

General Fractional Dynamics

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Abstract: General fractional dynamics (GFDynamics) can be viewed as an interdisciplinary science, in which the nonlocal properties of linear and nonlinear dynamical systems are studied by using general fractional calculus, equations with general fractional integrals (GFI) and derivatives (GFD), or general nonlocal mappings with discrete time. GFDynamics implies research and obtaining results concerning the general form of nonlocality, which can be described by general-form operator kernels and not by its particular implementations and representations. In this paper, the concept of “general nonlocal mappings” is proposed; these are the exact solutions of equations with GFI and GFD at discrete points. In these mappings, the nonlocality is determined by the operator kernels that belong to the Sonin and Luchko sets of kernel pairs. These types of kernels are used in general fractional integrals and derivatives for the initial equations. Using general fractional calculus, we considered fractional systems with general nonlocality in time, which are described by equations with general fractional operators and periodic kicks. Equations with GFI and GFD of arbitrary order were also used to derive general nonlocal mappings. The exact solutions for these general fractional differential and integral equations with kicks were obtained. These exact solutions with discrete timepoints were used to derive general nonlocal mappings without approximations. Some examples of nonlocality in time are described.



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1. Introduction

Fractional dynamics [1–4] is an interdisciplinary science in which the nonlocal properties of dynamical systems are studied by using methods of fractional calculus [5–12], integro-differential equations of non-integer orders and discrete nonlocal mappings. A fractional dynamical system is understood as a nonlocal system of any nature (physical, biological, economic, etc.), the state of which changes (discretely or continuously) in time. Fractional dynamics uses fractional differential equations and fractional discrete mappings to describe dynamical systems with nonlocality in space and time in, for example, physics [13,14], biology [15], and economics [16,17].

The processes with nonlocality in time are characterized by the dependence of behavior of dynamical systems at a given moment in time on the history of its behavior in a finite time interval in the past. To describe this dependence, the integral and integro-differential operators that form fractional calculus can be used. Fractional calculus is a branch of mathematical analysis that studies two types of integro-differential operators, for which generalized analogs of fundamental theorems hold, and therefore these types of operators are called fractional derivatives and fractional integrals. The characteristic property of fractional derivatives and integrals is nonlocality in space and time [18,19].

In fractional calculus, nonlocality is described by the kernels of the operators, which are fractional integrals (FI) and fractional derivatives (FD) of non-integer orders. To describe dynamical systems with various types of nonlocality in space and time, it is important to use integral and integro-differential operators with various types of kernels that allow us to describe various types of nonlocality. Therefore, it is important to have general fractional calculus that allows us to describe nonlocality in the most general form.

The concept of general fractional calculus (GFC) was suggested by Anatoly N. Kochubei in his work [20] in 2011 (see also [21–23]). In [20,21], general fractional derivatives (GFD) and general fractional integrals (GFI) are defined. For these operators, the general fundamental theorems are proved in [20,21]. This approach to GFC is based on the concept of kernel pairs, which was proposed by the Russian mathematician Nikolay Ya. Sonin (1849–1915) in his 1884 article [24] (see also [25]). Note that in the mathematical literature, his name “Sonin” [26] is mistakenly used in French transliteration as “Sonine” from French journals [25]. Then, very important results in constructing GFC were derived by Yuri Luchko in 2021 [27–29]. In [27,28], GFD and GFI of arbitrary order are suggested, and the general fundamental theorems for GFI and GFD are proved. Operational calculus for equations with general fractional derivatives is proposed in [29].

As a result, a mathematical basis was created for constructing general fractional dynamics. In the framework of general fractional dynamics, it is assumed and implied to obtain not only and not so much the results concerning studies of specific types of operator kernels, but, first of all, general results that do not depend on specific types (particular implementations) of operator kernels. In general fractional dynamics, all research and results should concern the general form of nonlocality, operator kernels of almost all types (all sets of kernels), or a wide subset of such kernels.

It should be noted that by now, solutions of some equations with general operators have already been described, which form the basis of general fractional dynamics and can be used to describe some dynamic processes in different fields of science. Let us note some ordinary and particular fractional differential equations with GFD and GFI, which have already been considered by now and can be used in GFDynamics. A solution of the relaxation equations with GFD with respect to time is described in [20,22]. The general growth equation with GFD which can be used in macroeconomic models for processes with memory and distributed lag is described in [23]. Equations with GFD are considered in [30,31]. Time-fractional diffusion equations with GFD are described in [20,22,32,33]. To solve a general fractional differential equation, general operational calculus was proposed in [29]. Integral equations of the first kind with kernels from the Sonin set and the GFI and GFD of the Liouville and Marchaud type are described in [34,35]. The partial differential equations containing GFD and GFI are considered in [36,37]. Some applications of GFC are described in recent published works (see [31–33,38,39] and references therein).

In nonlinear dynamics, discrete time description is usually derived from differential equations with integer-order derivatives and periodic kicks (Section 5 in [40], Sections 5.2 and 5.3 in [1] (pp. 60–68), Chapter 18 in [41] (pp. 432–482), Chapter 18 in [2] (pp. 409–453), and Section 1.2 in [42] (pp. 16–17)). This description is represented by mappings in which the value X_{n+1} is determined by the value X_n (or a fixed finite number of the values X_n and X_{n-1} , for example), i.e., $X_{n+1} = F(X_n)$ (or $X_{n+1} = F(X_n, X_{n-1})$, for example). These mappings cannot describe nonlocal dynamical systems since the original equations contain only derivatives of integer order and thus are local.

We should note that discrete general fractional calculus has not yet been created. However, it should be noted that discrete GFDynamics can be described by using the proposed concept of general nonlocal mappings. Such discrete nonlocal mappings must be described by kernels of the operators that are used in general fractional calculus. These mappings can be derived from equations with GFD and GFI without approximations. In fact, these general nonlocal mappings are the exact solutions of fractional differential equations at discrete points. The concept of “general nonlocal mappings” is based on the approach that was proposed for discrete fractional dynamics in [2,43–45].

In discrete fractional dynamics, the nonlocality in time is taken into account by the mappings, in which the value X_{n+1} is determined by all the past values, $X_{n+1} = F(X_n, X_{n-1}, \dots, X_1)$, where the number of variables in the function F increases with the number $n \in \mathbb{N}$. For the first time such discrete mappings were derived from fractional differential equations in [43]. Then, this approach was developed in [2,44,45], and it was applied in [17,46–50]. We should emphasize that nonlocal mappings were derived from fractional differential equations without approximations. The first computer simulations of these nonlocal mappings were proposed in [51,52]. Then, new types of chaotic behavior of systems with nonlocality in time were discovered [53–67]. Discrete fractional calculus [10] was also used to derive nonlocal mappings in [53–67]. All these mappings were described by discrete fractional dynamics with power-law nonlocality in time only.

Therefore, it is important to derive general fractional dynamics where we take into account the general form of nonlocality in time. General nonlocal mappings should be derived from equations with GFD and GFI. These nonlocal mappings can demonstrate new types of attractors and chaotic behavior of nonlocal systems. We should emphasize that these discrete GFDynamics can be derived from equations with GFD and GFI without approximations.

In this paper, using general fractional calculus, we considered fractional systems with nonlocality in time which are described by equations with general fractional derivatives, integrals and periodic kicks. The exact solutions for these nonlinear fractional differential and integral equations with kicks were obtained. These exact solutions for discrete timepoints used to derive mappings with nonlocality in time were described without approximations. The nonlocality of general nonlocal mappings was determined by the kernels that belong to the Sonin and Luchko sets of kernels which were used in GFI and GFD of the initial equations.

2. Equations of General Fractional Dynamical Systems

Fractional dynamics with continuous time is described by integral and integro-differential equations. To take into account nonlocality in time, we require these equations not to be represented as differential equations or systems of differential equations of integer order only.

To describe dynamical systems with the general form of nonlocality in time, it is possible to use the following equations

$$I_{(M)}^t[\tau] X(\tau) = F_I(t, X(t)), \quad (1)$$

$$D_{(K)}^t[\tau] X(\tau) = F_D(t, X(t)), \quad (2)$$

$$D_{(K)}^{t,*}[\tau] X(\tau) = F_{D*}(t, X(t)), \quad (3)$$

where the operators $I_{(M)}^t$, $D_{(K)}^t$, $D_{(K)}^{t,*}$ have the forms

$$I_{(M)}^t[\tau] X(\tau) = (M * X)(t) = \int_0^t d\tau M(t - \tau) X(\tau), \quad (4)$$

$$D_{(K)}^t[\tau] X(\tau) = \frac{d}{dt} (K * X)(t) = \frac{d}{dt} \int_0^t d\tau K(t - \tau) X(\tau), \quad (5)$$

$$D_{(K)}^{t,*}[\tau] X(\tau) = \left(K * X^{(1)} \right)(t) = \int_0^t d\tau K(t - \tau). \quad (6)$$

In this case, the nonlocality in time is characterized by the kernels $M(t - \tau)$, $K(t - \tau)$. The operators (4)–(6) are defined through the Laplace convolution $*$ with these kernels. The notation $X^{(1)}(t) = dX(t)/dt$ is used as a special form of the notation $X^{(n)}(\tau)$, which is used in Section 5 of the manuscript where GFI and GFD of arbitrary order are applied ($X^{(1)}(\tau)$ is $X^{(n)}(\tau)$ for $n = 1$).

Example 1. For the integral kernel

$$M(t - \tau) = \frac{(t - \tau)^{\alpha-1}}{\Gamma(\alpha)}, \quad (7)$$

Equation (1) gives the equation

$$I_{RL}^\alpha[\tau]X(\tau) = F_I(t, X(t)), \quad (8)$$

where I_{RL}^α is the Riemann–Liouville integral of the order $\alpha \in (0, \infty)$, [8].

Example 2. For the kernel

$$K(t - \tau) = \frac{(t - \tau)^{-\alpha}}{\Gamma(1 - \alpha)}, \quad (9)$$

Equations (2) and (3) give the equations

$$D_{RL}^\alpha[\tau]X(\tau) = F_D(t, X(t)), \quad D_C^\alpha[\tau]X(\tau) = F_{D*}(t, X(t)), \quad (10)$$

where D_{RL}^α and D_C^α are the Riemann–Liouville and Caputo fractional derivatives of the order $\alpha(0, 1)$, respectively, [8] (pp. 70,92).

Equations (8) and (10) describe dynamical systems with power-law nonlocality in time, which is interpreted as a fading memory, and the kernels are called the memory function.

