*mathematics*

*Article*

# Multi-Step Inertial Hybrid and Shrinking Tseng's Algorithm with Meir–Keeler Contractions for Variational Inclusion Problems

**Yuanheng Wang *, Mingyue Yuan and Bingnan Jiang**

College of Mathematics and Computer Science, Zhejiang Normal University, Jinhua 321004, China; ymybiubiu520@163.com (M.Y.); bnjiang_as123@163.com (B.J.)
* Correspondence: yhwang@zjnu.cn; Tel.: +86-579-8229-8258

**Abstract:** In our paper, we propose two new iterative algorithms with Meir–Keeler contractions that are based on Tseng's method, the multi-step inertial method, the hybrid projection method, and the shrinking projection method to solve a monotone variational inclusion problem in Hilbert spaces. The strong convergence of the proposed iterative algorithms is proven. Using our results, we can solve convex minimization problems.

**Keywords:** Meir–Keeler contractions; multi-step inertial method; hybrid projection method; shrinking projection method; variational inclusion problem

**MSC:** 47H09; 47H10; 47H04

check for updates

**Citation:** Wang, Y.; Yuan, M.; Jiang, B. Multi-Step Inertial Hybrid and Shrinking Tseng's Algorithm with Meir–Keeler Contractions for Variational Inclusion Problems. *Mathematics* **2021**, *9*, 1548. https://doi.org/10.3390/math9131548

Academic Editors: Mihai Postolache, Jen-Chih Yao and Yonghong Yao

Received: 31 May 2021
Accepted: 29 June 2021
Published: 1 July 2021



## 1. Introduction

### 1.1. Variational Inclusion Problem

In a real Hilbert space $H$ with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$, we assume that $G : H \to 2^H$ is a set-valued mapping while $F : H \to H$ is a single-valued mapping.

We consider the following variational inclusion problem: find an element $x^* \in H$ such that

$$0 \in Fx^* + Gx^*. \tag{1}$$

This problem has been studied by many scholars [1–9].

A classical algorithm to solve the problem (1) is the forward–backward splitting algorithm put forward by Passty [2] and by Lions and Mercier [3]. In 2000, Tseng [4] proposed a modified forward–backward splitting algorithm (Algorithm 1) about null points of maximal monotone mappings. This algorithm is weakly convergent under some conditions.

**Algorithm 1:** Modified forward–backward splitting algorithm.

$$y_n = (I + \gamma_n G)^{-1}(x_n - \gamma_n F x_n),$$
$$x_{n+1} = y_n - \gamma_n (F y_n - F x_n).$$

In 2015, an algorithm named inertial forward–backward algorithm (Algorithm 2) was proposed by Lorenz and Pock [5]. We notice that the algorithm is also weakly convergent.

**Algorithm 2:** Inertial forward–backward algorithm.

$$y_n = x_n + \alpha_n(x_n - x_{n-1}),$$
$$x_{n+1} = (I + \gamma_n G)^{-1}(y_n - \gamma_n F y_n).$$

In 2020, Tan et al. [6] introduced the inertial hybrid projection algorithm (Algorithm 3) and inertial shrinking projection method (Algorithm 4) by combining the two algorithms (Algorithms 1 and 2) with two classes of hybrid projection methods to solve the variational inclusion problem in Hilbert spaces, as follows:

**Algorithm 3:** Inertial hybrid projection algorithm.

$$
\begin{aligned}
&w_n = x_n + \alpha_n(x_n - x_{n-1}),\\
&y_n = (I + \gamma_n G)^{-1}(I - \gamma_n F)w_n,\\
&z_n = y_n - \gamma_n(Fy_n - Fw_n),\\
&C_n = \left\{u \in H : \|z_n - u\|^2 \le \|w_n - u\|^2 - \left(1 - \mu^2 \frac{\gamma_n^2}{\gamma_{n+1}^2}\right)\|w_n - y_n\|^2\right\},\\
&Q_n = \{u \in H : \langle x_n - u, x_n - x_0\rangle \le 0\},\\
&x_{n+1} = P_{C_n \cap Q_n} x_0.
\end{aligned}
$$

**Algorithm 4:** Inertial shrinking projection algorithm.

$$
\begin{aligned}
&w_n = x_n + \alpha_n(x_n - x_{n-1}),\\
&y_n = (I + \gamma_n G)^{-1}(I - \gamma_n F)w_n,\\
&z_n = y_n - \gamma_n(Fy_n - Fw_n),\\
&C_{n+1} = \left\{u \in C_n : \|z_n - u\|^2 \le \|w_n - u\|^2 - \left(1 - \mu^2 \frac{\gamma_n^2}{\gamma_{n+1}^2}\right)\|w_n - y_n\|^2\right\},\\
&x_{n+1} = P_{C_{n+1}} x_0.
\end{aligned}
$$

They proved these two algorithms are strongly convergent under certain conditions.

### 1.2. Fixed Point Problem

Assume that $D$ is a nonempty closed convex subset of $H$ and that $T : D \to D$ is a mapping. Let us recall that the fixed point problem is finding a point $\bar{x} \in D$ such that $T\bar{x} = \bar{x}$. We denote the set of fixed points of $T$ by $\mathrm{Fix}(T)$.

In the field of fixed point problems, many fruitful achievements were introduced by scholars [10–22]. One of the classic algorithms is the Krasnosel'skiĭ–Mann algorithm [10,11], which is defined as follows:

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T x_n.$$

Under some certain conditions, the sequence $\{x_n\}$ converges weakly to a fixed point of $T$. In 2019, Dong et al. [20] presented a multi-step inertial Krasnosel'skiĭ–Mann algorithm, which is defined as Algorithm 5.

