



Article Multiple Novels and Accurate Traveling Wave and Numerical Solutions of the (2+1) Dimensional Fisher-Kolmogorov-Petrovskii-Piskunov Equation

Mostafa M. A. Khater ^{1,2,*,†} and Aliaa Mahfooz Alabdali ^{3,†}

- ¹ Department of Mathematics, Faculty of Science, Jiangsu University, Zhenjiang 212013, China
- ² Department of Mathematics, Obour High Institute for Engineering and Technology, Cairo 11828, Egypt
 ³ Faculty of Computing and information Technology Rabigh, King Abdulaziz University, Rabigh,
- Makkah 21911, Saudi Arabia; amalabdaly@kau.edu.sa * Correspondence: 1000005364@ujs.edu.cn
- These authors did all this work equally.

Abstract: The analytical and numerical solutions of the (2+1) dimensional, Fisher-Kolmogorov-Petrovskii-Piskunov ((2+1) D-Fisher-KPP) model are investigated by employing the modified direct algebraic (MDA), modified Kudryashov (MKud.), and trigonometric-quantic B-spline (TQBS) schemes. This model, which arises in population genetics and nematic liquid crystals, describes the reaction–diffusion system by traveling waves in population genetics and the propagation of domain walls, pattern formation in bi-stable systems, and nematic liquid crystals. Many novel analytical solutions are constructed. These solutions are used to evaluate the requested numerical technique's conditions. The numerical solutions of the considered model are studied, and the absolute value of error between analytical and numerical is calculated to demonstrate the matching between both solutions. Some figures are represented to explain the obtained analytical solutions and the match between analytical and numerical results. The used schemes' performance shows their effectiveness and power and their ability to handle many nonlinear evolution equations.

Keywords: (2+1) D-Fisher-KPP model; computational and approximate solutions

1. Introduction

In the last century, and especially in the biological system, diffusion has been employed as one of the most famous models for spatial spread. It has been used for several services, such as invasion and pattern formation, ecology, motile cell populations, wound healing, the capillary growth network, the spatial movement of cell populations, and so on [1–5]. However, for studying closely packed cells such as epithelium [6], the linear diffusion model is not considered as an excellent idea where it contains a movement cell population; that is why the reaction–diffusion equation is a perfect bi-mathematical model [7]. Thus, the definition of the diffusion term in different cell populations is the ability of various cell populations to mix entirely. However, the fact shows in the different side where the cell will stop moving when it suddenly comes across another section that is well-known by contact inhibition of migration [8]. Thus, the wave propagation is considered as the ideal way of conveying biological information through the cells [9]. The simple model of the diffusion is given by [10–13]

$$\mathcal{C}_t = Y \, \mathcal{C}_{x \, x},\tag{1}$$

where C = C(x, t), Y respectively describe the nutrient or cell concentration and diffusion coefficient, while the exchanging information's time is evaluated by $Q\left(\frac{L^2}{Y}\right)$, where L is the length of the domain.



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Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). Fisher and Kolmogorov give the classical model for propagating an advantageous gene in a one-dimensional habitat [14]. This model is provided by [15,16]

$$C_t = Y C_{xx} + \zeta C (1 - C), \qquad (2)$$

where ζ demonstrates the growth rate of the chemical reaction. This model is known too with the extended Fisher–Kolmogorov equation [17] and the Fisher–KPP equation [18]. Equation (2) takes into account diffusion processes in accordance with Fick's law (term Y C_{xx}), population reproduction at a rate ζ linear term ζC and density-dependent losses (term $-\zeta C^2$). This model has also been used in several fields, such as auto-catalytic chemical reactions, flame propagation, neurophysiology, logistic growth models, and the nuclear reactor theory. Luther was the first researcher who found the wave speed of this model [19]. Many researchers have investigated the traveling wave solutions of this model, and they have also improved it into (2+1)-dimensional form, that is given by [20–23]

$$\mathcal{F}_t + \mu \left(\mathcal{F}_{xx} + 2 \mathcal{F}_{xy} + \mathcal{F}_{yy} \right) - \left(\mathcal{F}_x + \mathcal{F}_y \right) + \Im(\mathcal{F}) = 0, \tag{3}$$