The kernels (7) and (9) can be considered an approximation of a more general form of kernels [68] to describe dynamical systems with nonlocality in time. In article [68], using the generalization of the Taylor series that is proposed in [69] for the kernels, we proved that a wide class of kernels can be approximately considered power function.

For applications, it is important to describe dynamical systems with a more general form of nonlocality in time. Therefore, it is necessary to consider the general type of kernels for the integral and integro-differential operators (4)–(6).

To do this, we must have general integral operators and integro-differential operators which can be interpreted as a generalization of the standard integrals and derivatives of integer order. Moreover, these general fractional operators must form a certain calculus, and the fundamental theorems of this calculus must hold for these operators.

General fractional calculus (GFC) is a branch of mathematical analysis of the integral and integro-differential operators that are generalizations of integrals and derivatives of integer order for which the generalization of the fundamental theorems of the calculus are satisfied.

GFC is based on the concept of a pair of mutually associated kernels. This type of kernels (the set of operator kernels) was proposed by the Russian mathematician Nikolay Ya. Sonin (1849–1915) in 1884 [25,26] who published articles in French as “N. Sonine”.

The concept of mutually associated kernels of two operators which are defined through the Laplace convolution assumes that the Laplace convolution for the kernels of these operators is equal to one. For the operators (4)–(6), the Sonin condition for the kernels $M(t)$ and $K(t)$ requires that the relation

$$\int_0^t M(t - \tau)K(\tau)d\tau = 1 \quad (11)$$

holds for all $t \in (0, \infty)$.

In this work about GFDynamics, we followed Luchko’s approach to general fractional calculus that was proposed by Yuri Luchko in [27,28] in 2021.

Definition 1. The functions $M(t), K(t)$ are a Sonin pair of kernels if

$$M(t), K(t) \in C_{-1,0}(0, \infty) \quad (12)$$

and the Sonin condition (11) holds where

$$C_{a,b}(0, \infty) = \{X(t) : X(t) = t^p Y(t), t > 0, a < p < b, Y(t) \in C[0, \infty)\}. \quad (13)$$

The set of such kernels is denoted by $\mathcal{S}_{-1}$, and $(M(t), K(t)) \in \mathcal{S}_{-1}$ if

$$M(t), K(t) \in C_{-1,0}(0, \infty) \quad \text{and} \quad (M * K)(t) = 1 \quad (14)$$

for all $t \in (0, \infty)$.

Definition 2. Let $M(t) \in \mathcal{S}_{-1}$ and $X(t) \in C_{-1}(0, \infty) = C_{-1,\infty}(0, \infty)$. The general fractional integral with the kernel $M(t) \in C_{-1,0}(0, \infty)$ is the operator on the space $C_{-1}(0, \infty)$, that is,

$$I_{(M)}^t : C_{-1}(0, \infty) \rightarrow C_{-1}(0, \infty), \quad (15)$$

that is defined by the equation

$$I_{(M)}^t[\tau]X(\tau) = (M * X)(t) = \int_0^t d\tau M(t - \tau)X(\tau). \quad (16)$$

The following important property allows us define the repeated-action general operator (16). Let $M_1(t), M_2(t) \in \mathcal{S}_{-1}$ and $X(t) \in C_{-1}(0, \infty)$. Then, the equation

$$I_{(M_1)}^t[\tau] I_{(M_2)}^\tau[s]X(s) = I_{(M_1 * M_2)}^t[\tau]X(\tau) \quad (17)$$

holds for $t > 0$.

Let us assume that the kernels $M(t)$ and $K(t)$ form a Sonin pair of kernels. This allows us to define the general fractional derivatives $D_{(K)}^t$ and $D_{(K)}^{t,*}$ that are associated with the general fractional integral $I_{(M)}^t$.

The space $C_{-1}^m(0, \infty) \subset C_{-1}(0, \infty)$ where $m \in \mathbb{N}$ consists of the functions $X(t)$, for which $X^{(m)} \in C_{-1}(0, \infty)$.

Definition 3. Let $K(t) \in \mathcal{S}_{-1}$ and $X(t) \in C_{-1}^1(0, \infty)$. The general fractional derivative of the Riemann–Liouville type with the kernel $K(t) \in C_{-1,0}(0, \infty)$ which is associated with the GFI (16) is defined as

$$D_{(K)}^t[\tau]X(\tau) = \frac{d}{dt}(K * X)(t) = \frac{d}{dt} \int_0^t d\tau K(t - \tau)X(\tau) \quad (18)$$

for $t \in (0, \infty)$.

The general fractional derivative of the Caputo type with the kernel $K(t) \in C_{-1,0}(0, \infty)$ is defined as

$$D_{(K)}^{t,*}[\tau]X(\tau) = (K * X^{(1)})(t) = \int_0^t d\tau K(t - \tau)X^{(1)}(\tau) \quad (19)$$

for $t \in (0, \infty)$.

These GFI and GFD satisfy the fundamental theorems of GFC [27,28].

3. General Fractional Dynamics

General fractional dynamics is an interdisciplinary science, in which the nonlocal properties of nonlinear dynamical systems are studied. General fractional dynamics uses linear and nonlinear models to describe systems with general forms of nonlocality in space and time by using equations with operators of general fractional calculus.

We can conditionally distinguish the following three directions in general fractional dynamics with nonlocality in time.

(1) General fractional dynamics with continuous time is described by the equations with GFD and GFI with the kernels belonging to the Sonin set $\mathcal{S}_{-1}$. For example, the solution of integral equations with kernels from the Sonin set

$$\int_{-\infty}^t d\tau M(t-\tau)X(\tau) = F(t), \quad (20)$$

in which the lower limit is not zero but minus infinity, were proposed by Stefan G. Samko and Rogerio P. Cardoso [34,35], where general fractional integrals and derivatives are considered to be of the Marchaud type and the Liouville type.

For example, the solution of relaxation equations

$$D_{(K)}^{t,*}[\tau] X(\tau) = \lambda X(t), \quad (21)$$

where relaxation means that $\lambda < 0$, was derived by Anatoly N. Kochubei in [20,22]. The solution of the growth equation that is described by (21) with $\lambda > 0$ was derived by Anatoly N. Kochubei and Yuri Kondratiev [23] in 2019. Operational calculus for equations with general fractional derivatives with kernels from the Sonin set is proposed in [29].

(2) General fractional dynamics with discrete time could be described by a discrete analog of general fractional calculus. However, such a discrete general fractional calculus has not yet been created.

(3) Another way of describing general fractional dynamics with discrete time can be based on the use of discrete mappings obtained from exact solutions of general fractional differential and integral equations with periodic kicks. For the first time discrete mappings with nonlocality in time were derived from fractional differential equations in [43–45], in which the Riemann–Liouville and Caputo fractional derivatives were used (see also [2,46–49]). The proposed approach allows us to derive discrete time mappings with nonlocality in time from integro-differential equations of non-integer orders without approximation. This approach is a generalization of methods that is well-known in nonlinear dynamics and the theory of chaos (Section 5 in [40], Sections 5.2 and 5.3 in [1] (pp. 60–68), Chapter 18 in [41] (pp. 432–482), Chapter 18 in [2] (pp. 409–453), and Section 1.2 in [42] (pp. 16–17)), where the discrete time dynamical mappings are derived from ordinary differential equations of integer order with kicks.

Let us consider Equations (1)–(3) with GFD and GFI and periodic kicks by using

$$F_I(t, X(t)) = F_D(t, X(t)) = F_{D*}(t, X(t)) = \lambda G(t, X(t)) \sum_{k=1}^{\infty} \delta((t+\varepsilon)/T - k), \quad (22)$$

where T is the period of the periodic sequence of kicks, λ is the amplitude of these kicks, and $G(t, X)$ is the real-valued function.

Expression (22) contains delta functions that are distributions (generalized functions). Therefore, this expression and Equations (1)–(3) with function (22) are treated as continuous functionals in a space of test functions (see Section 8 [5] (pp. 145–160) and [70,71]).

We also use $t - \varepsilon$ where $0 < \varepsilon \ll T$ instead of t in the argument of the delta functions to make sense of the product of $G(t, X(t))$ and the delta function for the timepoints where $X(t - 0) \neq X(t + 0)$ [59].

We also assume that the function $G(t, X(t))$ is such that the product of the delta function is defined in the neighborhood of the points $t = Tk - \varepsilon$ where $k \in \mathbb{N}$ and $\varepsilon \rightarrow 0$.

We start first by considering the integro-differential Equation (3) with the GFD $D_{(K)}^{t,*}$ with the kernel $K(t) \in C_{-1,0}(0, \infty)$ and periodic kicks (22).

Theorem 1. Let $K(t) \in C_{-1,0}(0, \infty)$ and $X(t) \in C_{-1}^1(0, \infty)$. Then, the integro-differential equation

$$D_{(K)}^{t,*}[\tau]X(\tau) = \lambda G(t, X(t)) \sum_{k=1}^{\infty} \delta((t+\varepsilon)/T - k) \quad (23)$$

has the solution

$$X(t) = X(0) + \lambda T \sum_{k=1}^n M(t - (Tk - \varepsilon)) G(Tk - \varepsilon, X(Tk - \varepsilon)), \quad (24)$$

if $Tn < t < T(n + 1)$, where $M(t) \in C_{-1,0}(0, \infty)$ is a function that is a kernel associated with the kernel $K(t)$, i.e., the functions $K(t)$ and $M(t)$ form a pair of mutually associated kernels from the Sonin set $\mathcal{S}_{-1}$.

