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Differential Games for an Infinite 2-Systems of Differential Equations

Muminjon Tukhtasinov [1,†], Gafurjan Ibragimov [2,†], Sarvinoz Kuchkarova [1,†] and Risman Mat Hasim [2,*]

[1] National University of Uzbekistan, University Street, Al-Mazar District, Tashkent 1000174, Uzbekistan; mumin51@mail.ru (M.T.); kuchkarova11@yandex.ru (S.K.)

[2] Department of Mathematics, Faculty of Science, University Putra Malaysia, Serdang 43400, Malaysia; ibragimov@upm.edu.my

[*] Correspondence: risman@upm.edu.my

[†] These authors contributed equally to this work.

Abstract: A pursuit differential game described by an infinite system of 2-systems is studied in Hilbert space l_2. Geometric constraints are imposed on control parameters of pursuer and evader. The purpose of pursuer is to bring the state of the system to the origin of the Hilbert space l_2 and the evader tries to prevent this. Differential game is completed if the state of the system reaches the origin of l_2. The problem is to find a guaranteed pursuit and evasion times. We give an equation for the guaranteed pursuit time and propose an explicit strategy for the pursuer. Additionally, a guaranteed evasion time is found.

Keywords: pursuer; evader; constraints; strategy

1. Introduction

The notion of differential games was introduced by Isaacs [1]. Differential games were developed by Bercovitz [2], Elliot and Kalton [3], Fleming [4,5], Friedman [6], Hajek [7], Ho, Bryson, and Baron [8], Pontryagin [9], Krasovskii [10], Petrosyan [11], Pshenichnyi [12], Subbotin [13], Chikrii [14], and others.

There are many works which deal with the infinite dimensional spaces, such as [15–19]. Differential games studied in the works [17,19–22] are devoted to constructing of the optimal strategies of players.

In the paper of Satimov and Tukhtasinov [23], pursuit and evasion differential games were studied for the parabolic equation. Various cases of control constraints (integral, geometric) were analyzed. Two sets were specified such that pursuit can be completed if the initial state belongs to the first set, and evasion is possible if the initial state belongs to the second set. Note that, the works [24,25] also relates to differential games described by PDE.

In the game with countably many pursuers studied Ibragimov et al. [17] in the Hilbert space l_2, the duration of the game is prescribed. A formula for the value of the game was found and optimal strategies of the players were constructed explicitly. In the work by Salimi and Ferrara [26], an optimal approach of a finite or denumerable pursuers to one evader is studied. In that paper a formula for the value of the game and optimal strategies of players are proposed. A time-optimal problem of transition of the state of system into the origin was studied by Azamov and Ruziboev [27]. The main result of that paper is estimate of the optimal time from above.

In the present paper, we study differential game problems described by an infinite system of 2-systems of differential equations. We find a guaranteed pursuit time for the pursuit differential game and a guaranteed evasion time for the evasion differential game.

2. Motivation

The work [28] the first paper on time-optimal control problem for the parabolic type partial differential equations. The optimal control problems in systems with distributed

parameters is widely studied [29]. Interesting results were obtained by Albeverio and Alimov [30] for a time-optimal control problem for the parabolic differential equation where the control function is defined on the boundary, and by Chaves-Silva et al. [31] for the null controllability of evolution equations with memory terms, and by Philippe Martin et al. [32] for the structurally damped wave equation where the null controllability holds in some suitable Sobolev space and after a fixed positive time independent of the initial conditions.

Differential game problems described by partial differential equations are considered for the first time in the works [33,34]. One of the main tools in studying control or/and differential game problems for the systems described by partial differential equations is the method of Fourier. We can use this method to reduce differential game problems described by partial differential equations to differential game problems described by an infinite system of differential equations [23–25,27,35–40].

