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A Class of k-Symmetric Harmonic Functions Involving a Certain q-Derivative Operator

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Abstract: In this paper, we introduce a new class of harmonic univalent functions with respect to k-symmetric points by using a newly-defined q-analog of the derivative operator for complex harmonic functions. For this harmonic univalent function class, we derive a sufficient condition, a representation theorem, and a distortion theorem. We also apply a generalized q-Bernardi–Libera–Livingston integral operator to examine the closure properties and coefficient bounds. Furthermore, we highlight some known consequences of our main results. In the concluding part of the article, we have finally reiterated the well-demonstrated fact that the results presented in this article can easily be rewritten as the so-called (p,q)-variations by making some straightforward simplifications, and it will be an inconsequential exercise, simply because the additional parameter p is obviously unnecessary.

Keywords: univalent functions; harmonic functions; q-derivative (or q-difference) operator

MSC: Primary 30C45; 30C50; 30C80; Secondary 11B65; 47B38

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1. Introduction, Definitions and Motivation

Let the complex-valued function f, given by

$$f(z) = \mathfrak{u}(x,y) + \Im\mathfrak{v}(x,y),$$

be continuous and defined in a simply-connected complex domain $\mathbb{D} \subset \mathbb{C}$. Then, f is said to be harmonic in $\mathbb{D}$ if both $\mathfrak{u}(x,y)$ and $\mathfrak{v}(x,y)$ are real harmonic functions in $\mathbb{D}$. Suppose that there exist functions $\mathfrak{U}(z)$ and $\mathfrak{V}(z)$, analytic in $\mathbb{D}$, such that

$$\mathfrak{u}(x,y) = \Re\big(\mathfrak{U}(z)\big) \qquad \text{and} \qquad \mathfrak{v}(x,y) = \Im\big(\mathfrak{V}(z)\big).$$

Then, for

$$h(z) = \frac{1}{2}[\mathfrak{U}(z) + \mathfrak{V}(z)] \qquad \text{and} \qquad g(z) = \frac{1}{2}[\mathfrak{U}(z) - \mathfrak{V}(z)],$$

the harmonic function $f = h + \overline{g}$ can be expressed as follows (see, for details, [1]; see also [2–4]):

$$f(z) = h(z) + \overline{g(z)} \qquad (z \in \mathbb{D}),$$

in which h is called the analytic part of f and g is called the co-analytic part of f. In fact, if g is identically zero, the f reduces to the analytic case.

A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathbb{D}$ is that (see in [2])

$$|h'(z)| > |g'(z)| \qquad (z \in \mathbb{D}).$$

Thus, for $f = h + \overline{g} \in \mathcal{S}^*\mathcal{H}$, where $\mathcal{S}^*\mathcal{H}$ is the class of normalized starlike harmonic functions in the open unit disk:

$$\mathbb{U} = \{z : z \in \mathbb{C} \qquad \text{and} \qquad |z| < 1\},$$

we may write

$$h(z) = z + \sum_{n=2}^{\infty} a_n\, z^n \qquad \text{and} \qquad g(z) = \sum_{n=1}^{\infty} b_n\, z^n \qquad (|b_1| < 1). \tag{1}$$

We note that $\mathcal{S}^*\mathcal{H}$ reduces to the familiar class $\mathcal{S}^*$ of normalized starlike univalent functions in $\mathbb{U}$ if the co-analytic part of $f = h + \overline{g}$ is identically zero. We use the abbreviation $\mathcal{SH}$ in our notation for the subclasses of $\mathcal{S}^*\mathcal{H}$ consisting of functions f that map the open unit disk $\mathbb{U}$ onto a starlike domain.

