq U \in \mathcal{P}(\mathbb{N})$, we define

$$U + k = \{m + k \mid m \in U\}$$

and define $[\varepsilon]_\emptyset = \varepsilon$. For convenience, when we write a set $U = \{m_1, m_2, ..., m_r\} \in \mathcal{P}(\mathbb{N})$, we always assume $m_1 < m_2 < ... < m_r$. Moreover, the cardinality of the set U is denoted by $|U|$, and we simply denote $[u]_{\{m\}}$ by $[u]_m$.

Let $Tri\langle X\rangle$ be the free trialgebra generated by X. Then, by [1], $Tri\langle X\rangle$ is the free $\mathbf{k}$-module with a $\mathbf{k}$-basis $[X^+]_{\mathcal{P}(\mathbb{N})}$ and for all $[u]_U, [v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})}$, we have

$$[u]_U \vdash [v]_V = [uv]_{\ell(u)+V}, \quad [u]_U \dashv [v]_V = [uv]_U, \quad [u]_U \perp [v]_V = [uv]_{U \cup (\ell(u)+V)}.$$

Moreover, with the above products, $([X^+]_{\mathcal{P}(\mathbb{N})}, \dashv, \vdash, \perp)$ forms the free trisemigroup generated by X [3]. Though $[\varepsilon]_\emptyset$ is not an element in $[X^+]_{\mathcal{P}(\mathbb{N})}$, we still extend the operations $\vdash$ and $\dashv$ involving $[\varepsilon]_\emptyset$ to make formulas in the sequel simplified. More precisely, we extend them with the following convention:

$$[\varepsilon]_\emptyset \vdash [u]_U = [u]_U \dashv [\varepsilon]_\emptyset = [u]_U \perp [\varepsilon]_\emptyset = [\varepsilon]_\emptyset \perp [u]_U = [u]_U$$

for every $[u]_U \in [X^+]_{\mathcal{P}(\mathbb{N})}$.

The following lemma shows that every triword can be written as a leftnormed product of triwords.

Lemma 1. *Let $[u]_U = [u_1u_2...u_n]_U \in [X^+]_{\mathcal{P}(\mathbb{N})}$ with $u_1, ..., u_n \in X^+$. Then, there exist some operations $\delta_1, ..., \delta_{n-1} \in \{\dashv, \vdash, \perp\}$ such that*

$$[u]_U = (\dots(([u_1]_{U_1}\delta_1[u_2]_{U_2})\delta_2[u_3]_{U_3})...)\delta_{n-1}[u_n]_{U_n} \qquad \textit{(leftnormed bracketing).}$$

Proof. We use induction on n to prove the claim. For $n = 1$, there is nothing to prove. Assume $n > 1$ and $U = \{m_1, ..., m_r\}$. There are several subcases to consider:

Case 1. If $\ell(u_1...u_{n-1}) \geq m_r$, then we have $[u]_U = [u_1...u_{n-1}]_U \dashv [u_n]_1$. By induction hypothesis, we obtain

$$[u]_U = (...(([u_1]_{U_1}\delta_1[u_2]_{U_2})\delta_2[u_3]_{U_3})...) \dashv [u_n]_1.$$

Case 2. If $\ell(u_1...u_{n-1}) < m_1$, then we have $[u]_U = [u_1...u_{n-1}]_1 \vdash [u_n]_{-\ell(u_1...u_{n-1})+U}$. By induction hypothesis, we obtain

$$[u]_U = (...(([u_1]_{U_1}\delta_1[u_2]_{U_2})\delta_2[u_3]_{U_3})...) \vdash [u_n]_{-\ell(u_1...u_{n-1})+U}.$$

Case 3. If $m_i \leq \ell(u_1...u_{n-1}) < m_{i+1}$ for some $i \in \{1, ..., r-1\}$, then we have

$$[u]_U = [u_1...u_{n-1}]_{\{m_1,...,m_i\}} \perp [u_n]_{-\ell(u_1...u_{n-1})+\{m_{i+1},...,m_r\}}.$$

By induction hypothesis, we obtain

$$[u]_U = (...(([u_1]_{U_1}\delta_1[u_2]_{U_2})\delta_2[u_3]_{U_3})...) \perp [u_n]_{-\ell(u_1...u_{n-1})+\{m_{i+1},...,m_r\}}.$$

The proof is completed. □

3. Composition-Diamond Lemma for Trialgebras

In this section, we establish a method of Gröbner–Shirshov bases for trialgebras. By Proposition 1, $[X^+]_{\mathcal{P}(\mathbb{N})}$ forms a linear basis of the free trialgebra $Tri\langle X\rangle$ generated by X.

We first introduce a good ordering on X^+. Let X be a well-ordered set. We define the *deg-lex ordering* on X^+ as the following: for $u = x_{j_1}x_{j_2}...x_{j_n}$, $v = x_{i_1}x_{i_2}...x_{i_m} \in X^+$, where x_{i_l}, $x_{j_t} \in X$, we define

$$u > v \quad \text{if } (\ell(u), x_{j_1}, x_{j_2}, ..., x_{j_n}) > (\ell(v), x_{i_1}, x_{i_2}, ..., x_{i_m}) \text{ lexicographically.}$$

A well ordering $>$ on X^+ is called *monomial* if, for all $u, v, w \in X^+$, we have

$$u > v \Rightarrow uw > vw \text{ and } u > v \Rightarrow wu > wv.$$

Clearly, the above deg-lex ordering on X^+ is monomial.

We proceed to define a well ordering on $\mathcal{P}(\mathbb{N})\backslash\{\emptyset\}$. For all $U = \{m_1, ..., m_r\}$ and $V = \{n_1, ..., n_t\} \in \mathcal{P}(\mathbb{N})\backslash\{\emptyset\}$, we define

$$U > V \quad \text{if } (r, m_1, ..., m_r) > (t, n_1, ..., n_t) \text{ lexicographically.}$$

Fix a monomial ordering $>$ on X^+. Then, we define an order on $[X^+]_{\mathcal{P}(\mathbb{N})}$ as follows.

Definition 3. *For all $[u]_U$, $[v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})}$,*

$$[u]_U > [v]_V \quad \text{if } (u, U) > (v, V) \text{ lexicographically,} \tag{2}$$

where we compare u and v by the fixed ordering on X^+. This order is called the monomial-centers ordering.

Though we use the same notation $>$ for orderings on X^+, $\mathcal{P}(\mathbb{N})\backslash\{\emptyset\}$ and $[X^+]_{\mathcal{P}(\mathbb{N})}$, no confusion will arise because the monomials under consideration are always clear. It is clear that a monomial-centers ordering is a well ordering on $[X^+]_{\mathcal{P}(\mathbb{N})}$. Finally, if $>$ is the deg-lex ordering on X^+, then we call the ordering defined by (2) the *deg-lex-centers ordering* on $[X^+]_{\mathcal{P}(\mathbb{N})}$.

For all $[u]_U, [v]_V, [u']_{U'}, [v']_{V'} \in [X^+]_{\mathcal{P}(\mathbb{N})}$ and $\delta, \delta' \in \{\dashv, \vdash, \perp\}$, assume $[u]_U\delta[v]_V = [w]_W$ and $[u']_{U'}\delta'[v']_{V'} = [w']_{W'}$. Then, by $[u]_U\delta[v]_V > [u']_{U'}\delta'[v']_{V'}$, we mean $[w]_W > [w']_{W'}$.

From now on, we always assume that $>$ is a monomial-centers ordering $>$ on $[X^+]_{\mathcal{P}(\mathbb{N})}$.

We observe that the monomial-centers ordering $>$ on $[X^+]_{\mathcal{P}(\mathbb{N})}$ is monomial in the following sense:

Lemma 2. *Let $[u]_U$, $[v]_V$ and $[w]_W \in [X^+]_{\mathcal{P}(\mathbb{N})}$ with $[u]_U > [v]_V$. Then, we have*

$$\begin{aligned}
[w]_W \vdash [u]_U &> [w]_W \vdash [v]_V, & [u]_U \dashv [w]_W &> [v]_V \dashv [w]_W,\\
[u]_U \perp [w]_W &> [v]_V \perp [w]_W, & [w]_W \perp [u]_U &> [w]_W \perp [v]_V,\\
[u]_U \vdash [w]_W &\geq [v]_V \vdash [w]_W, & [w]_W \dashv [u]_U &\geq [w]_W \dashv [v]_V.
\end{aligned}$$

Moreover, if $u > v$, then $[u]_U \vdash [w]_W > [v]_V \vdash [w]_W$ and $[w]_W \dashv [u]_U > [w]_W \dashv [v]_V$.

For every polynomial $f = \sum_{i=1}^{n} \alpha_i [u_i]_{U_i} \in Tri\langle X\rangle$, where $0 \neq \alpha_i \in \mathbf{k}$, $[u_i]_{U_i} \in [X^+]_{\mathcal{P}(\mathbb{N})}$ and $[u_1]_{U_1} > [u_2]_{U_2} > ... > [u_n]_{U_n}$, we call $[u_1]_{U_1}$ the *leading monomial* of f, denoted by $\overline{f}$, and we denote by $\widetilde{f}$ the associative word of $\overline{f}$; finally, α_1 the *leading coefficient* of $\overline{f}$, denoted by $lc(f)$; A polynomial f is called *monic* if $lc(f) = 1$, and a nonempty subset S of $Tri\langle X\rangle$ is called *monic* if every element in S is monic. We call a nonzero polynomial $f \in Tri\langle X\rangle$ *strong* if $\widetilde{f} > \widetilde{r_f}$, where $r_f := f - lc(f)\bar{f}$.

For convenience, we define $\bar{0} = \widetilde{0} = 0$, $\widetilde{0} < u$ and $\bar{0} < [u]_U$ for any $[u]_U \in [X^+]_{\mathcal{P}(\mathbb{N})}$.

From Lemma 2, it follows that

Lemma 3. *Let $0 \neq h \in Tri\langle X\rangle$ and $[w]_W \in [X^+]_{\mathcal{P}(\mathbb{N})}$. Then, we have*

$$\begin{aligned}
\overline{([w]_W \vdash h)} &= [w]_W \vdash \bar{h}, & \overline{(h \dashv [w]_W)} &= \bar{h} \dashv [w]_W,\\
\overline{([w]_W \perp h)} &= [w]_W \perp \bar{h}, & \overline{(h \perp [w]_W)} &= \bar{h} \perp [w]_W,\\
\overline{([w]_W \dashv h)} &\leq [w]_W \dashv \bar{h}, & \overline{(h \vdash [w]_W)} &\leq \bar{h} \vdash [w]_W.
\end{aligned}$$

Moreover, if h is strong, then we obtain $\overline{([w]_W \dashv h)} = [w]_W \dashv \bar{h}$ and $\overline{(h \vdash [w]_W)} = \bar{h} \vdash [w]_W$.

Now, we begin to study elements of an ideal generated by a subset of $Tri\langle X\rangle$. We begin with the following notation. For every $[u]_U = [x_{i_1}...x_{i_t}...x_{i_n}]_U \in [X^+]_{\mathcal{P}(\mathbb{N})}$ such that $x_{i_1}, \ldots, x_{i_n}$ lie in X, by Lemma 1, we may assume that $[u]_U = ([v]_V \delta_1 x_{i_t})\delta_2 [w]_W$. Then, for every polynomial $f \in Tri\langle X\rangle$, we define

$$[u]_U|_{x_{i_t} \mapsto f} = ([v]_V \delta_1 f)\delta_2 [w]_W, \tag{3}$$

where, by convention, if exactly one of $[u]_U$ and $[v]_V$ is $[\varepsilon]_\varnothing$, then we define $[u]_U \delta [v]_V = [uv]_{U\cup V}$, in particular, the formula (3) makes sense. Clearly, the resulting polynomial $([v]_V \delta_1 f)\delta_2 [w]_W$ is independent of the choice of $[v]_V, [w]_W$ and δ_1, δ_2. For simplicity, we usually denote by (vfw) a polynomial of the form (3).