Indeed, let a controlled distributed system be described by the following parabolic equation

$$\frac{\partial z}{\partial t} + Az = w, \qquad z(x,0) = z_0(x),\ x \in D, \qquad z(x,t) = 0,\ x \in \partial D,\ 0 < t < T, \quad (1)$$

where $z = z(x,t)$ is the state of the system, $z_0(x) \in L_2(D)$, $x = (x_1, x_2, \ldots, x_n) \in D \subset \mathbb{R}^n$, $n \geq 1$, D is a bounded domain, and it is assumed that the boundary ∂D of D is piecewise smooth, $t \in [0, T]$, and $T > 0$ is a given number, $w = w(x,t)$, $w(x,t) \in L_2(C_T)$, is the control function, C_T is the following open cylinder

$$C_T = \{(x,t)\mid x \in D,\ 0 < t < T\} \subset \mathbb{R}^{n+1},$$

operator Az is defined by the equation

$$Az = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial z}{\partial x_j}\right), \quad a_{ij}(x) = a_{ji}(x),$$

$a_{ij}(x)$ are assumed to be bounded measurable functions. Additionally, $\sum_{i,j=1}^{n} a_{ij}(x)\eta_i\eta_j \geq k\sum_{i=1}^{n}\eta_i^2$ for all $(\eta_1, \eta_2, \ldots, \eta_n) \in \mathbb{R}^n$, $x \in D$, and for some positive number k.

Then, for any $w(x,t) \in L_2(C_T)$ and $z_0(x) \in L_2(D)$, problem (1) has the only generalized solution $z = z(x,t)$ in the set $\overset{\circ}{W}_2^{1,0}(C_T)$ [41]. Moreover, the solution can be represented in the form ([41], III.3)

$$z(x,t) = \sum_{i=1}^{\infty} z_i(t)v_i(x), \tag{2}$$

where $z_i(t)$, $t \in [0, T]$, $i = 1, 2, \ldots$, are solutions of the following initial value problems

$$\dot{z}_i = \lambda_i z_i + w_i(t), \qquad z_i(0) = z_{i0}, \qquad i = 1, 2, \ldots, \tag{3}$$

the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots$ are positive and they are the generalized eigenvalues of the operator A [37], and $\lambda_i \to +\infty$ as $i \to \infty$, $v_1(x), v_2(x), \ldots, v_i(x), \ldots$ are the generalized eigenfunctions of A, which form a complete orthonormal system in $L_2(D)$, and $w_i(t)$ and z_{i0} are the Fourier coefficients of of the functions $w(x,t)$ and $z_0(x)$, respectively, relative to the system $\{v_i(x)\}$,

$$w(x,t) = \sum_{i=1}^{\infty} w_i(t)v_i(x), \quad z_0(x) = \sum_{i=1}^{\infty} z_{i0}v_i(x).$$

Additionally, the series (2) uniformly converges in $L_2(C_T)$, and its sum $z(x,t)$ belongs to the Sobolev space $\overset{\circ}{W}_2^1(D)$ for every t, $0 \leq t \leq T$, and is a continuous in t in $\overset{\circ}{W}_2^1(D)$ [41].

For example, in the works [23,24,38,39], a differential game for a PDE of the form

$$\frac{\partial z}{\partial t} = Az + u - v, \quad Az = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial z}{\partial x_i}\right),$$

was studied by reducing to the following infinite system

$$\dot{z}_k + \lambda_k z_k = u_k - v_k, \ k = 1, 2, \dots, \tag{4}$$

where u_k and v_k are control parameters of pursuer and evader, respectively, $z_k, u_k, v_k \in \mathbb{R}$, and coefficients $\lambda_k, k = 1, 2, \dots$, satisfy the condition $0 < \lambda_1 \le \lambda_2 \le \dots \to \infty$.

Thus, differential games for the infinite system of differential Equation (3) are closely related to those for partial differential Equation (1). Therefore, we study differential games for the infinite system of differential Equation (3) separately assuming that $\lambda_1, \lambda_2, \dots$ are any numbers. Note that the differential equations we'll consider in the following section correspond to complex numbers λ_i.

