mathematics

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Application of Induced Preorderings in Score Function-Based Method for Solving Decision-Making with Interval-Valued Fuzzy Soft Information

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Abstract: Ranking interval-valued fuzzy soft sets is an increasingly important research issue in decision making, and provides support for decision makers in order to select the optimal alternative under an uncertain environment. Currently, there are three interval-valued fuzzy soft set-based decision-making algorithms in the literature. However, these algorithms are not able to overcome the issue of comparable alternatives and, in fact, might be ignored due to the lack of a comprehensive priority approach. In order to provide a partial solution to this problem, we present a group decision-making solution which is based on a preference relationship of interval-valued fuzzy soft information. Further, corresponding to each parameter, two crisp topological spaces, namely, lower topology and upper topology, are introduced based on the interval-valued fuzzy soft topology. Then, using the preorder relation on a topological space, a score function-based ranking system is also defined to design an adjustable multi-steps algorithm. Finally, some illustrative examples are given to compare the effectiveness of the present approach with some existing methods.

Keywords: interval-valued fuzzy soft sets; interval-valued fuzzy soft topology; preference relationship; decision-making

Citation: Ali, M.; Kılıçman, A.; Zahedi Khameneh, A. Application of Induced Preorderings in Score Function-Based Method for Solving Decision-Making with Interval-Valued Fuzzy Soft Information. *Mathematics* **2021**, *9*, 1575. https://doi.org/10.3390/math9131575

Academic Editor: Sorin Nadaban

Received: 18 May 2021
Accepted: 29 June 2021
Published: 4 July 2021

1. Introduction

Dealing with vagueness and uncertainty, rather than exactness, in most real-world situations is the main problem in data-analysis sciences and decision-making. Many mathematical theories and tools such as probability theory, fuzzy set theory [1], interval-valued fuzzy set theory [2], intuitionist fuzzy set theory [3], rough set theory [4] and soft set theory [5] have been implemented to handle this problem, with the latter allowing researchers to deal with parametric data. Nowadays, soft sets theory contributes to a vast range of applications, particularly in decision-making. In this regard, many important results have been achieved, from parameter reduction to new ranking models.

Many soft set extensions and their applications have been discussed in previous studies, such as fuzzy soft sets [6–13] intuitionistic fuzzy sets [14–17], rough soft sets [18,19] and fuzzy soft topology [20–23]. The interval-valued fuzzy soft method was first used for decision-making problems by Son [24]. He applied this method by using the comparison table. Yang et al. [25] developed the method presented in [7] for an interval-valued fuzzy soft set and then, applied the concept of interval-valued fuzzy choice values to propose an approach for solving decision-making problems. The notion of level set in decision-making based on interval-valued fuzzy soft sets was introduced by Feng et al. [26] and then, the level soft set for interval-valued fuzzy soft sets was developed, further see [27]. Khameneh et al. [28–30] introduced the preference relationship for both fuzzy soft sets and intuitionistic fuzzy soft sets and then selected an optimal option for group decision-making problems by defining a new function value. In addition, interval-valued fuzzy soft sets have also been applied to various fields, for example information measure [31–34], decision making [35–38], matrix theory [39–41], and parameter reduction [37,38,42].

Recently, Ma et al. [43] introduced an average and an antitheses table for interval-valued fuzzy soft sets and then selected an optimal option for group decision-making problems through the score value. Ma et al. [44] developed two methods [26,45] to solve decision-making problems by providing a new efficient decision-making algorithm and also considering added objects. However, these methods did not address the problem of incomparable alternatives because they lack a comprehensive priority approach. In order to solve these issues, this paper proposes an application of the induced preorderings based method for solving decision-making with interval-valued fuzzy soft information.Our contributions are as follows:

1 Proposing application of induced preordering based method for solving decision-making with interval-valued fuzzy soft information.
2 Proposing a novel score function of interval-valued fuzzy soft sets that selects an optimal option for group decision-making problems.
3 A real-life example is given to compare the effectiveness of this approach with some existing methods.

2. Preliminaries

In this section, we recall some definitions and properties of interval-valued fuzzy sets (IVF) and interval-valued fuzzy soft sets $(IVFS)$. Note that, throughout this paper, X and E denote the sets of objects and parameters, respectively. $\mathbb{I}^X$ and $[\mathbb{I}]^X$, where $\mathbb{I} = [0,1]$ and $[\mathbb{I}] = \{[a,b], a \leq b, a, b \in \mathbb{I}\}$ denote, respectively, the set of all fuzzy subsets and the set of all interval-valued fuzzy subsets of X.

Definition 1. *Ref. [2] A pair (f, X), is called an IVF subset of X if f is a mapping given by $f : X \to [\mathbb{I}]$ such that for any $x \in X, f(x) = [f^-(x), f^+(x)]$ is a closed subinterval of $[0,1]$ where $f^-(x)$ and $f^+(x)$ are referred to as the lower and upper degrees of membership x to f and $0 \leq f^-(x) \leq f^+(x) \leq 1$.*

In 1999, Molodtsov [5] defined the concept of soft sets (SS) for the first time as a pair of (f, E) or f_E such that E is a parameter set and f is the mapping $f : E \to 2^X$ where for any $e \in E, f(e)$ is a subset of X. By combining the concepts of soft sets and interval-valued fuzzy sets, a new hybrid tool was defined as the following.

Definition 2. *Ref. [25] A pair (f, E) is called an $IVFS$ set over X if the mapping f is given by $f : E \to [\mathbb{I}]^X$ where for any $e \in E$ and $x \in X$, $f(e)(x) = [f^-(e)(x), f^+(e)(x)]$.*

Consider two $IVFSs$ f_E, g_E over the common universe X. The union of f_E and g_E, denoted by $f_E \tilde{\vee} g_E$, is the $IVFSs$ $(f \tilde{\vee} g)_E$, where $\forall e \in E$ and any $x \in X$, we have $(f \tilde{\vee} g)(e)(x) = [\max\{f_e^-(x), g_e^-(x)\}, \max\{f_e^+(x), g_e^+(x)\}]$. The intersection of f_E and g_E, denoted by $f_E \tilde{\wedge} g_E$, is the $IVFSs$ $(f \tilde{\wedge} g)_E$, where $\forall e \in E$ and $\forall x \in X$, we have $(f \tilde{\wedge} g)(e)(x) = [\min\{f_e^-(x), g_e^-(x)\}, \min\{f_e^+(x), g_e^+(x)\}]$. The complement of f_E is denoted by f_E^c and is defined by $f^c : E \to [\mathbb{I}]^X$ where $\forall e \in E$ and any $x \in X$, $f^c(e)(x) = [1 - f_e^+(x), 1 - f_e^-(x)]$. The null $IVFSs$, denoted by $\emptyset_E$, is defined as an $IVFSs$ over X such that $f_e^-(x) = f_e^+(x) = 0$ for all $e \in E$ and any $x \in X$. The absolute $IVFSs$, denoted by X_E, is defined as an $IVFSs$ over X where $f_e^-(x) = f_e^+(x) = 1, \forall e \in E$ and any $x \in X$.

Using the matrix form of interval-valued fuzzy relations, authors in [39] represented a finite IVFSs f_E as the following $n \times m$ matrix

$$f_E = \left[[f_{ij}^-, f_{ij}^+]\right]_{n \times m} = \begin{bmatrix} [f_{e_1}^-(x_1), f_{e_1}^+(x_1)] & \dots & [f_{e_1}^-(x_m), f_{e_1}^+(x_m)] \\ \vdots & \dots & \vdots \\ [f_{e_n}^-(x_1), f_{e_1}^+(x_1)] & \dots & [f_{e_n}^-(x_m), f_{e_1}^+(x_m)] \end{bmatrix}_{n \times m}$$

where $\mid E \mid= n$, $\mid X \mid= m$ and $f_{ij}^- = f_{e_i}^-(x_j)$, $f_{ij}^+ = f_{e_i}^+(x_j)$ for $i = 1, \dots, n$ and $j = 1, \dots, m$.