Definition 4. *Let S be a monic subset of $Tri\langle X\rangle$. Then, for every $[u]_U = [x_{i_1}...x_{i_t}...x_{i_n}]_U$ in $[X^+]_{\mathcal{P}(\mathbb{N})}$ such that $x_{i_1}, \ldots, x_{i_n}$ lie in X and for every $s \in S$, $[u]_U|_{x_{i_t} \mapsto s}$ is called an* s-polynomial *or* S-polynomial, *and it is called* normal *if either $t \in U$ or s is strong.*

Remark 1. *By Lemma 1 and Definition 4, it follows that*

(i) Every S-polynomial (asb) has an expression:

$$(asb) = ([a]_A \delta_1 s)\delta_2 [b]_B \tag{4}$$

for some $\delta_1, \delta_2 \in \{\dashv, \vdash, \perp\}$ and $a, b \in X^*$. In (4), by convention, we always assume $\delta_1 \in \{\vdash, \perp\}$ (resp. $\delta_2 \in \{\dashv, \perp\}$) in case $[a]_A = [\varepsilon]_\varnothing$ (resp. $[b]_B = [\varepsilon]_\varnothing$). Then, $([a]_A \delta_1 s)\delta_2 [b]_B$ is a normal S-polynomial if and only if one of the following conditions holds:

(a) $\delta_1 \in \{\vdash, \perp\}$ *and* $\delta_2 \in \{\dashv, \perp\}$ *hold;*
(b) *s is strong.*

Moreover, if $([a]_A\delta_1 s)\delta_2[b]_B$ *is normal and* $\overline{([a]_A\delta_1 s)\delta_2[b]_B} = [w]_W$*, then we denote*

$$[asb]_W := ([a]_A\delta_1 s)\delta_2[b]_B.$$

(ii) *If* $(asb) = ([a]_A\delta_1 s)\delta_2[b]_B$ *is a normal S-polynomial, then* $([u]_U\delta_1 s)\delta_2[v]_V$ *is still a normal S-polynomial for all* $[u]_U, [v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})}$.

(iii) *Let* $([a]_A\delta_1 s)\delta_2[b]_B$ *be a normal S-polynomial and assume that s is not strong. Then, both*

$$[a']_{A'}\delta_3(([a]_A\delta_1 s)\delta_2[b]_B) \text{ and } ([a]_A\delta_1 s)\delta_2[b]_B)\delta_4[b']_{B'}$$

are normal S-polynomials if and only if $\delta_3 \in \{\vdash, \perp\}$ *and* $\delta_4 \in \{\dashv, \perp\}$.

The following lemma follows from the definition of normal S-polynomials.

Lemma 4. *Let* $(asb) = ([a]_A\delta_1 s)\delta_2[b]_B$ *be a normal S-polynomial. Assume* $\overline{s} = [u]_U$ *and* $\overline{(asb)} = [w]_W$. *Then, we have*

$$(\ell(a) + U) \subseteq W \subseteq (\{1, ..., \ell(a), \ell(a\widetilde{s}) + 1, ..., \ell(a\widetilde{s}) + \ell(b)\} \cup (\ell(a) + U)),$$

or

$$\emptyset \neq W \subseteq \{1, ..., \ell(a), \ell(a\widetilde{s}) + 1, ..., \ell(a\widetilde{s}) + \ell(b)\}.$$

Moreover, if W is a nonempty subset of $\{1, ..., \ell(a), \ell(a\widetilde{s}) + 1, ..., \ell(a\widetilde{s}) + \ell(b)\}$*, then s is strong. Finally, for every such a set W satisfying the above conditions, there exists a normal S-polynomial* (asb) *such that* $\overline{(asb)} = [w]_W$.

In view of Lemma 4, for every normal S-polynomial $(asb) = ([a]_A\delta_1 s)\delta_2[b]_B$ with $\overline{s} = [u]_U$ and $\overline{(asb)} = [w]_W$, we define $P([asb])$ to be the set of all the possible W for a normal S-polynomial of the form (asb) as in Lemma 4; in other words, we have

$$P([asb]) = \begin{cases} \{(\ell(a) + U) \cup W, W \mid W \subseteq \{1, ..., \ell(a), \ell(a\widetilde{s}) + 1, ..., \ell(a\widetilde{s}) + \ell(b)\}\} \setminus \{\emptyset\}, \\ \text{if } s \text{ is strong;} \\ \\ \{(\ell(a) + U) \cup W \mid W \subseteq \{1, ..., \ell(a), \ell(a\widetilde{s}) + 1, ..., \ell(a\widetilde{s}) + \ell(b)\}\}, \\ \text{if } s \text{ is not strong.} \end{cases}$$

In particular, we have $P(s) = U$.

By Lemma 2, we immediately obtain the following lemma.

Lemma 5. *Let* (asb) *be a normal s-polynomial and* $[u]_U, [v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})}$. *Then,*

$$[u]_U \vdash [asb]_C \dashv [v]_V = [uasbv]_{\ell(u)+C}, \quad [u]_U \vdash [asb]_C \perp [v]_V = [uasbv]_{(\ell(u)+C)\cup\ell(ua\widetilde{s}b)+V)}$$

$$[u]_U \perp [asb]_C \dashv [v]_V = [uasbv]_{U\cup(\ell(u)+C)}, \quad [u]_U \perp [asb]_C \perp [v]_V = [uasbv]_{U\cup(\ell(u)+C)\cup(\ell(ua\widetilde{s}b)+V)}.$$

The following lemma shows that the set

$$Irr(S) := \{[v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})} \mid [v]_V \neq \overline{[csd]_L} \text{ for any normal } S\text{-polynomial } [csd]_L\}$$

is a linear generating set of the quotient trialgebra $Tri\langle X|S\rangle := Tri\langle X\rangle / Id(S)$, where $Id(S)$ is the ideal of $Tri\langle X\rangle$ generated by S.

Lemma 6. *Let S be a monic subset of $Tri\langle X\rangle$. Then, for every nonzero polynomial $h \in Tri\langle X\rangle$, we have*

$$h = \sum \alpha_i [v_i]_{V_i} + \sum \beta_j [a_j s_j b_j]_{C_j},$$

for some $[v_i]_{V_i} \in Irr(S)$, $\alpha_i, \beta_j \in \mathbf{k}$, $a_j, b_j \in X^*$, $s_j \in S$, $[v_i]_{V_i} \leq \overline{h}$ and $\overline{[a_j s_j b_j]_{C_j}} \leq \overline{h}$.

Proof. Let $h = lc(h)\overline{h} + r_h$. If $\overline{h} \in Irr(S)$, then we define $h_1 = h - lc(h)\overline{h}$. If $\overline{h} \notin Irr(S)$, then we obtain $\overline{h} = \overline{[asb]_C}$ for some normal S-polynomial $[asb]_C$. In addition, we define $h_1 = h - lc(h)[asb]_C$. In both cases, we have $\overline{h_1} < \overline{h}$ and the result follows by induction on $\overline{h}$. $\square$

Now, we shall introduce some conditions such that the set $Irr(S)$ is a linear basis of a $Tri\langle X|S\rangle$. Our first step is to introduce the notation of composition.

Definition 5. *Let S be a monic subset of $Tri\langle X\rangle$. For all $g, h \in S$, $g \neq h$, we define* compositions *as follows:*

(i) *If g is not strong, then, for all $x \in X$ and $[u]_{\ell(u)} \in [X^+]_{\mathcal{P}(\mathbb{N})}$, we call $x \dashv g$ a* left multiplication composition *of g and call $g \vdash [u]_{\ell(u)}$ a* right multiplication composition *of g.*

(ii) *Let (chd) be a normal S-polynomial and suppose that $w = \widetilde{g} = c\widetilde{h}d$ for some words $c, d \in X^*$.*

  (a) *If $P(g) \in P([chd])$, then we call*

$$(g,h)_{\overline{g}} = g - [chd]_{P(g)}$$

  an inclusion composition *of S.*

  (b) *If $P(g) \notin P([chd])$ and both g and h are strong, then, for every $x \in X$, we call*

$$(g,h)_{[xw]_1} = [xg]_1 - [xchd]_1$$

  a left multiplicative inclusion composition *of S, and call*

$$(g,h)_{[wx]_{\ell(wx)}} = [gx]_{\ell(wx)} - [chdx]_{\ell(wx)}$$

  a right multiplicative inclusion composition *of S.*

(iii) *Let (ga) be a normal g-polynomial and let (ch) be a normal h-polynomial. Suppose that there exists a word $w = \widetilde{g}a = c\widetilde{h}$ for some words $a, c \in X^*$ such that $|\widetilde{g}| + |\widetilde{h}| > \ell(w)$.*

  (a) *If $P([ga]) \cap P([ch]) \neq \emptyset$, then, for every $W \in P([ga]) \cap P([ch])$, we call*

$$(g,h)_{[w]_W} = [ga]_W - [ch]_W$$

  an intersection composition *of S.*

  (b) *If $P([ga]) \cap P([ch]) = \emptyset$ and both g and h are strong, then, for every $x \in X$, we call*

$$(g,h)_{[xw]_1} = [xga]_1 - [xch]_1$$

  a left multiplicative intersection composition *of S, and call*

$$(g,h)_{[wx]_{\ell(wx)}} = [gax]_{\ell(wx)} - [chx]_{\ell(wx)}$$

  a right multiplicative intersection composition *of S.*

For all $f, f' \in Tri\langle X\rangle$, $[w]_W \in [X^+]_{\mathcal{P}(\mathbb{N})}$, we denote by

$$f \equiv f' \ mod\ (S) \ \ (resp.\ mod\ (S, [w]_W)),$$

if $f - f' = \sum \alpha_i [a_i s_i b_i]_{C_i}$, *where each* $\alpha_i \in \mathbf{k}$, $s_i \in S$, $a_i, b_i \in X^*$ *and* $\overline{[a_i s_i b_i]_{C_i}} \leq \overline{f - f'}$ *(resp.* $\overline{[a_i s_i b_i]_{C_i}} < [w]_W$*). Furthermore, f is called* trivial modulo *S (resp.* $(S, [w]_W)$*), if*

$$f \equiv 0 \ mod\ (S) \quad (resp.\ mod\ (S, [w]_W)).$$

A monic set S is said to be closed under left *(resp.* right*)* multiplication compositions *if every left (resp. right) multiplication composition* $x \dashv g$ *(resp.* $g \vdash [u]_{\ell(u)}$*) of S is trivial modulo S. A monic set S is called a* Gröbner–Shirshov basis *in* $Tri\langle X \rangle$ *if S is closed under left and right multiplication compositions and every composition* $(g, h)_{[u]_U}$ *of S is trivial modulo S.*

We shall prove that, to some extent, the ordering $<$ is compatible with the normal S-polynomials and normal triwords.

Lemma 7. *Let S be a monic subset of* $Tri\langle X \rangle$ *that is closed under left multiplication compositions and assume* $g \in S$*. If g is not strong, then, for every* $[v]_1 \in [X^+]_{\mathcal{P}(\mathbb{N})}$*, we have* $[v]_1 \dashv g \equiv 0 \ mod\ (S)$*.*

Proof. We shall use induction on $(v\widetilde{g}, \ell(v))$ to prove the claim. If $\ell(v) = 1$, then it is clear. Assume $\ell(v) \geq 2$ and $[v]_1 = [ux]_1$, $u \in X^+$, $x \in X$. Then, $[v]_1 \dashv g = [u]_1 \dashv (x \dashv g)$ can be written as a linear combination of S-polynomials of the form $[u]_1 \dashv [csd]_L$, where $s \in S$ and $[c\widetilde{s}d]_L \leq \overline{(x \dashv g)}$. Thus, we obtain

$$\overline{([u]_1 \dashv [csd]_L)} \leq [u]_1 \dashv [c\widetilde{s}d]_L \leq [u]_1 \dashv \overline{(x \dashv g)} = \overline{([v]_1 \dashv g)} \text{ and } c\widetilde{s}d \leq x\widetilde{g}.$$

If s is strong, then $[u]_1 \dashv [csd]_L$ is already a normal S-polynomial, and we are done. Now, we assume that s is not strong. If c is the empty word, then we have

$$[u]_1 \dashv [csd]_L = ([u]_1 \dashv s) \dashv [d]_1$$

and $(u\widetilde{s}, \ell(u)) < (v\widetilde{g}, \ell(v))$. If c is not the empty word, then we have $[csd]_L = [c]_1 \delta [sd]_{-\ell(c)+L}$, where δ lies in $\{\vdash, \perp\}$. Thus, we obtain

$$[u]_1 \dashv [csd]_L = ([uc]_1 \dashv s) \dashv [d]_1$$

and $L > \{1\}$. Since $[c\widetilde{s}d]_L \leq [x\widetilde{g}]_1$, we obtain $c\widetilde{s}d < x\widetilde{g}$ and $(uc\widetilde{s}, \ell(uc)) < (v\widetilde{g}, \ell(v))$. By induction, $[u]_1 \dashv [csd]_L$ is a linear combination of S-polynomials of the form $[as'b]_{L'} \dashv [d]_1$, where $s' \in S$ and $[a\widetilde{s'}b]_{L'} \leq \overline{([uc]_1 \dashv s)}$. By Lemma 5, $[as'b]_{L'} \dashv [d]_1$ is a normal S-polynomial. Thus, we deduce

$$[a\widetilde{s'}b]_{L'} \dashv [d]_1 \leq \overline{([uc]_1 \dashv s)} \dashv [d]_1 = \overline{([u]_1 \dashv [csd]_L)} \leq \overline{([v]_1 \dashv g)}.$$

The proof is completed. □

Let $g \in S$ be a polynomial that is not strong, and assume that $g \vdash x$ is trivial modulo S for every $x \in X$. Then, the following example shows that $g \vdash [u]_{\ell(u)}$ may not be trivial modulo S for some $u \in X^+$.