Proof. Let us use the general fractional integral

$$I_{(M)}^s[\tau]X(\tau) = (M * X)(t) = \int_0^t d\tau M(t - \tau)X(\tau) \quad (25)$$

with the kernel $M(t) \in C_{-1,0}(0, \infty)$ that is associated with the kernel $K(t)$, so that $K(t)$, $M(t)$ are a pair of mutually associated kernels from the Sonin set. Applying the integral operator (25) to Equation (23), we obtain

$$I_{(M)}^s[t]D_{(K)}^{t,*}[\tau]X(\tau) = \lambda I_{(M)}^s[t]G(t, X(t)) \sum_{k=1}^{\infty} \delta((t + \varepsilon)/T - k), \quad (26)$$

where $s > t > \tau > 0$. Using the second fundamental theorem of general fractional operators in the form of equation

$$I_{(M)}^s[t]D_{(K)}^{t,*}[\tau]X(\tau) = X(s) - X(0) \quad (27)$$

that holds for $X(t) \in C_{-1}^1(0, \infty)$ (see Equation (60) of Theorem 4 in [27] (p. 11) and Equation (33) of Theorem 2 in [28] (p. 7)), we derive

$$X(s) - X(0) = \lambda I_{(M)}^s[t]G(t, X(t)) \sum_{k=1}^{\infty} \delta((t + \varepsilon)/T - k). \quad (28)$$

Using (25), Equation (28) is written as

$$X(s) - X(0) = \lambda \int_0^s dt M(s - t)G(t, X(t)) \sum_{k=1}^{\infty} \delta((t + \varepsilon)/T - k). \quad (29)$$

For $Tn < s < T(n + 1)$, Equation (29) is

$$X(s) - X(0) = \lambda \sum_{k=1}^n \int_0^s dt M(s - t)G(t, X(t)) \delta((t + \varepsilon)/T - k). \quad (30)$$

Using the equation

$$\int_0^s dt f(t) \delta(t - a) = f(a), \quad (31)$$

which holds for $0 < a < s$ and $f(t) \in C^\infty(\Omega_a)$ where Ω_a is the neighborhood of point $t = a$, Equation (30) gives

$$X(s) - X(0) = \lambda T \sum_{k=1}^n M(s - (kT - \varepsilon)) G(kT - \varepsilon, X((kT - \varepsilon))). \quad (32)$$

Equation (32) leads to solution (24). $\square$

Let us consider the solution for the discrete time points $t = Tk$ with $k \in \mathbb{N}$.

Theorem 2. Let $K(t) \in C_{-1,0}(0, \infty)$ and $X(t) \in C_{-1}^1(0, \infty)$. The solution of the equation

$$D_{(K)}^{t,*}[\tau]X(\tau) = \lambda G(t, X(t)) \sum_{k=1}^{\infty} \delta((t + \varepsilon)/T - k) \quad (33)$$

for the time points $s = Tk - \varepsilon$ at $\varepsilon \rightarrow 0+$ and the variables

$$X_k = \lim_{\varepsilon \rightarrow 0+} X(Tk - \varepsilon), \quad (k = 1, \dots, n+1) \quad (34)$$

is the non-local mapping

$$X_{n+1} = X_n + \lambda TM(T)G(Tn, X_n) + \lambda T \sum_{k=1}^{n-1} (M(T(n+1-k)) - M(T(n-k)))G(Tk, X_k) \quad (35)$$

where $M(t) \in C_{-1,0}(0, \infty)$ is a function that is a kernel to the kernel $K(t)$, i.e., the functions $K(t)$, and $M(t)$ form a pair of mutually associated kernels from the Sonin set $\mathcal{S}_{-1}$.

Proof. Solution (24), which is derived in Theorem 1, for $t = T(n+1) - \varepsilon$ and $t = Tn - \varepsilon$ with $0 < \varepsilon \ll T$ is given by the equations

$$X(T(n+1) - \varepsilon) = X(0) + \lambda \sum_{k=1}^n M(T(n+1) - Tk)G(Tk - \varepsilon, X(Tk - \varepsilon)), \quad (36)$$

$$X(Tn - \varepsilon) = X(0) + \lambda T \sum_{k=1}^{n-1} M(Tn - Tk)G(Tk - \varepsilon, X(Tk - \varepsilon)). \quad (37)$$

Using the variables

$$X_k = \lim_{\varepsilon \rightarrow 0+} X(Tk - \varepsilon), \quad (k = 1, \dots, n+1), \quad (38)$$

solutions (36) and (37) at the limit $\varepsilon \rightarrow 0+$ give

$$X_{n+1} = X(0) + \lambda T \sum_{k=1}^n M(T(n+1-k))G(Tk, X_k), \quad (39)$$

$$X_n = X(0) + \lambda T \sum_{k=1}^{n-1} M(T(n-k))G(Tk, X_k). \quad (40)$$

Subtracting Equation (40) from Equation (39) gives

$$X_{n+1} - X_n = \lambda T M(T)G(Tn, X_n) + \lambda T \sum_{k=1}^{n-1} (M(T(n+1-k)) - M(T(n-k)))G(Tk, X_k). \quad (41)$$

Equation (41) leads to (35). $\square$

Let us consider equation with GFD of the Riemann–Liouville type.

Theorem 3. Let $K(t) \in C_{-1,0}(0, \infty)$ and $X(t) \in C_{-1}^1(0, \infty)$. The equation

$$D_{(K)}^t[\tau]X(\tau) = \lambda G(t, X(t)) \sum_{k=1}^{\infty} \delta((t+\varepsilon)/T - k) \quad (42)$$

has the solution

$$X(t) = \lambda T \sum_{k=1}^n M(t - (Tk - \varepsilon))G(Tk - \varepsilon, X(Tk - \varepsilon)) \quad (43)$$

if $Tn < t < T(n+1)$. For the time points $s = Tk - \varepsilon$ at $\varepsilon \rightarrow 0+$ and the variables

$$X_k = \lim_{\varepsilon \rightarrow 0+} X(Tk - \varepsilon), \quad (k = 1, \dots, n+1), \quad (44)$$

solution (43) is represented by the nonlocal mapping

$$X_{n+1} = X_n + \lambda T M(T)G(T(n+1), X_n) + \lambda T \sum_{k=1}^{n-1} (M(T(n+1-k)) - M(T(n-k)))G(kT, X_k), \quad (45)$$

where $M(t) \in C_{-1,0}(0, \infty)$ is a function that is a kernel associated with the kernel $K(t)$, i.e., the functions $K(t)$ and $M(t)$ form a pair of mutually associated kernels that belong to the set $\mathcal{S}_{-1}$.

Proof. The proof of this theorem is similar to the proofs of Theorems 1 and 2. In this proof, we use the second fundamental theorem of general fractional operators in the form of equation

$$I_{(M)}^s[t]D_{(K)}^t[\tau]X(\tau) = X(s) \quad (46)$$

that holds for $X(t) \in C_{-1}^1(0, \infty)$ (see Equation (61) of Theorem 4 in [27] (p. 11), Equation (34) of Theorem 2 in [28] (p. 7)).

Applying the GFI $I_{(M)}^s[t]$ with the kernel $M(t)$ which is associated with the kernel $G(t)$ of the GFD $D_{(K)}^t$, to Equation (42), and using (46), we derive

$$X(s) = \lambda I_{(M)}^s[t]G(t, X(t)) \sum_{k=1}^{\infty} \delta((t + \varepsilon)/T - k). \quad (47)$$

Using (25), Equation (47) is written as

$$X(s) = \lambda \int_0^s dt M(s - t)G(t, X(t)) \sum_{k=1}^{\infty} \delta((t + \varepsilon)/T - k). \quad (48)$$

For $Tn < s < T(n + 1)$, Equation (48) is

$$X(s) = \lambda \sum_{k=1}^n \int_0^s dt M(s - t)G(t, X(t))\delta((t + \varepsilon)/T - k). \quad (49)$$

Using Equation (31), Equation (49) gives

$$X(s) = \lambda T \sum_{k=1}^n M(s - (kT - \varepsilon))G(kT - \varepsilon, X(kT - \varepsilon)). \quad (50)$$

Using (50) for $s = T(n + 1) - \varepsilon$ and $s = Tn - \varepsilon$, we obtain

$$X(T(n + 1) - \varepsilon) = \lambda T \sum_{k=1}^n M(T(n + 1) - Tk)G(Tk - \varepsilon, X(Tk - \varepsilon)), \quad (51)$$

$$X(Tn - \varepsilon) = \lambda T \sum_{k=1}^{n-1} M(Tn - Tk)G(Tk - \varepsilon, X(Tk - \varepsilon)). \quad (52)$$

Using variables (44), solutions (51) and (52) at the limit $\varepsilon \rightarrow 0+$ give

$$X_{n+1} = \lambda T \sum_{k=1}^n M(T(n + 1 - k))G(Tk, X_k), \quad (53)$$

$$X_n = \lambda T \sum_{k=1}^{n-1} M(T(n - k))G(Tk, X_k). \quad (54)$$

Subtracting Equation (54) from (53), we obtain (45). $\square$

Let us consider equation with general fractional integral operator and kicks.

Theorem 4. Let $M(t) \in C_{-1,0}(0, \infty)$ and $X(t) \in C_{-1}(0, \infty)$. Then, the integral equation

$$I_{(M)}^t[\tau]X(\tau) = \lambda G(t, X(t)) \sum_{k=1}^{\infty} \delta((t + \varepsilon)/T - k) \quad (55)$$

has the solution

$$X(t) = \lambda T \sum_{k=1}^n K^{(1)}(t - (Tk - \varepsilon))G(Tk - \varepsilon, X(Tk - \varepsilon)) \quad (56)$$

for $Tn < t < T(n + 1)$, where $K^{(1)}(z) = (dK(t)/dt)_{t=z}$ and $K(t) \in C_{-1,0}^1(0, \infty)$ is a kernel associated with the kernel $M(t)$, i.e., the functions $K(t)$ and $M(t)$ form a pair of mutually associated kernels that belong to the Sonin set $\mathcal{S}_{-1}$.