where μ is a positive diffusion coefficient, while $\mathcal{F} = \mathcal{F}(x, y, t)$ is a function of the chemical concentration. Handling Equation (3) through the following wave transformation $\mathcal{F}(x, y, t) = \mathcal{K}(\mathfrak{S}), \mathfrak{S} = x + y + r t$, where *r* is an arbitrary constant to be evaluated later, converts Equation (3) into the following nonlinear ordinary differential equation

$$(r-2)\mathcal{K}' + \mathcal{K}^3 + 4\,\mu\,\mathcal{K}'' - \mathcal{K} = 0,\tag{4}$$

where $(\Im(\mathcal{F}) = \mathcal{F}^3 - \mathcal{F})$ [17]. In this context, investigating this model is considered a primary icon in various fields [24–26]. Based on the computer revolution that has a significant effect on deriving computational, semi-analytical, and numerical schemes, many schemes have derived, such as the sech-tanh expansion method, auxiliary equation method, direct algebraic equation method, iteration method, exponential expansion method, B-spline schemes, Kudryashov methods, Adomian decomposition method, Khater methods, $\left(\frac{\Theta'}{\Theta}\right)$ -expansion methods, and so on [27–33]. This paper employs three recent analytical and numerical techniques [34–38] to investigate novel analytical wave solutions of Equation (3)'s accuracy of the used analytical schemes. colorred Our results in the case n = 1 are proved using the homogeneous balance principles and the well-known frameworks of the MDA and MKud. methods. Thus, the general solutions of Equation (4) are evaluated by [39,40]

$$\mathcal{K}(\mathfrak{S}) = \begin{cases} \sum_{i=-n}^{n} a_i \,\mathfrak{Z}(\mathfrak{S})^i = a_1 \,\mathfrak{Z}(\mathfrak{S}) + \frac{a_{-1}}{\mathfrak{Z}(\mathfrak{S})} + a_0, \\ \\ \sum_{i=0}^{n} a_i \,\mathfrak{G}(\mathfrak{S})^i = a_1 \,\mathfrak{G}(\mathfrak{S}) + a_0, \end{cases}$$
(5)

where a_0 , a_1 , a_{-1} are arbitrary constants.

The other sections in this paper are given in the following order; Section 2 shows the novel and accurate solutions of the considered model through the above-mentioned analytical and numerical schemes. Section 3 demonstrates the novelty and originality of our paper. Section 4 gives a summary of the manuscript.

2. Analytical and Numerical Matching for the (2+1) D-Fisher-KPP Model

Here, the matching between the analytical and numerical solutions is discussed through implementing two recent analytical (MDA and MKud.) and one numerical (TQBS) scheme to Equation (4). This study also shows the accuracy of the obtained computational solutions and strategies used.

2.1. MDA Analytical Versus TQBS Numerical Techniques along (2+1) D-Fisher-KPP Model

Applying the MDA scheme's framework and its auxiliary $\left(Y'(\mathfrak{S}) = l_3 Y(\mathfrak{S})^2 + l_2 Y(\mathfrak{S})\right)$

 $+l_1$) where l_i , i = 1, 2, 3 are arbitrary constants to be determined later, to Equation (4), we obtain the following sets of the above-mentioned parameters:

Set I

$$a_{0} = \frac{\sqrt{(l_{2}^{2} - 4l_{1}l_{3})^{2}} - l_{2}\sqrt{l_{2}^{2} - 4l_{1}l_{3}}}{2(l_{2}^{2} - 4l_{1}l_{3})}, a_{-1} = -\frac{l_{1}}{\sqrt{l_{2}^{2} - 4l_{1}l_{3}}}, a_{1} = 0, r = 2 - \frac{3\sqrt{(l_{2}^{2} - 4l_{1}l_{3})^{2}}}{2(l_{2}^{2} - 4l_{1}l_{3})^{3/2}}, \mu = -\frac{1}{4(l_{2}^{2} - 4l_{1}l_{3})}.$$