Accordingly, the concepts of union, intersection, complement, etc., can be represented in a matrix format in the finite case.

Definition 3. *Ref. [46] A triplet (X, E, τ) is called an interval-valued fuzzy soft topological space (IVFST) if τ is a collection of interval-valued fuzzy soft subsets of X containing absolute and null $IVFSs$ and closed under arbitrary union and finite intersection.*

Preorders and Topologies

In this subsection, we present some basic properties about the connection between preorders and topologies proposed by [47].

Topological structures and classical order structures are well recognised to have close relationships, which can be summarised as follows:

(1) A subset A of X is called an upper set of X if $A = \uparrow A$, where $\uparrow A$ defined by $\uparrow A = \{y \in X : \exists x \in A, x \leq y\}$, and X is a preordered set, and B is called a lower set $B = \downarrow B = \{y \in X : \exists x \in X, y \leq x\}$.
(2) The family of all upper subsets of x is a topology for a preorder set $(X, \leq)$, which is called the Alexandrov topology induced in $(X, \leq)$.
(3) A topological space (X, τ) is defined by $x \leq y$ if and only if $x \in U$, then $y \in U$ for each open set U of X, or equivalently $x \in c\{y\}$, where $c\{y\}$, is the closure of $\{y\}$. Then, $\leq$ is a preorder on X, called the specialization order (X, τ) on X.

3. Construction Tow Preorderings in Lower and Upper Spaces

By using the notion of $[\alpha_1, \alpha_2]$-level sets of interval-valued fuzzy soft open sets in (X, E, τ), this section, introduces two topological spaces, known as lower and upper spaces, by which two preordering relations over the universal set X are investigated.

Definition 4. *Let f_E be an IVFS set over X. Corresponding to each parameter $e \in E$,, we define two crisp sets, called α-upper-e crisp set and β-lower-e crisp set, where $\alpha = [\alpha_1, \alpha_2] \subset \mathbb{I}, \beta = [\beta_1, \beta_2] \subset \mathbb{I}$ as the following:*

$$\begin{aligned} U.C.S^f_\alpha(e) &= \{x \in X : [f^-_e(x), f^+_e(x)] > \alpha, \alpha \subseteq [0,1)\} \\ &= \{x \in X : f^-_e(x) > \alpha_1, f^+_e(x) > \alpha_2, \alpha_1, \alpha_2 \in [0,1)\} \\ L.C.S^f_\beta(e) &= \{x \in X : [f^-_e(x), f^+_e(x)] < \beta, \beta \subseteq (0,1]\} \\ &= \{x \in X : f^-_e(x) < \beta_1, f^+_e(x) < \beta_2, \beta_1, \beta_2 \in (0,1]\} \end{aligned}$$

Proposition 1. *Let X be the set of objects, E be the set of parameters and f_E, g_E be two $IVFSs$ over X. Suppose that the threshold intervals $\alpha_1, \alpha_2, \subseteq [0,1)$, and $\beta_1, \beta_2, \subseteq (0,1]$ are given such that $\alpha_1 = [\alpha_1^\star, \alpha_1^{\star\star}], \alpha_2 = [\alpha_2^\star, \alpha_2^{\star\star}], \beta_1 = [\beta_1^\star, \beta_1^{\star\star}]$ and $\beta_2 = [\beta_2^\star, \beta_2^{\star\star}]$. Consider the parameter $e \in E$.*

1. *If $\alpha_1 \geq \alpha_2$, then $U.C.S^f_{\alpha_1}(e) \subseteq U.C.S^f_{\alpha_2}(e)$. If $\beta_1 \geq \beta_2$, then $L.C.S^f_{\beta_2}(e) \subseteq L.C.S^f_{\beta_1}(e)$.*
2. *If $f_E \tilde{\leq} g_E$, then $U.CS^f_{\alpha_1}(e) \subseteq U.C.S^g_{\alpha_1}(e)$ and $L.C.S^f_{\beta_1}(e) \subseteq L.C.S^g_{\beta_1}(e)$.*
3. *If $f_E = X_E$, then $U.C.S^f_{\alpha_1}(e) = X$ and $L.C.S^f_{\beta_1}(e) = \emptyset$. Moreover, if $f_E = \emptyset_E$ then, $U.CS^f_{\alpha_1}(e) = \emptyset$ and $L.CS^f_{\beta_1}(e) = X$.*
4. *$U.C.S^f_{\alpha_1}(\neg e) = L.CS^f_{[1-\alpha_1^{\star\star}, 1-\alpha_1^\star]}(e)$ and $L.CS^f_{\alpha_1}(\neg e) = U.CS^f_{[1-\alpha_1^{\star\star}, 1-\alpha_1^\star]}(e)$.*
5. *$U.CS^{\neg f}_{\alpha_1}(e) = L.CS^f_{[1-\alpha_1^{\star\star}, 1-\alpha_1^\star]}(e)$ and $L.CS^{\neg f}_{\alpha_1}(\neg e) = U.Des^f_{[1-\alpha_1^{\star\star}, 1-\alpha_1^\star]}(e)$.*

Proof. It is straightforward. □

Theorem 1. *Let (X, E, τ) be an $IVFSTS$. Suppose that the threshold intervals $\alpha_1, \alpha_2, \subseteq [0,1)$, and $\beta_1, \beta_2, \subseteq (0,1]$ are given such that $\alpha = [\alpha_1, \alpha_2]$ and $\beta = [\beta_1, \beta_2]$, then*

1. *The collection $\{U.C.S^f_\alpha(e) : f_E \in \tau, e \in E, \alpha \subseteq [0,1)\}$, denoted by $\tau^u_{e,\alpha}$, is a topology over X.*
2. *The collection $\mathfrak{B}^l_\beta(e) = \{L.C.S^f_\beta(e) : f_E \in \tau, e \in E, \beta \subseteq (0,1]\}$, is a base for a topology over X, denoted by $\tau^l_{e,\beta}$.*

Proof. 1. (a) By Proposition 1, $X, \emptyset \in \tau^u_{e,\alpha}$, since $X_E, \emptyset_E \in \tau$.

(b) Let $\{U.C.S^{f_i}_\alpha(e)\}_{i\in I}$ be a subfamily of $\tau^u_{e,\alpha}$. Then, we have $\bigcup_i U.C.S^{f_i}_\alpha(e) = U.C.S^{(\tilde{\bigtriangledown}_{i\in I} f_i)}_\alpha(e) \in \tau^u_{e,\alpha}$, since $\tilde{\bigtriangledown}_{i\in I} f_{iE} \in \tau$.

(c) Let $U.C.S^f_\alpha(e)$ and $U.C.S^g_\alpha(e)$ be two open sets in $\tau^u_{e,\alpha}$. Then, we have $U.CS^f_\alpha(e) \cap U.C.S^g_\alpha(e) = U.C.S^{(f\tilde{\bigtriangleup}g)}_\alpha(e) \in \tau^u_{e,\alpha}$, since $f_E\tilde{\bigtriangleup}g_E \in \tau$. This completes the proof.

2. (a) That $X \in \mathfrak{B}^l_\beta(e)$ is implied from $\emptyset_E$ is in τ.

(b) Let $L.C.S^f_\beta(e)$ and $L.C.S^g_\beta(e)$ in $\mathfrak{B}^l_\beta(e)$. Then, we have $L.C.S^f_\beta(e) \cap L.C.S^g_\beta(e) = L.C.S^{f\tilde{\bigtriangledown}g}_\beta(e) \in \mathfrak{B}^l_\beta(e)$ that is implied form $f\tilde{\bigtriangledown}g \in \tau$.