Proof. Let us use the general fractional derivative of the Riemann–Liouville type

$$D_{(K)}^s[t]X(t) = \frac{d}{ds}(K * X)(s) = \frac{d}{ds} \int_0^s dt K(s - t)X(t) \quad (57)$$

with the kernel $K(t) \in C_{-1,0}(0, \infty)$ that is a kernel associated with the kernel $M(t)$ of GFI so that the pair of functions $K(t)$, $M(t)$ belongs to the Sonin set S_{-1} . Applying operator (57) to Equation (55), we obtain

$$D_{(K)}^s[t]I_{(M)}^t[\tau]X(\tau) = \lambda D_{(K)}^s[t]G(t, X(t)) \sum_{k=1}^{\infty} \delta((t + \varepsilon)/T - k), \quad (58)$$

where $s > t > \tau > 0$. Using the first fundamental theorem of general fractional operators in the form of equation

$$D_{(M)}^s[t]I_{(K)}^t[\tau]X(\tau) = X(s) \quad (59)$$

that holds for $X(t) \in C_{-1}^1(0, \infty)$ (see Equation (51) of Theorem 3 in [27] (p. 9) and Equation (31) of Theorem 1 in [28] (p. 6)), Equation (58) takes the form

$$X(s) = \lambda D_{(K)}^s[t]G(t, X(t)) \sum_{k=1}^{\infty} \delta((t + \varepsilon)/T - k). \quad (60)$$

Then, using expression (57), we derive

$$X(s) = \lambda \frac{d}{ds} \int_0^s dt K(s-t)G(t, X(t)) \sum_{k=1}^{\infty} \delta((t + \varepsilon)/T - k). \quad (61)$$

For $Tn < s < T(n+1)$, Equation (61) is

$$X(s) = \lambda \sum_{k=1}^n \frac{d}{ds} \int_0^s dt K(s-t)G(t, X(t)) \delta((t + \varepsilon)/T - k). \quad (62)$$

Using Equation (31), Equation (62) gives

$$X(s) = \lambda T \frac{d}{ds} \sum_{k=1}^n K(s - (kT - \varepsilon))G(kT - \varepsilon, X(kT - \varepsilon)). \quad (63)$$

For $Tn < t < T(n+1)$, we obtain

$$X(s) = \lambda T \sum_{k=1}^n K^{(1)}(s - (kT - \varepsilon))G(kT - \varepsilon, X(kT - \varepsilon)). \quad (64)$$

Equation (63) leads to solution (56). $\square$

Theorem 5. Let $M(t) \in C_{-1,0}(0, \infty)$ and $X(t) \in C_{-1}(0, \infty)$. The solution of the equation

$$I_{(M)}^t[\tau]X(\tau) = \lambda G(t, X(t)) \sum_{k=1}^{\infty} \delta((t + \varepsilon)/T - k) \quad (65)$$

for the timepoints $s = Tk - \varepsilon$ at $\varepsilon \rightarrow 0+$ and the variables

$$X_k = \lim_{\varepsilon \rightarrow 0+} X(Tk - \varepsilon), \quad (k = 1, \dots, n+1) \quad (66)$$

is represented as the nonlocal mapping

$$X_{n+1} = X_n + \lambda T K^{(1)}(T) G(Tn, X_n) + \lambda T \sum_{k=1}^n \left(K^{(1)}(T(n+1-k)) - K^{(1)}(T(n-k)) \right) G(Tk, X_k), \quad (67)$$

where $K^{(1)}(z) = (dK(t)/dt)_{t=z}$ and $K(t) \in C_{-1,0}^1(0, \infty)$ is a function that is a kernel associated with the kernel $M(t)$, i.e., the functions $K(t)$ and $M(t)$ are a pair of mutually associated kernels that belong to the Sonin set S_{-1} .

Proof. Solution (56) which is derived in Theorem 4 for $t = T(n+1) - \varepsilon$ and $t = Tn - \varepsilon$ with $0 < \varepsilon \ll T$ is given by the equations

$$X(T(n+1) - \varepsilon) = \lambda T \sum_{k=1}^n K^{(1)}(T(n+1) - Tk)G(Tk - \varepsilon, X(Tk - \varepsilon)), \quad (68)$$

$$X(Tn - \varepsilon) = \lambda T \sum_{k=1}^{n-1} K^{(1)}(Tn - Tk)G(Tk - \varepsilon, X(Tk - \varepsilon)). \quad (69)$$

Using variables (66), Equations (68) and (69) at the limit $\varepsilon \rightarrow 0+$ give

$$X_{n+1} = \lambda T \sum_{k=1}^n K^{(1)}(T(n+1-k))G(Tk, X_k), \quad (70)$$

$$X_n = \lambda T \sum_{k=1}^{n-1} K^{(1)}(T(n-k))G(Tk, X_k). \quad (71)$$

Subtraction of Equation (71) from Equation (70) gives

$$X_{n+1} - X_n = \lambda T K^{(1)}(T) G(Tn, X_n) + \lambda T \sum_{k=1}^n \left(K^{(1)}(T(n+1-k)) - K^{(1)}(T(n-k)) \right) G(Tk, X_k). \quad (72)$$

Equation (72) gives (67). $\square$

4. Examples of Nonlocality in the Form of Kernels from the Sonin Set

In this section, examples of the Sonin pairs of kernels are presented.

Note that if the kernel $M(t)$ is associated with the kernel $K(t)$, then the kernel $K(t)$ is associated with $M(t)$. Therefore, if we have the operators

$$I_{(M)}^t[\tau]X(\tau) = (M * X)(t), D_{(K)}^{t,*}[\tau]X(\tau) = (K * X^{(1)})(t), \quad (73)$$

where $M(t), K(t) \in \mathcal{S}_{-1}$, then we can use the operators

$$I_{(K)}^t[\tau]X(\tau) = (K * X)(t), D_{(M)}^{t,*}[\tau]X(\tau) = (M * X^{(1)})(t). \quad (74)$$

Let us give examples of the kernels that satisfy the Sonin condition

$$(M * K)(t) = \int_0^t M(t-\tau)K(\tau)d\tau = 1. \quad (75)$$

Example 3. The following kernels,

$$M(t) = h_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad (76)$$

$$K(t) = h_{1-\alpha}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad (77)$$

where $0 < \alpha < 1$, are a pair of mutually associated kernels that belong to the Sonin set $\mathcal{S}_{-1}$. For these kernels, GFI and GFD are well-known Riemann–Liouville and Caputo fractional operators [8].

Example 4. The Sonin pair of kernels [34] (p. 3628),

$$M(t) = h_{\alpha,\lambda}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda t}, \quad (78)$$

$$K(t) = h_{1-\alpha,\lambda}(t) + \frac{\lambda^\alpha}{\Gamma(1-\alpha)} \gamma(1-\alpha, \lambda t), \quad (79)$$

and vice versa, where $0 < \alpha < 1$ and $\lambda \geq 0, t > 0$, and $\gamma(\beta, t)$, is the incomplete gamma function

$$\gamma(\beta, t) = \int_0^t \tau^{\beta-1} e^{-\tau} d\tau, \quad (80)$$

where $t > 0$.

Example 5. The following pair of kernels,

$$M(t) = (\sqrt{t})^{\alpha-1} J_{\alpha-1}(2\sqrt{t}), \quad (81)$$

$$K(t) = (\sqrt{t})^{-\alpha} I_{-\alpha}(2\sqrt{t}), \quad (82)$$

and vice versa are kernels from the Sonin set (see [25,26], [34] (p. 3627)) if $0 < \alpha < 1$, where

$$I_\nu(t) = \sum_{k=0}^{\infty} \frac{(-1)^k (t/2)^{2k+\nu}}{k! \Gamma(k+\nu+1)}, \quad I_\nu(t) = \sum_{k=0}^{\infty} \frac{(t/2)^{2k+\nu}}{k! \Gamma(k+\nu+1)} \quad (83)$$

are the Bessel and the modified Bessel functions, respectively.

Example 6. As a special case of kernels (81) and (82), we can consider the pair

$$M(t) = \frac{\cos(2\sqrt{t})}{\sqrt{\pi t}}, \quad (84)$$

$$K(t) = \frac{\cosh(2\sqrt{t})}{\sqrt{\pi t}}, \quad (85)$$

and vice versa.

Example 7. The example of the Sonin pair is the kernel pair [38]

$$M(t) = h_{1-\beta+\alpha}(t) + h_{1-\beta}(t), \quad (86)$$

$$K(t) = t^{\beta-1} E_{\alpha,\beta}[-t^\alpha], \quad (87)$$

where $0 < \alpha < \beta < 1$ and $E_{\alpha,\beta}[z]$ is the two-parameter Mittag-Leffler function

$$E_{\alpha,\beta}[z] = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (88)$$

where $\alpha > 0$ and $\beta, z \in \mathbb{C}$.

Example 8. The following Sonin pair is described by the kernels (see Equation (7.15) in [34] (p. 3629))

$$M(t) = t^{\alpha-1} \Phi(\beta, \alpha; -\lambda t), \quad (89)$$

$$K(t) = \frac{\sin(\pi\alpha)}{\pi} t^{-\alpha} \Phi(-\beta, 1-\alpha; -\lambda t) \quad (90)$$

or vice versa, where $0 < \alpha < 1$ and $\Phi(\beta, \alpha; z)$ is Kummer's function

$$\Phi(\beta, \alpha; z) = \sum_{k=0}^{\infty} \frac{(\beta)_k}{(\alpha)_k} \frac{z^k}{k!}. \quad (91)$$

Example 9. The following pair of kernels belongs to the Sonin set $\mathcal{S}_{-1}$ (see Equations (7.16) and (7.18) in [34] (p. 3629)),

$$M(t) = 1 + \frac{\lambda}{\Gamma(\alpha)\sqrt{t}}, \quad (92)$$

$$K(t) = \frac{1}{\sqrt{\pi t}} - \lambda e^{\lambda^2 t} \operatorname{erfc}\left(\lambda \sqrt{t}\right), \quad (93)$$

or vice versa, where $\lambda > 0$ and $\operatorname{erfc}(z)$ is the complementary error function

$$\operatorname{erfc}(z) = 1 - \operatorname{erf}(z) = 1 - \frac{2}{\sqrt{\pi}} \int_0^t e^{-z^2} dz. \quad (94)$$

Example 10. The following Sonin pair of kernels (see Equations (7.17) and (7.19) in [34] (pp. 3629–3630)),

$$M(t) = 1 - \frac{\lambda}{\Gamma(\alpha)} t^{\alpha-1}, \quad (95)$$

$$K(t) = \lambda t^{-\alpha} E_{1-\alpha, 1-\alpha} [\lambda t^{1-\alpha}], \quad (96)$$

or vice versa, where $\lambda > 0$.

Example 11. As a Sonin pair of kernels, we can use some functions with power-logarithmic singularities at the origin [34] (pp. 3627–3630). The following pair of kernels (see Equations (7.22)–(7.24) in [34] (p. 3630)) with power-logarithmic singularities,

$$M(t) = \frac{A - \ln(t)}{\Gamma(\alpha)} t^{\alpha-1}, \quad (97)$$

$$K(t) = \mu_{\alpha, h}(t) = \int_0^\infty \frac{t^{z-\alpha} e^{hz}}{\Gamma(z+1-\alpha)} dz, \quad (98)$$

or vice versa, where $\mu_{\alpha, h}(t)$ is Volterra's function with

$$h = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - A.$$

We assume that nonlocal discrete mappings which are derived from equations with GFD and GFI for these kernels are described by the same equations as for the kernels that belong to $C_{-1}(0, \infty)$.

