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Article

Perturbation of Wavelet Frames of Quaternionic-Valued Functions

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Abstract: Let $L^2(\mathbb{R}, \mathbb{H})$ denote the space of all square integrable quaternionic-valued functions. In this article, let $\Phi \in L^2(\mathbb{R}, \mathbb{H})$. We consider the perturbation problems of wavelet frame $\{\Phi_{m,n,a_0,b_0}, m, n \in \mathbb{Z}\}$ about translation parameter b_0 and dilation parameter a_0. In particular, we also research the stability of irregular wavelet frame $\{\sqrt{S_m}\Phi(S_m x - nb), m, n \in \mathbb{Z}\}$ for perturbation problems of sampling.

Keywords: wavelet frame; quaternionic-valued function; perturbation

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1. Introduction

Frame theory plays a significant role in both harmonic analysis and wavelet theory [1]. There are a number of mathematicians who have contributed a considerable amount of work on frame theory and perturbation theory, see [2–8]. The study of frames has attracted interest in recent years because of their applications in several areas of applied mathematics and engineering, like sampling [9] and signal processing [10].

Since the quaternion was discovered by Hamilton, some properties of quaternions and the theory of quaternionic-valued functions space have been widely studied. He [11,12] established the continuous wavelet transform theory of $L^2(\mathbb{R}, \mathbb{H})$ and $L^2(\mathbb{C}, \mathbb{H})$ associated with the affine group. Cheng and Kou [13] acquired the properties of the quaternion Fourier transform of square integrable function. It is known that quaternions have important applications in signal processing [14] and image processing [15]. Moreover, quaternions can be used to represent the three-dimensional rotation group SO(3) which has many applications in physics such as crystallography and kinematics of rigid body motion. For more details about this, we refer readers to see [16].

With the maturity of the quaternion theory, some researchers began to study the stability problems of frames of quaternionic-valued functions. He et al. [17] studied the stability of wavelet frames for perturbation problems of mother wavelet and sampling. The wavelet function Φ here is needed to satisfy some conditions. They obtained some useful results. In particular, they posed a question in their article for the stability of wavelet frames for $L^2(\mathbb{R}, \mathbb{H})$ when a_0 or b_0 has perturbation. Therefore, motivated by [17], our paper aims at studying the perturbation problems of wavelet frames about translation and dilation parameters b_0 and a_0. In practice, the sampling points may not be regular. This leads to the study of irregular frames. We also study sampling perturbation of irregular wavelet frames of quaternionic-valued functions. Our results show that a small perturbation does not change the stability of a wavelet frame when Φ satisfies some conditions, and we can reconstruct uniquely and stably any element through a wavelet transform.

The organization of this paper is as follows. In Section 2, we state notations and review some elementary facts of the Fourier transform for quaternionic-valued functions including the concept of frame. Section 3 contains the main theorems and their proofs. Finally, we show the conclusions in Section 4.

2. Preliminaries

First of all, we review some facts of quaternions, which are required throughout the paper. Relevant knowledge can be found in [17–19]. Write:

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\},$$

where $ij = -ji = k, jk = -kj = i, ki = -ik = j$ and $i^2 = j^2 = k^2 = -1$. Let $q \in \mathbb{H}$, it can be denoted by:

$$q = a + bi + cj + dk = (a + ib) + j(c - id) = u + jv.$$

The conjugation of q is:

$$\overline{q} = a - bi - cj - dk = (a - ib) - j(c - id) = \overline{u} - jv.$$

Suppose that $q_1, q_2 \in \mathbb{H}, q_1 = u_1 + jv_1, q_2 = u_2 + jv_2$. We introduce a mapping $\langle \cdot, \cdot \rangle$ from $\mathbb{H} \times \mathbb{H}$ to $\mathbb{H}$ as follows:

$$\langle q_1, q_2 \rangle_{\mathbb{H}} = q_1 \overline{q}_2 = (u_1 + jv_1)(\overline{u}_2 - jv_2) = (u_1 \overline{u}_2 + \overline{v}_1 v_2) + j(v_1 \overline{u}_2 - \overline{u}_1 v_2).$$

Clearly, $\langle \cdot, \cdot \rangle$ can be regarded as the inner product on $\mathbb{H}$ (see [11]). Quaternionic-valued function defined on $\mathbb{R}$ is given by:

$$F(x) = f_1(x) + jf_2(x), \quad f_1(x), f_2(x) \in L^2(\mathbb{R}).$$

Let $F(x) = f_1(x) + jf_2(x), G(x) = g_1(x) + jg_2(x) \in L^2(\mathbb{R}, \mathbb{H})$, the inner product $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R},\mathbb{H})}$ is defined by:

$$\begin{aligned}\langle F, G \rangle_{L^2(\mathbb{R},\mathbb{H})} &= \int_{\mathbb{R}} \langle F, G \rangle_{\mathbb{H}} \mathrm{d}x = \int_{\mathbb{R}} F(x)\overline{G(x)} \mathrm{d}x \\ &= \int_{\mathbb{R}} \Big[f_1(x)\overline{g_1(x)} + \overline{f_2(x)} g_2(x) + j\Big(f_2(x)\overline{g_1(x)} - \overline{f_1(x)} g_2(x) \Big) \Big] \mathrm{d}x.\end{aligned}$$

Specially, if $F = G$, then the norm of F is:

$$\| F \|_{L^2(\mathbb{R},\mathbb{H})} = \left\{ \int_{\mathbb{R}} (| f_1(x) |^2 + | f_2(x) |^2) \mathrm{d}x \right\}^{\frac{1}{2}}.$$

Let $F(x) = f_1(x) + jf_2(x) \in L^2(\mathbb{R}, \mathbb{H})$, we define the Fourier transform for F by:

$$\hat{F}(\omega) = \hat{f}_1(\omega) + j\hat{f}_2(\omega), \quad \omega \in \mathbb{R},$$

where $\hat{f}_t(\omega) = \int_{\mathbb{R}} f_t(x) e^{-i\omega x} \mathrm{d}x,\ t = 1, 2.$

Naturally, the frame of square integrable quaternionic-valued functions is defined as follows: A family of functions $\{\Phi_{m,n,a,b} : n, m \in \mathbb{Z}\} \subset L^2(\mathbb{R}, \mathbb{H})$ is called a frame if there exist two positive constants A and B with $0 < A \le B < \infty$ such that:

$$A \| F \|^2_{L^2(\mathbb{R},\mathbb{H})} \le \sum_{n,m \in \mathbb{Z}} | \langle F, \Phi_{m,n,a,b} \rangle |^2 \le B \| F \|^2_{L^2(\mathbb{R},\mathbb{H})},$$

where $\Phi_{m,n,a,b}(x) = a^{\frac{m}{2}} \Phi(a^m x - nb) = a^{\frac{m}{2}} (\varphi_1(a^m x - nb) + j\varphi_2(a^m x - nb)) \in L^2(\mathbb{R}, \mathbb{H})$, and A and B are called bounds of the frame. If $A = B$, we say that it is a tight frame.