Example 1 ([13] Example 3.12)**.** *Let* $X = \{x_1, x_2\}, x_1 > x_2$*. Assume that the characteristic of the underlying field* $\mathbf{k}$ *is not 2. Let* $S = \{f, g, h\}$*, where* $f = [x_1x_2]_2 + [x_1x_2]_1$, $g = [x_1x_2x_1]_3 - \frac{1}{2}[x_1x_2x_1]_2 - \frac{1}{2}[x_1x_2x_1]_1$, $h = [x_1x_2x_2]_3 - \frac{1}{2}[x_1x_2x_2]_2 - \frac{1}{2}[x_1x_2x_2]_1$*. These three polynomials are not strong. By a direct calculation, we have* $g \vdash x_i = 0$, $h \vdash x_i = 0$, $i = 1, 2$*, and* $f \vdash x_1 = 2g + f \dashv x_1 \equiv 0 \ mod\ (S)$, $f \vdash x_2 = 2h + f \dashv x_2 \equiv 0 \ mod\ (S)$*. However,* $f \vdash [x_1x_1]_2 = 2[x_1x_2x_1x_1]_4$ *is not trivial modulo S because* $f \vdash [x_1x_1]_2$ *is not normal, and, for every polynomial* $f' \in \{g, h\}$*, we have* $f' \vdash x_1 = 0$*.*

Lemma 8. *Let S be a monic subset of $Tri\langle X\rangle$ that is closed under left and right multiplication compositions. Then, for all normal S-polynomial $[asb]_C$ and normal triword $[u]_U \in [X^+]_{\mathcal{P}(\mathbb{N})}$, we have*

$$[u]_U\delta[asb]_C \equiv 0 \; mod \; (S) \;\; and \;\; [asb]_C\delta[u]_U \equiv 0 \; mod \; (S),$$

where $\delta \in \{\dashv, \vdash, \perp\}$. Moreover, for every normal triword $[w]_W \in [X^+]_{\mathcal{P}(\mathbb{N})}$, if $a\widetilde{s}b < w$, then we have

$$[u]_U\delta[asb]_C \equiv 0 \; mod \; (S, \; [u]_U\delta[w]_W) \;\; and \;\; [asb]_C\delta[u]_U \equiv 0 \; mod \; (S, \; [w]_W\delta[u]_U),$$

where $\delta \in \{\dashv, \vdash, \perp\}$.

Proof. By Lemma 5, it suffices to show that $[u]_U \dashv [asb]_C$ and $[asb]_C \vdash [u]_U$ are trivial modulo S, where s is not strong. Thus, we assume that s is not strong.

We first prove that $[u]_U \dashv [asb]_C$ is trivial modulo S. By Lemma 1, obviously we have

$$\begin{aligned}[u]_U \dashv [asb]_C &= (([u]_U \dashv [a]_A) \dashv s) \dashv [b]_B = (([u_1]_{U_1}\delta_1[u_2]_1) \dashv s)) \dashv [b]_B \\ &= ([u_1]_{U_1}\delta_1([u_2]_1 \dashv s)) \dashv [b]_B,\end{aligned}$$

where $\delta_1 \in \{\vdash, \perp\}$ and $ua = u_1u_2$ with $u_1 \in X^*$ and $u_2 \in X^+$. If $[u_1]_{U_1} = [\varepsilon]_\emptyset$, then we have $\delta_1 = \vdash$ by convention. By Lemmas 7, 5 and 2, the result follows. Moreover, if $a\widetilde{s}b < w$, then we obtain

$$ua\widetilde{s}b < uw \text{ and } \overline{([u]_U\delta[asb]_C)} < [u]_U\delta[w]_W,$$

where $\delta \in \{\dashv, \vdash, \perp\}$. Therefore, we deduce $[u]_U\delta[asb]_C \equiv 0 \; mod \; (S, \; [u]_U\delta[w]_W)$.

The proof for the case of $[asb]_C \vdash [u]_U$ is similar to the above case. More precisely, by Lemma 1, we have

$$[asb]_C \vdash [u]_U = [a]_A \vdash (s \vdash ([b]_B \vdash [u]_U)) = [a]_A \vdash ((s \vdash [u_1]_{\ell(u_1)})\delta_2[u_2]_{U_2}),$$

where $\delta_2 \in \{\dashv, \perp\}$ and $bu = u_1u_2$ with $u_1 \in X^+$, $u_2 \in X^*$. Since S is closed under right multiplication compositions, the results follow by Lemmas 5 and 2. Moreover, if $a\widetilde{s}b < w$, then we have

$$a\widetilde{s}bu < wu \text{ and } \overline{([asb]_C\delta[u]_U)} < [w]_W\delta[u]_U,$$

where $\delta \in \{\dashv, \vdash, \perp\}$. Therefore, we deduce $[asb]_C\delta[u]_U \equiv 0 \; mod \; (S, \; [w]_W\delta[u]_U)$. □

The following corollary is useful in the sequel, which shows that, if we replace certain "subtriword" in a triword with a "small" normal S-polynomial, then we shall obtain a linear combination of "small" normal S-polynomials.

Corollary 1. *Let S be a monic subset of $Tri\langle X\rangle$ that is closed under left and right multiplication compositions. Let $[w]_W$ be a normal triword such that $([a]_A\delta_1[u]_U)\delta_2[b]_B = [w]_W$, and let f be a normal S-polynomial with $\overline{f} < [u]_U$. If $\widetilde{f} < u$, or if $\delta_1 \in \{\vdash, \perp\}$ and $\delta_2 \in \{\dashv, \perp\}$, then we have*

$$([a]_A\delta_1 f)\delta_2[b]_B \equiv 0 \; mod \; (S, \; [w]_W).$$

Proof. If $\delta_1 \in \{\vdash, \perp\}$ and $\delta_2 \in \{\dashv, \perp\}$, then by Lemmas 5 and 2, $([a]_A\delta_1 f)\delta_2[b]_B$ is a normal S-polynomial with

$$\overline{([a]_A\delta_1 f)\delta_2[b]_B} < ([a]_A\delta_1[u]_U)\delta_2[b]_B = [w]_W,$$

the result follows.

Now, we assume $\widetilde{f} < u$. By Lemma 8, $([a]_A\delta_1 f)\delta_2[b]_B$ can be written as a linear combination of S-polynomials of the form $[cs'd]_L\delta_2[b]_B$, where $s' \in S$ and $c\widetilde{s'}d < au$. In addition, for every S-polynomial $[cs'd]_L\delta_2[b]_B$, by Lemma 8 and by the fact that $c\widetilde{s'}db < aub = w$, we have $[cs'd]_L\delta_2[b]_B \equiv 0 \; mod \; (S, \; [w]_W)$. □

Now, we show that, if a monic set S is closed under left and right multiplication compositions, then the elements of the ideal $Id(S)$ of $Tri\langle X\rangle$ can be written as linear combinations of normal S-polynomials.

Corollary 2. *Let S be a monic subset of $Tri\langle X\rangle$ that is closed under left and right multiplication compositions. Then, every S-polynomial (asb) has an expression of the form:*

$$(asb) = \sum \alpha_i [a_i s_i b_i]_{C_i},$$

where each $\alpha_i \in \mathbf{k},\ s_i \in S,\ a_i, b_i \in X^*$.

Proof. Let $[u]_U$ be a triword such that $\overline{s} < [u]_U$. In addition, assume $(asb) = ([a]_A \delta_1 s)\delta_2 [b]_B$ and $[w]_W = ([a]_A \delta_1 [u]_U)\delta_2 [b]_B$. Then, by Corollary 1, we obtain

$$(asb) = ([a]_A \delta_1 s)\delta_2 [b]_B \equiv 0 \ mod\ (S,\ [w]_W).$$

The proof is completed. □

Lemma 9. *Let S be a Gröbner–Shirshov basis in $Tri\langle X\rangle$. Suppose that $[a_1 s_1 b_1]_{C_1}, [a_2 s_2 b_2]_{C_2}$ are two normal S-polynomials with $\overline{[a_1 s_1 b_1]_{C_1}} = \overline{[a_2 s_2 b_2]_{C_2}} = [w]_W$. Then, we have*

$$[a_1 s_1 b_1]_{C_1} - [a_2 s_2 b_2]_{C_2} \equiv 0 \ mod\ (S, [w]_W).$$

Proof. Since $[w]_W = \overline{[a_1 s_1 b_1]_{C_1}} = \overline{[a_2 s_2 b_2]_{C_2}}$, we obtain $w = a_1 \widetilde{s_1} b_1 = a_2 \widetilde{s_2} b_2$ and $W = C_1 = C_2$. We have to consider the following three cases:

Case 1. Without loss of generality, we can assume $b_1 = a\widetilde{s_2} b_2$ and $a_2 = a_1 \widetilde{s_1} a$; here, a may be the empty word. Assume $s_1 = \overline{s_1} + \sum \beta_i [u_i]_{U_i}$ and $s_2 = \overline{s_2} + \sum \beta'_j [v_j]_{V_j}$. Then, by Lemma 1, we have

$$\begin{aligned}
& [a_1 s_1 b_1]_{C_1} - [a_2 s_2 b_2]_{C_2} \\
= \ & [a_1 s_1 a \widetilde{s_2} b_2]_W - [a_1 \widetilde{s_1} a s_2 b_2]_W \\
= \ & ((([a_1]_{A_1} \delta_1 s_1)\delta_2 [a]_A)\delta_3 \overline{s_2})\delta_4 [b_2]_{B_2} - ((([a_1]_{A_1} \delta_1 \overline{s_1})\delta_2 [a]_A)\delta_3 s_2)\delta_4 [b_2]_{B_2} \\
= \ & ((([a_1]_{A_1} \delta_1 s_1)\delta_2 [a]_A)\delta_3 (\overline{s_2} - s_2))\delta_4 [b_2]_{B_2} - ((([a_1]_{A_1} \delta_1 (\overline{s_1} - s_1))\delta_2 [a]_A)\delta_3 s_2)\delta_4 [b_2]_{B_2} \\
= \ & -\sum \beta'_j ((([a_1]_{A_1} \delta_1 s_1)\delta_2 [a]_A)\delta_3 [v_j]_{V_j})\delta_4 [b_2]_{B_2} \\
& + \sum \beta_i ((([a_1]_{A_1} \delta_1 [u_i]_{U_i})\delta_2 [a]_A)\delta_3 s_2)\delta_4 [b_2]_{B_2}
\end{aligned}$$

for some $\delta_1, \delta_2, \delta_3, \delta_4 \in \{\dashv, \vdash, \perp\}$.