The discrete mappings with an arbitrary function $N = G(t, X(t))$ are called universal in nonlinear dynamics. We can also call the proposed mappings of the GFDynamics universal general mappings with nonlocality in time. The universality of the mappings is due to the use of an arbitrary nonlinear function $N = G(t, X(t))$, and the generality is due to the use of almost arbitrary kernels of operators $M(t)$ and $K(t)$ that belong to the Sonin set $\mathcal{S}_{-1}$.

5. General Fractional Dynamics of Arbitrary Order

In this section, general fractional dynamics is described by the equations with GFI and GFD of arbitrary order proposed by Yu. Luchko [28].

5.1. General Fractional Calculus of Arbitrary Order

In [28], the Sonin set of kernel pairs is defined as follows:

Definition 4. Let the functions $\mu(t)$, $\nu(t)$ belong to the function space

$$C_{-1,0}(0, \infty) := \{X : X(t) = t^p Y(t), \quad t \in (0, \infty), \quad -1 < p < 0, \quad Y(t) \in C[0, \infty)\}, \quad (99)$$

and the Sonin condition

$$(\mu * \nu)(t) = \{1\} \quad (100)$$

hold for $t \in (0, \infty)$. The set of such kernel pairs is called the Sonin set $\mathcal{S}_{-1}$. For the kernel $\mu(t)$, the kernel $\nu(t)$ is called its associated kernel.

In [28], generalization of Definition 4 was proposed.

Definition 5. Let the function $M(t)$, $K(t)$ belong to the function spaces

$$M(t) \in C_{-1}(0, \infty), \quad K(t) \in C_{-1,0}(0, \infty), \quad (101)$$

where $C_{-1,0}(0, \infty)$ is defined by (99),

$$C_{-1}(0, \infty) := \{X : X(t) = t^p Y(t), t \in (0, \infty), p > -1, Y(t) \in C[0, \infty)\}, \quad (102)$$

and the Luchko condition

$$(M * K)(t) = \{1\}^n = h_n(t) = \frac{t^{n-1}}{(n-1)!} \quad (103)$$

holds for $t \in (0, \infty)$. The set of such kernel pairs is called the Luchko set $\mathcal{L}_{n,0}$.

In [28], an approach to constructing a pair $M(t)$, $K(t)$ of the kernels from the Luchko set $\mathcal{L}_{n,0}$ by using the kernels $\mu(t)$, $\nu(t)$ from the Sonin set $\mathcal{S}_{-1}$ was proposed. This approach is formulated by the following theorem that was proved in [28].

Theorem 6. Let $\mu(t)$, $\nu(t)$ be a Sonin pair of kernels from $\mathcal{S}_{-1}$. Then, the pair $M(t)$, $K(t)$ of the kernels

$$M(t) = (\{1\}^{n-1} * \mu)(t), \quad K(t) = \nu(t) \quad (104)$$

belongs to the set $\mathcal{L}_{n,0}$.

Theorem 6 was proved by Yu. Luchko in [28].

Using Definition 5 and Theorem 6, GFI and GFD of arbitrary order are defined in [28].

Definition 6. Let $M(t)$, $K(t)$ be a pair of kernels (104) from the Luchko set $\mathcal{L}_{n,0}$. GFI with the kernel $M(t)$ is

$$I_{(M)}^{t,n}[\tau]X(\tau) := \int_0^t d\tau M(t-\tau)X(\tau) = \int_0^t d\tau (\{1\}^{n-1} * \mu)(t-\tau)X(\tau), \quad (105)$$

where $t > 0$. GFD with the kernel $K(t)$ are

$$D_{(K)}^{t,n}[\tau]X(\tau) := \frac{d^n}{d\tau^n} \int_0^t d\tau K(t-\tau)X(\tau) = \frac{d^n}{d\tau^n} \int_0^t d\tau \nu(t-\tau)X(\tau), \quad (106)$$

$$D_{(K)}^{t,n,*}[\tau]X(\tau) := \int_0^t d\tau K(t-\tau)X^{(n)}(\tau) = \int_0^t d\tau \nu(t-\tau)X^{(n)}(\tau), \quad (107)$$

where $t > 0$ and $X^{(n)}(\tau) = d^n X(\tau) / d\tau^n$.

In [28], the following fundamental theorems of GFC for GFI (105) and GFD (106) and (107) are proved:

Theorem 7. (The first fundamental theorem for GFC of arbitrary order.)

Let $M(t)$, $K(t)$ be a kernel pair from the Luchko set $\mathcal{L}_{n,0}$.

If $X(t) \in C_{-1}(0, \infty)$, then

$$D_{(K)}^{t,n}[\tau] I_{(M)}^{\tau,n}[s] X(s) = X(t). \quad (108)$$

If $X(t) \in C_{-1,(K)}(0, \infty)$, then

$$D_{(K)}^{t,n,*}[\tau] I_{(M)}^{\tau,n}[s] X(s) = X(t). \quad (109)$$

The function $X(t)$ belongs to $C_{-1,(K)}(0, \infty)$ if it can be represented as $X(t) = I_{(K)}^{t,n}[\tau] Y(\tau)$ where $Y(t) \in C_{-1}(0, \infty)$.

Theorem 8. (The second fundamental theorem for GFC of arbitrary order.)

Let $M(t)$, $K(t)$ be a kernel pair from the Luchko set $\mathcal{L}_{n,0}$.

If $X(t) \in C_{-1}^n(0, \infty)$, then

$$I_{(M)}^{t,n}[\tau] D_{(K)}^{\tau,n,*}[s] X(s) = X(t) - \sum_{k=0}^{n-1} X^{(k)}(0) h_{k+1}(t). \quad (110)$$

If $X(t) \in C_{-1,(M)}^n(0, \infty)$, then

$$I_{(M)}^{t,n}[\tau] D_{(K)}^{\tau,n}[s] X(s) = X(t). \quad (111)$$

The function $X(t)$ belongs to $C_{-1,M}^n(0, \infty)$ if it can be represented in the form $X(t) = I_{(M)}^{t,n}[\tau] Y(\tau) \in C_{-1}^{-1}(0, \infty)$ where $Y(t) \in C_{-1}(0, \infty)$.

Theorem 7 and 8 were proved by Yu. Luchko in [28] (pp. 11–12).

5.2. General Momenta of Arbitrary Order

In fractional dynamics, to describe nonlocal mappings of arbitrary order, we should define generalized momenta [2]. For generalized fractional dynamics, we should also use the generalized momenta. We used the following definitions.

Definition 7. Let $M(t)$, $K(t)$ be a pair of kernels from the Luchko set $\mathcal{L}_{n,0}$ in the form

$$M(t) = (\{1\}^{n-1} * \mu)(t), \quad K(t) = \nu(t). \quad (112)$$

Then, the generalized momentum $V_k(t)$ is defined as

$$V_k(t) := I_{(h)}^{t,k}[\tau] X(\tau) = \int_0^t d\tau h_k(t - \tau) X(\tau), \quad (113)$$

where $X(t) \in C_{-1}(0, \infty)$. The generalized momenta $P_k(t)$, $P_k^*(t)$ are defined by the equations

$$P_k(t) := D_{(K)}^{t,k}[\tau] X(\tau) = \frac{d^k}{dt^k} \int_0^t d\tau K(t - \tau) X(\tau), \quad (114)$$

$$P_k^*(t) := \frac{d^k}{dt^k} X(t), \quad (115)$$

where $k = 1, \dots, n-1$, $t > 0$, and

$$h_n(t) = \frac{t^{n-1}}{(n-1)!}. \quad (116)$$

To simplify, we can define

$$V_0(t) := X(t), \quad P_0(t) := X(t), \quad P_0^*(t) := X(t), \quad (117)$$

where $X(t) \in C_{-1}^k(0, \infty)$.

Let us prove the properties of the generalized momenta that describe connections of GFI and GFD of the variable $X(t)$ with the momenta $V_k(t)$, $P_k(t)$, and $P_k^*(t)$.

Theorem 9. Let $V_k(t)$ be defined by (113). Then, the equation

$$I_{(M)}^{t,n}[\tau] X(\tau) = I_{(M)}^{t,n-k}[\tau] V_k(\tau) \quad (118)$$

holds if $X(t) \in C_{-1}(0, \infty)$ and $V_k(t) \in C_{-1}(0, \infty)$.

Let $P_k(t)$ and $P_k^*(t)$ be defined by (114) and (115), respectively. Then, the equations

$$D_{(K)}^{t,n}[\tau]X(\tau) = \frac{d^{n-k}}{dt^{n-k}}P_k(t), \quad (119)$$

$$D_{(K)}^{t,n,*}[\tau]X(\tau) = D_{(K)}^{t,n-k,*}[\tau]P_k^*(\tau), \quad (120)$$

are satisfied if $X(t) \in C_{-1}(0, \infty)$ and $P_k(t), P_k^*(t) \in C_{-1}^k(0, \infty)$.

Proof. (A) Let us first prove Equation (118) for GFI. We can use the fact that the standard integration of order $n \in \mathbb{N}$ (n -fold integral) has the form [5] (p. 33)

$$\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \dots \int_0^{\tau_{n-1}} d\tau_n X(\tau_n) = \int_0^t d\tau h_n(t-\tau)X(\tau) = I_{(h)}^{t,n}[\tau]X(\tau), \quad (121)$$

where

$$h_n(t-\tau) = \frac{(t-\tau)^{n-1}}{(n-1)!}. \quad (122)$$

Therefore, the GFI that is given by (105) with $n > 1$ is

$$I_{(M)}^{t,n}[\tau]X(\tau) = \int_0^t d\tau (h_{n-1} * \mu)(t-\tau)X(\tau) = ((h_{n-1} * \mu) * X)(t) = (h_{n-1} * (\mu * X))(t) = \int_0^t d\tau h_{n-1}(t-\tau)(\mu * X)(\tau) = I_{(h)}^{t,n}[\tau](\mu * X)(\tau). \quad (123)$$

Using

$$I_{(h)}^{t,n}[\tau]X(\tau) = I_{(h)}^{t,n-k}[\tau] I_h^{\tau,k}[s]X(s), \quad (124)$$

we get

$$I_{(M)}^{t,n-k}[\tau]V_k(\tau) = I_{(M)}^{t,n-k}[\tau]I_h^{\tau,k}[s]X(s) = I_{(M)}^{t,n}[\tau]X(\tau). \quad (125)$$

(B) Let us prove Equation (119) by using definitions (114) and (106). Equation (119) is proved by the following transformations:

$$\frac{d^{n-k}}{dt^{n-k}}P_k(t) = \frac{d^{n-k}}{dt^{n-k}}D_{(K)}^{t,k}[\tau]X(\tau) = \frac{d^{n-k}}{dt^{n-k}}\frac{d^k}{dt^k}\int_0^t d\tau K(t-\tau)X(\tau) = \frac{d^n}{dt^n}\int_0^t d\tau K(t-\tau)X(\tau) = D_{(K)}^{t,n}[\tau]X(\tau). \quad (126)$$

Let us prove Equation (120) by using definitions (115) and (107). Equation (120) is proved by the following transformations:

$$D_{(K)}^{t,n-k,*}[\tau]P_k^*(t) = \int_0^t d\tau K(t-\tau)(P_k^*)^{(n-k)}(\tau) = \int_0^t d\tau K(t-\tau)\left(X^{(k)}\right)^{(n-k)}(\tau) = \int_0^t d\tau K(t-\tau)X^{(n)}(\tau) = D_{(K)}^{t,n}[\tau]X(\tau). \quad (127)$$

□

Theorem 9 allows us to derive exact solutions of equations with GFI and GFD of arbitrary order and periodic kicks for $V_k(t)$, $P_k(t)$, and $P_k^*(t)$ with $k = 1, \dots, n-1$ and the variable $X(t)$.