In this paper, we use C to denote constant and do not distinguish different constants. $\mathbb{N}$ is the set of all positive integers.

For any $F(x) = f_1(x) + jf_2(x) \in L^2(\mathbb{R}, \mathbb{H})$, let:

$$\begin{aligned}K_\Phi(F) := \sum_{m,n\in\mathbb{Z}} \Big(& \int_{\mathbb{R}} \hat{f}_1(a_0^m\omega)\overline{\hat{\varphi}_1(\omega)} e^{inb_0\omega} \mathrm{d}\omega \int_{\mathbb{R}} \hat{f}_2(a_0^m\omega)\overline{\hat{\varphi}_2(\omega)} e^{inb_0\omega} \mathrm{d}\omega \\ & + \int_{\mathbb{R}} \overline{\hat{f}_1(a_0^m\omega)}\hat{\varphi}_1(\omega) e^{-inb_0\omega} \mathrm{d}\omega \int_{\mathbb{R}} \overline{\hat{f}_2(a_0^m\omega)}\hat{\varphi}_2(\omega) e^{-inb_0\omega} \mathrm{d}\omega \\ & - \int_{\mathbb{R}} \overline{\hat{f}_2(a_0^m\omega)}\hat{\varphi}_1(\omega) e^{-inb_0\omega} \mathrm{d}\omega \int_{\mathbb{R}} \overline{\hat{f}_1(a_0^m\omega)}\hat{\varphi}_2(\omega) e^{-inb_0\omega} \mathrm{d}\omega \\ & - \int_{\mathbb{R}} \hat{f}_2(a_0^m\omega)\overline{\hat{\varphi}_1(\omega)} e^{inb_0\omega} \mathrm{d}\omega \int_{\mathbb{R}} \hat{f}_1(a_0^m\omega)\overline{\hat{\varphi}_2(\omega)} e^{inb_0\omega} \mathrm{d}\omega \Big).\end{aligned}$$

Set $\mathscr{W} := \{\Phi : K_\Phi(F) = 0 \text{ for all } F(x) \in L^2(\mathbb{R}, \mathbb{H}), a_0 > 1, b_0 > 0\}$. As shown in [17], if one of φ_1 and φ_2 equals to 0, or $\varphi_1 = \kappa\varphi_2$, where $\kappa \in \mathbb{C} \setminus \{0\}$, then $\Phi \in \mathscr{W}$. That is to say $\mathscr{W} \neq \emptyset$. Evidently, $\mathscr{W}$ is a linear subspace of $L^2(\mathbb{R}, \mathbb{H})$. In the next discussion we need to assume that wavelet function $\Phi \in \mathscr{W}$.

3. Main Results and the Proofs

In this section, we will present our results and their proofs. The following lemmas are useful.

Lemma 1 ([17])**.** *Let $a > 1, b > 0$ and $\{\Phi_{m,n,a,b}\}$ is a frame for $L^2(\mathbb{R}, \mathbb{H})$ with bounds A and B. If $\Phi \in \mathscr{W}$, then for a.e. ω,*

$$\sum_m \Big(|\hat{\varphi}_1(a^m w)|^2 + |\hat{\varphi}_2(a^m w)|^2 \Big) \le 2Bb.$$

Lemma 2. *Let $\{\Phi_{m,n,a,b}\}$ be a frame for $L^2(\mathbb{R}, \mathbb{H})$ with frame bounds A and B. If,*

$$| \sum_{m,n} | \langle F, \Phi_{m,n,a,b} \rangle |^2 - \sum_{m,n} | \langle F, \Psi_{m,n,a,b} \rangle |^2 | \le M \| F \|^2 < A \| F \|^2,$$

then $\{\Psi_{m,n,a,b}\}$ is a frame with frame bounds A-M and B+M.

Proof. Using the triangle inequality, the lemma obviously holds. □

We are now in a position to show the main theorems. We first consider the perturbation of translation parameter b_0 in Theorems 1 and 2.

Theorem 1. *Let $\Phi, \Psi \in \mathscr{W}$. Assume that $\{\Phi_{m,n,a_0,b_0}\}$ is a wavelet frame for $L^2(\mathbb{R}, \mathbb{H})$ with bounds A and B, $\hat{\varphi}_1$, $\hat{\varphi}_2$ are continuous and bounded by:*

$$| \hat{\varphi}_t(\omega) | \le C \frac{| \omega |^\alpha}{(1+ | \omega |)^{1+\nu}}, \quad t = 1, 2,$$

for $\nu > \alpha > 0$. Then there exists a $\delta > 0$ such that for any b with $|b - b_0| < \delta$, $\{\Phi_{m,n,a_0,b}\}$ is a wavelet frame for $L^2(\mathbb{R}, \mathbb{H})$.

Proof. We define a unitary operator by:

$$U_b : L^2(\mathbb{R}, \mathbb{H}) \to L^2(\mathbb{R}, \mathbb{H}), \quad (U_b\varphi_t)(x) = (\frac{b}{b_0})^{\frac{1}{2}} \varphi_t(\frac{b}{b_0}x) = \psi_t(x).$$

Obviously, $\hat{\Psi}(\omega) = (\frac{b}{b_0})^{-\frac{1}{2}} \hat{\Phi}(\frac{b_0}{b}\omega)$, $\quad U_b\Phi_{m,n,a_0,b} = \Psi_{m,n,a_0,b_0}$.

Therefore, $\{\Phi_{m,n,a_0,b}\}$ is a frame if and only if $\{\Psi_{m,n,a_0,b_0}\}$ is a frame.