A function f is said to be starlike of order α $(0 \leqq \alpha < 1)$ in $\mathbb{U}$ denoted by $\mathcal{SH}(\alpha)$ (see in [5]) if

$$\begin{aligned}\frac{\partial}{\partial\theta}\left\{\arg\left(f\left(re^{i\theta}\right)\right)\right\} &= \Im\left(\frac{\frac{\partial}{\partial\theta}\left\{f\left(re^{i\theta}\right)\right\}}{f\left(re^{i\theta}\right)}\right) \\ &= \Re\left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right) \geqq \alpha \qquad (|z| = r < 1).\end{aligned}$$

A normalized univalent analytic function f is said to be starlike with respect to symmetrical points in $\mathbb{U}$ if it satisfies the following condition:

$$\Re\left(\frac{2zf'(z)}{f(z) - f(-z)}\right) > 0 \qquad (z \in \mathbb{U}).$$

This function class was introduced and studied by Sakaguchi [6] in 1959. Some other related function classes were also studied by Shanmugam et al. [7]. In 1979, Chand and Singh [8] defined the class of starlike functions with respect to k-symmetric points of order α $(0 \leqq \alpha < 1)$ (see also in [9]). Ahuja and Jahangiri [10] discussed the class $\mathcal{SH}(\alpha)$ of complex-valued and sense-preserving harmonic univalent functions f of the form (1) and satisfying the following condition:

$$\Im\left(\frac{2\,\frac{\partial}{\partial\theta}\left\{f\left(re^{i\theta}\right)\right\}}{f\left(re^{i\theta}\right) - f\left(-re^{i\theta}\right)}\right) \geqq \alpha \qquad (0 \leqq \alpha < 1).$$

Al-Shaqsi and Darus [11] introduced the class $\mathcal{SH}_k(\alpha)$ of complex-valued and sense-preserving harmonic univalent functions f of the form (1) as follows:

$$\Im\left(\frac{\frac{\partial}{\partial\theta}\{f(re^{i\theta})\}}{f_k(re^{i\theta})}\right) \geqq \alpha \qquad (0 \leqq \alpha < 1),$$

where

$$h_k(z) = z + \sum_{n=2}^{\infty} \varphi_n a_n z^n \quad \text{and} \quad g_k(z) = \sum_{n=1}^{\infty} \varphi_n b_n z^n \quad (|b_1| < 1) \tag{2}$$

and

$$\varphi_n = \frac{1}{k}\sum_{v=0}^{k-1} \epsilon^{(n-1)v} \qquad (k \geqq 1;\ \epsilon^k = 1). \tag{3}$$

From the definition (3) of φ_n, we have

$$\varphi_n = \begin{cases} 1 & (n = lk+1) \\ 0 & (n \neq lk+1), \end{cases}$$

where $n \geqq 2$ and $l, k \geqq 1$.

Next, for a function d, given by

$$d(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (\forall\, z \in \mathbb{U}),$$

and another function v, given by

$$v(z) = z + \sum_{n=2}^{\infty} b_n z^n \qquad (\forall\, z \in \mathbb{U}),$$

the convolution (or the Hadamard product) of d and v is defined, as usual, by

$$d(z) * v(z) = (d * v)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (v * d)(z).$$

The fractional q-calculus is the q-extension of the ordinary fractional calculus, which dates back to early twentieth century. The theory of the q-calculus operators are used in many diverse areas of science such as fractional q-calculus, optimal control, q-difference, and q-integral equations. This also in the geometric function theory of complex analysis as is described by Srivastava [12] in his recent survey-cum-expository review article [12].

Initially in 1908, Jackson [13] defined the q-analogs of the ordinary derivative and integral operators, and presented some of their applications. More recently, Ismail et al. [14] gave the idea of a q-extension of the familiar class of starlike functions in $\mathbb{U}$. Historically, however, Srivastava [15] studied the q-calculus in the context of the univalent function theory in 1989 and also applied the generalized basic (or q-) hypergeometric functions in the univalent function theory. Many researchers have since studied the q-calculus in the context of Geometric Functions Theory.

The survey-cum-expository review article by Srivastava [12] is potentially useful for those who are interested in Geometric Function Theory. Such various applications of the fractional q-calculus as, for example, the fractional q-derivative operator and the q-derivative operator in Geometric Function Theory were systematically highlighted in Srivastava's survey-cum-expository review article [12]. Moreover, the triviality of the so-called (p,q)-calculus involving an obviously redundant and inconsequential additional parameter p was revealed and exposed (see, for details, in [12] (p. 340)).