Theorem 10. Let the functions $K(t)$ and $M(t)$ be a Luchko pair of kernels from $\mathcal{L}_{n,0}$ and $X(t) \in C_{-1}(0, \infty)$. Then, the equations

$$I_{(M)}^{t,n}[\tau]X(\tau) = \lambda G(t, X(t)) \sum_{j=1}^{\infty} \delta((t+\varepsilon)/T-j), \quad (128)$$

$$I_{(M)}^{t,n-k}[\tau]V_k(\tau) = \lambda G(t, X(t)) \sum_{j=1}^{\infty} \delta((t+\varepsilon)/T-j), \quad (129)$$

where $k = 1, \dots, n-1$, have the solutions

$$X(t) = \lambda T \sum_{j=1}^m K^{(n)}(t-(Tj+\varepsilon)) G(Tj-\varepsilon, X(Tj-\varepsilon)), \quad (130)$$

$$V_k(t) = \lambda T \sum_{j=1}^m K^{(n-k)}(t-(Tj+\varepsilon)) G(Tj-\varepsilon, X(Tj-\varepsilon)) \quad (131)$$

if $Tm < t < T(m+1)$. For the time points $s = Tj - \varepsilon$ at $\varepsilon \rightarrow 0+$ and the variables

$$X_j = \lim_{\varepsilon \rightarrow 0+} X(Tj - \varepsilon), \quad V_{k,j} = \lim_{\varepsilon \rightarrow 0+} V_k(Tj - \varepsilon), \quad (j = 1, \dots, m+1), \quad (132)$$

solutions (130) and (131) are represented by the general nonlocal mappings

$$X_{m+1} = X_m + \lambda T K^{(n)}(T)G(Tm, X_m) + \lambda T \sum_{j=1}^{m-1} \Omega_{(K)}^{(n)}(T, m-j)G(jT, X_j), \quad (133)$$

$$V_{k,m+1} = V_{k,m} + \lambda T K^{(n-k)}(T)G(Tm, X_m) + \lambda T \sum_{j=1}^{m-1} \Omega_{(K)}^{(n-k)}(T, m-j)G(jT, X_j), \quad (134)$$

where function $K(t) \in C_{-1,0}(0, \infty)$ is a kernel associated with the kernel $M(t)$ and

$$\Omega_{(K)}(T, z) := K(T(z+1)) - K(Tz), \quad \Omega_{(K)}^{(k)}(T, z) := \frac{1}{T^k} \frac{d^k}{dz^k} \Omega_{(K)}(T, z). \quad (135)$$

Proof. In this proof, we use the first fundamental theorem for GFC of arbitrary order that was proved in [28], which states that if $M(t)$, $K(t)$ is a kernel pair from the Luchko set $\mathcal{L}_{n,0}$ and $X(t) \in C_{-1}(0, \infty)$, then

$$D_{(K)}^{t,n}[\tau] I_{(M)}^{t,n}[s] X(s) = X(t) \quad (136)$$

holds for $t > 0$.

Applying GFD $D_{(K)}^{s,n-k}[t]$ with the kernel $K(t)$, which is associated with the kernel $M(t)$ of GFI $I_{(M)}^{t,n}$, to Equations (128) and (129), we derive

$$D_{(K)}^{s,n-k}[t] I_{(M)}^{t,n-k}[\tau] V_k(\tau) = \lambda D_{(K)}^{s,n-k}[t] G(t, X(t)) \sum_{j=1}^{\infty} \delta((t+\varepsilon)/T - j), \quad (137)$$

where $k = 1, \dots, n-1$ and $V_k(\tau) := X(\tau)$ for $k = 0$. Using (136), Equation (137) is written as

$$V_k(s) = \lambda D_{(K)}^{s,n-k}[t] G(t, X(t)) \sum_{j=1}^{\infty} \delta((t+\varepsilon)/T - j). \quad (138)$$

Using the definition of $D_{(K)}^{s,n-k}[t]$, we get

$$V_k(s) = \lambda \frac{d^{n-k}}{ds^{n-k}} \int_0^s dt K(s-t) G(t, X(t)) \sum_{j=1}^{\infty} \delta((t+\varepsilon)/T - j). \quad (139)$$

For $Tm < s < T(m+1)$, Equation (139) is

$$V_k(s) = \lambda \sum_{j=1}^m \frac{d^{n-k}}{ds^{n-k}} \int_0^s dt K(s-t) G(t, X(t)) \delta((t+\varepsilon)/T - j). \quad (140)$$

Using Equation (31), Equation (140) gives

$$V_k(s) = \lambda T \sum_{j=1}^m \frac{d^{n-k}}{ds^{n-k}} K(s - (Tj - \varepsilon)) G(Tj - \varepsilon, X(Tj - \varepsilon)) \quad (141)$$

that can be written as

$$V_k(s) = \lambda T \sum_{j=1}^m K^{(n-k)}(s - (Tj - \varepsilon)) G(Tj - \varepsilon, X(Tj - \varepsilon)), \quad (142)$$

where $K^{(k)}(z) = d^{(k)}K(z)/dz^k$. Equation (142) gives solutions (130) and (131).

Using (142) for $s = T(m+1) - \varepsilon$ and $s = Tm - \varepsilon$, we obtain

$$V_k(T(m+1) - \varepsilon) = \lambda T \sum_{j=1}^m K^{(n-k)}(T(m+1) - Tj) G(Tj - \varepsilon, X(Tj - \varepsilon)), \quad (143)$$

$$V_k(Tm - \varepsilon) = \lambda T \sum_{j=1}^m K^{(n-k)}(Tm - Tj)G(Tj - \varepsilon, X(Tj - \varepsilon)). \quad (144)$$

Using variables (132), solutions (143) and (144) at the limit $\varepsilon \rightarrow 0+$ give

$$V_{k,m+1} = \lambda T \sum_{j=1}^m K^{(n-k)}(T(m+1-j))G(Tj, X_j), \quad (145)$$

$$V_{k,m} = \lambda T \sum_{j=1}^{m-1} K^{(n-k)}(T(m-j))G(Tj, X_j). \quad (146)$$

Subtracting Equation (146) from Equation (145), we obtain

$$V_{k,m+1} - V_{k,m} = \lambda T K^{(n-k)}(T)G(Tm, X_m) + \lambda T \sum_{j=1}^{m-1} (K^{(n-k)}(T(m+1-j)) - K^{(n-k)}(T(m-j)))G(Tj, X_j). \quad (147)$$

where $k = 0, 1, \dots, n-1$. Then, using (135), Equation (147) takes form (133), (134). $\square$

Let us derive general nonlocal mappings from the equations with GFD $D_{(K)}^{t,n}$ of arbitrary order and periodic kicks.

Theorem 11. Let $K(t)$ and $M(t)$ be a pair of kernels from the Luchko set $\mathcal{L}_{n,0}$ and $X(t) \in C_{-1}^m(0, \infty)$. Then, the equations

$$D_{(K)}^{t,n}[\tau]X(\tau) = \lambda G(t, X(t)) \sum_{j=1}^{\infty} \delta((t+\varepsilon)/T - j), \quad (148)$$

$$\frac{d^{n-k}}{dt^{n-k}} P_k(\tau) = \lambda G(t, X(t)) \sum_{j=1}^{\infty} \delta((t+\varepsilon)/T - j), \quad (149)$$

where $k = 1, \dots, n-1$, have the solutions

$$X(t) = \lambda T \sum_{j=1}^m M(t - Tj + \varepsilon)G(Tj - \varepsilon, X(Tj - \varepsilon)), \quad (150)$$

$$P_k(t) = \sum_{j=0}^{n-k-1} P_k^{(j)}(0)h_{j+1}(t) + \lambda T \sum_{j=1}^m h_{n-k}(t - Tj + \varepsilon)G(Tj - \varepsilon, X(Tj - \varepsilon)) \quad (151)$$

if $Tm < t < T(m+1)$. For the time points $s = Tj - \varepsilon$ at $\varepsilon \rightarrow 0+$ and the variables

$$X_j = \lim_{\varepsilon \rightarrow 0+} X(Tj - \varepsilon), \quad P_{k,j} = \lim_{\varepsilon \rightarrow 0+} P_k(Tj - \varepsilon), \quad (j = 1, \dots, m+1), \quad (152)$$

solutions of Equations (148) and (149) are represented by the general nonlocal mappings

$$X_{m+1} = X_m + \lambda T M(T)G(Tm, X_m) + \lambda T \sum_{j=1}^{m-1} \Omega_{(M)}(T, m-j)G(jT, X_j), \quad (153)$$

$$P_{k,m+1} = P_{k,m} + \sum_{j=0}^{n-k-1} P_k^{(j)}(0)\Omega_{(h_{j+1})}(T, m) + \lambda T h_{n-k}(T)G(Tm, X_m) + \lambda T \sum_{j=1}^{m-1} \Omega_{(h_{n-k})}(T, m-j)G(jT, X_j), \quad (154)$$

where $M(t) \in C_{-1}(0, \infty)$ is a function that is a kernel associated with the kernel $K(t)$, the function $\Omega_{(M)}(T, z)$ is defined by (135), and $h_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$ with $\alpha > 0$.