Set II

$$a_{0} = \frac{\sqrt{(l_{2}^{2} - 4l_{1}l_{3})^{2}} - l_{2}\sqrt{l_{2}^{2} - 4l_{1}l_{3}}}{2(l_{2}^{2} - 4l_{1}l_{3})}, a_{-1} = 0, a_{1} = -\frac{l_{3}}{\sqrt{l_{2}^{2} - 4l_{1}l_{3}}}, r = \frac{3\sqrt{(l_{2}^{2} - 4l_{1}l_{3})^{2}}}{2(l_{2}^{2} - 4l_{1}l_{3})^{3/2}} + 2, \mu = -\frac{1}{4(l_{2}^{2} - 4l_{1}l_{3})}.$$

Set III

$$a_0 = \frac{l_2}{\sqrt{l_2^2 - 4l_1l_3}}, a_{-1} = 0, a_1 = \frac{2l_3}{\sqrt{l_2^2 - 4l_1l_3}}, r = 2, \mu = \frac{1}{4l_1l_3 - l_2^2}$$

Therefore, the computational wave solutions of the (2+1) D-(2+1) D-Fisher-KPP model are constructed in the following formulas:

For $l_2 = 0$, $l_1 l_3 > 0$, we get

$$\mathcal{F}_{I,1}(x,y,t) = \frac{\sqrt{-l_1 l_3} \sqrt{l_1 l_3}}{2l_1 l_3} \cot\left(\sqrt{l_1 l_3} \left(\eta - \frac{3\sqrt{l_1^2 l_3^2 t}}{4(-l_1 l_3)^{3/2}} + 2t + x + y\right)\right) - \frac{\sqrt{l_1^2 l_3^2}}{2l_1 l_3}, \quad (6)$$

$$\mathcal{F}_{I,2}(x,y,t) = \frac{\sqrt{-l_1 l_3} \sqrt{l_1 l_3}}{2l_1 l_3} \tan\left(\sqrt{l_1 l_3} \left(\eta - \frac{3\sqrt{l_1^2 l_3^2} t}{4(-l_1 l_3)^{3/2}} + 2t + x + y\right)\right) - \frac{\sqrt{l_1^2 l_3^2}}{2l_1 l_3}, \quad (7)$$

$$\mathcal{F}_{\mathrm{II},1}(x,y,t) = \frac{\sqrt{-l_1 l_3}\sqrt{l_1 l_3}}{2l_1 l_3} \tan\left(\sqrt{l_1 l_3} \left(\eta + \frac{3\sqrt{l_1^2 l_3^2}t}{4(-l_1 l_3)^{3/2}} + 2t + x + y\right)\right) - \frac{\sqrt{l_1^2 l_3^2}}{2l_1 l_3}, \quad (8)$$

$$\mathcal{F}_{\text{II},2}(x,y,t) = \frac{\sqrt{-l_1 l_3} \sqrt{l_1 l_3}}{2l_1 l_3} \cot\left(\sqrt{l_1 l_3} \left(\eta + \frac{3\sqrt{l_1^2 l_3^2} t}{4(-l_1 l_3)^{3/2}} + 2t + x + y\right)\right) - \frac{\sqrt{l_1^2 l_3^2}}{2l_1 l_3}, \quad (9)$$

$$\mathcal{F}_{\text{III},1}(x,y,t) = -\frac{\sqrt{-l_1 l_3} \sqrt{l_1 l_3}}{l_1 l_3} \tan\left(\sqrt{l_1 l_3} (\eta + 2t + x + y)\right),\tag{10}$$

$$\mathcal{F}_{\text{III},2}(x,y,t) = -\frac{\sqrt{-l_1 l_3}\sqrt{l_1 l_3}}{l_1 l_3} \cot\left(\sqrt{l_1 l_3}(\eta + 2t + x + y)\right). \tag{11}$$