3. Statement of Problem

We study a differential game for the following system

$$\begin{aligned}
\dot{x}_i &= -\alpha_i x_i - \beta_i y_i + u_{i1} - v_{i1}, & x_i(0) &= x_{i0}, \\
\dot{y}_i &= \beta_i x_i - \alpha_i y_i + u_{i2} - v_{i2}, & y_i(0) &= y_{i0},
\end{aligned} \tag{5}$$

in Hilbert space l_2, where α_i, β_i are real numbers, $\alpha_i \ge 0$, $(x_{10}, x_{20}, \dots), (y_{10}, y_{20}, \dots) \in l_2$, pursuer's control parameter $u = (u_1, u_2, \dots)$ and evader's control parameter $v = (v_1, v_2, \dots)$ consist of 2-vectors $u_i = (u_{i1}, u_{i2})$ and $v_i = (v_{i1}, v_{i2})$, $i = 1, 2, \dots$, respectively. Throughout the paper we assume that $0 \le t \le T$, where T is a sufficiently large number, and $z_0 = (x_{10}, y_{10}, x_{20}, y_{20}, \dots) \ne 0$.

Let ρ and σ be given positive numbers.

Definition 1. *An admissible control of pursuer is a function $u(t) = (u_1(t), u_2(t), \dots), t \in [0, T]$, whose coordinates $u_i(t)$ are measurable and satisfy the condition*

$$\sum_{i=1}^{\infty}(u_{i1}^2(t) + u_{i2}^2(t)) \le \rho^2, \ 0 \le t \le T. \tag{6}$$

Definition 2. *An admissible control of evader is a function $v(t) = (v_1(t), v_2(t), \dots), t \in [0, T]$, whose coordinates $v_i(t)$ are measurable and satisfy the condition*

$$\sum_{i=1}^{\infty}(v_{i1}^2(t) + v_{i2}^2(t)) \le \sigma^2, \ 0 \le t \le T. \tag{7}$$

It is assumed that $\rho > \sigma$. Let

$$z_i(t) = \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix}, \quad z_{i0} = \begin{bmatrix} x_{i0} \\ y_{i0} \end{bmatrix}, \quad U_i = \begin{bmatrix} U_{i1} \\ U_{i2} \end{bmatrix}, \quad v_i = \begin{bmatrix} v_{i1} \\ v_{i2} \end{bmatrix},$$

Definition 3. *A strategy of pursuer is a function of the form*

$$U(t, v) = U^0(t) + v = (U_1^0(t) + v_1, U_2^0(t) + v_2, \dots), \quad U_i^0(t) = (U_{i1}^0(t), U_{i2}^0(t)),$$

where $U^0(t) = (U_1^0(t), U_2^0(t), \dots)$ has measurable coordinates $U_i^0(t)$, $0 \le t \le T$, that satisfy the condition

$$\sum_{i=1}^{\infty}\left(\left(U_{i1}^0(t)\right)^2 + \left(U_{i2}^0(t)\right)^2\right) \le (\rho - \sigma)^2, \ 0 \le t \le T.$$

We denote

$$A_i(t) = e^{-\alpha_i t}[rr \cos \beta_i t - \sin \beta_i t \sin \beta_i t \cos \beta_i t], \ i = 1, 2, \dots.$$

We can easily see that the matrix $A_i(t)$ has the following properties:

$$A_i(t+h) = A_i(t)A_i(h), \ A_i^{-1}(t) = A_i(-t) = e^{\alpha_i t}[rr \cos \beta_i t \sin \beta_i t - \sin \beta_i t \cos \beta_i t].$$

Each equation of the system (5) has the unique solution $z(t) = (z_1(t), z_2(t), \dots)$ defined by

$$z_i(t) = A_i(t)z_{i0} + \int_0^t A_i(t-s)(u_i(s) - v_i(s))ds, \ i = 1, 2, \dots,$$

hence,

$$z_i(t) = A_i(t)\left[z_{i0} + \int_0^t A_i(-s)(u_i(s) - v_i(s))ds\right], \ i = 1, 2, \dots \tag{8}$$

The relations (8) imply that $z(t) = 0$ is equivalent to $\eta(t) = 0$, where

$$\eta(t) = (\eta_1(t), \eta_2(t), \dots), \ \eta_i(t) = z_{i0} + \int_0^t A_i(-s)(u_i(s) - v_i(s))ds, \ i = 1, 2, \dots \tag{9}$$

Definition 4. *We call the number a guaranteed pursuit time if for some strategy of pursuer U and for any admissible control of the evader, $z(t') = 0$ at some t', $0 \le t' \le$, where $z(t) = (z_1(t), z_2(t), \dots)$ is the solution of the initial value problem (5).*

The pursuer is interested in minimizing the guaranteed pursuit time.