Proof. In the proof, we use the second fundamental theorem for GFC of arbitrary order that was proved in [28]. This theorem states that if $M(t)$, $K(t)$ is a kernel pair from the Luchko set $\mathcal{L}_{n,0}$ and $X(t) \in C_{-1,(M)}^n(0, \infty)$, then

$$I_{(M)}^{t,n}[\tau]D_{(K)}^{\tau,n}[s]X(s) = X(t). \quad (155)$$

The function $X(t)$ belongs to $C_{-1,(M)}^n(0, \infty)$ if it can be represented in the form $X(t) = I_{(M)}^{t,n}[\tau]Y(\tau) \in C_{-1}(0, \infty)$ where $Y(t) \in C_{-1}(0, \infty)$.

The action of the integral operator $I_{(M)}^{s,n}[t]$ on Equation (148) and of the operator $I_{(h)}^{t,n-k}[t]$ on (149) give

$$I_{(M)}^{s,n}[t]D_{(K)}^{t,n}[\tau]X(\tau) = \lambda I_{(M)}^{s,n}[t]G(t, X(t)) \sum_{j=1}^{\infty} \delta((t+\varepsilon)/T-j), \quad (156)$$

$$I_{(h)}^{s,n-k}[t] \frac{d^{n-k}}{dt^{n-k}} P_k(\tau) = \lambda I_{(h)}^{s,n-k}[t]G(t, X(t)) \sum_{j=1}^{\infty} \delta((t+\varepsilon)/T-j), \quad (157)$$

where $k = 1, \dots, n-1$, and

$$I_{(h)}^{s,n-k}[t]X(t) = \int_0^s dt h_{n-k}(s-t)X(t).$$

Using the second fundamental theorems for $D_{(K)}^{t,n}$ in form (155) and d^{n-k}/dt^{n-k} , Equations (156) and (157) give

$$X(s) = \lambda I_{(M)}^{s,n}[t]G(t, X(t)) \sum_{j=1}^{\infty} \delta((t+\varepsilon)/T-j), \quad (158)$$

$$P_k(s) - \sum_{j=0}^{n-k-1} P_k^{(j)}(0)h_{j+1}(s) = \lambda I_{(h)}^{s,n-k}[t]G(t, X(t)) \sum_{j=1}^{\infty} \delta((t+\varepsilon)/T-j). \quad (159)$$

Using the definition of $I_{(M)}^{s,n-k}$ and $I_{(h)}^{s,n-k}$, Equations (158) and (159) are written as

$$X(s) = \lambda \int_0^s dt M_n(s-t)G(t, X(t)) \sum_{j=1}^{\infty} \delta((t+\varepsilon)/T-j), \quad (160)$$

$$P_k(s) - \sum_{j=0}^{n-k-1} P_k^{(j)}(0)h_{j+1}(s) = \lambda \int_0^s dt h_{n-k}(s-t)G(t, X(t)) \sum_{j=1}^{\infty} \delta((t+\varepsilon)/T-j), \quad (161)$$

where

$$M_n(t) = (h_{n-1} * \mu)(t).$$

For $Tm < s < T(m+1)$, Equations (160) and (161) are

$$X(s) = \lambda \sum_{j=1}^m \int_0^s dt M_n(s-t)G(t, X(t))\delta((t+\varepsilon)/T-j), \quad (162)$$

$$P_k(s) - \sum_{j=0}^{n-k-1} P_k^{(j)}(0)h_{j+1}(s) = \lambda \sum_{j=1}^m \int_0^s dt h_{n-k}(s-t)G(t, X(t))\delta((t+\varepsilon)/T-j). \quad (163)$$

Using Equation (31), Equations (162) and (163) give

$$X(s) = \lambda T \sum_{j=1}^m M_n(s - (Tj - \varepsilon))G(Tj - \varepsilon, X(Tj - \varepsilon)), \quad (164)$$

$$P_k(s) - \sum_{j=0}^{n-k-1} P_k^{(j)}(0)h_{j+1}(s) = \lambda T \sum_{j=1}^m h_{n-k}(s - (Tj - \varepsilon))G(Tj - \varepsilon, X(Tj - \varepsilon)). \quad (165)$$

Equations (164) and (165) give solutions (150) and (151).

Using (164) and (165) for $s = T(m+1) - \varepsilon$ and $s = Tm - \varepsilon$, we obtain

$$X(T(m+1) - \varepsilon) = \lambda T \sum_{j=1}^m M_n(T(m+1) - Tj)G(Tj - \varepsilon, X(Tj - \varepsilon)), \quad (166)$$

$$P_k(T(m+1) - \varepsilon) - \sum_{j=0}^{n-k-1} P_k^{(j)}(0)h_{j+1}(T(m+1) - \varepsilon) = \lambda T \sum_{j=1}^m h_{n-k}(T(m+1) - Tj)G(Tj - \varepsilon, X(Tj - \varepsilon)), \quad (167)$$

and

$$X(Tm - \varepsilon) = \lambda T \sum_{j=1}^{m-1} M_n(Tm - Tj)G(Tj - \varepsilon, X(Tj - \varepsilon)), \quad (168)$$

$$P_k(Tm - \varepsilon) - \sum_{j=0}^{n-k-1} P_k^{(j)}(0)h_{j+1}(Tm - \varepsilon) = \lambda T \sum_{j=1}^{m-1} h_{n-k}(Tm - Tj)G(Tj - \varepsilon, X(Tj - \varepsilon)). \quad (169)$$

Using variables (152), solutions (166), (167), (168) and (169) at the limit $\varepsilon \rightarrow 0+$ give

$$X_{m+1} = \lambda T \sum_{j=1}^m M_n(T(m-j+1))G(Tj, X_j), \quad (170)$$

$$P_{k,m+1} - \sum_{j=0}^{n-k-1} P_k^{(j)}(0)h_{j+1}(T(m+1) - \varepsilon) = \lambda T \sum_{j=1}^m h_{n-k}(T(m-j+1))G(Tj, X_j), \quad (171)$$

and

$$X_m = \lambda T \sum_{j=1}^{m-1} M_n(T(m-j))G(Tj, X_j), \quad (172)$$

$$P_{k,m} - \sum_{j=0}^{n-k-1} P_k^{(j)}(0)h_{j+1}(Tm) = \lambda T \sum_{j=1}^{m-1} h_{n-k}(T(m-j))G(Tj, X_j). \quad (173)$$

Subtracting Equations (172) and (173) from Equations (170) and (171), we obtain

$$X_{m+1} - X_m = \lambda T M_n(T)G(Tm, X_m) + \lambda T \sum_{j=1}^{m-1} (M_n(T(m-j+1)) - M_n(T(m-j)))G(Tj, X_j), \quad (174)$$

$$P_{k,m+1} - P_{k,m} = \sum_{j=0}^{n-k-1} P_k^{(j)}(0)(h_{j+1}(T(m+1)) - h_{j+1}(Tm)) + \lambda T h_{n-k}(T)G(Tm, X_m) + \lambda T \sum_{j=1}^{m-1} (h_{n-k}(T(m-j+1)) - h_{n-k}(T(m-j)))G(Tj, X_j), \quad (175)$$

where $k = 1, \dots, n-1$. Then, using (135), Equations (174) and (175) take form (153), (154). $\square$

Let us derive general nonlocal mappings from the equations with GFD $D_{(K)}^{t*,n}$ of arbitrary order and periodic kicks.

Theorem 12. Let functions $K(t)$ and $M(t)$ be a pair of kernels from the Luchko set $\mathcal{L}_{n,0}$ and $X(t) \in C_{-1}^m(0, \infty)$. Then, the equations

$$D_{(K)}^{t*,n}[\tau]X(\tau) = \lambda G(t, X(t)) \sum_{j=1}^{\infty} \delta((t+\varepsilon)/T - j), \quad (176)$$

$$D_{(K)}^{t*,n-k}[\tau]P_k^*(\tau) = \lambda G(t, X(t)) \sum_{j=1}^{\infty} \delta((t+\varepsilon)/T - j), \quad (177)$$

where $k = 1, \dots, n-1$, have the solutions

$$X(t) = \sum_{j=0}^{n-1} X^{(j)}(0)h_{j+1}(t) + \lambda T \sum_{j=1}^m M(t - Tj + \varepsilon)G(Tj - \varepsilon, X(Tj - \varepsilon)), \quad (178)$$

$$P_k^*(t) = \sum_{j=0}^{n-k-1} P_k^{(j)}(0)h_{j+1}(t) + \lambda T \sum_{j=1}^m M_{n-k}(t - Tj + \varepsilon)G(Tj - \varepsilon, X(Tj - \varepsilon)) \quad (179)$$

if $Tm < t < T(m+1)$, where

$$M_{n-k}(t) := (h_{n-k-1} * \mu)(t), \quad M_0(t) := \mu(t). \quad (180)$$

For the time points $s = Tj - \varepsilon$ at $\varepsilon \rightarrow 0+$ and the variables

$$X_j = \lim_{\varepsilon \rightarrow 0+} X(Tj - \varepsilon), \quad P_{k,j}^* = \lim_{\varepsilon \rightarrow 0+} P_k^*(Tj - \varepsilon), \quad (j = 1, \dots, m+1), \quad (181)$$

solutions (178) and (179) are represented by the general nonlocal mappings

$$X_{m+1} = X_m + \sum_{j=0}^{n-1} X^{(j)}(0)\Omega_{(h_{j+1})}(T, m) + \lambda T M_n(T)G(T(m+1), X_m) + \lambda T \sum_{j=1}^{n-1} \Omega_{(M_{n-k})}(T, m-j)G(jT, X_j), \quad (182)$$

$$P_{k,m+1}^* = P_{k,m}^* + \sum_{j=0}^{n-k-1} P_k^{*(j)}(0)\Omega_{(h_{j+1})}(T, m) + \lambda T M_{n-k}(T)G(Tm, X_m) + \lambda T \sum_{j=1}^{m-1} \Omega_{(M_{n-k})}(T, m-j)G(jT, X_j), \quad (183)$$

where $M(t) \in C_{-1}(0, \infty)$ is a function that is a kernel associated with the kernel $K(t)$.