**Algorithm 5:** Multi-step inertial Krasnosel'skiĭ–Mann algorithm.

$$
\begin{aligned}
&y_n = x_n + \textstyle\sum_{k \in \mathcal{S}_n} a_{k,n}(x_{n-k} - x_{n-k-1}),\\
&z_n = x_n + \textstyle\sum_{k \in \mathcal{S}_n} b_{k,n}(x_{n-k} - x_{n-k-1}),\\
&x_{n+1} = (1 - \lambda_n)y_n + \lambda_n T z_n.
\end{aligned}
$$

Under suitable conditions, the sequence $\{x_n\}$ converges weakly to a point in $\mathrm{Fix}(T)$. In addition, Yao et al. [21] proposed a projected fixed point algorithm in real Hilbert spaces in 2017, which is defined as Algorithm 6. The sequence $\{x_n\}$ converges strongly to the unique fixed point of $P_{\mathrm{Fix}(T)} f$ under some conditions.

**Algorithm 6:** Projected fixed point algorithm.

$$y_n = (1 - \lambda_n)x_n + \lambda_n T x_n,$$
$$C_{n+1} = \{u \in C_n : \|(1 - \alpha_n)x_n + \alpha_n T y_n - u\| \leq \|x_n - u\|\},$$
$$x_{n+1} = P_{C_{n+1}} f(x_n).$$

Motivated by the results of [6,20,21], we construct two new algorithms to solve variational inclusion problems and obtain two strong convergence theorems. By using our results, we can solve convex minimization problems in Hilbert spaces as applications.

## 2. Preliminaries

Now, we present some necessary definitions and lemmas in the following for our convergence analysis.

**Definition 1** ([23–27])**.** *Let $S : H \to H$ be a nonlinear mapping.*

*(i)* *$S$ is nonexpansive if*
$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in H.$$
*(ii)* *$S$ is firmly nonexpansive if*
$$\langle Sx - Sy, x - y\rangle \geq \|Sx - Sy\|^2, \quad \forall x, y \in H.$$
*It is obvious to see that a firmly nonexpansive mapping is nonexpansive.*
*(iii)* *$S$ is contractive if*
$$\|Sx - Sy\| \leq \rho\|x - y\|, \quad \forall x, y \in H,$$
*where $\rho \in [0, 1)$ is a real number.*
*(iv)* *$S$ is Meir–Keeler contractive if, for any $\epsilon > 0$, there exists $\delta > 0$ such that*
$$\|x - y\| < \epsilon + \delta \quad \textit{implies} \quad \|Sx - Sy\| < \epsilon, \quad \forall x, y \in H.$$
*It it obvious to see that a contractive mapping is Meir–Keeler contractive.*
*(v)* *$S$ is L-Lipschitz continuous ($L > 0$) if*
$$\|Sx - Sy\| \leq L\|x - y\|, \quad \forall x, y \in H.$$
*(vi)* *$S$ is monotone if*
$$\langle Sx - Sy, x - y\rangle \geq 0, \quad \forall x, y \in H.$$

**Lemma 1** ([23,28–30])**.** *A Meir–Keeler contractive mapping has a unique fixed point on a complete metric space.*

**Lemma 2** ([31])**.** *Let $D$ be a convex subset of a Banach space $E$ and $S$ be a Meir–Keeler contractive mapping on $D$. Then, there exists $\rho \in (0, 1)$ for each $\epsilon > 0$, such that*
$$\|x - y\| \geq \epsilon \quad \textit{implies} \quad \|Sx - Sy\| \leq \rho\|x - y\|, \quad \forall x, y \in D.$$

Recall the metric projection operator $P_D$, defined as follows:
$$P_D x = \arg\min_{y \in D} \|x - y\|, \quad x \in H.$$

**Lemma 3** ([32,33])**.** *Given $x \in H$ and $q \in D$, we have*

*(i)* *$q = P_D x$ if and only if*
$$\langle x - q, q - y\rangle \geq 0, \quad \forall y \in D;$$

*(ii)* $P_D$ *is firmly nonexpansive, i.e.,*

$$\langle P_D u - P_D v, u - v \rangle \geq \|P_D u - P_D v\|^2, \quad \forall u, v \in H;$$

*(iii)* $\|x - P_D x\|^2 \leq \|x - y\|^2 - \|y - P_D x\|^2, \quad \forall y \in D.$

**Definition 2** ([34]). *Let $A : H \to 2^H$ be a set-valued mapping.* $\mathrm{Dom}(A) = \{x \in H : Ax \neq \emptyset\}$ *is the effective domain of $A$. The graph of $A$ is denoted by* $\mathrm{Gra}(A)$*, i.e.,* $\mathrm{Gra}(A) = \{(x, u) \in H \times H : u \in Ax\}$*. A set-valued mapping $A : H \to 2^H$ is called monotone if*

$$\langle x - y, u - v \rangle \geq 0, \quad \forall (x, u), (y, v) \in \mathrm{Gra}(A).$$

*A monotone set-valued mapping $A$ is called maximal monotone if, for each $(x, u) \in H \times H$, $(x, u) \in \mathrm{Gra}(A)$ if and only if*

$$\langle x - y, u - v \rangle \geq 0, \quad \forall (y, v) \in \mathrm{Gra}(A).$$

For a maximal monotone set-valued mapping $A : H \to 2^H$ and $r > 0$, we can define a mapping as

$$J_r = (I + rA)^{-1}.$$

It is worth noticing that $J_r$ is single-valued and firmly nonexpansive. The mapping $J_r$ is called the resolvent of $A$ for $r$.

**Lemma 4** ([35]). *Let $A$ be a maximal monotone mapping on $H$ into $2^H$ and $B : H \to H$ be a mapping. Then, for any $r > 0$, $(A + B)^{-1}(0) = \mathrm{Fix}(J_r(I - rB))$, where $J_r$ is the resolvent of $A$ for $r$.*

**Lemma 5** ([35,36]). *Let $A : H \to 2^H$ be a maximal monotone mapping. For $r, s > 0$,*

$$\|J_r x - J_s x\| \leq \frac{|r - s|}{r} \|x - J_r x\|, \quad \forall x \in H,$$

*where $J_r$ is the resolvent of $A$ for $r$ and $J_s$ is the resolvent of $A$ for $s$.*

Let $\{x_n\} \subset H$ be a sequence. We use $x_n \to x$ and $x_n \rightharpoonup x$ to indicate that $\{x_n\}$ converges strongly and weakly to $x$, respectively.