Definition 5. *A number τ is called a guaranteed evasion time if for any number τ', $0 \le \tau' < \tau$, we can construct an admissible control $v_0(t)$ such that, for the evader such that for any admissible control of the pursuer, we have $z(t) \ne 0$ for all $0 \le t \le \tau'$ and $i = 1, 2, \dots.$*

The evader is interested in minimizing the guaranteed evasion time.

Problem 1. *Find an equation for a guaranteed pursuit time θ and a guaranteed evasion time τ in the game (5).*

Note that a differential game with integral constraints described by the system (5) was studied in [22]. The present paper dealt with the differential game where the controls of players are subjected to geometric constraints (6) and (7).

4. Results

In this section we find a guaranteed pursuit time θ and construct a strategy for the pursuer that guarantees the time θ.

Theorem 1. *The number θ that satisfy the equation*

$$\sum_{\alpha_i > 0} \frac{\alpha_i^2 |z_{i0}|^2}{\sinh^2(\alpha_i \theta)} + \frac{1}{\theta^2} \sum_{\alpha_i = 0} |z_{i0}|^2 = (\rho - \sigma)^2 \tag{10}$$

is a guaranteed pursuit time in the game (5).

It should be noted that the series on the left hand side of Equation (6) is a decreasing continuous function of on $(0, \infty)$, approaches $+\infty$ as $\to 0+$, and approaches 0 as $\to +\infty$.

Proof. Let

$$W_i(\theta) = \int_0^\theta A_i(-s)A_i^T(-s)ds.$$

We have

$$W_i(\theta) = \int_0^\theta e^{2\alpha_i s} \begin{bmatrix} \cos\beta_i s & \sin\beta_i s \\ -\sin\beta_i s & \cos\beta_i s \end{bmatrix} \begin{bmatrix} \cos\beta_i s & -\sin\beta_i s \\ \sin\beta_i s & \cos\beta_i s \end{bmatrix} ds$$

$$= \int_0^\theta e^{2\alpha_i s} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ds = \varphi_i(\theta)I,$$

where

$$\varphi_i(\theta) = \begin{cases} \frac{1}{2\alpha_i}(e^{2\alpha_i\theta} - 1), & \alpha_i \neq 0 \\ \theta, & \alpha_i = 0 \end{cases}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We define the strategy for the pursuer as follows

$$U_i(t) = \begin{cases} -A_i^T(-t)W_i^{-1}(\theta)z_{i0} + v_i(t), & 0 \le t \le \theta, \\ v_i(t), & t > \theta \end{cases} \tag{11}$$

Note that $U_i^0(t)$, $i = 1, 2, ...,$ in Definition 3 are defined by the equations

$$U_i^0(t) = \begin{cases} -A_i^T(-t)W_i^{-1}(\theta)z_{i0}, & 0 \le t \le \theta \\ 0, & t > \theta \end{cases}, \quad i = 1, 2, ... \tag{12}$$

To show that the strategy (11) is admissible, we show that

$$\sum_{i=1}^\infty |U_i^0(t)|^2 = \sum_{i=1}^\infty |-A_i^T(-t)W_i^{-1}(\theta)z_{i0}|^2 \le (\rho - \sigma)^2.$$

Indeed, since

$$-A_i^T(-t)W_i^{-1}(\theta)z_{i0} = -\frac{e^{\alpha_i t}}{\varphi_i(\theta)} \begin{bmatrix} \cos\beta_i t & \sin\beta_i t \\ -\sin\beta_i t & \cos\beta_i t \end{bmatrix} z_{i0},$$

we have

$$\left| A_i^T(-t)W_i^{-1}(\theta)z_{i0} \right| = \frac{e^{\alpha_i t}}{\varphi_i(\theta)} \left| \begin{bmatrix} x_{i0}\cos\beta_i t + y_{i0}\sin\beta_i t \\ -x_{i0}\sin\beta_i t + y_{i0}\cos\beta_i t \end{bmatrix} \right|$$

$$= \frac{e^{\alpha_i t}}{\varphi_i(\theta)} \cdot \sqrt{x_{i0}^2 + y_{i0}^2}$$

$$\le \frac{e^{\alpha_i}}{\varphi_i(\theta)} \cdot |z_{i0}| = \begin{cases} \frac{\alpha_i |z_{i0}|}{\sinh(\alpha_i\theta)} & \alpha_i > 0, \\ 1 & \alpha_i = 0. \end{cases}$$