$$
\begin{aligned}
&\sum_{m,n\in\mathbb{Z}} \mid \langle F,\Phi_{m,n,a_0,b_0}\rangle \mid^2 \\
&= \sum_{m,n\in\mathbb{Z}} a_0^m \bigg| \int_{\mathbb{R}} \Big[f_1(x)\overline{\varphi_1(a_0^m x - nb_0)} + \overline{f_2(x)}\varphi_2(a_0^m x - nb_0) \\
&\qquad +j\Big(f_2(x)\overline{\varphi_1(a_0^m x - nb_0)} - \overline{f_1(x)}\varphi_2(a_0^m x - nb_0)\Big)\Big]\mathrm{d}x\bigg|^2 \\
&= \sum_{m,n\in\mathbb{Z}} \frac{1}{a_0^m(2\pi)^2}\bigg|\int_{\mathbb{R}}\Big[\hat{f}_1(\omega)\overline{\hat{\varphi}_1(a_0^{-m}\omega)}e^{ia_0^{-m}nb_0\omega} + \overline{\hat{f}_2(\omega)}\hat{\varphi}_2(a_0^{-m}\omega)e^{-ia_0^{-m}nb_0\omega} \\
&\qquad +j\Big(\hat{f}_2(\omega)\overline{\hat{\varphi}_1(a_0^{-m}\omega)}e^{ia_0^{-m}nb_0\omega} - \overline{\hat{f}_1(\omega)}\hat{\varphi}_2(a_0^{-m}\omega)e^{-ia_0^{-m}nb_0\omega}\Big)\Big]\mathrm{d}\omega\bigg|^2 \\
&= \sum_{m,n\in\mathbb{Z}} a_0^m(2\pi b_0)^{-2}\bigg|\int_{\mathbb{R}}\bigg[\hat{f}_1(\frac{a_0^m\omega}{b_0})\overline{\hat{\varphi}_1(\frac{\omega}{b_0})}e^{in\omega} + \overline{\hat{f}_2(\frac{a_0^m\omega}{b_0})}\hat{\varphi}_2(\frac{\omega}{b_0})e^{-in\omega} \\
&\qquad +j\bigg(\hat{f}_2(\frac{a_0^m\omega}{b_0})\overline{\hat{\varphi}_1(\frac{\omega}{b_0})}e^{in\omega} - \overline{\hat{f}_1(\frac{a_0^m\omega}{b_0})}\hat{\varphi}_2(\frac{\omega}{b_0})e^{-in\omega}\bigg)\bigg]\mathrm{d}\omega\bigg|^2 \\
&= \sum_{m,n\in\mathbb{Z}} a_0^m(2\pi b_0)^{-2}\bigg[\bigg|\int_{\mathbb{R}}\hat{f}_1(\frac{a_0^m\omega}{b_0})\overline{\hat{\varphi}_1(\frac{\omega}{b_0})}e^{in\omega}\mathrm{d}\omega\bigg|^2 + \bigg|\int_{\mathbb{R}}\overline{\hat{f}_2(\frac{a_0^m\omega}{b_0})}\hat{\varphi}_2(\frac{\omega}{b_0}) \\
&\qquad \times e^{-in\omega}\mathrm{d}\omega\bigg|^2 + \bigg|\int_{\mathbb{R}}\hat{f}_2(\frac{a_0^m\omega}{b_0})\overline{\hat{\varphi}_1(\frac{\omega}{b_0})}e^{in\omega}\mathrm{d}\omega\bigg|^2 + \bigg|\int_{\mathbb{R}}\overline{\hat{f}_1(\frac{a_0^m\omega}{b_0})}\hat{\varphi}_2(\frac{\omega}{b_0}) \\
&\qquad \times e^{-in\omega}\mathrm{d}\omega\bigg|^2\bigg] \\
&= I_1+I_2+I_3+I_4.
\end{aligned}
$$

A direct computation gives:

$$
\begin{aligned}
I_1 &= \sum_{m,n\in\mathbb{Z}} a_0^m(b_0)^{-2}\bigg|\frac{1}{2\pi}\sum_{l\in\mathbb{Z}}\int_{(2l-1)\pi}^{(2l+1)\pi}\hat{f}_1(\frac{a_0^m\omega}{b_0})\overline{\hat{\varphi}_1(\frac{\omega}{b_0})}e^{in\omega}\mathrm{d}\omega\bigg|^2 \\
&= \sum_{m\in\mathbb{Z}} a_0^m(b_0)^{-2}\sum_{n\in\mathbb{Z}}\bigg|\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{l\in\mathbb{Z}}\hat{f}_1(\frac{a_0^m(\omega+2l\pi)}{b_0})\overline{\hat{\varphi}_1(\frac{\omega+2l\pi}{b_0})}e^{in\omega}\mathrm{d}\omega\bigg|^2 \\
&=\frac{1}{2\pi}\sum_{m\in\mathbb{Z}} a_0^m(b_0)^{-2}\int_{-\pi}^{\pi}\bigg|\sum_{l\in\mathbb{Z}}\hat{f}_1(\frac{a_0^m(\omega+2l\pi)}{b_0})\overline{\hat{\varphi}_1(\frac{\omega+2l\pi}{b_0})}\bigg|^2\mathrm{d}\omega \\
&=\frac{1}{2\pi}\sum_{m\in\mathbb{Z}} a_0^m(b_0)^{-2}\int\hat{f}_1(\frac{a_0^m\omega}{b_0})\overline{\hat{\varphi}_1(\frac{\omega}{b_0})}\sum_{l'\in\mathbb{Z}}\overline{\hat{f}_1(\frac{a_0^m(\omega+2l'\pi)}{b_0})}\hat{\varphi}_1(\frac{\omega+2l'\pi}{b_0})\mathrm{d}\omega \\
&=(2\pi b_0)^{-1}\sum_{m,l'\in\mathbb{Z}}\int\hat{f}_1(\omega)\overline{\hat{\varphi}_1(a_0^{-m}\omega)}\sum_{l'\in\mathbb{Z}}\overline{\hat{f}_1(\omega+\frac{2l'\pi a_0^m}{b_0})}\hat{\varphi}_1(a_0^{-m}\omega+\frac{2l'\pi}{b_0})\mathrm{d}\omega \\
&\le(2\pi b_0)^{-1}\sum_{m,l'\in\mathbb{Z}}\bigg(\int\mid\hat{f}_1(\omega)\mid^2\mid\hat{\varphi}_1(a_0^{-m}\omega)\hat{\varphi}_1(a_0^{-m}\omega+\frac{2l'\pi}{b_0})\mid\mathrm{d}\omega\bigg)^{\frac{1}{2}} \\
&\qquad\times\bigg(\int\mid\hat{f}_1(\omega+\frac{2l'\pi a_0^m}{b_0})\mid^2\mid\hat{\varphi}_1(a_0^{-m}\omega)\hat{\varphi}_1(a_0^{-m}\omega+\frac{2l'\pi}{b_0})\mid\mathrm{d}\omega\bigg)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
\leq &(2\pi b_0)^{-1}\left(\sum_{m,l'\in\mathbb{Z}}\int \mid \hat{f}_1(\omega)\mid^2 \mid \hat{\varphi}_1(a_0^{-m}\omega)\hat{\varphi}_1(a_0^{-m}\omega+\frac{2l'\pi}{b_0})\mid \mathrm{d}\omega\right)^{\frac{1}{2}}\\
&\times\left(\sum_{m,l'\in\mathbb{Z}}\int \mid \hat{f}_1(\omega)\mid^2 \mid \hat{\varphi}_1(a_0^{-m}\omega-\frac{2l'\pi}{b_0})\hat{\varphi}_1(a_0^{-m}\omega)\mid \mathrm{d}\omega\right)^{\frac{1}{2}}\\
= &(2\pi b_0)^{-1}\sum_{m,l'\in\mathbb{Z}}\int \mid \hat{f}_1(\omega)\mid^2 \mid \hat{\varphi}_1(a_0^{-m}\omega)\hat{\varphi}_1(a_0^{-m}\omega+\frac{2l'\pi}{b_0})\mid \mathrm{d}\omega.
\end{aligned}
$$