For $l_2 = 0$, $l_1 l_3 < 0$, we get

$$\mathcal{F}_{\mathrm{L},3}(x,y,t) = \frac{1}{2} \operatorname{coth}\left(\sqrt{-l_1 l_3} \left(-\frac{3\sqrt{l_1^2 l_3^2} t}{4(-l_1 l_3)^{3/2}} + 2t + x + y\right) \mp \frac{\log(\eta)}{2}\right) - \frac{\sqrt{l_1^2 l_3^2}}{2l_1 l_3}, \quad (12)$$

$$\mathcal{F}_{\mathrm{I},4}(x,y,t) = \frac{1}{2} \tanh\left(\sqrt{-l_1 l_3} \left(-\frac{3\sqrt{l_1^2 l_3^2} t}{4(-l_1 l_3)^{3/2}} + 2t + x + y\right) \mp \frac{\log(\eta)}{2}\right) - \frac{\sqrt{l_1^2 l_3^2}}{2l_1 l_3}, \quad (13)$$

$$\mathcal{F}_{\mathrm{II},3}(x,y,t) = -\frac{1}{2} \tanh\left(\sqrt{-l_1 l_3} \left(\frac{3\sqrt{l_1^2 l_3^2} t}{4(-l_1 l_3)^{3/2}} + 2t + x + y\right) \mp \frac{\log(\eta)}{2}\right) - \frac{\sqrt{l_1^2 l_3^2}}{2l_1 l_3}, \quad (14)$$

$$\mathcal{F}_{\mathrm{II},4}(x,y,t) = -\frac{1}{2} \operatorname{coth}\left(\sqrt{-l_1 l_3} \left(\frac{3\sqrt{l_1^2 l_3^2} t}{4(-l_1 l_3)^{3/2}} + 2t + x + y\right) \mp \frac{\log(\eta)}{2}\right) - \frac{\sqrt{l_1^2 l_3^2}}{2l_1 l_3}, \quad (15)$$

$$\mathcal{F}_{\mathrm{III,3}}(x,y,t) = \tanh\left(\sqrt{-l_1 l_3}(2t+x+y) \mp \frac{\log(\eta)}{2}\right),\tag{16}$$

$$\mathcal{F}_{\mathrm{III},4}(x,y,t) = \operatorname{coth}\left(\sqrt{-l_1 l_3}(2t+x+y) \mp \frac{\log(\eta)}{2}\right). \tag{17}$$

For $l_1 = 0$, $l_2 > 0$, we get

$$\mathcal{F}_{\mathrm{II},5}(x,y,t) = \frac{\sqrt{l_2^2}}{l_2 \left(l_3 \exp\left(l_2 \left(\eta + \frac{3\sqrt{l_2^4}t}{2(l_2^2)^{3/2}} + 2t + x + y \right) \right) - 1 \right)} + \frac{\sqrt{l_2^2}}{2l_2} + \frac{\sqrt{l_2^4}}{2l_2^2}, \quad (18)$$

$$\mathcal{F}_{\text{III},5}(x,y,t) = -\frac{l_3\sqrt{l_2^2}e^{l_2(\eta+2t+x+y)}}{l_2(l_3e^{l_2(\eta+2t+x+y)}-1)} - \frac{\sqrt{l_2^2}}{l_2(l_3e^{l_2(\eta+2t+x+y)}-1)}.$$
(19)

For
$$l_1 = 0, l_2 < 0$$
, we get

$$\mathcal{F}_{\mathrm{II},6}(x,y,t) = -\frac{l_3}{\sqrt{l_2^2} \left(l_3 \exp\left(l_2 \left(\eta + \frac{3\sqrt{l_2^4}t}{2(l_2^2)^{3/2}} + 2t + x + y \right) \right) + 1 \right)} - \frac{\sqrt{l_2^2}}{2l_2} + \frac{\sqrt{l_2^4}}{2l_2^2} + \frac{l_3}{\sqrt{l_2^2}},\tag{20}$$

$$\mathcal{F}_{\text{III,6}}(x,y,t) = \frac{2l_3}{\sqrt{l_2^2} (l_3 e^{l_2(\eta + 2t + x + y)} + 1)} + \frac{\sqrt{l_2^2}}{l_2} - \frac{2l_3}{\sqrt{l_2^2}}.$$
(21)