Proof. In the proof, we use the second fundamental theorem for GFC of arbitrary order that was proved in [28]. This theorem states that if $M(t)$, $K(t)$ is a kernel pair from the Luchko set $\mathcal{L}_{n,0}$ and $X(t) \in C_{-1}^n(0, \infty)$, then

$$I_{(M)}^{s,n}[t]D_{(K)}^{t*,n}[\tau]X(\tau) = X(s) - \sum_{j=0}^{n-1} X^{(j)}(0)h_{j+1}(s). \quad (184)$$

Let us consider Equations (176) and (177) in the form

$$D_{(K)}^{t*,n-k}[\tau]P_k^*(\tau) = \lambda G(t, X(t)) \sum_{j=1}^{\infty} \delta((t + \varepsilon)/T - j), \quad (185)$$

where $k = 1, \dots, n-1$ and $P_k^*(\tau) = X(t)$ for $k = 0$. The action of the integral operator $I_{(M)}^{s,n-k}[t]$ on Equation (185) gives

$$I_{(M)}^{s,n-k}[t]D_{(K)}^{t*,n-k}[\tau]P_k^*(\tau) = \lambda I_{(M)}^{s,n-k}[t]G(t, X(t)) \sum_{j=1}^{\infty} \delta((t + \varepsilon)/T - j). \quad (186)$$

Using second fundamental theorems (184) for $D_{(K)}^{t*,n}$, we get

$$P_k^*(s) - \sum_{j=0}^{n-k-1} P_k^{*,(j)}(0)h_{j+1}(s) = \lambda I_{(M)}^{s,n-k}[t]G(t, X(t)) \sum_{j=1}^{\infty} \delta((t + \varepsilon)/T - j). \quad (187)$$

Using the definition of $I_{(M)}^{s,n-k}$, we write Equation (187) as

$$P_k^*(s) - \sum_{j=0}^{n-k-1} P_k^{*,(j)}(0)h_{j+1}(s) = \lambda \int_0^s dt M_{n-k}(s-t)G(t, X(t)) \sum_{j=1}^{\infty} \delta((t + \varepsilon)/T - j), \quad (188)$$

where $M_{n-k}(t)$ is defined by (180). For $Tm < s < T(m+1)$, Equation (188) is

$$P_k^*(s) - \sum_{j=0}^{n-k-1} P_k^{*,(j)}(0)h_{j+1}(s) = \lambda \sum_{j=1}^m \int_0^s dt M_{n-k}(s-t)G(t, X(t))\delta((t + \varepsilon)/T - j). \quad (189)$$

Using Equation (31), Equation (189) gives

$$P_k^*(s) - \sum_{j=0}^{n-k-1} P_k^{*,(j)}(0)h_{j+1}(s) = \lambda T \sum_{j=1}^m M_{n-k}(s - Tj + \varepsilon)G(Tj - \varepsilon, X(Tj - \varepsilon)). \quad (190)$$

Equation (190) gives solutions (178) and (179).

Using (190) for $s = T(m+1) - \varepsilon$ and $s = Tm - \varepsilon$, we obtain

$$P_k^*(T(m+1) - \varepsilon) - \sum_{j=0}^{n-k-1} P_k^{*,(j)}(0)h_{j+1}(T(m+1) - \varepsilon) = \lambda T \sum_{j=1}^m M_{n-k}(T(m+1) - Tj)G(Tj - \varepsilon, X(Tj - \varepsilon)), \quad (191)$$

$$P_k^*(Tm - \varepsilon) - \sum_{j=0}^{n-k-1} P_k^{*,(j)}(0)h_{j+1}(Tm - \varepsilon) = \lambda T \sum_{j=1}^{m-1} M_{n-k}(Tm - Tj)G(Tj - \varepsilon, X(Tj - \varepsilon)). \quad (192)$$

Using variables (181), solutions (191) and (192) at the limit $\varepsilon \rightarrow 0+$ give

$$P_{(k,m+1)}^* - \sum_{j=0}^{n-k-1} P_k^{*,(j)}(0)h_{j+1}(T(m+1)) = \lambda T \sum_{j=1}^m M_{n-k}(T(m-j+1))G(Tj, X_j), \quad (193)$$

$$P_{k,m}^* - \sum_{j=0}^{n-k-1} P_k^{*,(j)}(0)h_{j+1}(Tm) = \lambda T \sum_{j=1}^{m-1} M_{n-k}(T(m-j))G(Tj, X_j). \quad (194)$$

Subtracting Equation (194) from Equation (193), we obtain

$$P_{k,m+1}^* - P_{k,m}^* = \sum_{j=0}^{n-k-1} P_k^{*,(j)}(0)(h_{j+1}(T(m+1)) - h_{j+1}(Tm)) + \lambda T M_{n-k}(T)G(Tm, X_m) + \lambda T \sum_{j=1}^{m-1} (M_{n-k}(T(m-j+1)) - M_{n-k}(T(m-j)))G(Tj, X_j), \quad (195)$$

where $k = 0, 1, \dots, n-1$ with $P_0(t) = X(t)$. Then, using function (135), Equation (195) takes form (182) and (183). $\square$

6. Conclusions

This paper proposes a new direction of research that can be called general fractional dynamics (GFDynamics). It can be considered an interdisciplinary science, in which the nonlocal properties of linear and nonlinear dynamical systems are studied in the most general form. Therefore, it is important to have general fractional calculus that allows us to describe nonlocality in the most general form. The mathematical tools are general fractional calculus, equations with general fractional integrals (GFI) and derivatives (GFD), or general nonlocal mappings with discrete time. The most general form means that the operator kernels should belong to the most general set of kernels, for which general fractional calculus exists. At present, such a set is the Luchko set of kernel pairs that is used

in the proposed work. The distinction of Luchko's approach in comparison with other approaches is the construction of general fractional calculus of arbitrary order for kernels from the Luchko set. In GFDynamics, the results should be derived for the general form of nonlocality that is described by general-form operator kernels. This involves obtaining general results which are independent of the particular representations of the kernel.

In this paper, the concept of "general nonlocal mappings" that are the exact solutions of equations with GFI and GFD at discrete points is also proposed. In these mappings, the nonlocality is described by the kernels of GFD and GFI from the Luchko set.

We considered fractional dynamical systems with general nonlocality in time which are described by equations with GFI, GFD and periodic kicks. The exact solutions for these equations were obtained. These exact solutions with discrete timepoints were used to derive general nonlocal mappings without approximations. Equations with GFI and GFD of arbitrary order were also used to derive general nonlocal mappings. It should be emphasized that all the results were derived for the general form of kernels $M(t)$, $K(t)$ that belong to the Luchko set.

In this work, we derived general fractional dynamics with discrete time from general fractional dynamics with continuous time. Starting from equations with general fractional integrals and derivatives with kernels belonging to the Sonin set, we derived the exact solutions of these equations. Then, using these solutions for discrete timepoints, we obtained general universal mappings with nonlocality in time without approximations. The universality of the mappings is due to the use of an arbitrary nonlinear function $N = G(t, X(t))$, and the generality is due to the general operators, kernels $M(t)$ and $K(t)$ of GFC from the Luchko set $L_{n,0}$.

We assume that the proposed general nonlocal mappings can be studied by the methods and equations proposed in [67]. This possibility is due to the following. Note that the proposed general nonlocal mappings, which are derived from general fractional differential and integral equations, can be represented by Equations (40), (54) and (71), which have the form

$$X_n = X(0) + \lambda T \sum_{k=1}^{n-1} M(T(n-k))G(Tk, X_k), \quad (196)$$

$$X_n = \lambda T \sum_{k=1}^{n-1} M(T(n-k))G(Tk, X_k), \quad (197)$$

$$X_n = \lambda T \sum_{k=1}^{n-1} K^{(1)}(T(n-k))G(Tk, X_k). \quad (198)$$

We see that all the proposed general universal mappings (196), (197) and (198) with nonlocality in time and $G(t, X) = G(X)$, which are derived from equations with GFD and GFI, can be represented as

$$X_n = X_0 - \sum_{k=1}^{n-1} U(n-k)G^0(X_k) \quad (199)$$

with the function

$$G^0(X_k) = -\lambda T G(X_k), \quad (200)$$

and the kernel

$$U(n-k) = M(T(n-k)), \quad (201)$$

$$U(n-k) = K^{(1)}(T(n-k)), \quad (202)$$

where $M(t)$ and $K(t)$ are the Sonin pair of kernels of general fractional integral $I_{(M)}^t$ and general fractional derivatives $D_{(K)}^t, D_{(K)}^{t,*}$.

We see that general nonlocal mappings (196), (197) and (198) are described by discrete convolution. Equations of type (199) are important to study the chaotic and regular behavior of fractional systems that are nonlocal in time.

Equation (199) coincides with Equation (6) in [67], which is the starting point for the study of fractional mappings with nonlocality in time in the framework of the approach proposed by Mark Edelman in [67].

The general nonlocal mappings of arbitrary order, which are derived from equations with GFI and GFD of arbitrary order and described by Theorems (10)–(12), can also be represented by equations with discrete convolution. These mappings can be given by Equations (146), (172), (173) and (194) in the form

$$V_{k,m} = \lambda T \sum_{j=1}^{m-1} K^{(n-k)}(T(m-j))G(Tj, X_j), \quad (203)$$

where $k = 0, 1, \dots, n-1$ and $V_{0,m} = X_m$:

$$X_m = \lambda T \sum_{j=1}^{m-1} M_n(T(m-j))G(Tj, X_j), \quad (204)$$

$$P_{k,m} - \sum_{j=0}^{n-k-1} P_k^{(j)}(0) h_{j+1}(Tm) = \lambda T \sum_{j=1}^{m-1} h_{n-k}(T(m-j))G(Tj, X_j), \quad (205)$$

where $k = 1, \dots, n-1$, and

$$P_{k,m}^* - \sum_{j=0}^{n-k-1} P_k^{*(j)}(0) h_{j+1}(Tm) = \lambda T \sum_{j=1}^{m-1} M_{n-k}(T(m-j))G(Tj, X_j), \quad (206)$$

where $k = 0, 1, \dots, n-1$ and $P_{0,m}^* = X_m$.

The chaotic and regular behavior of the systems described by such general nonlocal mappings of arbitrary order can be investigated by generalizing the Edelman method [67] to nonlocal mappings of arbitrary order.

The behavior of fractional dynamical systems with nonlocality in time can be very different from the behavior of dynamical systems with locality in time. To study and describe the chaotic and regular behavior of dynamical systems, it is important to know periodic points. Fractional dynamical systems have only fixed points, but these systems can have asymptotically periodic points (sinks) [67]. For the first time, a method and equations which allow one to find asymptotically periodic points for nonlinear fractional systems with nonlocality in time were proposed in [67]. In [67], the equations which can be used to calculate coordinates of the asymptotically periodic sinks were derived.

Note that the proposed equations and mappings can be used to describe economic processes with memory [17,47], for non-Markovian quantum processes [48], processes in the dynamics of populations [49], and many other processes.

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