If s_1 and s_2 are both strong, then all the resulting polynomials

$$((([a_1]_{A_1} \delta_1 s_1)\delta_2 [a]_A)\delta_3 [v_j]_{V_j})\delta_4 [b_2]_{B_2} \ \text{and}\ ((([a_1]_{A_1} \delta_1 [u_i]_{U_i})\delta_2 [a]_A)\delta_3 s_2)\delta_4 [b_2]_{B_2}$$

are normal S-polynomials; if neither s_1 nor s_2 are strong, then, by Remark 1, we deduce $\delta_3 = \perp$, $\delta_1 \in \{\vdash, \perp\}$ and $\delta_2, \delta_4 \in \{\dashv, \perp\}$, which implies that the above resulting S-polynomials are normal; If only one of s_1 and s_2 is not strong, say, s_1 is not strong, then, by Remark 1, we deduce $\delta_1 \in \{\vdash, \perp\}$ and $\delta_2, \delta_3, \delta_4 \in \{\dashv, \perp\}$. It follows that the resulting S-polynomials are normal. In all subcases, by Lemmas 2 and 3, the leading monomials of the resulting normal S-polynomials are less than $[w]_W$.

Case 2. Without loss of generality, we may assume that $\widetilde{s_1} = a\widetilde{s_2} b$, $a_2 = a_1 a$ and $b_2 = b b_1$. If $P(s_1) \in P([as_2 b])$, then, since S is a Gröbner–Shirshov basis, we may assume

$$s_1 - [as_2 b]_{P(s_1)} = \sum \alpha'_i [c'_i s'_i d'_i]_{L'_i}$$

satisfying $\overline{[c_i's_i'd_i']_{L_i'}} \leq \overline{s_1 - [as_2b]_{P(s_1)}} < \overline{s_1}$ for every i. Thus, we have

$$\begin{aligned} [a_1s_1b_1]_{C_1} - [a_2s_2b_2]_{C_2} &= ([a_1]_{A_1}\delta_1(s_1 - [as_2b]_{P(s_1)}))\delta_2[b_1]_{B_1} \\ &= \sum \alpha_i'([a_1]_{A_1}\delta_1[c_i's_i'd_i']_{L_i'})\delta_2[b_1]_{B_1}. \end{aligned}$$

If one of s_1 and s_2 is not strong, then we deduce $\delta_1 \in \{\vdash, \perp\}$ and $\delta_2 \in \{\dashv, \perp\}$; and, if s_1 and s_2 are strong, then we obtain $c_i'\widetilde{s_i'}d_i' < \widetilde{s_1}$ for all i. In either of these subcases, by Corollary 1, we obtain

$$[a_1s_1b_1]_{C_1} - [a_2s_2b_2]_{C_2} = \sum \alpha_i'([a_1]_{A_1}\delta_1[c_i's_i'd_i']_{L_i'})\delta_2[b_1]_{B_1} \equiv 0 \; mod \; (S, [w]_W).$$

If $P(s_1) \notin P([as_2b])$, then we deduce that s_1, s_2 are strong and $\ell(a_1) + \ell(b_1) \geq 1$. Thus, we have either

$$[a_1s_1b_1]_{C_1} - [a_2s_2b_2]_{C_2} = ([a_1']_{A_1'}\delta_1'([xs_1]_1 - [xas_2b]_1))\delta_2[b_1]_{B_1}$$

or

$$[a_1s_1b_1]_{C_1} - [a_2s_2b_2]_{C_2} = ([a_1]_{\ell(a_1)}\delta_1([s_1y]_{\ell(\widetilde{s_1}y)} - [as_2by]_{\ell(\widetilde{s_1}y)}))\delta_2'[b_1']_{B_1'},$$

where we have $a_1 = a_1'x$ and $b_1 = yb_1'$ for some words $a_1', b_1' \in X^*$ and $x, y \in X$. Then, by the fact that S is a Gröbner–Shirshov basis and by Lemmas 5 and 8, we deduce

$$[a_1s_1b_1]_{C_1} - [a_2s_2b_2]_{C_2} \equiv 0 \; mod \; (S, [w]_W).$$

Case 3. Without loss of generality, we assume $a_2 = a_1a$, $b_1 = bb_2$ and $w' = \widetilde{s_1}b = a\widetilde{s_2}$. If $P([s_1b]) \cap P([as_2]) \neq \emptyset$, then, since S is a Gröbner–Shirshov basis, we may assume

$$[s_1b]_C - [as_2]_C = \sum \alpha_i'[c_i's_i'd_i']_{L_i'}$$

satisfying $\overline{[c_i's_i'd_i']_{L_i'}} \leq \overline{[s_1b]_C - [as_2]_C} < [w']_C$ for every i. Thus, we have

$$\begin{aligned} [a_1s_1b_1]_{C_1} - [a_2s_2b_2]_{C_2} &= ([a_1]_{A_1}\delta_1([s_1b]_C - [as_2]_C))\delta_2[b_2]_{B_2} \\ &= \sum \alpha_i'([a_1]_{A_1}\delta_1[c_i's_i'd_i']_{L_i'})\delta_2[b_2]_{B_2}. \end{aligned}$$

If one of s_1 and s_2 is not strong, then we deduce $\delta_1 \in \{\vdash, \perp\}$ and $\delta_2 \in \{\dashv, \perp\}$; and, if s_1 and s_2 are strong, then we obtain $c_i'\widetilde{s_i'}d_i' < \widetilde{w'}$ for all i. In either of these subcases, by Corollary 1, we obtain

$$[a_1s_1b_1]_{C_1} - [a_2s_2b_2]_{C_2} = \sum \alpha_i'([a_1]_{A_1}\delta_1[c_i's_i'd_i']_{L_i'})\delta_2[b_2]_{B_2} \equiv 0 \; mod \; (S, [w]_W).$$

If $P([s_1b]) \cap P([as_2) = \emptyset$, then we deduce that s_1, s_2 are strong and $\ell(a_1) + \ell(b_1) \geq 1$. Thus, we have either

$$[a_1s_1b_1]_{C_1} - [a_2s_2b_2]_{C_2} = ([a_1']_{A_1'}\delta_1'([xs_1b]_1 - [xas_2]_1))\delta_2[b_2]_{B_2}$$

or

$$([a_1]_{\ell(a_1)}\delta_1([s_1by]_{\ell(w'y)} - [as_2y]_{\ell(w'y)}))\delta_2'[b_2']_{B_2'},$$

where we have $a_1 = a_1'x$, $b_2 = yb_2'$ for some $a_1', b_2' \in X^*$ and $x, y \in X$. Then, by the fact that S is a Gröbner–Shirshov basis and by Lemmas 5 and 8, we deduce

$$[a_1s_1b_1]_{C_1} - [a_2s_2b_2]_{C_2} \equiv 0 \; mod \; (S, [w]_W).$$

The proof is completed. $\square$

Theorem 1. *(Composition-Diamond lemma for trialgebras) Let $>$ be a monomial-center ordering on $[X^+]_{\mathcal{P}(\mathbb{N})}$, and let S be a monic subset of $Tri\langle X\rangle$ and $Id(S)$ the ideal of $Tri\langle X\rangle$ generated by S. Then, the following statements are equivalent.*

(i) S *is a Gröbner–Shirshov basis in* $Tri\langle X\rangle$.

(ii) $0 \neq h \in Id(S) \Rightarrow \overline{h} = \overline{[csd]_L}$ *for some normal* S*-polynomial* $[csd]_L$.

(iii) $Irr(S) = \{[v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})} \mid [v]_V \neq \overline{[csd]_L}$ *for any normal* S*-polynomial* $[csd]_L\}$ *is a* $\mathbf{k}$*-basis of the quotient trialgebra* $Tri\langle X|S\rangle = Tri\langle X\rangle/Id(S)$.

Proof. (i) $\Rightarrow$ (ii) Let $0 \neq h \in Id(S)$. Then, by Corollary 2, we may assume $h = \sum_{i=1}^{n} \alpha_i [c_i s_i d_i]_{L_i}$, where each $\alpha_i \in \mathbf{k}$, $c_i, d_i \in X^*$, $s_i \in S$. Define $[u_i]_{L_i} = \overline{[c_i s_i d_i]_{L_i}} = [c_i \widetilde{s_i} d_i]_{L_i}$, $1 \leq i \leq n$. Then, we may assume without loss of generality that

$$[u_1]_{L_1} = [u_2]_{L_2} = ... = [u_l]_{L_l} > [u_{l+1}]_{L_{l+1}} \geq [u_{l+2}]_{L_{l+2}} \geq ....$$

Now, we use induction on $[u_1]_{L_1}$ to show $\overline{h} = \overline{[csd]_L}$ for some normal S-polynomial $[csd]_L$. For $[u_1]_{L_1} = \overline{h}$, there is nothing to prove. For $[u_1]_{L_1} > \overline{h}$, we have $\sum_{i=1}^{l} \alpha_i = 0$ and

$$\begin{aligned} h &= \sum_{i=1}^{l} \alpha_i [c_i s_i d_i]_{L_i} + \sum_{i=l+1}^{n} \alpha_i [c_i s_i d_i]_{L_i} \\ &= (\sum_{i=1}^{l} \alpha_i)[c_1 s_1 d_1]_{L_1} - \sum_{i=2}^{l} \alpha_i ([c_1 s_1 d_1]_{L_1} - [c_i s_i d_i]_{L_i}) + \sum_{i=l+1}^{n} \alpha_i [c_i s_i d_i]_{L_i} \\ &= 0 + \sum \beta_j [a_j s_j' b_j]_{C_j} + \sum_{i=l+1}^{n} \alpha_i [c_i s_i d_i]_{L_i}, \end{aligned}$$

where each $[a_j s_j' b_j]_{C_j}$ is a normal S-polynomial and $\overline{[a_j s_j' b_j]_{C_j}} < [u_1]_{L_1}$ by Lemma 9. Thus, the result follows by induction hypothesis.

(ii) $\Rightarrow$ (iii) By Lemma 6, the set $Irr(S)$ is a linear generator of the space $Tri\langle X|S\rangle$. Assume that $g = \sum \alpha_i [v_i]_{V_i} = 0$ in $Tri\langle X|S\rangle$, where $\alpha_i \in \mathbf{k}$, $[v_i]_{V_i} \in Irr(S)$ for every i and $[v_1]_{V_1} > [v_2]_{V_2} > \cdots$. This implies that $g \in Id(S)$. Then, $\alpha_i = 0$ for every i. Otherwise, $\overline{g} = [v_j]_{V_j}$ for some j, which is a contradiction.

(iii) $\Rightarrow$ (i) Assume that g is a composition of elements of S. We have $g \in Id(S)$. By Lemma 6, $g = \sum_i \alpha_i [v_i]_{V_i} + \sum_j \beta_j [c_j s_j d_j]_{L_j}$, where each $\alpha_i, \beta_j \in \mathbf{k}$, $c_j, d_j \in X^*$, $[v_i]_{V_i} \in Irr(S)$, $s_j \in S$, and $[v_i]_{V_i} \leq \overline{g}$, $\overline{[c_j s_j d_j]_{L_j}} \leq \overline{g}$. Clearly, $\sum_i \alpha_i [v_i]_{V_i} \in Id(S)$. By (iii), we obtain $\alpha_i = 0$ for every i, and thus we have $g \equiv 0 \ mod\ (S)$. $\square$

Shirshov algorithm If a monic subset $S \subseteq Tri\langle X\rangle$ is not a Gröbner–Shirshov basis, then one can add to S all nontrivial compositions. Continuing this process repeatedly, we finally obtain a Gröbner–Shirshov basis S^{comp} that contains S and generates the same ideal, that is, $Id(S^{comp}) = Id(S)$.

Similarly, we may introduce the Gröbner–Shirshov bases for trirings, which may be useful when one would like to construct an R-basis for some trisemigroup-trirings over an associative and commutative ring R with a unit.

Definition 6. *A* triring *is a quinary* $(E, +, \dashv, \vdash, \perp)$ *such that all of* $(E, +, \vdash)$, $(E, +, \dashv)$ *and* $(E, +, \perp)$ *are associative rings such that the identities in* (1) *hold in* E.