For $4 l_1 l_3 > l_2^2$, we get

$$\mathcal{F}_{I,5}(x,y,t) = \frac{2l_1l_3}{\sqrt{l_2^2 - 4l_1l_3} \left(l_2 - \sqrt{4l_1l_3 - l_2^2} \tan\left(\frac{1}{2}\sqrt{4l_1l_3 - l_2^2} \left(\eta - \frac{3\sqrt{(l_2^2 - 4l_1l_3)^2}t}{2(l_2^2 - 4l_1l_3)^{3/2}} + 2t + x + y\right)\right)\right)} + \frac{l_2^2}{2\sqrt{(l_2^2 - 4l_1l_3)^2}} - \frac{l_2}{2\sqrt{l_2^2 - 4l_1l_3}} - \frac{2l_1l_3}{\sqrt{(l_2^2 - 4l_1l_3)^2}},$$
(22)

$$\mathcal{F}_{I,6}(x,y,t) = \frac{2l_1l_3}{\sqrt{l_2^2 - 4l_1l_3} \left(l_2 - \sqrt{4l_1l_3 - l_2^2} \cot\left(\frac{1}{2}\sqrt{4l_1l_3 - l_2^2} \left(\eta - \frac{3\sqrt{(l_2^2 - 4l_1l_3)^2}t}{2(l_2^2 - 4l_1l_3)^{3/2}} + 2t + x + y\right)\right) \right)} + \frac{l_2^2}{2\sqrt{(l_2^2 - 4l_1l_3)^2}} - \frac{l_2}{2\sqrt{l_2^2 - 4l_1l_3}} - \frac{2l_1l_3}{\sqrt{(l_2^2 - 4l_1l_3)^2}},$$
(23)

$$\mathcal{F}_{\text{II,7}}(x,y,t) = -\frac{\sqrt{-(l_2^2 - 4l_1l_3)^2} \tan\left(\frac{1}{2}\sqrt{4l_1l_3 - l_2^2}\left(\eta + \frac{3\sqrt{(l_2^2 - 4l_1l_3)^2t}}{2(l_2^2 - 4l_1l_3)^{3/2}} + 2t + x + y\right)\right)}{2(l_2^2 - 4l_1l_3)} + \frac{\sqrt{(l_2^2 - 4l_1l_3)^2}}{2(l_2^2 - 4l_1l_3)},$$
(24)

$$\mathcal{F}_{\text{II},8}(x,y,t) = -\frac{\sqrt{-(l_2^2 - 4l_1l_3)^2} \cot\left(\frac{1}{2}\sqrt{4l_1l_3 - l_2^2}\left(\eta + \frac{3\sqrt{(l_2^2 - 4l_1l_3)^2t}}{2(l_2^2 - 4l_1l_3)^{3/2}} + 2t + x + y\right)\right)}{2(l_2^2 - 4l_1l_3)} + \frac{\sqrt{(l_2^2 - 4l_1l_3)^2}}{2(l_2^2 - 4l_1l_3)^2},$$
(25)

$$\mathcal{F}_{\text{III,7}}(x,y,t) = \frac{\sqrt{4l_1 l_3 - l_2^2}}{\sqrt{l_2^2 - 4l_1 l_3}} \tan\left(\frac{1}{2}\sqrt{4l_1 l_3 - l_2^2}(\eta + 2t + x + y)\right),\tag{26}$$

$$\mathcal{F}_{\text{III,8}}(x,y,t) = \frac{\sqrt{4l_1l_3 - l_2^2}}{\sqrt{l_2^2 - 4l_1l_3}} \cot\left(\frac{1}{2}\sqrt{4l_1l_3 - l_2^2}(\eta + 2t + x + y)\right).$$
(27)

Matching between Analytical and Numerical

Applying the TQBS numerical technique with the following initial condition $\mathcal{F}(x,0,0) = \frac{1}{2} \tanh(25x) + \frac{1}{2}$ gets the following shown analytical and numerical solutions' values with respect to different values of *x* in Table 1.