**Definition 3** ([21,37]). *Let $D_n \subset H$ be a nonempty closed convex subsets, $n = 1, 2, \cdots$. We define* $s\text{-}\mathrm{Li}_n D_n$ *and* $w\text{-}\mathrm{Ls}_n D_n$ *as follows:*

$$s\text{-}\mathrm{Li}_n D_n = \{x \in H : x_n \in D_n, x_n \to x\},$$

$$w\text{-}\mathrm{Ls}_n D_n = \{x \in H : \{D_{n_k}\} \subset \{D_n\}, x_{n_k} \in D_{n_k}, x_{n_k} \rightharpoonup x\}.$$

*If there exists a set $D_0 \subset H$ such that $D_0 = s\text{-}\mathrm{Li}_n D_n = w\text{-}\mathrm{Ls}_n D_n$, we say that $\{D_n\}$ converges to $D_0$ in the sense of Mosco and denote by $M\text{-}\lim_{n\to\infty} D_n = D_0$. It is obvious to prove that, if $\{D_n\}$ is non-increasing with respect to inclusion, then $\{D_n\}$ converges to $\bigcap_{n=1}^{\infty} D_n$ in the sense of Mosco.*

**Lemma 6** ([21,38]). *Let $D_n \subset H$ be a nonempty closed convex subsets, $n = 1, 2, \cdots$. If $D_0 = M\text{-}\lim_{n\to\infty} D_n$ exists and is nonempty, then $\forall x \in H$, $P_{D_n} x \to P_{D_0} x$.*

## 3. Algorithms

In this section, we present two algorithms to find the solutions to variational inclusion problems in Hilbert spaces.

The following conditions are assumed to be true.

**(A1)** $F : H \to H$ is $L$-Lipschitz continuous ($L > 0$) and monotone.
**(A2)** $G : H \to 2^H$ is maximal monotone.
**(A3)** $f : H \to H$ is a Meir–Keeler contraction.
**(A4)** $\Omega = (F + G)^{-1}(0) \neq \emptyset$.
We need the following lemma.

**Lemma 7** ([6]). *The sequence $\{\gamma_n\}$ generated by the algorithm is non-increasing and*

$$\lim_{n\to\infty} \gamma_n = \gamma \geq \min\Big\{\gamma_1, \frac{\mu}{L}\Big\}.$$

## 4. Main Results

In this section, we analyze the strong convergence of Algorithms 7 and 8.

---

**Algorithm 7:** Multi-step inertial hybrid Tseng's algorithm.

**Initialization:** Choose $x_0, x_1 \in H$, $\gamma_1 > 0$, $\mu \in (0,1)$ arbitrarily. For each $i = 1, 2, \cdots, s$ (where $s$ is a chosen positive integer), choose a bounded sequence $\{\alpha_{i,n}\} \subset \mathbb{R}$. Let $\{\varepsilon_n\}$ be a nonnegative number sequence with $\lim_{n\to\infty} \varepsilon_n = 0$.

**Iterative step:** Compute $x_{n+1}$ via

$$\begin{aligned}
&w_n = x_n + \textstyle\sum_{i=1}^{\min\{s,n\}} \alpha_{i,n}(x_{n-i+1} - x_{n-i}),\\
&y_n = J_{\gamma_n}(I - \gamma_n F)w_n,\\
&z_n = y_n - \gamma_n(Fy_n - Fw_n),\\
&C_n = \Big\{u \in H : \|z_n - u\|^2 \leq \|w_n - u\|^2 - \Big(1 - \mu^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\Big)\|y_n - w_n\|^2 + \varepsilon_n\Big\},\\
&Q_n = \begin{cases} H, & \text{if } n = 1,\\ \{u \in Q_{n-1} : \langle x_n - f(x_{n-1}), x_n - u\rangle \leq 0\}, & \text{if } n \geq 2,\end{cases}\\
&x_{n+1} = P_{C_n \cap Q_n} f(x_n),
\end{aligned}$$

where

$$\gamma_{n+1} = \begin{cases} \min\Big\{\gamma_n, \frac{\mu\|w_n - y_n\|}{\|Fw_n - Fy_n\|}\Big\}, & \text{if } Fw_n \neq Fy_n,\\ \gamma_n, & \text{otherwise.}\end{cases}$$

---

**Algorithm 8:** Multi-step inertial shrinking Tseng's algorithm.

**Initialization:** Choose $x_0, x_1 \in H$, $\gamma_1 > 0$, $\mu \in (0,1)$ arbitrarily. Let $C_1 = H$. For each $i = 1, 2, \cdots, s$ (where $s$ is a chosen positive integer), choose a bounded sequence $\{\alpha_{i,n}\} \subset \mathbb{R}$. Let $\{\varepsilon_n\}$ be a nonnegative number sequence with $\lim_{n\to\infty} \varepsilon_n = 0$.

**Iterative step:** Compute $x_{n+1}$ via

$$\begin{aligned}
&w_n = x_n + \textstyle\sum_{i=1}^{\min\{s,n\}} \alpha_{i,n}(x_{n-i+1} - x_{n-i}),\\
&y_n = J_{\gamma_n}(I - \gamma_n F)w_n,\\
&z_n = y_n - \gamma_n(Fy_n - Fw_n),\\
&C_{n+1} = \Big\{u \in C_n : \|z_n - u\|^2 \leq \|w_n - u\|^2 - \Big(1 - \mu^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\Big)\|y_n - w_n\|^2 + \varepsilon_n\Big\},\\
&x_{n+1} = P_{C_{n+1}} f(x_n),
\end{aligned}$$

where the computation of $\gamma_{n+1}$ is the same as in Algorithm 7.

---

**Theorem 1.** *Assume that the conditions (A1)–(A4) are satisfied. Then, the sequence $\{x_n\}$ generated by Algorithm 7 converges strongly to $x^* \in \Omega$, where $x^*$ is the unique fixed point of $P_\Omega f$.*

**Proof.** The proof is divided into four steps.

*Step 1.* $\forall n \in \mathbb{N}$, and $C_n$ and $Q_n$ are closed and convex.