□

Theorem 2. *Let (X, E, τ) be an $IVFSTS$. Suppose that the threshold intervals $\alpha_1, \alpha_2, \subseteq [0, 1)$, and $\beta_1, \beta_2, \subseteq (0, 1]$ are given such that $\alpha = [\alpha_1, \alpha_2]$ and $\beta = [\beta_1, \beta_2]$.*

1. *The binary relation $\succsim^\tau_{e,\alpha}$ on X defined by*

$$y \succeq^\tau_{e,\alpha} x \Leftrightarrow [\forall V \in \tau^u_{e,\alpha} : x \in V \Rightarrow y \in V]$$

is a preorder relation called α-upper-e preorder relation on X.

2. *The binary relation $\preceq^{\tau,\beta}_e$ on X defined by*

$$y \preceq^{\tau,\beta}_e x \Leftrightarrow [\forall U \in \tau^l_{e,\beta} : x \in U \Rightarrow y \in U]$$

is a preorder relation called β-lower-e preorder relation on X.

Proof. 1. For all $x \in X$, obviously, $x \succsim^u_{e,\alpha} x$, that is, "$\succsim^u_{e,\alpha}$" is reflexive. Now, for all $x, y, z \in X$, if $y \succsim^u_{e,\alpha} x$, and $z \succsim^u_{e,\alpha} y$, then, if for all $V \in \tau^u_{e,\alpha}$-open set, $x \in V$, then $y \in V$ and $z \in V$, so, $z \succsim^u_{e,\alpha} x$, that is, "$\succsim^u_{e,\alpha}$" is transitive. Thererfore, $(X, \succsim^u_{e,\alpha})$ is a preordered set.

2. A similar technique is used to prove the second part.

□

Theorem 3. *Let (X, E, τ) be an $IVFSTS$. Suppose that the threshold intervals $\alpha_1, \alpha_2, \subseteq [0, 1)$, and $\beta_1, \beta_2, \subseteq (0, 1]$ are given such that $\alpha = [\alpha_1, \alpha_2]$ and $\beta = [\beta_1, \beta_2]$.*

1. *The binary relation $\simeq^\tau_{e,\alpha}$,defined by*

$$y \simeq^\tau_{e,\alpha} x \Leftrightarrow [y \succsim^\tau_{e,\alpha} x, x \succsim^\tau_{e,\alpha} y]$$

is an equivalence relation over X. If $y \simeq^\tau_{e,\alpha} x$, then we say x and y are α-upper equivalent with to respect to the parameter e.
The equivalence relation $\simeq^\tau_{e,\alpha}$, generates the partition $P^\tau_{e,\alpha}$ of X where the equivalence classes are defined as $[x]^\tau_{e,\alpha} = \{z \in X : z \simeq^\tau_{e,\alpha} x$ and are called α-upper-e equivalence classes.

2. *The binary relation $\simeq^{\tau,\beta}_e$, where $\beta = [\beta_1, \beta_2]$,*

$$y \simeq^{\tau,\beta}_e x \Leftrightarrow [y \precsim^{\tau,\beta}_e x, x \precsim^{\tau,\beta}_e y]$$

is an equivalence relation over X. If $y \simeq^\tau_{e,\beta} x$, then we say x and y are $[\beta_1, \beta_2]$-lower equivalent with to respect to the parameter e. The equivalence relation $\simeq^{\tau,\beta}_e$, generates the partition $P^{\tau,\beta}_e$ of X where the equivalence classes are defined as $[x]^{\tau,\beta}_e = \{z \in X : z \simeq^{\tau,\beta}_e x$ and are called β-lower-e equivalence classes.

Proof. It is straightforward. □

Preorder and Equivalence Matrices

Now, let the finite sets $X = \{x_1, \cdots, x_m\}$ and $E = \{e_1, \cdots, e_n\}$ be given as the sets of objects and parameters. Then, the previous properties can be represented by using the matrix form of $IVFS$ sets as the following.

Take an $IVFS$ set f_E over X. First, for any $1 \leq i \leq m$ and $1 \leq t \leq n$, the concepts of α-upper-e_t and β-lower-e_t matrices of f_E, where $\alpha, \beta \subseteq \mathbb{I}$, can be formulated as the following two matrices (or row vectors)

$$U.C.S_\alpha e_t^f = [u_i^f(e_t, \alpha)]_{1\times m} = \begin{cases} 1 & \text{if } f_{e_t}^-(x_i) > \alpha_1, f_{e_t}^+(x_i) > \alpha_2 \\ 0 & \text{if } f_{e_t}^-(x_i) \leq \alpha_1, f_{e_t}^+(x_i) \leq \alpha_2 \end{cases} \tag{1}$$

and

$$L.C.S_\beta e_t^f = [l_i^f(e_t, \beta)]_{1\times m} = \begin{cases} 0 & \text{if } f_{e_t}^-(x_i) \geq \beta_1, f_{e_t}^+(x_i) \geq \beta_2 \\ 1 & \text{if } f_{e_t}^-(x_i) < \beta_1, f_{e_t}^+(x_i) < \beta_2 \end{cases} \tag{2}$$

where $\alpha = [\alpha_1, \alpha_1]$ and $\beta = [\beta_1, \beta_2]$ are the given threshold vectors.

Then, obviously, for any $e_t \in E$, the topologies $\tau_{e_t,\alpha}^u$ and $\tau_{e_t,\beta}^l$ can be represented by the collections

$$\tau_{e_t,\alpha}^u = \{[u_i^f(e_t, \alpha)]_{1\times m} : \alpha \subseteq [0,1), f_E \in \tau, 1 \leq i \leq m\}$$

and

$$\tau_{e_t,\beta}^l = \{[l_i^f(e_t, \beta)]_{1\times m} : \beta \subseteq (0,1], f_E \in \tau, 1 \leq i \leq m\}$$

where τ is the $IVFST$ on X.

Accordingly, the preorderings $\succeq_{e_t,\alpha}^\tau$ and $\preceq_{e_t}^{\tau,\beta}$ can be represented by

$$x_i \succeq_{e_t,\alpha}^\tau x_j \Leftrightarrow [\forall f_E \in \tau : u_j^f(e_t, \alpha) = 1 \Rightarrow u_i^f(e_t, \alpha) = 1]$$

and

$$x_i \preceq_{e_t}^{\tau,\beta} x_j \Leftrightarrow [\forall f_E \in \tau : l_j^f(e_t, \beta) = 1 \Rightarrow l_i^f(e_t, \beta) = 1]$$

where $x_i, x_j \in X$.

The matrix forms of the preorderings $\succsim_{e_t,\alpha}^\tau$ and $\precsim_{e_t}^{\tau,\beta}$ are used to define two comparison matrices $G_\alpha(e_t) = [g_\alpha(e_t)_{ij}]_{m\times m}$ and $S_\beta(e_t) = [s_\beta(e_t)_{ij}]_{m\times m}$, which are two square matrices whose rows and columns are labeled by the objects of X, as below.