Value of <i>x</i>	Analytical	Numerical	Error	Value of <i>x</i>	Analytical	Numerical	Error
0	0.5	0.5	0	0.515625	0.925947	0.999254	0.073307
0.015625	0.519127	0.596515	0.077388	0.53125	0.931028	0.999203	0.068176
0.03125	0.538199	0.705602	0.167403	0.546875	0.935784	0.999146	0.063362
0.046875	0.557159	0.77828	0.221121	0.5625	0.940233	0.999082	0.058849
0.0625	0.575954	0.835791	0.259837	0.578125	0.944392	0.999013	0.05462
0.078125	0.594532	0.878033	0.283501	0.59375	0.948278	0.998936	0.050658
0.09375	0.612843	0.909745	0.296901	0.609375	0.951906	0.998854	0.046947
0.109375	0.630841	0.933241	0.3024	0.625	0.955292	0.998764	0.043472
0.125	0.648482	0.950674	0.302192	0.640625	0.95845	0.998667	0.040217
0.140625	0.665726	0.963567	0.297841	0.65625	0.961393	0.998561	0.037168
0.15625	0.682539	0.973094	0.290555	0.671875	0.964136	0.998448	0.034312
0.171875	0.698889	0.980124	0.281235	0.6875	0.966691	0.998325	0.031634
0.1875	0.714748	0.985307	0.270559	0.703125	0.96907	0.998192	0.029123
0.203125	0.730095	0.989125	0.25903	0.71875	0.971283	0.998049	0.026765
0.21875	0.744911	0.991936	0.247025	0.734375	0.973343	0.997894	0.024551
0.234375	0.759182	0.994003	0.234821	0.75	0.975259	0.997727	0.022468
0.25	0.772897	0.995521	0.222623	0.765625	0.97704	0.997547	0.020506
0.265625	0.786052	0.996634	0.210582	0.78125	0.978696	0.997352	0.018656
0.28125	0.798644	0.997448	0.198805	0.796875	0.980235	0.997142	0.016907
0.296875	0.810672	0.998042	0.18737	0.8125	0.981665	0.996915	0.015251
0.3125	0.822143	0.998474	0.176331	0.828125	0.982993	0.996671	0.013678
0.328125	0.833061	0.998784	0.165724	0.84375	0.984226	0.996407	0.012181
0.34375	0.843437	0.999006	0.155569	0.859375	0.985372	0.996122	0.010751
0.359375	0.853281	0.999161	0.14588	0.875	0.986435	0.995815	0.00938
0.375	0.862607	0.999267	0.13666	0.890625	0.987422	0.995484	0.008062
0.390625	0.87143	0.999336	0.127906	0.90625	0.988338	0.995126	0.006788
0.40625	0.879765	0.999377	0.119611	0.921875	0.989188	0.99474	0.005552
0.421875	0.88763	0.999396	0.111766	0.9375	0.989977	0.994326	0.004349
0.4375	0.895041	0.999399	0.104357	0.953125	0.990709	0.993871	0.003162
0.453125	0.902018	0.999388	0.09737	0.96875	0.991387	0.993411	0.002023
0.46875	0.908578	0.999367	0.090789	0.984375	0.992017	0.992795	0.000778
0.484375	0.914741	0.999337	0.084596	1	0.992601	0.992601	$5.55 imes 10^{-17}$

 Table 1. Analytical and numerical matching through MDA and TQBS schemes.

2.2. Kud Analytical vs. TQBS Numerical Techniques along (2+1) D-Fisher-KPP Model

Applying the MKud. scheme's framework and its auxiliary $\int \Gamma'(\mathfrak{S}) = \ln(k)(\Gamma(\mathfrak{S})^2)$

 $-\Gamma(\mathfrak{S})$) where *k* is an arbitrary constant to be determined later, to Equation (4), we obtain the following sets of the above-mentioned parameters:

Set I

$$a_0 = -1, a_1 = 2, r = 2, \mu = -\frac{1}{\log^2(k)}.$$

Set II

$$a_0 = 0, a_1 = 1, r = \frac{4\log(k) - 3}{2\log(k)}, \mu = -\frac{1}{4\log^2(k)}.$$

Set III

$$a_0 = 1, a_1 = -1, r = \frac{4\log(k) + 3}{2\log(k)}, \mu = -\frac{1}{4\log^2(k)}.$$