Obviously, for each $n \in \mathbb{N}$, $C_n$ is a half-space, so $C_n$ is closed and convex.

For $n = 1$, $Q_1 = H$ is closed and convex. Suppose that $x_k$ is given and that $Q_k$ is closed and convex for some $k \in \mathbb{N}$. It is clear that $\{u \in H : \langle x_{k+1} - f(x_k), x_k - u\rangle \leq 0\}$ is a half-space, so it is closed and convex. Hence, $Q_{k+1}$ is closed and convex. $\forall n \in \mathbb{N}$, and $Q_n$ is closed and convex by induction.

*Step 2.* We prove that $\Omega \subset C_n \cap Q_n$ for each $n \in \mathbb{N}$.

Let $p \in \Omega$. We see that

$$\begin{aligned}
& \|z_n - p\|^2 \\
= \; & \|(y_n - p) - \gamma_n(Fy_n - Fw_n)\|^2 \\
= \; & \|y_n - p\|^2 + \gamma_n^2\|Fy_n - Fw_n\|^2 - 2\gamma_n\langle y_n - p, Fy_n - Fw_n\rangle \\
= \; & \|w_n - p\|^2 + \|y_n - w_n\|^2 + 2\langle w_n - p, y_n - w_n\rangle + \gamma_n^2\|Fy_n - Fw_n\|^2 \\
& -2\gamma_n\langle y_n - p, Fy_n - Fw_n\rangle \\
\leq \; & \|w_n - p\|^2 - \|y_n - w_n\|^2 + \gamma_n^2\|Fy_n - Fw_n\|^2 \\
& -2\langle y_n - p, w_n - y_n + \gamma_n(Fp - Fw_n)\rangle \\
\leq \; & \|w_n - p\|^2 - \|y_n - w_n\|^2 + \mu^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\|y_n - w_n\|^2 \\
& -2\langle y_n - p, w_n - y_n + \gamma_n(Fp - Fw_n)\rangle \\
= \; & \|w_n - p\|^2 - \left(1 - \mu^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\right)\|y_n - w_n\|^2 \\
& -2\langle y_n - p, w_n - y_n + \gamma_n(Fp - Fw_n)\rangle.
\end{aligned} \tag{2}$$

Since $y_n = J_{\gamma_n}(I - \gamma_n F)w_n = (I + \gamma_n G)^{-1}(I - \gamma_n F)w_n$, we have

$$(I - \gamma_n F)w_n \in (I + \gamma_n G)y_n.$$

Hence,

$$\frac{1}{\gamma_n}(w_n - \gamma_n Fw_n - y_n) \in Gy_n. \tag{3}$$

On the other hand, since $p \in \Omega = (F + G)^{-1}(0)$, we have

$$0 \in (F + G)p.$$

Hence

$$-Fp \in Gp. \tag{4}$$

From the maximal monotonicity of $G$, we deduce

$$\left\langle \frac{1}{\gamma_n}(w_n - \gamma_n Fw_n - y_n) + Fp, y_n - p \right\rangle \geq 0,$$

which means

$$\langle w_n - y_n + \gamma_n(Fp - Fw_n), y_n - p\rangle \geq 0. \tag{5}$$

Substituting (5) into (2), we conclude

$$\begin{aligned}
\|z_n - p\|^2 \leq \; & \|w_n - p\|^2 - \left(1 - \mu^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\right)\|y_n - w_n\|^2 \\
\leq \; & \|w_n - p\|^2 - \left(1 - \mu^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\right)\|y_n - w_n\|^2 + \varepsilon_n.
\end{aligned} \tag{6}$$

This means that $p \in C_n$. Hence, $\Omega \subset C_n$ for each $n \in \mathbb{N}$.

For $n = 1$, $Q_1 = H$, which yields $\Omega \subset C_1 \cap Q_1$.

Assume that $x_k$ is given and that $\Omega \subset C_k \cap Q_k$ for some $k \in \mathbb{N}$. From Lemma 3, we obtain

$$\langle y - x_{k+1}, f(x_k) - x_{k+1} \rangle \leq 0, \quad \forall y \in C_k \cap Q_k.$$

Since $\Omega \subset C_k \cap Q_k$, we have

$$\langle y - x_{k+1}, f(x_k) - x_{k+1} \rangle \leq 0, \quad \forall y \in \Omega. \tag{7}$$

From the expression of $Q_n$, we obtain $\Omega \subset Q_{k+1}$. Hence, $\Omega \subset C_{k+1} \cap Q_{k+1}$.

Therefore, $\forall n \in \mathbb{N}$, $\Omega \subset C_n \cap Q_n$ by induction.

*Step 3.* We prove that $\{x_n\}$ converges strongly to $z$, where $z$ is the unique fixed point of $P_{\bigcap_{n=1}^{\infty} Q_n} f$.

From the expression of $Q_n$, we know that $\emptyset \neq \Omega \subset \bigcap_{n=1}^{\infty} Q_n = \text{M-}\lim_{n\to\infty} Q_n$. Set $v_n = P_{Q_n} f(z)$. It follows from Lemma 6 that

$$v_n \to P_{\bigcap_{n=1}^{\infty} Q_n} f(z) = z. \tag{8}$$

Suppose the contrary, i.e., $\limsup_{n\to\infty} \|x_n - z\| > 0$. One can choose a real number $\epsilon > 0$ such that $\limsup_{n\to\infty} \|x_n - z\| > \epsilon$. Continue to choose a real number $\delta_1$ such that $\limsup_{n\to\infty} \|x_n - z\| > \epsilon + \delta_1$. Since $f$ is a Meir–Keeler contraction, there exists $\delta_2 > 0$ such that $\|x - y\| < \epsilon + \delta_2$ implies, $\forall x, y \in H$, $\|f(x) - f(y)\| < \epsilon$. Take $\delta = \min\{\delta_1, \delta_2\}$, we have

$$\limsup_{n\to\infty} \|x_n - z\| > \epsilon + \delta \tag{9}$$

and

$$\|x - y\| < \epsilon + \delta \quad \text{implies} \quad \|f(x) - f(y)\| < \epsilon, \quad \forall x, y \in H. \tag{10}$$

Since $v_n \to z$, there exists $n_0 \in \mathbb{N}$ such that

$$\|v_n - z\| < \delta, \quad \forall n \geq n_0. \tag{11}$$

The following two cases are considered now.