And by the same way, we can get the values of I_2, I_3, and I_4. Thus:

$$
\begin{aligned}
&\sum_{m,n\in\mathbb{Z}} \mid \langle F,\Phi_{m,n,a_0,b_0}\rangle \mid^2\\
&\leq 2\pi b_0^{-1}\sum_{m,l'\in\mathbb{Z}}\int\left[\mid \hat{f}_1(\omega)\mid^2+\mid \hat{f}_2(\omega)\mid^2\right]\left[\mid \hat{\varphi}_1(a_0^{-m}\omega)\hat{\varphi}_1(a_0^{-m}\omega+\frac{2l'\pi}{b_0})\mid\right.\\
&\qquad\left.+\mid \hat{\varphi}_2(a_0^{-m}\omega)\hat{\varphi}_2(a_0^{-m}\omega+\frac{2l'\pi}{b_0})\mid\right]\mathrm{d}\omega\\
&\leq b_0^{-1}\sup_{1\leq|\omega|\leq a_0}\sum_{m,l'\in\mathbb{Z}}\left[\mid \hat{\varphi}_1(a_0^{-m}\omega)\hat{\varphi}_1(a_0^{-m}\omega+\frac{2l'\pi}{b_0})\mid\right.\\
&\qquad\left.+\mid \hat{\varphi}_2(a_0^{-m}\omega)\hat{\varphi}_2(a_0^{-m}\omega+\frac{2l'\pi}{b_0})\mid\right]\parallel F\parallel^2.
\end{aligned}
$$

Substituting $\Phi-\Psi$ for Φ, we have:

$$
\begin{aligned}
&\sum_{m,n\in\mathbb{Z}} \mid \langle F,(\Phi-\Psi)_{m,n,a_0,b_0}\rangle \mid^2\\
\leq\, & b_0^{-1}\sup_{1\leq|\omega|\leq a_0}\sum_{m,l'\in\mathbb{Z}}\left\{\left|\left[\hat{\varphi}_1(a_0^{-m}\omega)-\hat{\psi}_1(a_0^{-m}\omega)\right]\right.\right.\\
&\times\left.\left[\hat{\varphi}_1(a_0^{-m}\omega+\frac{2l'\pi}{b_0})-\hat{\psi}_1(a_0^{-m}\omega+\frac{2l'\pi}{b_0})\right]\right|+\left|\left[\hat{\varphi}_2(a_0^{-m}\omega)-\hat{\psi}_2(a_0^{-m}\omega)\right]\right.\\
&\times\left.\left.\left[\hat{\varphi}_2(a_0^{-m}\omega+\frac{2l'\pi}{b_0})-\hat{\psi}_2(a_0^{-m}\omega+\frac{2l'\pi}{b_0})\right]\right|\right\}\parallel F\parallel^2.
\end{aligned}
$$

For all m and ω,

$$
\sup_{1\leq|\omega|\leq a_0}\sum_{l'\in\mathbb{Z}}\mid \hat{\varphi}_1(a_0^{-m}\omega+\frac{2l'\pi}{b_0})\mid\leq C\sup_{1\leq|\omega|\leq a_0}\sum_{l'\in\mathbb{Z}}\frac{1}{(1+\mid a_0^{-m}\omega+\frac{2l'\pi}{b_0}\mid)^{1+\nu-\alpha}}\leq C.
$$

Similar argument shows that:

$$
\sup_{1\leq|\omega|\leq a_0}\sum_{l'\in\mathbb{Z}}\mid \hat{\psi}_1(a_0^{-m}\omega+\frac{2l'\pi}{b_0})\mid\leq C.
$$

For all $m'\in\mathbb{N}$,

$$
\begin{aligned}
&\sup_{1\leq|\omega|\leq a_0}\sum_{m\in\mathbb{Z}}|\hat{\varphi}_1(a_0^{-m}\omega)-\hat{\psi}_1(a_0^{-m}\omega)\mid\\
&\leq\sup_{1\leq|\omega|\leq a_0}\sum_{|m|\leq m'}\mid \hat{\varphi}_1(a_0^{-m}\omega)-(\frac{b}{b_0})^{-\frac{1}{2}}\hat{\varphi}_1(a_0^{-m}\frac{b_0}{b}\omega)\mid
\end{aligned}
$$

$$+ \sup_{1\le|\omega|\le a_0} \sum_{m<-m'} \left[|\hat{\varphi}_1(a_0^{-m}\omega)| + |\hat{\psi}_1(a_0^{-m}\omega)|\right]$$
$$+ \sup_{1\le|\omega|\le a_0} \sum_{m>m'} \left[|\hat{\varphi}_1(a_0^{-m}\omega)| + |\hat{\psi}_1(a_0^{-m}\omega)|\right]$$
$$= L_1 + L_2 + L_3.$$

For every $\varepsilon > 0$, choose m' such that $a_0^{-m'} < \varepsilon$. Since $1 \le| \omega |\le a_0$, $| m |\le m'$, and $\hat{\varphi}_1(a_0^{-m}\omega)$ is uniformly continuous on ω, we choose δ small enough so that if $| b - b_0 |< \delta$,

$$| \hat{\varphi}_1(a_0^{-m}\omega) - \hat{\varphi}_1(a_0^{-m}\frac{b_0}{b}\omega) |< \varepsilon, \quad \forall \mid m \mid\le m'.$$

Therefore,

$$\begin{aligned} L_1 \le \sup_{1\le|\omega|\le a_0} \sum_{|m|\le m'} &\Bigg[| 1 - (\frac{b}{b_0})^{-\frac{1}{2}} || \hat{\varphi}_1(a_0^{-m}\omega) | \\ &+(\frac{b}{b_0})^{-\frac{1}{2}} | \hat{\varphi}_1(a_0^{-m}\omega) - \hat{\varphi}_1(a_0^{-m}\frac{b_0}{b}\omega) |\Bigg] \\ \le& C(2m'+1)\left[|1 - (\frac{b}{b_0})^{-\frac{1}{2}}| + (\frac{b}{b_0})^{-\frac{1}{2}}\varepsilon\right] = o(1), \qquad b \to b_0. \end{aligned}$$

For L_2 and L_3, we will just estimate the first term in the series, since the other term can be handled similarly.

$$\begin{aligned} \sup_{1\le|\omega|\le a_0} \sum_{m<-m'} | \hat{\varphi}_1(a_0^{-m}\omega) |\le& \sup_{1\le|\omega|\le a_0} C \sum_{m<-m'} \frac{1}{(1+ | a_0^{-m}\omega |)^{1+\nu-\alpha}} \\ \le& C \sum_{m<-m'} a_0^{m(1+\nu-\alpha)} \le Ca_0^{-m'(1+\nu-\alpha)} = o(1), \quad m' \to +\infty. \end{aligned}$$

Finally,

$$\sup_{1\le|\omega|\le a_0} \sum_{m>m'} | \hat{\varphi}_1(a_0^{-m}\omega) |\le C \sum_{m>m'} | a_0^{-m}a_0 |^{\alpha} \le Ca_0^{-m'\alpha} = o(1), \quad m' \to +\infty.$$