Let $(E, \dashv, \vdash, \perp)$ be a trisemigroup, and T the free left R-module with R-basis E. Then, $(T, +, \dashv, \vdash, \perp)$ is a triring equipped with the following operations:

$$g \vdash h := \sum_{i,j} r_i r_j' (u_i \vdash v_j), \quad f \dashv g := \sum_{i,j} r_i r_j' (u_i \dashv v_j), \quad g \perp h := \sum_{i,j} r_i r_j' (u_i \perp v_j),$$

for all $g = \sum_i r_i u_i,\ h = \sum_j r'_j v_j \in T,\ r_i, r'_j \in R,\ u_i, v_j \in E$. Such a triring, denoted by $Tri_R(E)$, is called a *trisemigroup-triring* of E over R.

Let $Trisgp\langle X\rangle$ be the free trisemigroup generated by X; then, we obtain a trisemigroup-triring of $Trisgp\langle X\rangle$ over R, denoted by $Tri_R\langle X\rangle$, which is also called the *free triring* over R generated by X. In particular, $Tri_{\mathbf{k}}\langle X\rangle = Tri\langle X\rangle$ is the free trialgebra generated by X when $\mathbf{k}$ is a field.

An *ideal* I of $Tri_R\langle X\rangle$ is an R-submodule of $Tri_R\langle X\rangle$ such that $g \vdash h,\ h \vdash g,\ g \dashv h, h \dashv g,\ g \perp h,\ h \perp g \in I$ for every $g \in Tri_R\langle X\rangle$ and $h \in I$.

The proof of the following Theorem 2 is similar to Theorem 1.

Theorem 2. *(Composition-Diamond lemma for trirings) Let R be an associative and commutative ring with a unit. Let $>$ be a monomial-centers ordering on $[X^+]_{\mathcal{P}(\mathbb{N})}$, and let S be a monic subset of $Tri_R\langle X\rangle$ and $Id(S)$ the ideal of $Tri_R\langle X\rangle$ generated by S. Then, the following statements are equivalent.*

(i) *S is a Gröbner–Shirshov basis in $Tri_R\langle X\rangle$.*

(ii) *$0 \neq f \in Id(S) \Rightarrow \overline{f} = \overline{[csd]_L}$ for some normal S-polynomial $[csd]_L$.*

(iii) *$Irr(S) = \{[v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})} \mid [v]_V \neq \overline{[csd]_L}$ for any normal S-polynomial $[csd]_L$ $\}$ is an R-basis of the quotient triring $Tri_R\langle X|S\rangle := Tri_R\langle X\rangle / Id(S)$, i.e., $Tri_R\langle X|S\rangle$ is a free R-module with R-basis $Irr(S)$.*

Remark 2. *The Shirshov algorithm does not work generally in $Tri_R\langle X\rangle$.*

We now turn to the question on how to recognize whether two ideals of $Tri\langle X\rangle$ are the same or not. We begin with the notion of a minimal (resp. reduced) Gröbner–Shirshov basis.

Definition 7. *A Gröbner–Shirshov basis S in $Tri\langle X\rangle$ is* minimal *(resp.* reduced*) if, for every $s \in S$, we have $\bar{s} \in Irr(S\backslash\{s\})$ (resp. $supp(s) \subseteq Irr(S\backslash\{s\})$), where*

$$supp(s) := \{[u_1]_{U_1}, \ldots, [u_n]_{U_n}\}$$

for $s = \alpha_1[u_1]_{U_1} + \cdots + \alpha_n[u_n]_{U_n},\ 0 \neq \alpha_i \in \mathbf{k},\ [u_i]_{U_i} \in [X^+]_{\mathcal{P}(\mathbb{N})}$.

Suppose that I is an ideal of $Tri\langle X\rangle$ and $I = Id(S)$. If S is a reduced (resp. minimal) Gröbner–Shirshov basis in $Tri\langle X\rangle$, then we call S a reduced (resp. minimal) Gröbner–Shirshov basis for the ideal I or for the quotient dialgebra $Tri\langle X\rangle / I$.

It is known that every ideal of associative algebras (dialgebras) has a unique reduced Gröbner–Shirshov basis. Now, we show that an analogous result holds for trialgebras.

Lemma 10. *Let I be an ideal of $Tri\langle X\rangle$ and S a Gröbner–Shirshov basis for I. For every $E \subseteq S$, if $Irr(E) = Irr(S)$, then E is also a Gröbner–Shirshov basis for I.*

Proof. For every $g \in I$, since $Irr(E) = Irr(S)$ and S a Gröbner–Shirshov basis for I, by Theorem 1, we obtain $\overline{g} = \overline{[csd]_L} = \overline{[afb]_L}$ for some $s \in S,\ f \in E,\ a,b,c,d \in X^*$. Thus, we obtain $g_1 = g - lc(g)[afb]_L \in I$ and $\overline{g_1} < \overline{g}$. By induction on $\overline{g}$, we deduce that g is a linear combination of normal E-polynomials, i.e., $g \in Id(E)$. This shows that $I = Id(E)$. Now, the result follows from Theorem 1. □

Let S be a subset of $Tri\langle X\rangle$ and $[u]_U \in [X^+]_{\mathcal{P}(\mathbb{N})}$. We set

$$\overline{S} := \{\bar{s} \in [X^+]_{\mathcal{P}(\mathbb{N})} \mid s \in S\},\ \ S^{[u]_U} := \{s \in S \mid \bar{s} = [u]_U\},\ \ S^{<[u]_U} := \{s \in S \mid \bar{s} < [u]_U\}.$$

Theorem 3. *There is a unique reduced Gröbner–Shirshov basis for every ideal of the free trialgebra $Tri\langle X\rangle$.*

Proof. Let I be a ideal of $Tri\langle X\rangle$. We first prove the existence. It is clear that $S = \{lc(g)^{-1}g \mid 0 \neq g \in I\}$ is a Gröbner–Shirshov basis for I. For each $[u]_U \in \overline{S}$, we fix a polynomial $g^{[u]_U}$ in S such that $\overline{g^{[u]_U}} = [u]_U$. Define

$$S_0 = \{g^{[u]_U} \in S \mid [u]_U \in \overline{S}\}.$$

Then, the leading monomials of elements in S_0 are pairwise different. Since $I \supseteq S \supseteq S_0$ and $\overline{I} = \overline{S} = \overline{S_0}$, we have $Irr(S_0) = Irr(S) = [X^+]_{\mathcal{P}(\mathbb{N})} \backslash \overline{S}$. By Lemma 10, S_0 is a Gröbner–Shirshov basis for I.

Moreover, we may assume that, for every $s \in S_0$, we have

$$supp(s - \overline{s}) \subseteq Irr(S_0), \tag{5}$$

i.e., $supp(s-\overline{s}) \subseteq [X^+]_{\mathcal{P}(\mathbb{N})} \backslash \overline{S_0}$. If $supp(s-\overline{s}) \cap \overline{S_0} \neq \emptyset$ for some $s \in S_0$, then set $[w]_W = \max\{supp(s-\overline{s}) \cap \overline{S_0}\}$. Then, there exists an element $g \in S_0$ such that $\overline{g} = [w]_W$. Note that $\overline{s} > [w]_W = \overline{g}$ and $\overline{s - \alpha g} = \overline{s}$, where α is the coefficient of $[w]_W$ in s. Replace s by $s - \alpha g$ in S_0. Then, $supp(s - \alpha g - \overline{s - \alpha g}) \cap \overline{S_0} = \emptyset$ or $\max\{supp(s - \alpha g - \overline{s-\alpha g}) \cap \overline{S_0}\} < [u]_U$. Since $>$ is a well ordering on $[X^+]_{\mathcal{P}(\mathbb{N})}$, this process will terminate.

Noting that, for every $[u]_U \in \overline{S_0}$, there exists a unique $g \in S_0$ such that $[u]_U = \overline{g}$. Set $\min\{\overline{S_0}\} = \overline{s_0}$ with $s_0 \in S_0$. Define $S_{\overline{s_0}} := \{s_0\}$. Suppose that $g \in S_0$, $\overline{s_0} < \overline{g}$ and $S_{\overline{h}}$ has been defined for every $h \in S_0$ with $\overline{h} < \overline{g}$. Define

$$S_{\overline{g}} := \begin{cases} S_{<\overline{g}} & \text{if } \overline{g} \notin Irr(S_{<\overline{g}}), \\ S_{<\overline{g}} \cup \{g\} & \text{if } \overline{g} \in Irr(S_{<\overline{g}}), \end{cases} \qquad \text{where } S_{<\overline{g}} := \bigcup_{\overline{h}<\overline{g},\, h\in S_0} S_{\overline{h}}.$$

Let

$$S_1 := \bigcup_{g \in S_0} S_{\overline{g}}.$$

Then, for every $g \in S_0$, we have $g \in S_1 \Leftrightarrow \overline{g} \in Irr(S_{<\overline{g}}) \Leftrightarrow g \in S_{\overline{g}}$.

We first claim that $Irr(S_1) = Irr(S_0)$. Since $S_1 \subseteq S_0$, it suffices to show $Irr(S_1) \subseteq Irr(S_0)$. Assume that there exists a normal triword $[u]_U \in [X^+]_{\mathcal{P}(\mathbb{N})}$ such that $[u]_U \in Irr(S_1)$ and $[u]_U \notin Irr(S_0)$. Since $\overline{S_0} = \overline{I}$, it follows that $[u]_U = \overline{g}$ for some $g \in S_0 \backslash S_1$. If $\overline{g} \in Irr(S_{<\overline{g}})$, then $g \in S_{\overline{g}} \subseteq S_1$, a contradiction. If $\overline{g} \notin Irr(S_{<\overline{g}})$, then $\overline{g} = \overline{[asb]_C}$ for some $s \in S_{<\overline{g}} \subseteq S_1, a, b \in X^*$. This implies that $\overline{g} \notin Irr(S_1)$, a contradiction. Therefore, $Irr(S_1) = Irr(S_0)$. By Lemma 10, S_1 is a Gröbner–Shirshov basis for I.

If $g, h \in S_1$, $g \neq h$, $\overline{g} = \overline{[ahb]_C}$, then we have $\overline{h} < \overline{g}$, $h \in S_{\overline{h}} \subseteq S_{<\overline{g}}$. Thus, we deduce $\overline{g} \notin Irr(S_{<\overline{g}})$ and $g \notin S_1$, a contradiction. Thus, S_1 is a minimal Gröbner–Shirshov basis for I. By (5), for every $s \in S_1$, we have $supp(s) \subseteq Irr(S_1 \backslash \{s\})$, so S_1 is a reduced Gröbner–Shirshov basis for I.