**Case 1.** There exists $n_1 \geq n_0$ such that $\|x_{n_1} - z\| < \epsilon + \delta$.

From the expression of $Q_n$ and Lemma 3, we can obviously see that $x_{n+1} = P_{Q_{n+1}} f(x_n)$. Thus, from (10) and (11), we conclude

$$\begin{aligned}
& \|x_{n_1+1} - z\| \\
\leq\ & \|x_{n_1+1} - v_{n_1+1}\| + \|v_{n_1+1} - z\| \\
<\ & \|P_{Q_{n_1+1}} f(x_{n_1}) - P_{Q_{n_1+1}} f(z)\| + \delta \\
\leq\ & \|f(x_{n_1}) - f(z)\| + \delta \\
<\ & \epsilon + \delta.
\end{aligned}$$

By induction, we obtain

$$\|x_{n_1+m} - z\| < \epsilon + \delta, \quad \forall m \in \mathbb{N},$$

which implies that

$$\limsup_{n\to\infty} \|x_n - z\| \leq \epsilon + \delta.$$

This contradicts (9).

**Case 2.** $\|x_n - z\| \geq \epsilon + \delta$ for all $n \geq n_0$.

From Lemma 2, there exists $\rho \in (0, 1)$ such that

$$\|f(x_n) - f(z)\| \leq \rho \|x_n - z\|.$$

Thus, for $n \geq n_0$, we obtain

$$
\begin{aligned}
& \|x_{n+1} - z\| \\
\leq \; & \|x_{n+1} - v_{n+1}\| + \|v_{n+1} - z\| \\
< \; & \|P_{Q_{n+1}} f(x_n) - P_{Q_{n+1}} f(z)\| + \delta \\
\leq \; & \|f(x_n) - f(z)\| + \delta \\
\leq \; & \rho \|x_n - z\| + \delta \\
< \; & \rho^2 \|x_{n-1} - z\| + (1 + \rho)\delta \\
\cdots & \\
< \; & \rho^{n-n_0+1} \|x_{n_0} - z\| + (1 + \rho + \cdots + \rho^{n-n_0})\delta \\
< \; & \|x_{n_0} - z\| + \frac{\delta}{1-\rho},
\end{aligned}
$$

which means that $\limsup_{n\to\infty} \|x_n - z\|$ is a finite number. Therefore,

$$
\begin{aligned}
& \limsup_{n\to\infty} \|x_n - z\| \\
= \; & \limsup_{n\to\infty} \|x_{n+1} - z\| \\
\leq \; & \limsup_{n\to\infty} \|x_{n+1} - v_{n+1}\| + \lim_{n\to\infty} \|v_{n+1} - z\| \\
= \; & \limsup_{n\to\infty} \|P_{Q_{n+1}} f(x_n) - P_{Q_{n+1}} f(z)\| \\
\leq \; & \limsup_{n\to\infty} \|f(x_n) - f(z)\| \\
\leq \; & \rho \limsup_{n\to\infty} \|x_n - z\| \\
< \; & \limsup_{n\to\infty} \|x_n - z\|.
\end{aligned}
$$

This is a contradiction.
Hence, we obtain that $\{x_n\}$ converges strongly to $z$.
*Step 4.* We prove that $\{x_n\}$ converges strongly to $x^*$.
From Step 3, it is sufficient to prove that $z = x^*$. Since $x_n \to z$, we have

$$
x_{n+1} - x_n \to 0, \quad n \to \infty. \tag{12}
$$

From the computation of $w_n$, we deduce

$$
\begin{aligned}
& \|x_n - w_n\| \\
= \; & \left\| x_n - x_n - \sum_{i=1}^{s} \alpha_{i,n}(x_{n-i+1} - x_{n-i}) \right\| \\
= \; & \left\| \sum_{i=1}^{s} \alpha_{i,n}(x_{n-i+1} - x_{n-i}) \right\| \\
\leq \; & \sum_{i=1}^{s} |\alpha_{i,n}| \|x_{n-i+1} - x_{n-i}\|
\end{aligned} \tag{13}
$$

for $n \geq s$. From (12), (13), and the boundedness of $\{\alpha_{i,n}\}$, we have

$$
x_n - w_n \to 0, \quad n \to \infty. \tag{14}
$$

Hence, $w_n \to z$. Therefore $\{w_n\}$ is bounded, and so are $\{Fw_n\}$, $\{(I - \gamma F)w_n\}$, and $\{J_\gamma(I - \gamma F)w_n\}$, where $\gamma$ appears in Lemma 7. Combining (12) and (14), we conclude

$$
x_{n+1} - w_n \to 0, \quad n \to \infty. \tag{15}
$$

Since $x_{n+1} = P_{C_n \cap Q_n} f(x_n)$, we know that $x_{n+1} \in C_n$. Hence,

$$\|z_n - x_{n+1}\|^2 \leq \|w_n - x_{n+1}\|^2 - \left(1 - \mu^2 \frac{\gamma_n^2}{\gamma_{n+1}^2}\right) \|y_n - w_n\|^2 + \varepsilon_n. \tag{16}$$

By Lemma 7, we know that $\gamma_n \to \gamma > 0$. Therefore, $\frac{\gamma_n}{\gamma_{n+1}} \to 1$. Combining this with $\mu \in (0,1)$, we obtain

$$\lim_{n\to\infty} \left(1 - \mu^2 \frac{\gamma_n^2}{\gamma_{n+1}^2}\right) = 1 - \mu^2 \in (0,1) \tag{17}$$