We can deduce that:

$$\sup_{1\le|\omega|\le a_0} \sum_{m\in\mathbb{Z}} | \hat{\varphi}_1(a_0^{-m}\omega) - \hat{\psi}_1(a_0^{-m}\omega) |\le \varepsilon.$$

By the same way, we have:

$$\sup_{1\le|\omega|\le a_0} \sum_{m\in\mathbb{Z}} | \hat{\varphi}_2(a_0^{-m}\omega) - \hat{\psi}_2(a_0^{-m}\omega) |\le \varepsilon.$$

Based on the above argument, we conclude that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for $| b - b_0 |< \delta$,

$$\sum_{m,n\in\mathbb{Z}} | \langle F, (\Phi - \Psi)_{m,n,a_0,b_0} \rangle |^2 \le \varepsilon \parallel F \parallel^2,$$

which shows that $\{\Psi_{m,n,a_0,b_0}\}$ is a frame for b sufficiently close to b_0 by Theorem 3 of [2]. The proof is completed. □

By Theorem 1, we get a definite answer to the stability about translation parameter b_0. If $\hat{\Phi}$ has a small support, we can estimate the frame bounds as follows.

Theorem 2. *Let $\Phi \in \mathscr{W}$ and supp $\hat{\Phi} \subset [\frac{-\pi}{b'}, \frac{\pi}{b'}]$. Suppose that $\{\Phi_{m,n,a_0,b_0}\}$ is a wavelet frame for $L^2(\mathbb{R}, \mathbb{H})$ with bounds A and B. Then there exists a $\delta > 0$ such that for any b with $\mid b - b_0 \mid < \delta$ and $M = 2 \mid 1 - (\frac{b_0}{b}) \mid B < A$, $\{\Phi_{m,n,a_0,b}\}$ is a wavelet frame for $L^2(\mathbb{R}, \mathbb{H})$ with bounds A-M and B+M, $b' = max(b_0, b)$.*

Proof. Since $supp\ \hat{\Phi} \subset [\frac{-\pi}{b'}, \frac{\pi}{b'}]$,

$$
\begin{aligned}
&\sum_{m,n\in\mathbb{Z}} \mid \langle F, \Phi_{m,n,a_0,b_0} \rangle \mid^2 \\
=&\frac{1}{(2\pi b_0)^2} \sum_{m,n\in\mathbb{Z}} a_0^m \Bigg[\left| \int_{\mathbb{R}} \hat{f}_1(\frac{a_0^m \omega}{b_0}) \overline{\hat{\varphi}_1(\frac{\omega}{b_0})} e^{in\omega} \mathrm{d}\omega \right|^2 + \left| \int_{\mathbb{R}} \overline{\hat{f}_2(\frac{a_0^m \omega}{b_0})} \hat{\varphi}_2(\frac{\omega}{b_0}) e^{-in\omega} \mathrm{d}\omega \right|^2 \\
&+ \left| \int_{\mathbb{R}} \hat{f}_2(\frac{a_0^m \omega}{b_0}) \overline{\hat{\varphi}_1(\frac{\omega}{b_0})} e^{in\omega} \mathrm{d}\omega \right|^2 + \left| \int_{\mathbb{R}} \overline{\hat{f}_1(\frac{a_0^m \omega}{b_0})} \hat{\varphi}_2(\frac{\omega}{b_0}) e^{-in\omega} \mathrm{d}\omega \right|^2 \Bigg] \\
=&J_1 + J_2 + J_3 + J_4.
\end{aligned}
$$

The value of J_1 is:

$$
\begin{aligned}
J_1 =&(2\pi b_0)^{-2} \sum_{m,n\in\mathbb{Z}} a_0^m \left| \int_{\mathbb{R}} \hat{f}_1(\frac{a_0^m \omega}{b_0}) \overline{\hat{\varphi}_1(\frac{\omega}{b_0})} e^{in\omega} \mathrm{d}\omega \right|^2 \\
=&(2\pi)^{-1} b_0^{-2} \sum_{m\in\mathbb{Z}} a_0^m \int_{\mathbb{R}} \left| \hat{f}_1(\frac{a_0^m \omega}{b_0}) \overline{\hat{\varphi}_1(\frac{\omega}{b_0})} \right|^2 \mathrm{d}\omega \\
=&(2\pi b_0)^{-1} \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} \left| \hat{f}_1(\omega) \overline{\hat{\varphi}_1(a_0^{-m}\omega)} \right|^2 \mathrm{d}\omega.
\end{aligned}
$$

By performing the same calculations, we have:

$$
\begin{aligned}
&\sum_{m,n\in\mathbb{Z}} \mid \langle F, \Phi_{m,n,a_0,b_0} \rangle \mid^2 \\
&= (2\pi b_0)^{-1} \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} \Big(\mid \hat{f}_1(\omega) \mid^2 + \mid \hat{f}_2(\omega) \mid^2 \Big) \Big(\mid \hat{\varphi}_1(a_0^{-m}\omega) \mid^2 + \mid \hat{\varphi}_2(a_0^{-m}\omega) \mid^2 \Big) \mathrm{d}\omega.
\end{aligned}
$$

According to the conclusion of Lemma 1, we get:

$$
\begin{aligned}
&\mid \sum_{m,n\in\mathbb{Z}} \mid \langle F, \Phi_{m,n,a_0,b_0} \rangle \mid^2 - \sum_{m,n\in\mathbb{Z}} \mid \langle F, \Phi_{m,n,a_0,b} \rangle \mid^2 \mid \\
&= (2\pi)^{-1} \mid b_0^{-1} - b^{-1} \mid \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} \Big(\mid \hat{\varphi}_1(a_0^{-m}\omega) \mid^2 + \mid \hat{\varphi}_2(a_0^{-m}\omega) \mid^2 \Big) \\
&\times \Big(\mid \hat{f}_1(\omega) \mid^2 + \mid \hat{f}_2(\omega) \mid^2 \Big) \mathrm{d}\omega \\
&\leq 2 \mid b_0^{-1} - b^{-1} \mid B b_0 \parallel F \parallel^2 = 2B \mid 1 - \frac{b_0}{b} \mid \parallel F \parallel^2 .
\end{aligned}
$$

Choosing suitable b such that:

$$
M = 2B \mid 1 - \frac{b_0}{b} \mid < A.
$$

Applying Lemma 2, we obtain the result. □

Next, we shall consider the perturbation of dilation parameter a_0 in Theorems 3 and 4.