Now, we prove the uniqueness. Suppose that T is an arbitrary reduced Gröbner–Shirshov basis for I. Let $\overline{s_0} = \min \overline{S_1}$ and $\overline{t_0} = \min \overline{T}$, where $s_0 \in S_1, t_0 \in T$. By Theorem 1, we have $\overline{s_0} = \overline{[a't'b']_{C'}} \geq \overline{t'} \geq \overline{t_0}$ for some $t' \in T, a', b' \in X^*$. Similarly, $\overline{t_0} \geq \overline{s_0}$. Thus, we deduce $\overline{t_0} = \overline{s_0}$. We claim that $t_0 = s_0$. Otherwise, we have $0 \neq t_0 - s_0 \in I$. By the above argument again, we obtain that $\overline{t_0} > \overline{t_0 - s_0} \geq \overline{t''} \geq \overline{t_0}$ for some $t'' \in T$, a contradiction. Thus, we have

$$S_1^{\overline{s_0}} = \{s_0\} = \{t_0\} = T^{\overline{t_0}}.$$

For every $[u]_U \in \overline{S_1} \cup \overline{T}$ with $[u]_U > \overline{t_0}$, assume that $S_1^{<[u]_U} = T^{<[u]_U}$. To prove $T = S_1$, it suffices to show that $S_1^{[u]_U} \subseteq T^{[u]_U}$. For every $s \in S_1^{[u]_U}$, we have $\overline{s} = \overline{[c'td']_{L'}} \geq \overline{t}$ for some $t \in T, c', d' \in X^*$. Now, we claim that $[u]_U = \overline{s} = \overline{t}$. Otherwise, we have $[u]_U = \overline{s} > \overline{t}$. Then, $t \in T^{<[u]_U} = S_1^{<[u]_U}$ and $t \in S_1 \backslash \{s\}$. However, $\overline{s} = \overline{[c'td']_{L'}}$, which contradicts with the fact that S_1 is a reduced Gröbner–Shirshov basis. Now, we show

$s = t \in T^{[u]_U}$. If $s \neq t$, then $0 \neq s - t \in I$. By Theorem 1, $\overline{s-t} = \overline{[at_1b]_{L_1}} = \overline{[cs_1d]_{L_1}}$ for some $t_1 \in T, s_1 \in S_1, a, b, c, d \in X^*$ with $\overline{t_1}, \overline{s_1} \leq \overline{s-t} < \bar{s} = \bar{t}$. Thus, we deduce $s_1 \in S_1 \backslash \{s\}$ and $t_1 \in T \backslash \{t\}$. Noting that $\overline{s-t} \in supp(s) \cup supp(t)$, we may assume that $\overline{s-t} \in supp(s)$. As S_1 is a reduced Gröbner–Shirshov basis, we have $\overline{s-t} \in Irr(S_1 \backslash \{s\})$, which contradicts with the fact that $\overline{s-t} = \overline{[cs_1d]_{L_1}}$, where $s_1 \in S_1 \backslash \{s\}$. Thus, $s = t$. Therefore, we obtain $S_1^{[u]_U} \subseteq T^{[u]_U}$. It follows that we have $S \subseteq T$. Similarly, we have $T \subseteq S$, which proves the uniqueness. $\square$

Remark 3. *It is known that every Gröbner–Shirshov basis for an ideal of associative (polynomial) algebras can be reduced to a reduced Gröbner–Shirshov basis. However, this is neither the case for dialgebras ([13] Example 3.24), nor the case for trialgebras. It suffices to consider the trialgebra defined by the same generators and relations as those in ([13] Example 3.24) because the relations form a Gröbner–Shirshov basis for the considered trialgebra.*

By using Theorem 3, we have the following theorem.

Theorem 4. *Let I_1, I_2 be two ideals of $Tri\langle X\rangle$. Then, $I_1 = I_2$ if and only if I_1 and I_2 have the same reduced Gröbner–Shirshov basis.*

4. Applications

In this section, we apply Theorem 1 to give a method to find normal forms of elements of an arbitrary trisemigroup. As applications, we reconstruct normal forms of elements of a free commutative trisemigroup which is obtained in [2] and construct normal forms of elements of a free abelian trisemigroup. We also give some characterizations of the Gelfand–Kirillov dimensions of some trialgebras.

Denote by

$$Trisgp\langle X\rangle := ([X^+]_{\mathcal{P}(\mathbb{N})}, \dashv, \vdash, \perp)$$

the free trisemigroup generated by X [1,3]. Clearly, every trisemigroup T is a quotient of some free trisemigroup, say

$$T = Trisgp\langle X|S\rangle := [X^+]_{\mathcal{P}(\mathbb{N})}/\rho(S)$$

for some set X and $S \subseteq [X^+]_{\mathcal{P}(\mathbb{N})} \times [X^+]_{\mathcal{P}(\mathbb{N})}$, where $\rho(S)$ is the congruence on $([X^+]_{\mathcal{P}(\mathbb{N})}, \dashv, \vdash, \perp)$ generated by S. Thus, it is natural to ask the question: how can normal forms of elements of an arbitrary quotient trisemigroup of the form $Trisgp\langle X|S\rangle$ be found?

Let $>$ be a monomial-centers ordering on $[X^+]_{\mathcal{P}(\mathbb{N})}$ and

$$S = \{([v_i]_{V_i}, [w_i]_{W_i}) \mid [v_i]_{V_i} > [w_i]_{W_i},\ i \in I\}.$$

Consider the trialgebra $Tri\langle X|S\rangle$, where we identify the set S with the set $\{[v_i]_{V_i} - [w_i]_{W_i} | i \in I\}$. By the Shirshov algorithm, we have a Gröbner–Shirshov basis S^{comp} in $Tri\langle X\rangle$ and $Id(S^{comp}) = Id(S)$. It is clear that each element in S^{comp} is of the form $[u]_U - [v]_V$, $[u]_U, [v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})}$. Let

$$\begin{aligned} \sigma \quad : \quad & Tri\langle X|S\rangle \to Tri_{\mathbf{k}}([X^+]_{\mathcal{P}(\mathbb{N})}/\rho(S)), \\ & \sum \alpha_i[u_i]_{U_i} + Id(S) \mapsto \sum \alpha_i[u_i]_{U_i}\rho(S), \quad \alpha_i \in \mathbf{k},\ [u_i]_{U_i} \in [X^+]_{\mathcal{P}(\mathbb{N})}. \end{aligned}$$

Then, σ is obviously a trialgebra isomorphism. Noting that, by Theorem 1, $Irr(S^{comp})$ is a linear basis of $Tri\langle X|S\rangle$, we have that $\sigma(Irr(S^{comp}))$ is a linear basis of $Tri_{\mathbf{k}}([X^+]_{\mathcal{P}(\mathbb{N})}/\rho(S))$. It follows that $Irr(S^{comp})$ is exactly a set of normal forms of elements of the trisemigroup $Trisgp\langle X|S\rangle$.

Therefore, we obtain the following theorem.

Theorem 5. *Let $>$ be a monomial-centers ordering on $[X^+]_{\mathcal{P}(\mathbb{N})}$ and $T = Trisgp\langle X|S\rangle$, where $S = \{([v_i]_{V_i}, [w_i]_{W_i}) \mid [v_i]_{V_i} > [w_i]_{W_i},\ i \in I\}$ is a subset of $[X^+]_{\mathcal{P}(\mathbb{N})} \times [X^+]_{\mathcal{P}(\mathbb{N})}$. Then, $Irr(S^{comp})$ is a set of normal forms of elements of the trisemigroup $Trisgp\langle X|S\rangle$.*

If we can construct a set of normal forms of certain trialgebra, then we can know how fast the trialgebra grows by the tool of Gelfand–Kirillov dimension. The Gelfand–Kirillov dimension measures the asymptotic growth rate of algebras. Since it provides important structural information, this invariant has become one of the important tools in the study of algebras. In this section, we shall calculate some interesting examples and show how we can apply Gröbner–Shirshov bases in the calculation of Gelfand–Kirillov dimensions of certain trialgebras.

Let T be a trialgebra, and let $\mathcal{W}, \mathcal{W}_1$ and $\mathcal{W}_2$ be vector subspaces of T. We first define

$$\mathcal{W}_1 \dashv \mathcal{W}_2 = \mathsf{Span}_{\mathbf{k}}\{a \dashv b \mid a \in \mathcal{W}_1, b \in \mathcal{W}_2\},\ \mathcal{W}_1 \vdash \mathcal{W}_2 = \mathsf{Span}_{\mathbf{k}}\{a \vdash b \mid a \in \mathcal{W}_1, b \in \mathcal{W}_2\},$$

$$\text{and } \mathcal{W}_1 \perp \mathcal{W}_2 = \mathsf{Span}_{\mathsf{k}}\{a \perp b \mid a \in \mathcal{W}_1, b \in \mathcal{W}_2\}.$$

Then, we define $\mathcal{W}^1 = \mathcal{W}$ and $\mathcal{W}^n = \sum_{1\le i\le n-1}(\mathcal{W}^i \dashv \mathcal{W}^{n-i} + \mathcal{W}^i \vdash \mathcal{W}^{n-i} + \mathcal{W}^i \perp \mathcal{W}^{n-i})$ for every integer number $n \ge 2$. Finally, we define

$$\mathcal{W}^{\le n} := \mathcal{W}^1 + \mathcal{W}^2 + \ldots + \mathcal{W}^n.$$

Obviously, we have

$$\mathcal{W}^n = \mathsf{Span}_{\mathbf{k}}\{[a_1 \ldots a_n]_U \mid \emptyset \ne U \subseteq \{1, \ldots, n\}, a_1, \ldots, a_n \in \mathcal{W}\}$$

and

$$\mathcal{W}^{\le n} = \mathsf{Span}_{\mathbf{k}}\{[a_1 \ldots a_m]_U \mid \emptyset \ne U \subseteq \{1, \ldots, m\}, m \in \mathbb{N}, m \le n, a_1, \ldots, a_m \in \mathcal{W}\}.$$

Now, we are ready to introduce the Gelfand–Kirillov dimension of a trialgebra.

Definition 8. *Let T be a trialgebra over $\mathbf{k}$. Then, the Gelfand–Kirillov dimension of a trialgebra T is defined to be*

$$\mathsf{GKdim}(T) = \sup_{\mathcal{W}} \overline{\lim_{n\to\infty}} \log_n \mathsf{dim}(\mathcal{W}^{\le n}),$$

where the supremum is taken over all finite dimensional subspaces $\mathcal{W}$ of T.

We have the following obvious observation, which is well-known in the context [31], for example.

Lemma 11. *Let T be a trialgebra generated by a finite set X and $\mathbf{k}X$ the subspace of T spanned by X. Then, we have*

$$\mathsf{GKdim}(T) = \overline{\lim_{n\to\infty}} \log_n \mathsf{dim}((\mathbf{k}X)^{\le n}).$$

Let $X = \{x\}$. It is well known that $\mathsf{GKdim}(\mathbf{k}\langle X\rangle) = 1$ and $\mathsf{GKdim}(Di\langle X\rangle) = 2$, where $\mathbf{k}\langle X\rangle$ (resp. $Di\langle X\rangle$) is the free associative algebra (resp. dialgebra) generated by X. Note that a normal triword of length n in $Tri\langle X\rangle$ is of the form $[x\ldots x]_U$, where U is a nonempty subset of $\{1, \ldots, n\}$. Thus, by a direct calculation, we have $\mathsf{GKdim}(Tri\langle X\rangle) = +\infty$.

We shall show in Sections 4.1 and 4.2 that the Gelfand–Kirillov dimensions of finitely generated free commutative trialgebras and those of finitely generated free abelian trialgebras are positive integers.

From now on, let X be a well-ordered set and $>$ the deg-lex-centers ordering on $[X^+]_{\mathcal{P}(\mathbb{N})}$.

4.1. Normal Forms of Free Commutative Trisemigroups

The commutative trisemigroups are introduced and the free commutative trisemigroup generated by a set is constructed by [2]. In this subsection, we give another approach to normal forms of elements of a free commutative trisemigroup.

Definition 9 ([2])**.** *A trisemigroup (trialgebra)* $(T, \dashv, \vdash, \perp)$ *is* commutative *if* $\dashv, \vdash$ *and* $\perp$ *are commutative.*

Let T_c be the subset of $Tri\langle X\rangle$ consisting of the following polynomials:

$$[u]_U \vdash [v]_V - [v]_V \vdash [u]_U, \quad [u]_U \dashv [v]_V - [v]_V \dashv [u]_U, \quad [u]_U \perp [v]_V - [v]_V \perp [u]_U, \tag{6}$$

where $[u]_U, [v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})}$. Then,

$$Tri[X] := Tri\langle X|T_c\rangle$$

is clearly the free commutative trialgebra generated by X. In particular, a linear basis of $Tri[X]$ consisting of normal triwords over X is exactly a set of normal forms of elements of the free commutative trisemigroup generated by X.

Let $X = \{x_i \mid i \in I\}$ be a well-ordered set. For every $u = x_{i_1}x_{i_2}...x_{i_n} \in X^+,\ x_{i_k} \in X$, we define

$$\lfloor u \rfloor = \lfloor x_{i_1}x_{i_2}...x_{i_n} \rfloor := x_{j_1}x_{j_2}...x_{j_n},$$

where $x_{j_1}x_{j_2}...x_{j_n}$ is a reordering of $x_{i_1}x_{i_2}...x_{i_n}$ satisfying $x_{j_1} \le x_{j_2} \le \cdots \le x_{j_n}$.