Combining (15)–(17) and the conditions of $\{\varepsilon_n\}$, we deduce

$$x_{n+1} - z_n \to 0, \quad n \to \infty,$$

and hence

$$y_n - w_n \to 0, \quad n \to \infty,$$

i.e.,

$$J_{\gamma_n}(I - \gamma_n F)w_n - w_n \to 0, \quad n \to \infty. \tag{18}$$

Since $J_{\gamma_n}$ is nonexpansive, we conclude

$$\begin{aligned} & \|J_{\gamma_n}(I - \gamma_n F)w_n - J_{\gamma_n}(I - \gamma F)w_n\| \\ \leq\ & \|(I - \gamma_n F)w_n - (I - \gamma F)w_n\| \\ =\ & |\gamma_n - \gamma| \|F w_n\|. \end{aligned}$$

Hence,

$$J_{\gamma_n}(I - \gamma_n F)w_n - J_{\gamma_n}(I - \gamma F)w_n \to 0, \quad n \to \infty. \tag{19}$$

From Lemma 5, we have

$$\begin{aligned} & \|J_{\gamma_n}(I - \gamma F)w_n - J_{\gamma}(I - \gamma F)w_n\| \\ \leq\ & \frac{|\gamma_n - \gamma|}{\gamma} \|(I - \gamma F)w_n - J_{\gamma}(I - \gamma F)w_n\|. \end{aligned}$$

Hence,

$$J_{\gamma_n}(I - \gamma F)w_n - J_{\gamma}(I - \gamma F)w_n \to 0, \quad n \to \infty. \tag{20}$$

Combining (18)–(20), we obtain

$$w_n - J_{\gamma}(I - \gamma F)w_n \to 0, \quad n \to \infty. \tag{21}$$

By $w_n \to z$ and the continuity of $J_\gamma(I - \gamma F)$, we conclude $z = J_\gamma(I - \gamma F)z$. It follows from Lemma 4 that $z \in \Omega$. Since $\Omega \subset Q_{n+1}$, we see that

$$\langle x_{n+1} - f(x_n), x_{n+1} - y \rangle \leq 0, \quad \forall y \in \Omega. \tag{22}$$

Taking the limit in (22), we obtain

$$\langle z - f(z), z - y \rangle \leq 0, \quad \forall y \in \Omega. \tag{23}$$

It follows from Lemma 3 that $z = P_\Omega f(z)$. Since $P_\Omega f$ has the unique fixed point $x^*$, we obtain $z = x^*$. □

**Theorem 2.** *Assume that the conditions (A1)–(A4) are satisfied. Let the sequence $\{x_n\}$ be generated by Algorithm 8. Then, $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $x^*$ is the unique fixed point of $P_\Omega f$.*

**Proof.** It is obvious that $C_n$ is a closed convex subset of $H$ for each $n \in \mathbb{N}$ by induction. Using the same proof as in (2)–(6), we obtain that $\Omega \subset C_n$ for each $n \in \mathbb{N}$. Denote the unique fixed point of $P_{\bigcap_{n=1}^{\infty} C_n} f$ by $z$. From the expression of $C_n$, we know that $\emptyset \neq \Omega \subset \bigcap_{n=1}^{\infty} C_n = \text{M-}\lim_{n\to\infty} C_n$. Set $v_n = P_{C_n} f(z)$. It follows from Lemma 6 that $v_n$ is as follows:

$$v_n \to P_{\bigcap_{n=1}^{\infty} C_n} f(z) = z.$$

Using the similar proof of Theorem 1, we obtain $x_n \to x^*$. □

**Remark 1.** *If $s = 1$, $\varepsilon_n \equiv 0$, $x_0 = x_1$, and $f \equiv x_1$, then Algorithm 8 reduces to Algorithm 4.*

## 5. Applications

In this section, some applications for solving the nonsmooth composite convex minimization problems are introduced in Hilbert spaces.

Denote $\Gamma_0(H)$ by

$$\Gamma_0(H) = \{f : H \to (-\infty, \infty] : f \text{ is proper convex lower semi-continuous}\}.$$

Consider the following problem

$$\min_{x\in H}(g(x) + h(x)), \tag{24}$$

where $g, h \in \Gamma_0(H)$ and which satisfies the following conditions:

- $g$ is Gâteaux differentiable, and its gradient $\nabla g$ is Lipschitz continuous. $h$ may not be Gâteaux differentiable.
- $\Psi = \arg\min_{x\in H}(g(x) + h(x)) \neq \emptyset$.

We need the following definitions and lemmas.

**Definition 4** ([34])**.** *Let $h \in \Gamma_0(H)$. The proximal operator of $h$ of order $\lambda > 0$ is defined by*

$$\text{prox}_{\lambda h}(x) := \arg\min_{y\in H}\left\{\frac{1}{2\lambda}\|y - x\|^2 + h(y)\right\}, \quad \forall x \in H.$$

**Lemma 8** ([34])**.** *Let $h \in \Gamma_0(H)$. Then, $\partial h$ is maximal monotone and $(I + \lambda\partial h)^{-1} = \text{prox}_{\lambda h}$.*

**Lemma 9** ([34])**.** *Let $g, h \in \Gamma_0(H)$. Then, $\hat{x} \in \Psi$ if and only if $0 \in (\nabla g + \partial h)(\hat{x})$.*

Next, we apply our main results to solve problem (24).

**Theorem 3.** *Assume that the condition (A3) is satisfied. Let the sequence $\{x_n\}$ be generated by Algorithm 9. The $\{x_n\}$ converges strongly to $x^* \in \Psi$, where $x^*$ is the unique fixed point of $P_\Psi f$.*

**Proof.** $\nabla g$ is monotone because $g$ is convex. Let $F = \nabla g$ and $G = \partial h$ in Theorem 4.1. We can obtain the desired result using Lemmas 8 and 9. □

**Theorem 4.** *Assume that the condition (A3) is satisfied. Let the sequence $\{x_n\}$ be generated by Algorithm 10. Then, $\{x_n\}$ converges strongly to $x^* \in \Psi$, where $x^*$ is the unique fixed point of $P_\Psi f$.*

**Proof.** $\nabla g$ is monotone because $g$ is convex. Let $F = \nabla g$ and $G = \partial h$ in Theorem 4.2. We can obtain the desired result by Lemmas 8 and 9. □

**Algorithm 9:**

**Initialization:** Choose $x_0, x_1 \in H$, $\gamma_1 > 0$, $\mu \in (0,1)$ arbitrarily. For each $i = 1, 2, \cdots, s$ (where $s$ is a chosen positive integer), choose a bounded sequence $\{\alpha_{i,n}\} \subset \mathbb{R}$. Let $\{\varepsilon_n\}$ be a nonnegative number sequence with $\lim_{n\to\infty} \varepsilon_n = 0$.