Theorem 3. *Let $\Phi \in \mathscr{W}$. If $\{\Phi_{m,n,a_0,b_0}\}$ is a wavelet frame for $L^2(\mathbb{R},\mathbb{H})$ with bounds A and B, satisfying $supp\ \hat{\Phi} \subset [\frac{-\pi}{b_0},\frac{\pi}{b_0}]$, and $M < b_0 A$, where:*

$$M = esssup \mid \sum_{m\in\mathbb{Z}} \mid \hat{\varphi}_1(a_0^{-m}\omega) \mid^2 - \sum_{m\in\mathbb{Z}} \mid \hat{\varphi}_1(a^{-m}\omega) \mid^2 + \sum_{m\in\mathbb{Z}} \mid \hat{\varphi}_2(a_0^{-m}\omega) \mid^2 - \sum_{m\in\mathbb{Z}} \mid \hat{\varphi}_2(a^{-m}\omega) \mid^2 \mid,$$

then $\{\Phi_{m,n,a,b_0}\}$ is a wavelet frame for $L^2(\mathbb{R},\mathbb{H})$ with bounds $A - b_0^{-1}M$ and $B + b_0^{-1}M$.

Proof. In fact, it is not difficult to calculate that:

$$\begin{aligned}
&\mid \sum_{m,n\in\mathbb{Z}} \mid \langle F,\Phi_{m,n,a_0,b_0}\rangle \mid^2 - \sum_{m,n\in\mathbb{Z}} \mid \langle F,\Phi_{m,n,a,b_0}\rangle \mid^2 \mid \\
&= (2\pi b_0)^{-1} \left| \int_{\mathbb{R}} (\sum_{m\in\mathbb{Z}} \mid \hat{\varphi}_1(a_0^{-m}\omega) \mid^2 - \sum_{m\in\mathbb{Z}} \mid \hat{\varphi}_1(a^{-m}\omega) \mid^2 \right. \\
&\quad \left. + \sum_{m\in\mathbb{Z}} \mid \hat{\varphi}_2(a_0^{-m}\omega) \mid^2 - \sum_{m\in\mathbb{Z}} \mid \hat{\varphi}_2(a^{-m}\omega) \mid^2) \times (\mid \hat{f}_1(\omega) \mid^2 + \mid \hat{f}_2(\omega) \mid^2) \mathrm{d}\omega \right| \\
&\le esssup \mid \sum_{m\in\mathbb{Z}} \mid \hat{\varphi}_1(a_0^{-m}\omega) \mid^2 - \sum_{m\in\mathbb{Z}} \mid \hat{\varphi}_1(a^{-m}\omega) \mid^2 \\
&\quad + \sum_{m\in\mathbb{Z}} \mid \hat{\varphi}_2(a_0^{-m}\omega) \mid^2 - \sum_{m\in\mathbb{Z}} \mid \hat{\varphi}_2(a^{-m}\omega) \mid^2 \mid b_0^{-1} \parallel F \parallel^2 .
\end{aligned}$$

Setting:

$$M = esssup \mid \sum_{m\in\mathbb{Z}} \mid \hat{\varphi}_1(a_0^{-m}\omega) \mid^2 - \sum_{m\in\mathbb{Z}} \mid \hat{\varphi}_1(a^{-m}\omega) \mid^2 + \sum_{m\in\mathbb{Z}} \mid \hat{\varphi}_2(a_0^{-m}\omega) \mid^2 - \sum_{m\in Z} \mid \hat{\varphi}_2(a^{-m}\omega) \mid^2 \mid < b_0 A,$$

the conclusion follows from Lemma 2. □

Theorem 4. *Let $\Phi \in \mathscr{W}$ and $supp\ \hat{\Phi} \subset [\frac{-\pi}{b_0},\frac{\pi}{b_0}]$. Suppose that $\{\Phi_{m,n,a_0,b_0}\}$ is a wavelet frame for $L^2(\mathbb{R},\mathbb{H})$ with bounds A and B, and:*

$$\mid \hat{\varphi}_1(a^{-m}\omega) \mid \le \gamma_1 \mid \hat{\varphi}_1(a_0^{-m}\omega) \mid, \mid \hat{\varphi}_2(a^{-m}\omega) \mid \le \gamma_2 \mid \hat{\varphi}_2(a_0^{-m}\omega) \mid .$$

Then $\{\Phi_{m,n,a,b_0}\}$ is a wavelet frame for $L^2(\mathbb{R},\mathbb{H})$ with bounds $A - 2B\gamma$ and $B + 2B\gamma$, where $2B\gamma < A, \gamma = max\{1+\gamma_1^2, 1+\gamma_2^2\}$.

Proof. From the hypothesis above, we have:

$$\begin{aligned}
&\mid \sum_{m,n\in\mathbb{Z}} \mid \langle F,\Phi_{m,n,a_0,b_0}\rangle \mid^2 - \sum_{m,n\in\mathbb{Z}} \mid \langle F,\Phi_{m,n,a,b_0}\rangle \mid^2 \mid \\
&= \frac{1}{2\pi b_0} \left| \int_{\mathbb{R}} (\sum_{m\in\mathbb{Z}} |\hat{\varphi}_1(a_0^{-m}\omega) \mid^2 - \sum_{m\in\mathbb{Z}} \mid \hat{\varphi}_1(a^{-m}\omega) \mid^2 \right. \\
&\quad \left. + \sum_{m\in\mathbb{Z}} \mid \hat{\varphi}_2(a_0^{-m}\omega) \mid^2 - \sum_{m\in\mathbb{Z}} \mid \hat{\varphi}_2(a^{-m}\omega) \mid^2) \times (\mid \hat{f}_1(\omega) \mid^2 + \mid \hat{f}_2(\omega) \mid^2) \mathrm{d}\omega \right| \\
&\le \frac{\gamma}{2\pi b_0} \left| \int_{\mathbb{R}} \sum_{m\in\mathbb{Z}} (\mid \hat{\varphi}_1(a_0^{-m}\omega) \mid^2 + \mid \hat{\varphi}_2(a_0^{-m}\omega) \mid^2) \times (\mid \hat{f}_1(\omega) \mid^2 + \mid \hat{f}_2(\omega) \mid^2) \mathrm{d}\omega \right|.
\end{aligned}$$

Applying the conclusion of Lemma 1, we get:

$$\begin{aligned}&|\sum_{m,n\in Z} |\langle F,\Phi_{m,n,a_0,b_0}\rangle|^2-\sum_{m,n\in Z}|\langle F,\Phi_{m,n,a,b_0}\rangle|^2|\\&\leq 2B\gamma\parallel F\parallel^2.\end{aligned}$$

The result is derived by Lemma 2. □

For $\Phi\in L^2(\mathbb{R},\mathbb{H})$, the continuous wavelet transform of a function $F\in L^2(\mathbb{R},\mathbb{H})$ is defined by:

$$(\mathcal{W}_\Phi F)(s,p)=(T_{\varphi_1}f_1)(s,p)+\overline{(T_{\varphi_2}f_2)}(s,-p)+j\Big(-\overline{(T_{\varphi_2}f_1)}(s,-p)+(T_{\varphi_1}f_2)(s,p)\Big),$$

where $(T_\varphi f)(s,p)=\int_{-\infty}^{\infty}f(x)s^{-\frac{1}{2}}\overline{\varphi\left(\frac{x-p}{s}\right)}\mathrm{d}x$, $s>0$ and $p\in\mathbb{R}$ (see [20]).