We define

$$\lfloor X^+ \rfloor := \{\lfloor u \rfloor \mid u \in X^+\}; \ \ \lfloor u \rfloor_U := [\lfloor u \rfloor]_U,\ \emptyset \ne U \subseteq \{1, ..., \ell(u)\};$$

$$\lfloor X^+ \rfloor_{\mathcal{P}(\mathbb{N})} := \{\lfloor u \rfloor_U \mid u \in X^+,\ \emptyset \ne U \subseteq \{1, ..., \ell(u)\}\}.$$

For $u \in X^+$, $[u]_U$ is a normal triword, while $\lfloor u \rfloor_U$ is called a *commutative normal triword*. For instance, assume $u = x_2x_1x_2x_1x_2x_1 \in X^+$ and assume $x_1 < x_2$, where $x_1, x_2 \in X$. Then, we have $\lfloor u \rfloor = x_1x_1x_1x_2x_2x_2$, $\lfloor u \rfloor_{\{3,5\}} = [x_1x_1x_1x_2x_2x_2]_{\{3,5\}}$.

Proposition 2. *Let* $X = \{x_i \mid i \in I\}$ *be a well-ordered set. Then, we have the following:*

(i) $Tri[X] = Tri\langle X|S_c\rangle$, *where* S_c *consists of the following polynomials:*

$$[u]_U - \lfloor u \rfloor_U \quad ([u]_U \in [X^+]_{\mathcal{P}(\mathbb{N})},\ \ell(u) = 2 \text{ or } |U| = \ell(u) \ge 3),$$

$$[v]_V - \lfloor v \rfloor_1 \quad ([v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})},\ \ell(v) \ge 3 \text{ and } |V| < \ell(v)).$$

(ii) S_c *is a Gröbner–Shirshov basis in* $Tri\langle X\rangle$.

(iii) The set

$$\begin{aligned}\lfloor X^+ \rfloor_c := \{\lfloor v \rfloor_1 \mid \lfloor v \rfloor \in \lfloor X^+ \rfloor\} \ &\cup \ \{\lfloor u \rfloor_2 \mid \lfloor u \rfloor \in \lfloor X^+ \rfloor, \ell(u) = 2\} \\ &\cup \ \{\lfloor u \rfloor_{\{1,2,...,\ell(u)\}} \mid \lfloor u \rfloor \in \lfloor X^+ \rfloor\}.\end{aligned}$$

forms a **k***-basis of the free commutative trialgebra* $Tri[X]$.

Proof. (i) It suffices to show $S_c \subseteq Id(T_c)$ and $T_c \subseteq Id(S_c)$, where T_c consists of the elements described in (6). We first show $S_c \subseteq Id(T_c)$. Since $\dashv$, $\vdash$ and $\perp$ are commutative, we have

$$[x_ix_j]_2 - \lfloor x_ix_j \rfloor_2 \in Id(T_c), \ \ [u]_{\{1,...,\ell(u)\}} - \lfloor u \rfloor_{\{1,...,\ell(u)\}} \in Id(T_c) \text{ and } [v]_1 - \lfloor v \rfloor_1 \in Id(T_c),$$

where $x_i, x_j \in X,\ u, v \in X^+,\ |u|, |v| \geq 2$. It remains to prove that

$$[v]_V - \lfloor v \rfloor_1 \in Id(T_c),\ \text{where } [v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})},\ \ell(v) \geq 3,\ |V| < \ell(v) \text{ and } V \neq \{1\}.$$

There are two cases to consider:

Case 1. If $1 \notin V$, we assume $[v]_V = [v_0]_{\ell(v_0)} \vdash [v_1]_{V_1}$ for some $v_0, v_1 \in X^+$. Then, in $Tri\langle X|T_c\rangle$, we have

$$\begin{aligned} [v]_V - \lfloor v \rfloor_{\ell(v)} &= [v_1]_{V_1} \vdash [v_0]_{\ell(v_0)} - \lfloor v \rfloor_{\ell(v)} \\ &= [v_1 v_0]_{\ell(v)} - \lfloor v \rfloor_{\ell(v)} \\ &= 0. \end{aligned}$$

Assume $\lfloor v \rfloor_{\ell(v)} = (\lfloor v' \rfloor_1 \vdash x) \vdash y$ with $x, y \in X$ and $\lfloor v' \rfloor \in \lfloor X^+ \rfloor$. Then, in $Tri\langle X|T_c\rangle$, we obtain

$$\begin{aligned} \lfloor v \rfloor_{\ell(v)} - \lfloor v \rfloor_1 &= ([v']_{(|v'|)} \vdash x) \vdash y - \lfloor v \rfloor_1 \\ &= (\lfloor v' \rfloor_1 \dashv x) \vdash y - \lfloor v \rfloor_1 \\ &= y \vdash (\lfloor v' \rfloor_1 \dashv x) - \lfloor v \rfloor_1 \\ &= x \dashv (y \vdash \lfloor v' \rfloor_1) - \lfloor v \rfloor_1 \\ &= 0. \end{aligned}$$

It follows that $[v]_V - \lfloor v \rfloor_1 \in Id(T_c)$.

Case 2. If $1 \in V$, then, by Lemma 1, we may assume $[v]_V = ([v_0']_{V_0'} \perp [za]_1)\delta[v_1']_{V_1'}$ with $z \in X$, $a \in X^+$, $v_0', v_1' \in X^*$ and $\delta \in \{\dashv, \perp\}$. Then, in $Tri\langle X|T_c\rangle$, we have

$$\begin{aligned} [v]_V - \lfloor v \rfloor_1 &= [v_0']_{V_0'} \perp ([za]_1 \delta [v_1']_{V_1'}) - \lfloor v \rfloor_1 \\ &= [v_0']_{V_0'} \perp ([v_1']_{V_1'} \delta [za]_1) - \lfloor v \rfloor_1 \\ &= ([v_0']_{V_0'} \perp [v_1']_{V_1'})\delta(z \dashv [a]_1) - \lfloor v \rfloor_1 \\ &= (([v_0']_{V_0'} \perp [v_1']_{V_1'})\delta z) \dashv [a]_1 - \lfloor v \rfloor_1 \\ &= [a v_0' v_1' z]_1 - \lfloor v \rfloor_1 \\ &= 0. \end{aligned}$$

It follows that $[v]_V - \lfloor v \rfloor_1 \in Id(T_c)$.

Now, we show $T_c \subseteq Id(S_c)$. Clearly, we have

$$x \vdash y - y \vdash x \in Id(S_c),\ x \dashv y - y \dashv x \in Id(S_c),\ x \perp y - y \perp x \in Id(S_c),$$

where $x, y \in X$. Suppose that $[u]_U, [v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})}$ with $\ell(uv) > 2$. Then, in $Tri\langle X|S_c\rangle$, we have

$$\begin{aligned} &[u]_U \vdash [v]_V - [v]_V \vdash [u]_U = [uv]_{\ell(u)+V} - [vu]_{\ell(v)+U} = \lfloor uv \rfloor_1 - \lfloor vu \rfloor_1 = 0, \\ &[u]_U \dashv [v]_V - [v]_V \dashv [u]_U = [uv]_U - [vu]_V = \lfloor uv \rfloor_1 - \lfloor vu \rfloor_1 = 0, \end{aligned}$$

and $[u]_U \perp [v]_V - [v]_V \perp [u]_U = [uv]_{U \cup (\ell(u)+V)} - [vu]_{V \cup (\ell(v)+U)}$

$$= \begin{cases} \lfloor uv \rfloor_{\{1,2,\ldots,\ell(uv)\}} - \lfloor vu \rfloor_{\{1,2,\ldots,\ell(uv)\}} = 0, & \text{if } |U| = \ell(u) \text{ and } |V| = \ell(v), \\ \lfloor uv \rfloor_1 - \lfloor vu \rfloor_1 = 0, & \text{otherwise.} \end{cases}$$

This shows that $Id(T_c) = Id(S_c)$ and (i) holds.

(ii) It is easy to check that all possible left (right) multiplication compositions in S_c are equal to zero. For an arbitrary composition $(f,g)_{[w]_W}$ in S_c, we have $-r_f, -r_g \in [X^+]_{\mathcal{P}(\mathbb{N})}$,

$\ell(w) \geq 3$, $[w]_W = \overline{[afb]_W} = \overline{[cgd]_W}$ and $\lfloor w \rfloor = \lfloor a\widetilde{r_f}b \rfloor = \lfloor c\widetilde{r_g}d \rfloor$, where $f = \overline{f} + r_f$, $g = \overline{g} + r_g$, $a, b, c, d \in X^*$. Assume that $[afb]_W = [a\widetilde{f}b]_W - [a\widetilde{r_f}b]_{W_1}$ and $[cgd]_W = [c\widetilde{g}d]_W - [c\widetilde{r_g}d]_{W_2}$. Then, we deduce $|W| = \ell(w)$ if and only if $|W_1| = \ell(w) = |W_2|$. It follows that

$$(f, g)_{[w]_W} = [afb]_W - [cgd]_W = -[a\widetilde{r_f}b]_{W_1} + [c\widetilde{r_g}d]_{W_2}$$

$$\equiv \begin{cases} -\lfloor a\widetilde{r_f}b \rfloor_1 + \lfloor c\widetilde{r_g}d \rfloor_1 \equiv 0 \mod(S_c) & \text{if } |W| < \ell(w), \\ -\lfloor a\widetilde{r_f}b \rfloor_{\{1,\ldots,\ell(w)\}} + \lfloor c\widetilde{r_g}d \rfloor_{\{1,\ldots,\ell(w)\}} \equiv 0 \mod(S_c) & \text{if } |W| = \ell(w). \end{cases}$$

Then, all the compositions in S_c are trivial. Thus, S_c is a Gröbner–Shirshov basis in $Tri\langle X \rangle$.

(iii) The claim follows immediately from Theorem 1. □

From Theorem 1, Lemma 10 and Proposition 2, it follows that

Corollary 3. *Let $X = \{x_i \mid i \in I\}$ be a well-ordered set and $S_c' \subset Tri\langle X \rangle$ be a set consisting of the following polynomials:*

$$\begin{aligned} &[x_j x_k]_2 - [x_k x_j]_2, \ [x_j x_k]_1 - [x_k x_j]_1, \ [x_j x_k]_{\{1,2\}} - [x_k x_j]_{\{1,2\}} \ (j, k \in I, \ j > k); \\ &[x_j x_k x_l]_2 - [x_j x_k x_l]_1, \ [x_j x_k x_l]_3 - [x_j x_k x_l]_1, \ [x_j x_k x_l]_{\{1,2\}} - [x_j x_k x_l]_1, \\ &[x_j x_k x_l]_{\{1,3\}} - [x_j x_k x_l]_1, \ [x_j x_k x_l]_{\{2,3\}} - [x_j x_k x_l]_1 \ (j, k, l \in I, \ j \leq k \leq l). \end{aligned}$$

Then, S_c' is the reduced Gröbner–Shirshov basis for the free commutative trialgebra $Tri[X]$.

Now, by using Theorem 5 and Proposition 2, we have the following corollary.

Corollary 4. [2] *Let $Trisgp[X] := (\lfloor X^+ \rfloor_c, \dashv, \vdash, \perp)$, where $\lfloor X^+ \rfloor_c$ is defined as in Proposition 2. Then, $Trisgp[X]$ is the free commutative trisemigroup generated by X, where the operations $\vdash$, $\dashv$ and $\perp$ are as follows: for any $x, x' \in X$, $\lfloor u \rfloor_U, \lfloor v \rfloor_V \in \lfloor X^+ \rfloor_c$ with $\ell(u)\ell(v) > 1$,*

$$\begin{aligned} &\lfloor v \rfloor_V \vdash \lfloor u \rfloor_U = \lfloor u \rfloor_U \vdash \lfloor v \rfloor_V = \lfloor u \rfloor_U \dashv \lfloor v \rfloor_V = \lfloor v \rfloor_V \dashv \lfloor u \rfloor_U = \lfloor uv \rfloor_1; \\ &\lfloor v \rfloor_V \perp \lfloor u \rfloor_U = \lfloor u \rfloor_U \perp \lfloor v \rfloor_V = \lfloor uv \rfloor_{\{1,2,\ldots,\ell(uv)\}} \quad \textit{if } |U| = \ell(u) \textit{ and } |V| = \ell(v); \\ &\lfloor v \rfloor_V \perp \lfloor u \rfloor_U = \lfloor u \rfloor_U \perp \lfloor v \rfloor_V = \lfloor uv \rfloor_1 \quad \textit{if } |U| < \ell(u) \textit{ or } |V| < \ell(v); \\ &x \dashv x' = x' \dashv x = \lfloor xx' \rfloor_1, \ x \vdash x' = x' \vdash x = \lfloor xx' \rfloor_2, \ x \perp x' = x' \perp x = \lfloor xx' \rfloor_{\{1,2\}}. \end{aligned}$$

By Lemma 11 and Proposition 2, we can easily obtain the Gelfand–Kirillov dimension of $Tri[X]$ for every finite set X.