**Iterative step:** Compute $x_{n+1}$ via

$$
\begin{aligned}
&w_n = x_n + \textstyle\sum_{i=1}^{\min\{s,n\}} \alpha_{i,n}(x_{n-i+1} - x_{n-i}),\\
&y_n = \mathrm{prox}_{\gamma_n h}(I - \gamma_n \nabla g) w_n,\\
&z_n = y_n - \gamma_n(\nabla g(y_n) - \nabla g(w_n)),\\
&C_n = \left\{ u \in H : \|z_n - u\|^2 \le \|w_n - u\|^2 - \left(1 - \mu^2 \frac{\gamma_n^2}{\gamma_{n+1}^2}\right)\|y_n - w_n\|^2 + \varepsilon_n \right\},\\
&Q_n = \begin{cases} H, & \text{if } n = 1,\\ \{u \in Q_{n-1} : \langle x_n - f(x_{n-1}), x_n - u\rangle \le 0\}, & \text{if } n \ge 2, \end{cases}\\
&x_{n+1} = P_{C_n \cap Q_n} f(x_n),
\end{aligned}
$$

where

$$
\gamma_{n+1} = \begin{cases} \min\left\{\gamma_n, \frac{\mu\|w_n - y_n\|}{\|\nabla g(w_n) - \nabla g(y_n)\|}\right\}, & \text{if } \nabla g(w_n) \ne \nabla g(y_n),\\ \gamma_n, & \text{otherwise.} \end{cases}
$$

**Algorithm 10:**

**Initialization:** Choose $x_0, x_1 \in H$, $\gamma_1 > 0$, $\mu \in (0,1)$ arbitrarily. Let $C_1 = H$. For each $i = 1, 2, \cdots, s$ (where $s$ is a chosen positive integer), choose a bounded sequence $\{\alpha_{i,n}\} \subset \mathbb{R}$. Let $\{\varepsilon_n\}$ be a nonnegative number sequence with $\lim_{n\to\infty} \varepsilon_n = 0$.

**Iterative step:** Compute $x_{n+1}$ via

$$
\begin{aligned}
&w_n = x_n + \textstyle\sum_{i=1}^{\min\{s,n\}} \alpha_{i,n}(x_{n-i+1} - x_{n-i}),\\
&y_n = \mathrm{prox}_{\gamma_n h}(I - \gamma_n \nabla g) w_n,\\
&z_n = y_n - \gamma_n(\nabla g(y_n) - \nabla g(w_n)),\\
&C_{n+1} = \left\{ u \in C_n : \|z_n - u\|^2 \le \|w_n - u\|^2 - \left(1 - \mu^2 \frac{\gamma_n^2}{\gamma_{n+1}^2}\right)\|y_n - w_n\|^2 + \varepsilon_n \right\},\\
&x_{n+1} = P_{C_{n+1}} f(x_n),
\end{aligned}
$$

where the computation of $\gamma_{n+1}$ is the same as in Algorithm 9.

## 6. Conclusions

As known, the variational inclusion problems have always been a topic discussed by a large number of scholars. It not only plays an increasingly important role in the field of modern mathematics but also is widely used in many other fields, such as mechanics, optimization theory, nonlinear programming, etc. Tan et al. combined Tseng's algorithm and hybrid projection algorithm to obtain a new strongly convergent algorithm. Our work in this paper is based on the work conducted by Tan et al. combined with the multi-step inertial method and the Krasnosel'skiĭ–Mann algorithm for solving the variational inclusion problems in a real Hilbert space. Then, new strong convergence theorems are obtained. By using our results, we can solve the related problems in a Hilbert space. Our results extend and improve many recent correlative results of other authors [1–6,20,21]. For example, our Algorithm 8 extends and improves Algorithm 4 in [6] in the following ways:

(i) One-step inertia is generalized to multi-step inertia.
(ii) There is an $\varepsilon_n$ in the definition of $C_{n+1}$.

(iii) The anchor value $x_0$ is replaced with $f(x_n)$ for the last step of iteration, where $f$ is a Meir–Keeler contraction. This greatly expands the application scope of the iterative algorithm.

**Author Contributions:** Conceptualization, M.Y. and B.J.; Data curation, B.J.; Formal analysis, Y.W. and M.Y.; Funding acquisition, M.Y.; Methodology, Y.W. and B.J.; Project administration, Y.W.; Resources, B.J.; Supervision, Y.W.; Writing—original draft, M.Y. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** The authors thank the referees for their helpful comments, which notably improved the presentation of this paper. This work was supported by the National Natural Science Foundation of China (grant no. 11671365).