Definition 5. *Consider the binary relations $\succsim_{e_t,\alpha}^\tau$ and $\precsim_{e_t}^{\tau,\beta}$ and threshold intervals $\alpha = [\alpha_1, \alpha_2]$, $\beta = [\beta_1, \beta_2 \subseteq \mathbb{I}$. Then, we define*

$$G_\alpha(e_t) = [g_\alpha(e_t)_{ij}]_{m\times m} : g_{[\alpha_1,\alpha_2]}(e_t)_{ij} = \begin{cases} 1 & \text{if } x_i \succsim_{e_t,\alpha}^\tau x_j \\ 0 & \text{otherwis} \end{cases} \tag{3}$$

and

$$S_\beta(e_t) = [s_\beta(e_t)_{ij}]_{m\times m} : s_{[\beta_1,\beta_2]}(e_t)_{ij} = \begin{cases} 1 & \text{if } x_i \precsim_{e_t}^{\tau,\beta} x_j \\ 0 & \text{otherwis} \end{cases} \tag{4}$$

Proposition 2. *Let (X, E, τ) be an $IVFST$ and $G_\alpha(e)$ and $S_\beta(e)$ be two matrices defined in Equations (3) and (4). Then,*

1. *For $1 \leq i \leq m, g_\alpha(e_t)_{ii} = 1$ and $s_\beta(e_t)_{ii} = 1$,*
2. *If $g_\alpha(e_t)_{ij} = g_\alpha(e_t)_{jk} = 1$, then $g_\alpha(e_t)_{ik} = 1$. If $s_\beta(e_t)_{ij} = s_\beta(e_t)_{jk} = 1$, then $s_\beta(e_t)_{ik} = 1$.*
3. *$G_\alpha(e_t)$ and $S_\beta(e_t)$ are symmetric matrices.*

where $i, j, k \in \{1, \ldots, m\}$

Proof. It is straightforward. □

Proposition 3. *Let (X, E, τ) be an $IVFSTS$ and $\alpha, \beta \subseteq \mathbb{I}$, where $\alpha = [\alpha_1, \alpha_1]$ and $\beta = [\beta_1, \beta_2]$ are the threshold intervals, then*

1. *$G_\alpha(e_t)$ is an identity matrix if and only if $\neg(x_i \succeq_{e_t\alpha}^\tau x_j), \forall i, j = 1, \ldots, m$ and $i \neq j$.*
2. *$S_\beta(e_t)$ is an identity matrix if and only if $\neg(x_i \preceq_{e_t}^{\tau,\beta} x_j), \forall i, j = 1, \ldots, m$ and $i \neq j$.*
3. *$G_\alpha(e_t)$ is a unit matrix if and only if $x_i \succeq_{e_t,\alpha}^\tau x_j, \forall i, j = 1, \ldots, m$ and $i \neq j$.*
4. *$S_\beta(e_t)$ is a unit matrix if and only if $x_i \preceq_{e_t}^{\tau,\beta} x_j, \forall i, j = 1, \ldots, m$ and $i \neq j$.*

Proof. It is straightforward. □

Proposition 4. *Let (X, E, τ) be an IVFSTS and $\alpha, \beta \subseteq \mathbb{I}$, where $\alpha = [\alpha_1, \alpha_1]$, $\beta = [\beta_1, \beta_2]$ are the threshold intervals, then*

1. $G_\alpha(e_t) = I_m^U$ *if and only if we have* $x_1 \succeq_{e_t,\alpha}^{\tau} \cdots \succeq_{e_t,\alpha}^{\tau} x_m$.
2. $S_\beta(e_t) = I_m^U$ *if and only if* $x_1 \preceq_{e_t}^{\tau,\beta} \cdots \preceq_{e_t}^{\tau,\beta} x_m$.
3. $G_\alpha(e_t) = I_m^L$ *if and only if* $x_m \succeq_{e_t,\alpha}^{\tau} \cdots \succeq_{e_t,\alpha}^{\tau} x_1$.
4. $S_\beta(e_t) = I_m^L$ *if and only if* $x_m \preceq_{e_t}^{\tau,\beta} \cdots \preceq_{e_t}^{\tau,\beta} x_1$.

where I_m^U, I_m^L are the upper and lower triangular matrix, respectively.

Proof. It is straightforward. □

Analogously, the equivalence relations $\simeq_{e_t,\alpha}^{\tau}$ and $\simeq_{e_t}^{\tau,\beta}$ can be applied to compute the following two square matrices

$E_\alpha^U(e_t) = [e_\alpha^u(e_t)_{ij}]_{m\times m}$ and $E_\beta^L(e_t) = [e_\beta^l(e_t)_{ij}]_{m\times m}$, respectively, where $\alpha = [\alpha_1, \alpha_2]$, $\beta = [\beta_1, \beta_2] \subseteq \mathbb{I}$.

Definition 6. *Consider the binary relations $\simeq_{e_t,\alpha}^{\tau}$ and $\simeq_{e_t}^{\tau,\beta}$ and threshold intervals $\alpha = [\alpha_1, \alpha_2]$, $\beta = [\beta_1, \beta_2 \subseteq \mathbb{I}$. We define*

$$E_\alpha^U(e_t) = [e_\alpha^u(e_t)_{ij}]_{m\times m} : e_\alpha^u(e_t)_{ij} = \begin{cases} 1 & \text{if } x_i \simeq_{e_t,\alpha}^{\tau} x_j \\ 0 & \text{otherwis} \end{cases} \tag{5}$$

and

$$E_\beta^L(e_t) = [e_\beta^l(e_t)_{ij}]_{m\times m} : e_\beta^l(e_t)_{ij} = \begin{cases} 1 & \text{if } x_i \simeq_{e_t}^{\tau,\beta} x_j \\ 0 & \text{otherwis} \end{cases} \tag{6}$$

Proposition 5. *Let (X, E, τ) be an IVFST and $E_\alpha^U(e_t)$ and $E_\beta^L(e_t)$ be the comparison matrices defined in Equations (5) and (6). Then,*

1. *For any $1 \le i \le m$: $e_\alpha^u(e_t)_{ii} = 1$ and $e_\beta^l(e_t)_{ii} = 1$.*
2. *$E^U(e_t)$ and $E^L(e_t)$ are symmetric matrices.*
3. *If $e_\alpha^u(e_t)_{ik} = e_\alpha^u(e_t)_{jk} = 1$, then $e_\alpha^u(e_t)_{ij} = e_\alpha^u(e_t)_{ji} = 1$. If $e_\beta^l(e_t)_{ik} = e_\beta^l(e_t)_{jk} = 1$, then $e_\beta^l(e_t)_{ij} = e_\beta^l(e_t)_{ji} = 1$.*
4. *If $e_\alpha^u(e_t)_{ki} = e_\alpha^u(e_t)_{kj} = 1$, then $e_\alpha^u(e_t)_{ij} = e_\alpha^u(e_t)_{ji} = 1$. If $e_\beta^l(e_t)_{ki} = e_\beta^l(e_t)_{kj} = 1$, then $e_\beta^l(e_t)_{ij} = e_\beta^l(e_t)_{ji} = 1$.*

where $i, j, k \in \{1, \cdots, m\}$

Proof. It is straightforward. □

4. An Application in Decision-Making Problems

The main task in decision making methods is to rank the given candidates to find the optimum choice. Since the proposed preorderings, given in Section 3, are not total or linear, we define a score function S based on the entries of defined comparison matrices to obtain a new ranking system of objects according to preorderings $\succsim_{e_t,\alpha}^{\tau}$ and $\precsim_{e_t}^{\tau,\beta}$.

Definition 7. *Let X and E be the universal sets of objects and parameters, respectively, and $\alpha, \beta \subseteq \mathbb{I}$, where $\alpha = [\alpha_1, \alpha_1]$ and $\beta = [\beta_1, \beta_2]$, are the threshold intervals. The mapping $S = X \to \mathbb{R}$ defined by*

$$S(x_i) = S_i = \sum_{t=1}^{n}\left(\left[\sum_{j=1}^{m} g_\alpha(e_t)_{ij} - \sum_{j=1}^{m} e_\alpha^u(e_t)_{ij}\right] - \left[\sum_{j=1}^{m} s_\beta(e_t)_{ij} - \sum_{j=1}^{m} e_\beta^l(e_t)_{ij}\right]\right)$$

where $x_i \in X$ and S_i is score value of object x_i.