This implies that

$$\sum_{i=1}^\infty |U_i^0(t)|^2 = \sum_{i=1}^\infty |-A_i^T(-s)W_i^{-1}(\theta)z_{i0}|^2$$

$$\le \sum_{\alpha_i > 0} \frac{\alpha_i^2 |z_{i0}|^2}{\sinh^2(\alpha_i\theta)} + \frac{1}{\theta^2} \sum_{\alpha_i = 0} |z_{i0}|^2.$$

We are now in position to prove the admissibility of strategy (11). By the Minkowskii inequality and the definition of θ we have, for $0 \leq t \leq \theta$,

$$
\|U(t)\| = \|U_0(t) + v(t)\| \leq \|U_0(t)\| + \|v(t)\|
$$

$$
= \left(\sum_{i=1}^{\infty} |U_i^0(t)|^2 \right)^{1/2} + \left(\sum_{i=1}^{\infty} |v_i(t)|^2 \right)^{1/2}
$$

$$
\leq \left(\sum_{\alpha_i > 0} \frac{\alpha_i^2 |z_{i0}|^2}{\sinh^2(\alpha_i \theta)} + \frac{1}{\theta^2} \sum_{\alpha_i = 0} |z_{i0}|^2 \right)^{1/2} + \sigma = \rho - \sigma + \sigma = \rho.
$$

The proof of admissibility of strategy $U(t)$ is complete.

To show that θ is a guaranteed pursuit time, we show that $z(\theta) = 0$. To this end we show that

$$
\eta_i(\theta) = 0, \quad i = 1, 2, \ldots. \tag{13}
$$

Indeed, by (11)

$$
\eta_i(\theta) = z_{i0} + \int_0^{\theta} A_i(-s)(U_i(s) - v_i(s)) ds
$$

$$
= z_{i0} + \int_0^{\theta} A_i(-s) \left(-A_i^T(-s) W_i^{-1}(\theta) z_{i0} \right) ds
$$

$$
= z_{i0} - z_{i0} = 0,
$$

and so $z_i(\theta) = 0$, hence, $z(\theta) = 0$. Thus, pursuit is completed at the time θ. This completes the proof of the theorem. $\square$

5. Guaranteed Evasion Time

In this section, we study the evasion differential game and we find a guaranteed evasion time τ. We prove the following statement.

Theorem 2. *For any initial state $z_0 = (z_{10}, z_{20}, \cdots)$, the number*

$$
\tau = \sup_i \tau_i, \quad \tau_i = \begin{cases} \frac{1}{\alpha_i} \ln \left(\frac{\alpha_i |z_{i0}|}{\rho - \sigma} + 1 \right), & \alpha_i > 0 \\ \frac{|z_{i0}|}{\rho - \sigma}, & \alpha_i = 0 \end{cases}
$$

is a guaranteed evasion time in game (5).

Proof. Let pursuer apply an arbitrary admissible control $u = u(t)$. Let τ' be an arbitrary time that satisfies the condition $0 < \tau' < \tau$. To prove the theorem, we construct an admissible control for the evader, such that $z(t) \neq 0, 0 \leq t < \tau'$. Indeed, by definition of τ there exists $j \in \{1, 2, \cdots\}$ such that $\tau' < \tau_j$. Show that there is a control of the evader such that $z(t) \neq 0, 0 \leq t < \tau_j$. Observe $z_{j0} \neq 0$ since otherwise $\tau_j = 0$ contradicting positivity of τ_j. Let $e_j = \frac{z_{j0}}{|z_{j0}|}$ and

$$
v_j(t) = -\sigma \begin{bmatrix} \cos \beta_j t & \sin \beta_j t \\ -\sin \beta_j t & \cos \beta_j t \end{bmatrix} e_j, \quad v_i(t) = 0, \, i = 1, 2, \cdots, \, i \neq j, \, t \in [0, \tau_j). \tag{14}
$$