**Conflicts of Interest:** The author declare that they have no competing interests.

## References

1. Rockafellar, R.T. Monotone operators and the proximal point algorithms. *SIAM J. Control Optim.* **1976**, *14*, 877–898. [CrossRef]
2. Passty, G.B. Ergodic convergence to a zero of the sum of monotone operators in Hilbert spaces. *J. Math. Anal. Appl.* **1979**, *72*, 383–390. [CrossRef]
3. Lions, P.L.; Mercier, B. Splitting algorithms for the sum of two nonlinear operators. *SIAM J. Number. Anal.* **1979**, *16*, 964–979. [CrossRef]
4. Tseng, P. A modified forward-backward splitting method for maximal monotone mappings. *SIAM J. Control Optim.* **2000**, *38*, 431–446. [CrossRef]
5. Lorenz, D.; Pock, T. An inertial forward-backward algorithm for monotone inclusions. *J. Math. Imaging Vis.* **2015**, *51*, 311–325. [CrossRef]
6. Tan, B.; Zhou, Z.; Qin, X. Accelerated projection-based forward-backward splitting algorithms for monotone inclusion problems. *J. Appl. Anal. Comput.* **2020**, *10*, 2184–2197.
7. Dadashi, V.; Postolache, M. Forward-backward splitting algorithm for fixed point problems and zeros of the sum of monotone operators. *Arab. J. Math.* **2020**, *9*, 89–99. [CrossRef]
8. Bauschke, H.H.; Combettes, P.L.; Reich, S. The asymptotic behavior of the composition of two resolvents. *Nonlinear Anal.* **2005**, *60*, 283–301. [CrossRef]
9. Zhao, X.P.; Yao, J.C.; Yao, Y. A proximal algorithm for solving split monotone variational inclusions. *UPB Sci. Bull. Ser. A* **2020**, *82*, 43–52.
10. Mann, W.R. Mean value methods in iteration. *Proc. Am. Math. Soc.* **1953**, *4*, 506–510. [CrossRef]
11. Krasnosel'skiĭ, M.A. Two remarks on the method of successive approximations. *Usp. Mat. Nauk.* **1955**, *10*, 123–127.
12. Halpern, B. Fixed points of nonexpanding maps. *Bull. Amer. Math. Soc.* **1967**, *73*, 957–961. [CrossRef]
13. Reich, S. Weak convergence theorems for nonexpansive mappings in Banach spaces. *J. Math. Anal. Appl.* **1979**, *67*, 274–276. [CrossRef]
14. Moudafi, A. Viscosity approximation methods for fixed-points problems. *J. Math. Anal. Appl.* **2000**, *241*, 46–55. [CrossRef]
15. Nakajo, K.; Takahashi, W. Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups. *J. Math. Anal. Appl.* **2003**, *279*, 372–379. [CrossRef]
16. Takahashi, W.; Takeuchi, Y.; Kutoba, R. Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert space. *J. Math. Anal. Appl.* **2008**, *341*, 276–286. [CrossRef]
17. Marino, G.; Xu, H.K. A general iterative method for nonexpansive mappings in Hilbert spaces. *J. Math. Anal. Appl.* **2006**, *318*, 43–52. [CrossRef]
18. Tian, M. A general iterative algorithm for nonexpansive mappings in Hilbert spaces. *Nonlinear Anal.* **2010**, *73*, 689–694. [CrossRef]
19. Thakur, B.S.; Thakur, D.; Postolache, M. A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings. *Appl. Math. Comput.* **2016**, *275*, 147–155. [CrossRef]
20. Dong, Q.L.; Huang, J.Z.; Li, X.H.; Cho, Y.J.; Rassias, T.M. MiKM: Multi-step inertial Krasnosel'skiĭ-Mann algorithm and its applications. *J. Glob. Optim.* **2019**, *73*, 801–824. [CrossRef]
21. Yao, Y.; Shahzad, N.; Liou, Y.C.; Zhu, L.J. A projected fixed point algorithm with Meir-Keeler contraction for pseudocontractive mappings. *J. Nonlinear Sci. Appl.* **2017**, *10*, 483–491. [CrossRef]

22. Yao, Y.; Postolache, M.; Yao, J.C. Strong convergence of an extragradient algorithm for variational inequality and fixed point problems. *UPB Sci. Bull. Ser. A* **2020**, *82*, 3–12.
23. Meir, A.; Keeler, E. A theorem on contraction mappings. *J. Math. Anal. Appl.* **1969**, *28*, 326–329. [CrossRef]
24. Goebel, K.; Reich, S. *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*; Marcel Dekker: New York, NY, USA, 1984.
25. Xu, H.K. Iterative methods for the split feasibility problem in infinite-dimensional Hilbert space. *Inverse Probl.* **2010**, *26*, 10518. [CrossRef]
26. Yao, Y.; Liou, Y.C.; Yao, J.C. Iterative algorithms for the split variational inequality and fixed point problems under nonlinear transformations. *J. Nonlinear Sci. Appl.* **2017**, *10*, 843–854. [CrossRef]
27. Ceng, L.C.; Petrusel, A.; Yao, J.C.; Yao, Y. Systems of variational inequalities with hierarchical variational inequality constraints for Lipschitzian pseudocontractions. *Fixed Point Theory* **2019**, *20*, 113–133. [CrossRef]
28. Reich, S. Fixed points of contractive functions. *Boll. Un. Mat. Ital.* **1972**, *5*, 26–42.
29. Karapınar, E.; Samet, B.; Zhang, D. Meir-Keeler type contractions on JS-metric spaces and related fixed point theorems. *J. Fixed Point Theory Appl.* **2018**, *20*, 60. [CrossRef]
30. Li, C.Y.; Karapınar, E.; Chen, C.M. A discussion on random Meir-Keeler contractions. *Mathematics* **2020**, *8*, 245. [CrossRef]
31. Suzuki, T. Moudafi's viscosity approximations with Meir-Keeler contractions. *J. Math. Anal. Appl.* **2007**, *325*, 342–352. [CrossRef]
32. Xu, H.K. Averaged mappings and the gradient-projection algorithm. *J. Optim. Theory Appl.* **2011**, *150*, 360–378. [CrossRef]
33. Ceng, L.C.; Ansari, Q.H.; Yao, J.C. Some iterative methods for finding fixed points and for solving constrained convex minimization problems. *Nonlinear Anal.* **2011**, *74*, 5286–5302. [CrossRef]
34. Bauschke, H.H.; Combettes, P.L. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*; Springer: Berlin, Germany, 2011.
35. Lin, L.J.; Takahashi, W. A general iterative method for hierarchical variational inequality problems in Hilbert spaces and applications. *Positivity* **2012**, *16*, 429–453. [CrossRef]
36. Takahashi, W.; Xu, H.K.; Yao, J.C. Iterative methods for generalized split feasibility problems in Hilbert spaces. *Set-Valued Var. Anal.* **2015**, *23*, 205–221. [CrossRef]
37. Mosco, U. Convergence of convex sets and of solutions of variational inequalities. *Adv. Math.* **1969**, *3*, 510–585. [CrossRef]
38. Tsukada, M. Convergence of best approximations in a smooth Banach space. *J. Approx. Theory* **1984**, *40*, 301–309. [CrossRef]