Example 1. *Suppose that* $X = \{o_1, o_2, o_3, o_5\}$ *be a set of 5 hotels in Langkawi and* $E = \{e_1, \ldots, e_4\}$ *be a set of parameters where for any* $t = 1, \ldots, 4$ *the parameter* e_t *stands for "location", "cleanliness", "facilities", and " food", respectively. Reviewers are classified into three groups: couples, solo travelers, and a group of friends. We consider these groups of reviewers as three different decision-makers,* f_1, f_2, f_3*, characterized based on the criteria* $e_t \in E$*. These three groups provide the following three IVFS matrices* f_{1E}, f_{2E}, f_{3E}*.*

Step 1. The following three interval-valued fuzzy soft set $f_{sE}(s = 1, 2, 3)$ that are given in Tables 1–3.

Table 1. f_{1E}.

f_{1E}	o_1	o_2	o_3	o_4	o_5
e_1	$[0.1, 0.4]$	$[0.4, 0.4]$	$[0.4, 0.5]$	$[0, 0.5]$	$[0.0, 0.0]$
e_2	$[0.5, 0.6]$	$[0.3, 0.6]$	$[0.3, 1.0]$	$[0.7, 1.0]$	$[0.0, 0.7]$
e_3	$[0.0, 0.5]$	$[0.5, 0.8]$	$[0.1, 0.8]$	$[0.1, 0.9]$	$[0.3, 0.9]$
e_4	$[0.0, 0.8]$	$[0.7, 0.8]$	$[0.1, 0.7]$	$[0.1, 1.0]$	$[0.6, 1.0]$

Table 2. f_{2E}.

f_{2E}	o_1	o_2	o_3	o_4	o_5
e_1	$[0.2, 0.6]$	$[0.2, 0.6]$	$[0.6, 0.6]$	$[0.5, 0.6]$	$[0.5, 0.6]$
e_2	$[0.4, 0.8]$	$[0.0, 0.8]$	$[0.0, 0.6]$	$[0.6, 0.9]$	$[0.6, 0.9]$
e_3	$[0.1, 0.5]$	$[0.1, 0.8]$	$[0.6, 0.8]$	$[0.5, 0.6]$	$[0.5, 0.9]$
e_4	$[0.3, 0.6]$	$[0.3, 0.3]$	$[0.2, 0.3]$	$[0.2, 0.7]$	$[0.7, 0.7]$

Table 3. f_{3E}.

f_{3E}	o_1	o_2	o_3	o_4	o_5
e_1	$[0.5, 0.6]$	$[0.6, 0.5]$	$[0.1, 0.8]$	$[0.1, 0.5]$	$[0.5, 0.6]$
e_2	$[0.2, 0.8]$	$[0.2, 0.2]$	$[0.2, 0.2]$	$[0.2, 0.6]$	$[0.1, 0.6]$
e_3	$[0.1, 0.3]$	$[0.1, 0.2]$	$[0.2, 0.3]$	$[0.2, 0.3]$	$[0.2, 0.8]$
e_4	$[0.7, 0.8]$	$[0.0, 0.8]$	$[0.0, 0.1]$	$[0.1, 1.0]$	$[0.1, 0.3]$

Step 2. Assume that $[\alpha_1, \alpha_2] = [0.3, 0.6]$ and $[\beta, \beta_2 = [0.2, 0.4]$.
Step 3. The upper crisp matrices and lower crisp matrices, as below:

$$f_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}, f_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, f_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$f_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, f_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, f_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Step 4. The upper topology and lower topology are shown in Tables 4 and 5.

Table 4. α-Upper-e_t topology; $\alpha = [\alpha_1, \alpha_2], t = 1, \ldots, 4$.

		$\tau^{u}_{e_t,\alpha}$										
e_1	$\{[0]_{1\times 5}$	$[1]_{1\times 5}$	[0	1	0	0	0]}					
e_2	$\{[0]_{1\times 5}$	$[1]_{1\times 5}$	[0	0	0	1	0]	[1	0	0	1	1]}
e_3	$\{[0]_{1\times 5}$	$[1]_{1\times 5}$	[0	1	0	0	0]	[0	0	1	0	1]
	[0	1	1	0	1] }							
e_4	$\{[0]_{1\times 5}$	$[1]_{1\times 5}$	[0	1	0	0	1]	[0	0	0	0	1]
	[1	0	0	0	0]	[1	0	0	0	1]		
	[1	1	0	0	1] }							

Table 5. β-Lower-e_t topology; $\beta = [\beta_1, \beta_2], t = 1, \ldots, 4$.

		$\tau^{L}_{e_t,\beta}$					
e_1	$\{[0]_{1\times 5}$	$[1]_{1\times 5}$	[0	0	0	0	1]}
e_2	$\{[0]_{1\times 5}$	$[1]_{1\times 5}\}$					
e_3	$\{[0]_{1\times 5}$	$[1]_{1\times 5}$	[1	1	0	0	0] }
e_4	$\{[0]_{1\times 5}$	$[1]_{1\times 5}$	[0	0	1	0	1]}

Step 5. The comparison matrices $G(e_t, \alpha), S(e_t, \beta), E^U(e_t, \alpha)$ and $E^U(e_t, \alpha)$, over X where $\alpha = [\alpha_1, \alpha_2], \beta = [\beta_1, \beta_2], t = 1, \ldots, 4$ as below:

$$G(e_1, [0.3, 0.6]) = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix} S(e_1[0.2, 0.4) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$G(e_2, [0.3, 0.6]) = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} S(e_2[0.2, 0.4) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$G(e_3, [0.3, 0.6]) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix} S(e_3, [0.2, 0.4) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$G(e_4, [0.3, 0.6]) = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} S(e_4, [0.2, 0.4) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$E^U(e_1, [0.3, 0.6]) = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix} E^L(e_1, [0.2, 0.4) = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E^U(e_2,[0.3,0.6]) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix} E^L(e_2,[0.2,0.4]) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$E^U(e_3,[0.3,0.6]) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} E^L(e_3,[0.2,0.4]) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$E^U(e_4,[0.3,0.6]) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} E^L(e_4,[0.2,0.4]) = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Step 6. By using Definition (2), we have,

$$S_1 = r_1(e_1;[0.3,0.6],[0.2,0.4]) + r_1(e_2,[0.3,0.6],[0.2,0.4]) + r_1(e_3,[0.3,0.6],[0.2,0.4]) \\ +r_1(e_4,[0.3,0.6],[0.2,0.4] = 1.$$

Similarly, $S_2 = 5$, $S_3 = 0$, $S_4 = -1$, $S_6 = 5$.

Step 7. Then, the ordering is obtained as below

$$o_2 \backsimeq o_5 \succeq o_1 \succeq o_3 \succeq o_4$$

Steps 8 and 9. Accordingly, o_2 and o_5 can be the best objects (Acceptance region), while o_4 not be selected(Rejection region), and o_1, o_3 cannot be judged(Boundary region).

4.1. Comparison with Existing Methods

In this section, we will apply and compare present method and other methods [25,43,44] using real-life example via datasets given in [47] Table 8 from the www.weather.com.cn website. (accessed on 15 May 2021).