In the development of Geometric Function Theory, a number of researchers have been inspired by the aforementioned works [12,14]. Several convolution and fractional q-operators, that have been already defined, were surveyed in the above-cited work [12]. For example, Kanas and Răducanu [16] introduced the q-analog of the Ruscheweyh derivative operator and Zang et al. in [17] studied q-starlike functions related with a generalized conic domain $\Omega_{k,\alpha}$. By using the concept of convolution, Srivastava et al. [18] introduced the q-Noor integral operator and studied some of its applications. Furthermore, Srivastava et al. published a series of articles in which they concentrated upon the class of q-starlike functions from many different aspects and viewpoints (see in [18–22]). For some more recent investigations about the q-calculus, we may refer the interested reader to the recent works [23–37].

Recently, Jahangiri [38] applied certain q-operators to complex harmonic functions and obtained sharp coefficient bounds, distortion theorems, and covering results. On the other hand, Porwal and Gupta [39] discussed an application of the q-calculus to harmonic univalent functions. In this article, we apply the q-calculus in order to define a q-analog of the derivative operator which is applicable to complex harmonic functions, and to introduce and investigate new classes of harmonic univalent functions with respect to k-symmetric points.

For better understanding of this article, we recall some concept details and definitions of the q-difference calculus. We suppose throughout this paper that $0 < q < 1$ and that

$$\mathbb{N} = \{1, 2, 3, \cdots\} = \mathbb{N}_0 \setminus \{0\} \qquad (\mathbb{N}_0 := \{0, 1, 2, \cdots\}).$$

Definition 1. *The q-number $[\tau]_q$ is defined by*

$$[\tau]_q := \begin{cases} \dfrac{1-q^\tau}{1-q} & (\tau \in \mathbb{C}) \\ \sum\limits_{k=0}^{n-1} q^k & (\tau = n \in \mathbb{N}). \end{cases}$$

Definition 2. *The q-factorial $[n]_q!$ is defined by*

$$[n]_q! := \begin{cases} \prod\limits_{k=1}^{n} [k]_q & (n \in \mathbb{N}) \\ 1 & (n = 0). \end{cases}$$

Definition 3. *The generalized q-Pochhammer symbol $[\tau]_{n,q}$ is defined by*

$$[\tau]_{n,q} := \begin{cases} \dfrac{\Gamma_q(\tau+n)}{\Gamma_q(\tau)} = [\tau]_q[\tau+1]_q[\tau+2]_q \cdots [\tau+n-1]_q & (n \in \mathbb{N}) \\ 1 & (n = 0). \end{cases}$$

Furthermore, for $\tau > 0$, let the q-gamma function be defined as follows:

$$\Gamma_q(\tau+1) = [\tau]_q \Gamma_q(\tau) \qquad \textit{and} \qquad \Gamma_q(1) = 1,$$

where

$$\Gamma_q(\tau) = (1-q)^{1-\tau} \prod_{n=0}^{\infty} \left(\frac{1-q^{n+1}}{1-q^{n+\tau}} \right).$$

Definition 4 (see, for example, in [13]). *For $q \in (0,1)$, the q-derivative operator (or the q-difference operator) $\mathfrak{D}_q$, when applied to a given function f normalized by*

$$\mathfrak{f}(z) = z + \sum_{n=2}^{\infty} \mathfrak{a}_n \, z^n \qquad (z \in \mathbb{U}), \tag{4}$$

is defined as follows:

$$\begin{aligned} \mathfrak{D}_q \mathfrak{f}(z) &= \frac{f(z) - f(qz)}{(1-q)z} \qquad (z \neq 0;\ q \neq 1) \\ &= 1 + \sum_{n=2}^{\infty} [n]_q \, \mathfrak{a}_n \, z^{n-1} \qquad (z \in \mathbb{U}), \end{aligned} \tag{5}$$

so that, clearly, we have

$$\lim_{q \to 1-} \mathfrak{D}_q \mathfrak{f}(z) = \mathfrak{f}^*(z),$$

provided that the ordinary derivative $\mathfrak{f}^*(z)$ exists.