If $\{\Phi_{m,n,a,b},m,n\in\mathbb{Z}\}$ is a wavelet frame, then F is determined by the sampling values of continuous wavelet transform $\mathcal{W}_\Phi F$ on the set $\{(a^m,nba^m):m,n\in\mathbb{Z}\}$.

In practice, the sampling points may not be regular. Thus the following problem has been investigated: Suppose $\{\Phi_{m,n,a,b},m,n\in\mathbb{Z}\}$ is a wavelet frame, $\{S_m\}$ and $\{b_n\}$ are perturbations of $\{a^m\}$ and $\{nb\}$, respectively, in some sense. If $\{\sqrt{S_m}\Phi(S_m\cdot-b_n):m,n\in\mathbb{Z}\}$ is a frame, where $S_m>0$ and b_n are real numbers, then $\{\sqrt{S_m}\Phi(S_m\cdot-b_n)\}$ will be called an irregular wavelet frame. In this case, Φ can also be reconstructed by the sampling points $\{(S_m,b_nS_m):m,n\in\mathbb{Z}\}$ (see [21]).

Next, we still take $b_n=bn$, let $\{\sqrt{S_m}\Phi(S_m\cdot-nb):m,n\in\mathbb{Z}\}$ be a frame for $L^2(\mathbb{R},\mathbb{H})$, we study the sampling perturbation of the irregular wavelet frame, replacing the sequence of integers by a double sequence $\{\lambda_{m,n}\}$.

Theorem 5. *For $1<\beta\leq 2$, $\varepsilon>0$. Suppose that $\{\sqrt{S_m}\Phi(S_m\cdot-nb):m,n\in\mathbb{Z}\}$ is a frame for $L^2(\mathbb{R},\mathbb{H})$ with bounds A and B, $\hat{\varphi}_1,\hat{\varphi}_2\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ and $|\hat{\varphi}_1(\omega)|,|\hat{\varphi}_2(\omega)|\leq C|\omega|^{-\beta-\varepsilon}$. Then there exist some $a>1$ and $0<\eta<1$, such that $|S_m-a^m|\leq\eta a^m$, $\{\sqrt{S_m}\Phi(S_m\cdot-\lambda_{m,n}b):m,n\in\mathbb{Z}\}$ is a frame for $L^2(\mathbb{R},\mathbb{H})$ with bounds $(1-\sqrt{M\sigma/A})^2A$ and $(1+\sqrt{M\sigma/B})^2B$ whenever $\sigma=\sum_{m,n\in\mathbb{Z}}|n-\lambda_{m,n}|^\beta<\frac{A}{M}$, where:*

$$\begin{aligned}M=2^{2-\beta}b^\beta(\pi)^{-1}\Big(&\frac{\parallel\hat{\varphi}_1\parallel_1\parallel\hat{\varphi}_1\parallel_\infty+\parallel\hat{\varphi}_2\parallel_1\parallel\hat{\varphi}_2\parallel_\infty}{(1-a^{-\beta})(1-\eta)^\beta}\\&+C\,\frac{(\parallel\hat{\varphi}_1\parallel_1+\parallel\hat{\varphi}_2\parallel_1)(1+\eta)^\varepsilon}{1-a^{-\varepsilon}}\Big).\end{aligned}$$

Proof. We first calculate:

$$\begin{aligned}&\sum_{m,n\in\mathbb{Z}}|\langle F,\sqrt{S_m}\Phi(S_m\cdot-nb)-\sqrt{S_m}\Phi(S_m\cdot-\lambda_{m,n}b)\rangle|^2\\
=&\sum_{m,n\in\mathbb{Z}}\Big|(2\pi)^{-1}S_m^{-\frac{1}{2}}\int_{\mathbb{R}}\Big[\hat{f}_1(\omega)\overline{\hat{\varphi}_1(\omega/S_m)}e^{i\frac{\lambda_{m,n}b}{S_m}\omega}(e^{i\frac{(n-\lambda_{m,n})b}{S_m}\omega}-1)\\
&+\overline{\hat{f}_2(\omega)}\hat{\varphi}_2(\omega/S_m)e^{-i\frac{\lambda_{m,n}b}{S_m}\omega}(e^{-i\frac{(n-\lambda_{m,n})b}{S_m}\omega}-1)\\
&+j\Big(\hat{f}_2(\omega)\overline{\hat{\varphi}_1(\omega/S_m)}e^{i\frac{\lambda_{m,n}b}{S_m}\omega}(e^{i\frac{(n-\lambda_{m,n})b}{S_m}\omega}-1)\\
&-\overline{\hat{f}_1(\omega)}\hat{\varphi}_2(\omega/S_m)e^{-i\frac{\lambda_{m,n}b}{S_m}\omega}(e^{-i\frac{(n-\lambda_{m,n})b}{S_m}\omega}-1)\Big)\Big]\mathrm{d}\omega\Big|^2\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{m,n\in\mathbb{Z}} (2\pi^2)^{-1} S_m^{-1} \Bigg\{ \Bigg[\int_{\mathbb{R}} \mid \hat{f}_1(\omega) + j\hat{f}_2(\omega) \mid\mid \hat{\varphi}_1(\omega/S_m)(e^{i\frac{(n-\lambda_{m,n})b}{S_m}\omega} - 1) \mid \mathrm{d}\omega \Bigg]^2 \\
&\quad + \Bigg[\int_{\mathbb{R}} \mid \overline{\hat{f}_2(\omega)} - j\overline{\hat{f}_1(\omega)} \mid\mid \hat{\varphi}_2(\omega/S_m)(e^{i\frac{(\lambda_{m,n}-n)b}{S_m}\omega} - 1) \mid \mathrm{d}\omega \Bigg]^2 \Bigg\} \\
&\leq \sum_{m,n\in\mathbb{Z}} (2\pi^2)^{-1} \Bigg[\int_{\mathbb{R}} \mid \hat{f}_1(\omega) + j\hat{f}_2(\omega) \mid^2 \mid \hat{\varphi}_1(\omega/S_m) \mid\mid e^{i\frac{(n-\lambda_{m,n})b}{S_m}\omega} - 1 \mid^2 \mathrm{d}\omega \\
&\quad \times \int_{\mathbb{R}} S_m^{-1} \mid \hat{\varphi}_1(\omega/S_m) \mid \mathrm{d}\omega + \int_{\mathbb{R}} \mid \overline{\hat{f}_2(\omega)} - j\overline{\hat{f}_1(\omega)} \mid^2 \mid \hat{\varphi}_2(\omega/S_m) \mid \\
&\quad \times \mid e^{i\frac{(\lambda_{m,n}-n)b}{S_m}\omega} - 1 \mid^2 \mathrm{d}\omega \int_{\mathbb{R}} S_m^{-1} \mid \hat{\varphi}_2(\omega/S_m) \mid \mathrm{d}\omega \Bigg] \\
&= \sum_{m,n\in\mathbb{Z}} (2\pi^2)^{-1} \Bigg[\parallel \hat{\varphi}_1 \parallel_1 \int_{\mathbb{R}} (\mid \hat{f}_1(\omega) \mid^2 + \mid \hat{f}_2(\omega) \mid^2) \mid \hat{\varphi}_1(\omega/S_m) \mid \\
&\quad \times \mid e^{i\frac{(n-\lambda_{m,n})b}{S_m}\omega} - 1 \mid^2 \mathrm{d}\omega + \parallel \hat{\varphi}_2 \parallel_1 \int_{\mathbb{R}} (\mid \hat{f}_1(\omega) \mid^2 + \mid \hat{f}_2(\omega) \mid^2) \mid \hat{\varphi}_2(\omega/S_m) \mid \\
&\quad \times \mid e^{i\frac{(\lambda_{m,n}-n)b}{S_m}\omega} - 1 \mid^2 \mathrm{d}\omega \Bigg].
\end{aligned} \tag{1}
$$