Therefore, the computational wave solutions of the (2+1) D-Fisher-KPP model are constructed in the following formulas:

$$S_{\rm I}(x,y,t) = \frac{2}{1 \pm k^{2t+x+y}} - 1,$$
(28)

$$S_{\rm II}(x,y,t) = \frac{1}{1 \pm k^{-\frac{3t}{2\log(k)} + 2t + x + y}},\tag{29}$$

$$S_{\rm III}(x,y,t) = 1 - \frac{1}{1 + k^{\frac{3t}{2\log(k)} + 2t + x + y}}.$$
(30)

Semi-Analytical Solutions

Applying the TQBS numerical technique with the following initial condition $\mathcal{F}(x,0,0) = \frac{1}{e^x+1}$ gets the following shown analytical and numerical solutions' values with respect to different values of *x* in Table 2.

Table 2. Analytical and numerical matching along MKud. and TQBS techniques.

Value of <i>x</i>	Analytical	Numerical	Error	Value of <i>x</i>	Analytical	Numerical	Error
0	0.5	0.5	0	0.515625	0.373876	0.373876	1.62×10^{-11}
0.015625	0.496094	0.496094	1.2×10^{-12}	0.53125	0.370225	0.370225	1.59×10^{-11}
0.03125	0.492188	0.492188	$2.94 imes 10^{-12}$	0.546875	0.36659	0.36659	1.56×10^{-11}
0.046875	0.488283	0.488283	4.39×10^{-12}	0.5625	0.362969	0.362969	1.52×10^{-11}
0.0625	0.48438	0.48438	5.79×10^{-12}	0.578125	0.359364	0.359364	1.49×10^{-11}
0.078125	0.480479	0.480479	7.07×10^{-12}	0.59375	0.355775	0.355775	1.45×10^{-11}
0.09375	0.47658	0.47658	8.26×10^{-12}	0.609375	0.352202	0.352202	1.41×10^{-11}
0.109375	0.472683	0.472683	9.37×10^{-12}	0.625	0.348645	0.348645	1.37×10^{-11}
0.125	0.468791	0.468791	1.04×10^{-11}	0.640625	0.345105	0.345105	1.32×10^{-11}
0.140625	0.464902	0.464902	1.13×10^{-11}	0.65625	0.341582	0.341582	1.28×10^{-11}
0.15625	0.461017	0.461017	1.22×10^{-11}	0.671875	0.338077	0.338077	1.23×10^{-11}
0.171875	0.457137	0.457137	1.29×10^{-11}	0.6875	0.334589	0.334589	1.18×10^{-11}
0.1875	0.453262	0.453262	1.36×10^{-11}	0.703125	0.33112	0.33112	1.13×10^{-11}
0.203125	0.449393	0.449393	1.43×10^{-11}	0.71875	0.327668	0.327668	1.08×10^{-11}
0.21875	0.44553	0.44553	1.48×10^{-11}	0.734375	0.324235	0.324235	1.02×10^{-11}
0.234375	0.441673	0.441673	1.53×10^{-11}	0.75	0.320821	0.320821	9.68×10^{-12}
0.25	0.437823	0.437823	1.58×10^{-11}	0.765625	0.317426	0.317426	9.13×10^{-12}
0.265625	0.433981	0.433981	1.62×10^{-11}	0.78125	0.314051	0.314051	8.56×10^{-12}
0.28125	0.430147	0.430147	1.65×10^{-11}	0.796875	0.310694	0.310694	7.98×10^{-12}
0.296875	0.426322	0.426322	1.68×10^{-11}	0.8125	0.307358	0.307358	7.39×10^{-12}
0.3125	0.422505	0.422505	$1.7 imes 10^{-11}$	0.828125	0.304042	0.304042	$6.8 imes 10^{-12}$
0.328125	0.418697	0.418697	1.72×10^{-11}	0.84375	0.300746	0.300746	6.19×10^{-12}