Example 2. *Let an $IFVSs$ f_E describes a family who wants to go to a city in China. Suppose that the weather provides a forecast for fifteen cities in China during the holiday, $X = \{o_1, \ldots, o_{15}\}$, which is shown in Table 6. Suppose that the data of weather forecast describes five parameters $E = \{e_1, e_2, e_3, e_4, e_5\}$. Parameters $e_t, t = 1, \ldots, 5$, stand for "temperature", "air quality index", "levels of ultraviolet radiation", "wind speed", "precipitation", respectively.*

Step 1. The $IVFSs$ f_E is given in Table 6.
Step 2. Suppose that
$\alpha = [0.67, 0.92], [0.75, 0.94], [0.66, 0.92], [0.49, 0.75], [0.96, 0.99]$
$\beta = [0.14, 0.8], [0.37, 0.77], [0.25, 0.76], [0.26, 0.76], [0.67, 1]$,
where $\alpha = [\alpha_1, \alpha_2], \beta = [\beta_1, \beta_2]$

Table 6. Table for f_E.

f_E	e_1	e_2	e_3	e_4	e_5
o_1	[0.14, 0.86]	[0.21, 0.97]	[0.0, 0.47]	[0.25, 1.0]	[1.0, 1.0]
o_2	[0.43 ,0.82]	[0.45, 0.78]	[0.0, 0.33]	[0.25, 1.0]	[0.83, 1]
o_3	[0.64, 1.0]	[0.26, 0.63]	[0.0, 0.73]	[0.5, 1.0]	[1.0, 1.0]
o_4	[0.5, 0.82]	[0.45, 0.82]	[0.6, 0.93]	[0.0, 1.0]	[0.97, 1]
o_5	[0.39, 0.68]	[0.79, 0.88]	[0.67, 1.0]	[0.25, 1.0]	[0.83, 1]
o_6	[0.68, 0.93]	[0.6, 0.77]	[0.6, 0.93]	[0.5, 1.0]	[0.58, 1]
o_7	[0.36, 0.71]	[0.37, 0.96]	[0.67, 0.93]	[0.0, 0.75]	[0.96, 1]
o_8	[0.5, 0.89]	[0.76, 0.95]	[0.67, 1.0]	[0.5, 1.0]	[0.89, 1]
o_9	[0.25, 0.71]	[0.02, 1.0]	[0.67, 1.0]	[0.0, 0.75]	[0.58, 1]
o_{10}	[0.0, 0.71]	[0.53, 0.92]	[0.6, 0.93]	[0.5, 0.75]	[1.0, 1.0]
o_{11}	[0.0, 0.54]	[0.58, 1.0]	[0.73, 1.0]	[0.0, 0.75]	[0.67, 1]
o_{12}	[0.34, 0.89]	[0.0, 1.0]	[0.67, 1.0]	[0.25, 0.75]	[1.0, 1.0]
o_{13}	[0.25, 0.71]	[0.58, 1.0]	[0.73, 1.0]	[0.0, 0.75]	[0.67, 1]
o_{14}	[0.34, 0.89]	[0.53, 0.95]	[0.67, 0.93]	[0.25, 0.75]	[1.0, 1.0]
o_{15}	[0.25, 0.71]	[0.66, 0.97]	[0.6, 0.93]	[0.25, 1.0]	[0.0, 1.0]

Steps 3 and 4. The α-Upper-e_t Crisp and β-Lower-e_t Crisp; the α-Upper-e_t Topology and β-Lower-e_t Topology (where $(t = 1, \dots, 5)$) as shown in Tables 7–10.

Table 7. α-Upper-e_t; $t = 1, \dots, 5$.

		Upper-e_t Crisp													
x	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}
e_1	[0	0	0	0	0	1	0	0	0	0	0	0	0	0	0]
e_2	[0	0	0	0	0	0	0	1	0	0	0	0	0	0	0]
e_3	[0	0	0	0	1	1	1	1	1	0	1	1	1	1	0]
e_4	[0	0	1	0	0	0	0	1	0	1	0	0	0	0	0]
e_5	[1	0	1	1	0	0	0	0	0	1	0	1	0	1	0]

Table 8. β-Lower-e_t; $t = 1, \dots, 5$.

		Lower-e_t Crisp													
x	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}
e_1	[0	0	0	0	0	0	0	0	0	1	1	0	0	0	0]
e_2	[0	0	1	0	0	0	0	0	0	0	0	0	0	0	0]
e_3	[1	1	1	0	0	0	0	0	0	0	0	0	0	0	0]
e_4	[0	0	0	0	0	0	1	0	1	0	1	1	1	1	0]
e_5	[0	0	0	0	0	1	0	0	1	0	0	0	0	0	0]

Table 9. α-Upper-e_t topology; $t = 1, \dots, 5$.

		$\tau^{u}_{e_t,\alpha}$															
e_1	$\{[0]_{1\times15}$,	$[1]_{1\times15}$,	[0	0	0	0	0	1	0	0	0	0	0	0	0	0	0]}
e_2	$\{[0]_{1\times15}$,	$[1]_{1\times15}$,	[0	0	0	0	0	0	0	1	0	0	0	0	0	0	0]}
e_3	$\{[0]_{1\times15}$,	$[1]_{1\times15}$,	[0	0	0	0	1	1	1	1	1	0	1	1	1	1	0]}
e_4	$\{[0]_{1\times15}$,	$[1]_{1\times15}$,	[0	0	1	0	0	0	0	1	0	1	0	0	0	0	0]}
e_5	$\{[0]_{1\times15}$,	$[1]_{1\times15}$,	[1	0	1	1	0	0	0	0	0	1	0	1	0	1	0]}

Table 10. β-Lower-e_t topology; $t = 1, \ldots, 5$.

		$\tau^{l}_{e_t,\beta}$	
e_1	$\{[0]_{1\times 15},$	$[1]_{1\times 15},$	$[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0]\}$
e_2	$\{[0]_{1\times 15},$	$[1]_{1\times 15},$	$[0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]\}$
e_3	$\{[0]_{1\times 15},$	$[1]_{1\times 15},$	$[1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]\}$
e_4	$\{[0]_{1\times 15},$	$[1]_{1\times 15},$	$[0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 0]\}$
e_5	$\{[0]_{1\times 15},$	$[1]_{1\times 15},$	$[0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]\}$

Step 5. The comparison matrices $G(e_t, \alpha), S(e_t, \beta), E^U(e_t, \alpha)$ and $E^L(e_t, \beta)$, where $\alpha = [\alpha_1, \alpha_2], \beta = [\beta_1, \beta_2], t = 1, \ldots, 5$ are below:

$G(e_1, [0.67, 0.92])$ and $L(e_1, [0.13, 0.8])$

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix} \& \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}$$

$G(e_2, [0.75, 0.94])$ and $L(e_2, [0.37, 0.77])$

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 11 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix} \& \begin{bmatrix}
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}$$

$G(e_3, [0.66, 0.92])$ and $L(e_3, [0.25, 0.76])$

$$\begin{bmatrix}
1&1&1&1&0&0&0&0&0&1&0&0&0&0&1\\
1&1&1&1&0&0&0&0&0&1&0&0&0&0&1\\
1&1&1&1&0&0&0&0&0&1&0&0&0&0&1\\
1&1&1&1&0&0&0&0&0&1&0&0&0&0&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&0&0&0&0&0&1&0&0&0&0&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&0&0&0&0&0&1&0&0&0&0&1
\end{bmatrix}
\&
\begin{bmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1
\end{bmatrix}$$

$G(e_4,[0.48,0.74])$ and $L(e_4,[0.26,0.76])$

$$\begin{bmatrix}
1&1&0&1&1&1&1&0&1&0&1&1&1&1&1\\
1&1&0&1&1&1&1&0&1&0&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&0&1&1&1&1&0&1&0&1&1&1&1&1\\
1&1&0&1&1&1&1&0&1&0&1&1&1&1&1\\
1&1&0&1&1&1&1&0&1&0&1&1&1&1&1\\
1&1&0&1&1&1&1&0&1&0&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&0&1&1&1&1&0&1&0&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&0&1&1&1&1&0&1&0&1&1&1&1&1\\
1&1&0&1&1&1&1&0&1&0&1&1&1&1&1\\
1&1&0&1&1&1&1&0&1&0&1&1&1&1&1\\
1&1&0&1&1&1&1&0&1&0&1&1&1&1&1\\
1&1&0&1&1&1&1&0&1&0&1&1&1&1&1
\end{bmatrix}
\&
\begin{bmatrix}
1&1&1&1&1&1&0&1&0&1&0&0&0&0&1\\
1&1&1&1&1&1&0&1&0&1&0&0&0&0&1\\
1&1&1&1&1&1&0&1&0&1&0&0&0&0&1\\
1&1&1&1&1&1&0&1&0&1&0&0&0&0&1\\
1&1&1&1&1&1&0&1&0&1&0&0&0&0&1\\
1&1&1&1&1&1&0&1&0&1&0&0&0&0&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&0&1&0&1&0&0&0&0&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&0&1&0&1&0&0&0&0&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&0&1&0&1&0&0&0&0&1
\end{bmatrix}$$