On the other hand, for any $\omega \neq 0$, there exist some $m_0 \in \mathbb{Z}$ such that $\mid a^{m_0}\omega \mid \leq 1 < \mid a^{m_0+1}\omega \mid$. It follows from $(1-\eta)a^m \leq S_m \leq (1+\eta)a^m$ that:

$$
\begin{aligned}
&\sum_{m\in\mathbb{Z}} \mid \hat{\varphi}_1(\omega/S_m) \mid \cdot \mid \omega/S_m \mid^\beta \\
&= \sum_{m\geq -m_0} \mid \hat{\varphi}_1(\omega/S_m) \mid \cdot \mid \omega/S_m \mid^\beta + \sum_{m\leq -m_0-1} \mid \hat{\varphi}_1(\omega/S_m) \mid \cdot \mid \omega/S_m \mid^\beta \\
&\leq \sum_{m\geq -m_0} \parallel \hat{\varphi}_1 \parallel_\infty \mid \frac{a^{-m}\omega}{1-\eta} \mid^\beta + \sum_{m\leq -m_0-1} C \mid \omega/S_m \mid^{-\beta-\varepsilon} \mid \omega/S_m \mid^\beta \\
&\leq \sum_{m\geq -m_0} \parallel \hat{\varphi}_1 \parallel_\infty \mid \frac{a^{-m}\omega}{1-\eta} \mid^\beta + \sum_{m\leq -m_0-1} C \mid \frac{a^{-m}\omega}{1+\eta} \mid^{-\varepsilon} \\
&= \parallel \hat{\varphi}_1 \parallel_\infty \frac{\mid a^{m_0}\omega \mid^\beta}{(1-a^{-\beta})(1-\eta)^\beta} + C \frac{\mid a^{m_0+1}\omega \mid^{-\varepsilon}}{(1-a^{-\varepsilon})(1+\eta)^{-\varepsilon}} \\
&\leq \frac{\parallel \hat{\varphi}_1 \parallel_\infty}{(1-a^{-\beta})(1-\eta)^\beta} + C \frac{(1+\eta)^\varepsilon}{1-a^{-\varepsilon}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\sum_{m,n\in\mathbb{Z}} \mid \hat{\varphi}_1(\omega/S_m) \mid \cdot \mid e^{i\frac{(n-\lambda_{m,n})b}{S_m}\omega} - 1 \mid^2 \\
&\leq \sum_{m,n\in\mathbb{Z}} \mid \hat{\varphi}_1(\omega/S_m) \mid 2^{2-\beta} b^\beta \mid n - \lambda_{m,n} \mid^\beta \mid \omega/S_m \mid^\beta \\
&\leq \sum_{m,n\in\mathbb{Z}} 2^{2-\beta} b^\beta \left(\frac{\parallel \hat{\varphi}_1 \parallel_\infty}{(1-a^{-\beta})(1-\eta)^\beta} + C \frac{(1+\eta)^\varepsilon}{1-a^{-\varepsilon}} \right) \mid n - \lambda_{m,n} \mid^\beta .
\end{aligned} \tag{2}
$$

For the same reason, we have:

$$
\begin{aligned}
&\sum_{m,n\in\mathbb{Z}} \mid \hat{\varphi}_2(\omega/S_m) \mid \cdot \mid e^{i\frac{(\lambda_{m,n}-n)b}{S_m}\omega} - 1 \mid^2 \\
&\leq \sum_{m,n\in\mathbb{Z}} 2^{2-\beta} b^\beta \left(\frac{\parallel \hat{\varphi}_2 \parallel_\infty}{(1-a^{-\beta})(1-\eta)^\beta} + C \frac{(1+\eta)^\varepsilon}{1-a^{-\varepsilon}} \right) \mid \lambda_{m,n} - n \mid^\beta .
\end{aligned} \tag{3}
$$

Consequently, by (1)–(3), we get:

$$\sum_{m,n\in\mathbb{Z}} \mid \langle F, \sqrt{S_m}\Phi(S_m\cdot -nb) - \sqrt{S_m}\Phi(S_m\cdot -\lambda_n b)\rangle_{L^2(\mathbb{R},\mathbb{H})} \mid^2 < \sigma M \parallel F \parallel^2_{L^2(\mathbb{R},\mathbb{H})} .$$

We finish the proof by Theorem 3 of [2]. □

4. Conclusions

In this paper, we consider the perturbation problems of wavelet frames of quaternionic-valued functions about translation and dilation parameters. Let $\{\Phi_{m,n,a_0,b_0}, m, n \in \mathbb{Z}\}$ be a wavelet frame for $L^2(\mathbb{R},\mathbb{H})$. We prove that $\{\Phi_{m,n,a_0,b}, m, n \in \mathbb{Z}\}$ is still a wavelet frame when Φ satisfies certain conditions and b is sufficiently close to b_0. Moreover, if the Fourier transform $\hat{\Phi}$ has small support, we can estimate the frame bounds. Next, for wavelet functions whose Fourier transforms have small supports, we give a method to determine whether the perturbation system $\{\Phi_{m,n,a,b_0}, m, n \in \mathbb{Z}\}$ is a frame. We address the open issues raised in [17]. We also study a sampling perturbation of irregular wavelet frames of quaternionic-valued functions. Suppose that $\{\sqrt{S_m}\Phi(S_m x - nb), m, n \in \mathbb{Z}\}$ is an irregular wavelet frame for $L^2(\mathbb{R},\mathbb{H})$, then $\{\sqrt{S_m}\Phi(S_m x - \lambda_{m,n} b)\}$ is also a frame when Φ satisfies some conditions and $\sum_{m,n} |n - \lambda_{m,n}|^\beta$ is sufficiently small. Specific frame bounds of the sampling perturbation are given.

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