Value of <i>x</i>	Analytical	Numerical	Error	Value of <i>x</i>	Analytical	Numerical	Error
0.34375	0.414899	0.414899	1.73×10^{-11}	0.859375	0.29747	0.29747	5.58×10^{-12}
0.359375	0.411111	0.411111	1.74×10^{-11}	0.875	0.294215	0.294215	4.96×10^{-12}
0.375	0.407333	0.407333	1.74×10^{-11}	0.890625	0.290981	0.290981	4.34×10^{-12}
0.390625	0.403567	0.403567	1.74×10^{-11}	0.90625	0.287768	0.287768	3.71×10^{-12}
0.40625	0.399812	0.399812	1.74×10^{-11}	0.921875	0.284576	0.284576	3.07×10^{-12}
0.421875	0.396068	0.396068	1.73×10^{-11}	0.9375	0.281406	0.281406	2.44×10^{-12}
0.4375	0.392337	0.392337	1.72×10^{-11}	0.953125	0.278257	0.278257	1.79×10^{-12}
0.453125	0.388618	0.388618	1.71×10^{-11}	0.96875	0.27513	0.27513	$1.16 imes 10^{-12}$
0.46875	0.384912	0.384912	1.69×10^{-11}	0.984375	0.272025	0.272025	$4.57 imes 10^{-13}$
0.484375	0.38122	0.38122	$1.67 imes 10^{-11}$	1	0.268941	0.268941	5.55×10^{-17}

Table 2. Cont.

3. Results' Interpretation

This section studies the originality and novelty of this research paper. It also shows the accuracy of the obtained analytical solutions. The MDA and MKud. computational schemes have been applied to the (2+1) D-Fisher-KPP model for constructed novel analytical wave solutions. Many distinct analytical wave solutions have been obtained, and some of them have been demonstrated through sketches in 2D, 3D, and contour plots. These figures explain the dynamical characterization of the reaction–diffusion system by traveling waves in population genetics and propagating domain walls and pattern formation in bi-stable systems in nematic liquid crystals. Figures 1–4 show some novel analytical wave solutions with different values to the above–mentioned parameters. Comparing our solutions are completely different from those that have been evaluated in [41]. Still, some of our solutions match the obtained solutions in [17] when $\alpha = 1$, where in that paper, the authors in that paper studied the fractional form of the considered model.



Figure 1. Computation wave solutions of Equation (12) in 3D (a), 2D (b), and contour (c) plots.



Figure 2. Computational wave solutions of Equation (16) in 3D (a), 2D (b), and contour (c) plots.



Figure 3. Computational wave solutions of Equation (28) in 3D (a), 2D (b), and contour (c) plots.

The TQBS scheme has been applied to the considered model based on the obtained computational solutions. The absolute error between the analytical and numerical solutions has been calculated to show the accuracy of the obtained results and used methods (Tables 1 and 2 and Figures 5 and 6). This calculating shows the arrogance of the MKud. method over the MDA method, where its absolute values of error are much smaller than those obtained by the MDA method (Figure 7).



Figure 4. Computational wave solutions of Equation (29) in 3D (a), 2D (b), and contour (c) plots.



Figure 5. Two-dimensional plot (**a**), double y-column plot (**b**) for the matching between obtained analytical and numerical solutions, and two-dimensional plot of the absolute error (**c**) for Equation (3) with respect to Table 1.



Figure 6. Two-dimensional plot (**a**) and double y-column plot (**b**) for the matching between obtained analytical and numerical solutions, and two-dimensional plot of the absolute error (**c**) for Equation (3) with respect to Table 2.



Figure 7. Two-dimensional (**a**) and radar (**b**) plots for the calculated absolute error along the MDA, MKud. analytical schemes and the TQBS numerical scheme.

4. Conclusions

This research paper has successfully handled the nonlinear (2+1) D-Fisher-KPP model in population genetics and nematic liquid crystals through analytical and numerical investigations. The MDA, MKud., and TQBS schemes have been employed to find accurate novel analytical wave solutions of the considered model. Some obtained solutions have been represented in three different types of sketches. The accuracy of the MKud. method has been verified over the MDA method, where it is our fourth paper in keeping with the title's accuracy on the computational schemes.

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