Corollary 5. *Let $X = \{x_1, \ldots, x_r\}$ and $Tri[X]$ be the free commutative trialgebra generated by X. Then, we have $\mathsf{GKdim}(Tri[X]) = r$.*

4.2. Normal Forms of Free Abelian Trisemigroups

In this subsection, we first introduce a notion of abelian trisemigroups which is an analogy of abelian disemigroups introduced in [11]. Then, we construct a set of normal forms of elements of the free abelian trisemigroups.

Definition 10. *A trisemigroup (trialgebra) $(T, \dashv, \vdash, \perp)$ is* abelian *if $c \vdash d = d \dashv c$ and $c \perp d = d \perp c$ for all $c, d \in T$.*

Let X be an arbitrary set and T'_{ab} the subset of $[X^+]_{\mathcal{P}(\mathbb{N})} \times [X^+]_{\mathcal{P}(\mathbb{N})}$ consisting of the following:

$$([v]_V \vdash [w]_W, [w]_W \dashv [v]_V), \ ([v]_V \perp [w]_W, [w]_W \perp [v]_V),$$

where $[v]_V, [w]_W \in [X^+]_{\mathcal{P}(\mathbb{N})}$. Let T_{ab} be the set consisting of elements of the form

$$[v]_V \vdash [w]_W - [w]_W \dashv [v]_V, \;\; [v]_V \perp [w]_W - [w]_W \perp [v]_V, \tag{7}$$

where $[v]_V, [w]_W \in [X^+]_{\mathcal{P}(\mathbb{N})}$. Then, $Trisgp\langle X|T'_{ab}\rangle$ is clearly the free abelian trisemigroup generated by X, and $Tri\langle X|T_{ab}\rangle$ is the free abelian trialgebra generated by X. By Theorem 5, a linear basis of $Tri\langle X|T_{ab}\rangle$ consisting of normal triwords is a set of normal forms of elements of $Trisgp\langle X|T'_{ab}\rangle$.

Now, we shall try to construct a linear basis of $Tri\langle X|T_{ab}\rangle$ by the method of Gröbner–Shirshov bases. We introduce a method of writing down a new normal triword from a given one. Let $X = \{x_i \mid i \in I\}$ be a well-ordered set, and let $\dot{X} = \{\dot{x} \mid x \in X\}$ be a copy of X, where by $\dot{x}$ we mean a new symbol. We extend the ordering on X to a well-ordering on $X \cup \dot{X}$ in the following way: (i) $\dot{x}_i < x_i$, (ii) $x_i < x_j$ implies $\dot{x}_i < \dot{x}_j$, $\dot{x}_i < x_j$, $x_i < \dot{x}_j$.

We note that $[X^+]_{\mathcal{P}(\mathbb{N})}$ has a one-to-one correspondence with $(X \cup \dot{X})^+$, and we denote this correspondence by φ. More precisely, φ maps an arbitrary normal triword $[x_{i_1}...x_{i_m}]_U$ to a word in $y_1...y_m$ in $(X \cup \dot{X})^+$, such that, if $i_t \in U$, then $y_t = \dot{x}_{i_t}$, and if $i_t \notin U$, then, $y_t = x_{i_t}$ for every $t \leq m$. For instance, $\varphi([x_1x_2x_2x_1x_3]_{\{2,4\}}) = x_1\dot{x}_2x_2\dot{x}_1x_3$. Thus, we can identity elements in $[X^+]_{\mathcal{P}(\mathbb{N})}$ with those in $(X \cup \dot{X})^+$.

Recall that, for every $y_1y_2...y_t \in (X \cup \dot{X})^+$, where each y_i lies in $X \cup \dot{X}$, we have

$$\lfloor y_1y_2...y_t \rfloor = y_{i_1}y_{i_2}...y_{i_t},$$

where $y_{i_1}, y_{i_2}, ..., y_{i_t}$ is a reordering of $y_1, y_2, ..., y_t$ satisfying $y_{i_1} \leq y_{i_2} \leq ... \leq y_{i_t}$. Define

$$\pi: \quad (X \cup \dot{X})^+ \to (X \cup \dot{X})^+, \;\; y_{j_1}y_{j_2}...y_{j_t} \mapsto \lfloor y_{j_1}y_{j_2}...y_{j_t} \rfloor.$$

Finally, define

$$\tau := \varphi^{-1}\pi\varphi: \quad [X^+]_{\mathcal{P}(\mathbb{N})} \to [X^+]_{\mathcal{P}(\mathbb{N})}.$$

For instance,

$$\begin{aligned}\tau([x_1x_2x_2x_1x_3]_{\{2,4\}}) &= \varphi^{-1}\pi\varphi([x_1x_2x_2x_1x_3]_{\{2,4\}}) = \varphi^{-1}\pi(x_1\dot{x}_2x_2\dot{x}_1x_3)\\ &= \varphi^{-1}(\dot{x}_1x_1\dot{x}_2x_2x_3) = [x_1x_1x_2x_2x_3]_{\{1,3\}}.\end{aligned}$$

Roughly speaking, τ reorders the letters in $[u]_U$ such that the middle entries are preserved. Therefore, we immediately deduce that such a map τ satisfies some useful properties, the proof of which is quite easy and thus is omitted.

Lemma 12. *For all $[u]_U$, $[v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})}$, we have $\tau([uv]_{U\cup(\ell(u)+V)}) = \tau([vu]_{V\cup(\ell(v)+U)})$, $\tau([uv]_{\ell(u)+V}) = \tau([vu]_V)$ and $\tau(\tau([u]_U)) = \tau([u]_U)$.*

Proposition 3. *Let $X = \{x_i \mid i \in I\}$ be a well-ordered set, T_{ab} the subset of $Tri\langle X\rangle$ consisting of the elements described in* (7)*. Then, we have*

(i) $Tri\langle X|T_{ab}\rangle = Tri\langle X|S_{ab}\rangle$*, where* $S_{ab} = \{[u]_U - \tau([u]_U) \mid [u]_U \in [X^+]_{\mathcal{P}(\mathbb{N})}, \ell(u) \geq 2\}$*;*
(ii) S_{ab} *is a Gröbner–Shirshov basis in* $Tri\langle X\rangle$*;*
(iii) The set $\{\tau([u]_U) \mid [u]_U \in [X^+]_{\mathcal{P}(\mathbb{N})}\}$ *is a* $\mathbf{k}$*-basis of the free abelian trialgebra* $Tri\langle X|T_{ab}\rangle$*.*

Proof. (i) It suffices to show $T_{ab} \subseteq Id(S_{ab})$ and $S_{ab} \subseteq Id(T_{ab})$. We first show $T_{ab} \subseteq Id(S_{ab})$. In $Tri\langle X|S_{ab}\rangle$, for all $[u]_U$, $[v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})}$, clearly we have

$$\begin{aligned}
[u]_U \vdash [v]_V - [v]_V \dashv [u]_U &= [uv]_{\ell(u)+V} - [vu]_V \\
&= \tau([uv]_{\ell(u)+V}) - \tau([vu]_V), \\
&= 0, \\
[u]_U \perp [v]_V - [v]_V \perp [u]_U &= [uv]_{U\cup(\ell(u)+V)} - [vu]_{V\cup(\ell(v)+U)} \\
&= \tau([uv]_{U\cup(\ell(u)+V)}) - \tau([vu]_{V\cup(\ell(v)+U)}) \\
&= 0.
\end{aligned}$$

Now, we show $S_{ab} \subseteq Id(T_{ab})$. Note that, for an arbitrary normal triword, say $[u]_U = [x_{i_1}...x_{i_n}]_U$ for some letters $x_{i_1},...,x_{i_n} \in X$ such that $n \geq 2$, the normal triword $\tau([u]_U)$ contains the same letters (with repetitions) as those of $[u]_m$; moreover, the middle entries are preserved. Thus, it suffices to show that we can reorder x_{i_t} and $x_{i_{t+1}}$ with middle entries preserved. By Lemma 1, we may assume

$$[u]_U = ([v]_V \delta_1(x_{i_t}\delta_2 x_{i_{t+1}}))\delta_3[w]_W,$$

where, if $[v]_V = [\varepsilon]_\emptyset$, then $\delta_1 \in \{\vdash, \perp\}$, and, if $[w]_W = [\varepsilon]_\emptyset$, then $\delta_3 \in \{\dashv, \perp\}$. Then, by the relations in T_{ab}, we clearly can reorder x_{i_t} and $x_{i_{t+1}}$ with middle entries preserved. It follows that $S_{ab} \subseteq Id(T_{ab})$.

(ii) Clearly all possible left and right multiplication compositions in S_{ab} are equal to zero. Assume for every composition $(f,g)_{[w]_W}$ in S_{ab}, where $f = [u]_U - \tau([u]_U)$ and $g = [v]_V - \tau([v]_V)$. We may assume $[afb]_W = ([a]_A\delta_1 f)\delta_2[b]_B$, $[cgd]_W = ([c]_C\delta_3 g)\delta_4[d]_D$. Then, we have

$$\overline{[afb]_W} = \overline{[cgd]_W} = ([a]_A\delta_1[u]_U)\delta_2[b]_B = ([c]_C\delta_3[v]_V)\delta_4[d]_D.$$

It follows that

$$\tau(([a]_A\delta_1\tau([u]_U))\delta_2[b]_B) = \tau(([c]_C\delta_3\tau([v]_V))\delta_4[d]_D).$$

Thus, we obtain

$$\begin{aligned}
(f,g)_{[w]_W} &= [afb]_W - [cgd]_W = ([a]_A\delta_1\tau([u]_U))\delta_2[b]_B - ([c]_C\delta_3\tau([v]_V))\delta_4[d]_D \\
&\equiv \tau(([a]_A\delta_1\tau([u]_U))\delta_2[b]_B) - \tau(([c]_C\delta_3\tau([v]_V))\delta_4[d]_D) \\
&\equiv 0 \mod(S_{ab}).
\end{aligned}$$

Thus, all the compositions in S_{ab} are trivial, and thus S_{ab} is a Gröbner–Shirshov basis in $Tri\langle X\rangle$.

(iii) By Theorem 1, we get the result. □

From Theorem 1, Lemma 10, and Proposition 3, it follows that

Corollary 6. *Let $X = \{x_i \mid i \in I\}$ be a well-ordered set and $W_{ab} \subset Tri\langle X\rangle$ be a set consisting of the following polynomials:*

$$[x_ix_j]_2 - [x_jx_i]_1,\ [x_ix_j]_1 - [x_jx_i]_2,\ [x_ix_i]_2 - [x_ix_i]_1,\ [x_ix_j]_{\{1,2\}} - [x_jx_i]_{\{1,2\}},\quad (i,j \in I,\ i > j).$$

Then, W_{ab} is the reduced Gröbner–Shirshov basis for the free abelian trialgebra $Tri\langle X|T_{ab}\rangle$.

From Lemma 11 and Proposition 3, it follows that

Corollary 7. *Let $X = \{x_1, ..., x_r\}$ and let $Tri\langle X|T_{ab}\rangle$ be the free abelian trialgebra generated by X. Then, we have*

$$\mathsf{GKdim}(Tri\langle X|T_{ab}\rangle) = 2r.$$

Proof. Let $\dot{X} = \{\dot{x} \mid x \in X\}$ be a copy of X, and let $\mathbf{k}[X \cup \dot{X}]$ be the commutative ploynomial algebra generated by $X \cup \dot{X}$. It is obvious that $\mathbf{k}[X \cup \dot{X}]$ is isomorphism to $Tri\langle X|T_{ab}\rangle$ as a vector space. Thus, we obtain

$$\mathsf{GKdim}(Tri\langle X|T_{ab}\rangle) = \mathsf{GKdim}(\mathbf{k}[X \cup \dot{X}]) = 2r.$$

The proof is completed. □

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