Then, for any admissible control of the pursuer $u(\cdot)$ and time $t \in [0, \tau_j)$, we have

$$
\begin{aligned}
\eta_j(t) &= z_{j0} + \int_0^t A_i(-s)u_j(s)ds - \int_0^t A_i(-s)v_j(s)ds \\
&= z_{j0} + \int_0^t A_i(-s)u_j(s)ds + \sigma e_j \int_0^t e^{\alpha_j s}ds.
\end{aligned}
$$

From this using the inequality $|(A_i(-s)u_j(s), e_i)| \leq \rho e^{\alpha_j s}$ and the definition of τ_j we obtain

$$
\begin{aligned}
\eta_j(t)e_j &= |z_{j0}| + \int_0^t (A_i(-s)u_j(s), e_i)ds + \sigma \int_0^t e^{\alpha_j s}ds \\
&\geq |z_{j0}| - \rho \int_0^t e^{\alpha_j s}ds + \sigma \int_0^t e^{\alpha_j s}ds \\
&> |z_{j0}| - (\rho - \sigma) \int_0^{\tau_j} e^{\alpha_j s}ds = 0.
\end{aligned}
$$

Consequently, $\eta_j(t) \neq 0$ and hence $z_j(t) \neq 0$ by (9), therefore, $z(t) \neq 0$ for $t \in [0, \tau_j)$. In particular, $z(t) \neq 0$ on the interval $[0, \tau']$. This completes the proof of Theorem 2. $\square$

6. Discussion

We have studied a pursuit and evasion differential games for an infinite system of differential equations. We have obtained a formula for the guaranteed pursuit time. Additionally, we have constructed an explicit strategy for the pursuer that ensures the completion of the game by the guaranteed pursuit time.

We estimate the guaranteed pursuit time θ. Using the Taylor series, for $\alpha_i \neq 0$, we have

$$
\begin{aligned}
\frac{\alpha_i}{\sinh \alpha_i \theta} &= \frac{2\alpha_i}{e^{\alpha_i \theta} - e^{-\alpha_i \theta}} \\
&= \frac{2\alpha_i}{1 + \alpha_i\theta + \frac{\alpha_i^2\theta^2}{2!} + \frac{\alpha_i^3\theta^3}{3!} + \cdots - \left(1 - \alpha_i\theta + \frac{\alpha_i^2\theta^2}{2!} - \frac{\alpha_i^3\theta^3}{3!} + \ldots\right)} \\
&= \frac{2\alpha_i}{2\alpha_i\theta(1 + \frac{\alpha_i^2\theta^2}{3!} + \frac{\alpha_i^4\theta^4}{5!} + \ldots)} \leq \frac{1}{\theta}.
\end{aligned}
$$

Therefore, we obtain from (10) that

$$
\begin{aligned}
(\rho - \sigma)^2 &= \sum_{\alpha_i > 0} \frac{\alpha_i^2 |z_{i0}|^2}{\sinh^2(\alpha_i\theta)} + \sum_{\alpha_i = 0} \frac{|z_{i0}|^2}{\theta^2} \\
&\leq \sum_{\alpha_i > 0} \frac{|z_{i0}|^2}{\theta^2} + \sum_{\alpha_i = 0} \frac{|z_{i0}|^2}{\theta^2} = \frac{|z_0|^2}{\theta^2},
\end{aligned}
$$

where

$$
|z_0| = \left(\sum_{i=1}^\infty |z_{i0}|^2\right)^{1/2}.
$$

Hence,

$$
\theta \leq \frac{|z_0|}{\rho - \sigma}.
$$

Thus, the guaranteed pursuit time is less than or equal to $|z_0|/(\rho - \sigma)$. Hence, the strategy of pursuer (11) guarantees the completion of the game by the time $|z_0|/(\rho - \sigma)$.

Additionally, we have obtained a formula for the guaranteed evasion time.

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