$G(e_5,[0.96,0.99])$ and $L(e_5,[0.67,1])$

$$\begin{bmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&0&0&1&1&1&1&1&0&1&0&1&0&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&0&0&1&1&1&1&1&0&1&0&1&0&1\\
0&1&0&0&1&1&1&1&1&0&1&0&1&0&1\\
0&1&0&0&1&1&1&1&1&0&1&0&1&0&1\\
0&1&0&0&1&1&1&1&1&0&1&0&1&0&1\\
0&1&0&0&1&1&1&1&1&0&1&0&1&0&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&0&0&1&1&1&1&1&0&1&0&1&0&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&0&0&1&1&1&1&1&0&1&0&1&0&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&0&0&1&1&1&1&1&0&1&0&1&0&1
\end{bmatrix}
\&
\begin{bmatrix}
1&1&1&1&1&0&1&1&0&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&0&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&0&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&0&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&0&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&0&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&0&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&0&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&0&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&0&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&0&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&0&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&0&1&1&1&1&1&1
\end{bmatrix}$$

Now, we compute matrices $E^U(e_t,\alpha), E^L(e_t,\beta)\alpha=[\alpha_1,\alpha_2], \beta=[\beta_1,\beta_2], t=1,\ldots,5$
$E^U(e_1,[0.67,0.92])$ and $E^L(e_1,[0.13,0.8])$

$$\begin{bmatrix}
1&1&1&1&1&0&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&0&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&1&1&1&1&1&1&1
\end{bmatrix}
\;\&\;
\begin{bmatrix}
1&1&1&1&1&1&1&1&1&0&0&1&1&1&1\\
1&1&1&1&1&1&1&1&1&0&0&1&1&1&1\\
1&1&1&1&1&1&1&1&1&0&0&1&1&1&1\\
1&1&1&1&1&1&1&1&1&0&0&1&1&1&1\\
1&1&1&1&1&1&1&1&1&0&0&1&1&1&1\\
1&1&1&1&1&1&1&1&1&0&0&1&1&1&1\\
1&1&1&1&1&1&1&1&1&0&0&1&1&1&1\\
1&1&1&1&1&1&1&1&1&0&0&1&1&1&1\\
1&1&1&1&1&1&1&1&1&0&0&1&1&1&1\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0\\
1&1&1&1&1&1&1&1&1&0&0&1&1&1&1\\
1&1&1&1&1&1&1&1&1&0&0&1&1&1&1\\
1&1&1&1&1&1&1&1&1&0&0&1&1&1&1\\
1&1&1&1&1&1&1&1&1&0&0&1&1&1&1
\end{bmatrix}$$

$E^U(e_2, [0.75, 0.94])$ and $E^L(e_2, [0.37, 0.77])$

$$\begin{bmatrix}
1&1&1&1&1&1&1&0&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&0&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&0&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&0&1&1&1&1&1&1&11\\
1&1&1&1&1&1&1&0&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&0&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&0&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&0&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&0&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&0&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&0&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&0&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&0&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&0&1&1&1&1&1&1&1
\end{bmatrix}
\;\&\;
\begin{bmatrix}
1&1&0&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&0&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&0&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&0&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&0&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&0&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&0&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&0&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&0&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&0&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&0&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&0&1&1&1&1&1&1&1&1&1&1&1&1
\end{bmatrix}$$

$E^U(e_3, [0.66, 0.92])$ and $E^L(e_3, [0.25, 0.76])$

$$\begin{bmatrix}
1&1&1&1&0&0&0&0&0&1&0&0&0&0&1\\
1&1&1&1&0&0&0&0&0&1&0&0&0&0&1\\
1&1&1&1&0&0&0&0&0&1&0&0&0&0&1\\
1&1&1&1&0&0&0&0&0&1&0&0&0&0&1\\
0&0&0&0&1&1&1&1&1&0&1&1&1&1&0\\
0&0&0&0&1&1&1&1&1&0&1&1&1&1&0\\
0&0&0&0&1&1&1&1&1&0&1&1&1&1&0\\
0&0&0&0&1&1&1&1&1&0&1&1&1&1&0\\
0&0&0&0&1&1&1&1&1&0&1&1&1&1&0\\
1&1&1&1&0&0&0&0&0&1&0&0&0&0&1\\
0&0&0&0&1&1&1&1&1&0&1&1&1&1&0\\
0&0&0&0&1&1&1&1&1&0&1&1&1&1&0\\
0&0&0&0&1&1&1&1&1&0&1&1&1&1&0\\
0&0&0&0&1&1&1&1&1&0&1&1&1&1&0\\
1&1&1&1&0&0&0&0&0&1&0&0&0&0&1
\end{bmatrix}
\;\&\;
\begin{bmatrix}
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1
\end{bmatrix}$$

$E^U(e_4, [0.48, 0.74])$ and $E^L(e_4, [0.26, 0.76])$

$$\begin{bmatrix}
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1
\end{bmatrix} \& \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$E^U(e_5, [0.96, 0.99])$ and $E^L(e_5, [0.67, 1])$

$$\begin{bmatrix}
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & - & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
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1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
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1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{bmatrix} \& \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}$$

Step 6. By using Definition (7), we have:

$$\begin{aligned}
S_1 = {} & r_1(e_1; [0.67, 0.92], [0.13, 0.8]) + r_1(e_2, [0.75, 0.94], [0.37, 0.77]) + r_1 \\
& (e_3, [0.66, 0.92], [0.25, 0.76]) + r_1(e_4, [0.48, 0.74], [0.26, 0.76] + r_1(e_5, [0.96, 0.99], [0.67, 1] \\
& = 0 + 0 - 3 + 0 + 0 = -3
\end{aligned}$$

Similarly, we have:

$$\begin{aligned}
& S_2 = -12, S_3 = -5, S_4 = -9, S_5 = 6, S_6 = 7, S_7 = 15, S_8 = 42, S_9 = 2, \\
& S_8 = 42, S_9 = 2, S_{10} = 8, S_{11} = -16, S_{12} = 6, S_{13} = -3, S_{14} = -9, S_{15} = 0.
\end{aligned}$$

Step 7. We have the following ordering system on X :

$$o_8 \succeq o_7 \succeq o_{10} \succeq o_6 \succeq o_{12} \simeq o_5 \succeq o_9 \succeq o_{15} \succeq o_{13} \simeq o_1 \succeq o_{14} \simeq o_4 \succeq o_2 \succeq o_{11}.$$

Steps 8 and 9. Then, from the corresponding object, we obtain, o_8 to be the best object (Acceptance region), while o_{11} is not selected (Rejection region) and others options $(o_7, o_{10}, o_6, o_{12}, o_5, o_9, o_{15},$
$o_{13}, o_1, o_{14}, o_4, o_2)$ cannot be judged(Boundary region).

Example 3. *(Example 2) Let us discuss Example 2 compared to existing methods proposed in [25,43,44] according to the ranking of objects.*

Yang et al. [25] defined the function score value as simply the total of lower and upper membership degrees of objects concerning each parameter. Ma et al. [44] applied Yang's Algorithm 1, which is given in [25] to solve Example 2 and showed the score value as follows: $o_8 \succeq o_6 \succeq o_{14} \succeq o_5 \succeq o_4 \succeq o_{10} \succeq o_{12} \succeq o_3 \succeq o_7 \succeq o_{13} \succeq o_{15} \succeq o_{11} \succeq o_9 \succeq o_2 \succeq o_1$.

Ma et al. [44] proposed a new efficient decision-making algorithm by using added objects. By using Algorithm 3 Section 4 in [44], Example 2 was solved and the score value for all objects was obtained as follows $o_8 \succeq o_6 \succeq o_{14} \succeq o_5 \succeq o_4 \succeq o_{10} \succeq o_{12} \succeq o_3 \succeq o_7 \succeq o_{13} \succeq o_{15} \succeq o_{11} \succeq o_9 \succeq o_2 \succeq o_1$.

Ma et al. [43] applied a new decision-making algorithm, based on the average table and the antithesis table—the antithesis the table has symmetry between the objects. Applying Algorithm in [43], Section 3, to solve the Example 2, the following ranking of objects is obtained $o_8 \succeq o_6 \succeq o_5 \succeq o_{14} \succeq o_4 \succeq o_{12} \succeq o_{10} \succeq o_3 \succeq o_{13} \succeq o_7 \succeq o_{15} \succeq o_{11} \succeq o_2 \succeq o_9 \succeq o_1$.

The comparison results among the present method and methods in [25,43,44] are given in Figure 1.

Algorithm 1: Rangking Objects by Interval-Valued Fuzzy Soft Topology

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Input:
$|E| = n, |X| = m, |D| = k, f_{sE}, 1 \leq s \leq k$,
threshold intervals $\alpha, \beta \subseteq \mathbb{I}$,
where n, m the number of parameters, object, respectively, and k shows the number of decision makers, and f_{sE} shows the matrix of $IVFSs$.
Output: Optimal objects and worst objects.
begin
while $t = 1, 2, \ldots, n, i = 1, 2, \ldots m$, *and* $s = 1, 2, \ldots k$ **do**
  Step 1. Compute crisp sets $U.C.S_\alpha f_s(e_t), L.C.S_\beta f_s(e_t)$ (see Matrices (1) and (2)).
  Step 2. Compute topological $(X, \tau^u_{e_t,\alpha}), (X, \tau^l_{e_t,\beta})$ ( ).
  Step 3. Compute $G_\alpha(e_t), S_\beta(e_t), E^U_\alpha(e_t)$, and $E^L_\beta(e_t)$(see (3)–(6)), for all $t = 1, 2, \ldots n$;
  if $G_\alpha(e_t) = I^U_m$ *and* $S_\beta(e_t) = I^L_m$ **then**
    x_1 is the optimum decision and x_m is the worst one ;
  else
    if $G_\alpha(e_t) = I^L_m$ *and* $S_\beta(e_t) = I^U_m$ **then**
      x_m is the optimum decision and x_1 is the worst one;
    else
      if $E^U_\alpha(e_t) = E^L_\beta(e_t) = I_m$, *where* I_m *is the identiy matrix* **then**
        there is no optimal over X;
      else
        if $E^U_\alpha(e_t) = E^L_\beta(e_t) = J_m$, *where* J_m *is the unit matrix* **then**
          all objects of X can be selected as an optimal choice;
        else
          Go to the step 6.
        Step 4. Calculate the score function $S_i \forall i$(Definition 7).
        Step 5.Rank all objects according to the values S_i.
      Step 6.The optimal alternative is to choose any one of the alternatives x_o such that $S_o = \max_i S_i$.
      The alternative x_l such that $S_l = \min_i S_i$ should not be selected.
      Step 7.**if** *the number of elements that* S_o *is maximum is more one* **then**
        any one of x_o may be chosen.
      else

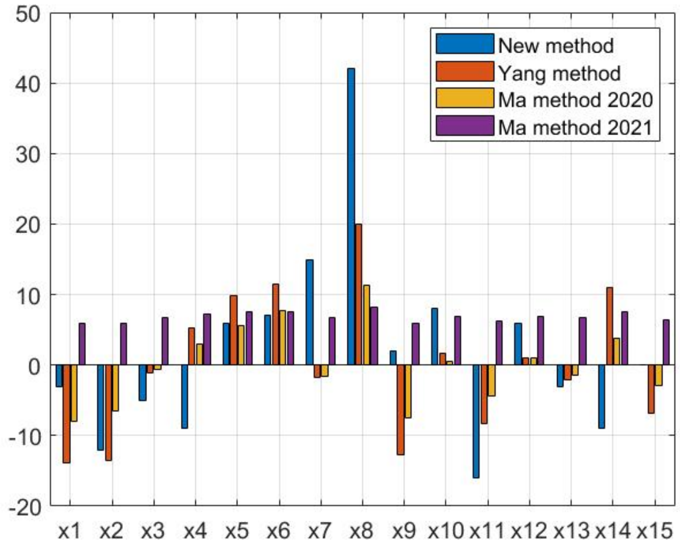

Figure 1. Comparison methods.

5. Discussion

According to the present method and the methods proposed in [25,43,44], to reach the process consensus, Yang et al. [25] use the "AND" operator, while methods in [43,44] did not discuss the aggregation problem. In addition, Example 2 shows that all the methods have the same option o_8, which is the best object. Consequently, algorithms in methods [25,43,44] select just one option, which is the optimum, and do not select the worst option, while the proposed algorithm selects two options—the optimum and as well as the worst option. However, the methods in [25,43,44], rank the objects based on a linear ordering system (see Example 3), while the present method ranks the objects based on preorder relation and a preference relationship, which allows one to have some incomparable objects (nonlinear ordering system). For example, in the Example 2, the objects o_{12} and o_5 have the same overall score values, which means that these objects cannot be compared with all of the others.

This is the same for the objects o_{13}, o_1 and o_{14}, o_4 (see Figure 2). The comparison results between the new proposed method and methods in [25,43,44] are also given in Table 11.

Table 11. Comparison of Existing Methods

Methods	Output Comparision	Aggregation Methodology	Ranking Methodology	Rank the Objects
[25]	optimal option	AND operator	fuzzy choice values	a linear ordering system
[44]	optimal option	Not discussed	choice values	a linear ordering system
[43]	optimal option	Not discussed	score function computed from an average and an antithesis tables	a linear ordering system
present method	optimal option and worst	$IVFST$	A collective preference relationship in topological space	a nonlinear ordering system

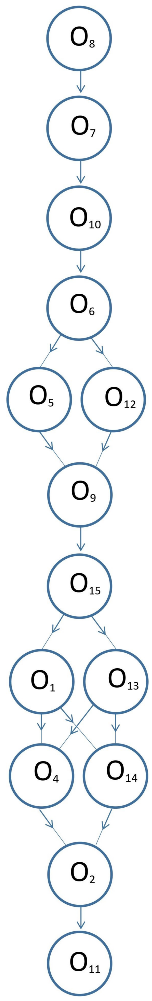

Figure 2. Nonlinear ordering system.

6. Conclusions

The interval-valued soft set is a useful tool to deal with fuzziness and uncertainties in decision-making problems. In this paper, we constructed two crisp topological spaces over the set of objects, and then presented two different preorder relations in these topological spaces. By using a new method for ranking data, we proposed an approach for solving multi-attribute group decision-making problems by using a new method for ranking data. Finally, a real-life example has been presented to verify the proposed method approach and to demonstrate the effectiveness by comparing the results with those of some of the existing approaches.

For future research, it would be of merit to apply the decision-making methods into practical applications such as evaluation systems, recommender systems, and conflict handling.

Author Contributions: Formal analysis, A.Z.K.; methodology, A.K. and A.Z.K.; supervision, A.K. and A.Z.K.; validation, A.K.; visualization, A.K.; writing—original draft, M.A.; writing—review & editing, A.K. All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Funding: The second and third authors gratefully acknowledge the Fundamental Research Grant Schemes, Reference No: FRGS/1/2018/STG06/UPM/01/3, awarded by the Ministry of Higher Education Malaysia.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the referees and editors for the useful comments and valuable remarks, which improved the current manuscript substantially.

Conflicts of Interest: The authors declare that they have no competing interests.

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