Definition 5. *We define the q-analog of the derivative operator for the harmonic function $f = h + \overline{g}$ given by (1) as follows:*

$$D^{\sigma,s}_{\lambda,\delta,q} f(z) = D^{\sigma,s}_{\lambda,\delta,q} h(z) + (-1)^s \overline{D^{\sigma,s}_{\lambda,\delta,q} g(z)},$$

where

$$D^{\sigma,s}_{\lambda,\delta,q} h(z) = z + \sum_{n=2}^{\infty} \psi_n(\lambda,\sigma,\delta,s,q) a_n z^n,$$

$$D^{\sigma,s}_{\lambda,\delta,q} g(z) = \sum_{n=1}^{\infty} \psi_n(\lambda,\sigma,\delta,s,q) b_n z^n$$

and

$$\psi_n(\lambda,\sigma,\delta,s,q) = [n]_q^s \left(\frac{[\delta+1]_{n-1}}{[n-1]_q!} \left\{ 1 + \lambda \Big([n]_q - 1 \Big) \right\} \right)^{\sigma} \qquad (\lambda,\delta,\sigma,s \in \mathbb{N}_0). \tag{6}$$

Remark 1. *First of all, it is easy to see that, for*

$$s = 0 = \lambda \quad \text{and} \quad \sigma = 1,$$

we have the q-Ruscheweyh derivative for harmonic functions (see in [38]). Second, for $\sigma = 0$, we obtain the q-Sălăgean operator for harmonic functions (see [38]). Third, if we take

$$s = 0 \quad \text{and} \quad \sigma = 1,$$

and let $q \to 1-$, we obtain the operator for harmonic functions studied by Al-Shaqsi and Darus [40].

Definition 6. *Let $\mathcal{MH}^{\sigma,s}_{k,q}(\lambda,\delta,\alpha)$ denote the class of complex-valued and sense-preserving harmonic univalent functions f of the form (1) which satisfy the following condition:*

$$\Im\left(\frac{\frac{\partial}{\partial\theta}\left\{D^{\sigma,s}_{\lambda,\delta,q}f\left(re^{i\theta}\right)\right\}}{D^{\sigma,s}_{\lambda,\delta,q}f_k\left(re^{i\theta}\right)}\right)$$

$$= \Re\left(\frac{z\mathfrak{D}_q D^{\sigma,s}_{\lambda,\delta,q}h(z) - (-1)^s\overline{z\mathfrak{D}_q D^{\sigma,s}_{\lambda,\delta,q}g(z)}}{D^{\sigma,s}_{\lambda,\delta,q}h_k(z) + (-1)^s\overline{D^{\sigma,s}_{\lambda,\delta,q}g_k(z)}}\right) \geqq \alpha, \tag{7}$$

where

$$z = re^{i\theta} \qquad (0 \leqq r < 1;\ 0 \leqq \theta < \pi;\ 0 \leqq \alpha < 1),$$

$$D^{\sigma,s}_{\lambda,\delta,q}h_k(z) = z + \sum_{n=2}^{\infty}\psi_n(\lambda,\sigma,\delta,s,q)\varphi_n\, a_n\, z^n$$

and

$$D^{\sigma,s}_{\lambda,\delta,q}g_k(z) = \sum_{n=1}^{\infty}\psi_n(\lambda,\sigma,\delta,s,q)\varphi_n\, b_n\, z^n. \tag{8}$$

Furthermore, we denote by $\overline{\mathcal{MH}^{\sigma,s}_{k,q}(\lambda,\delta,\alpha)}$ the subclass of the class $\mathcal{MH}^{\sigma,s}_{k,q}(\lambda,\delta,\alpha)$ such that the functions h and g in $f = h + \overline{g}$ are of the following form:

$$h(z) = z - \sum_{n=2}^{\infty}|a_n|z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty}|b_n|z^n \quad (|b_1| < 1) \tag{9}$$

and the functions h_k and g_k in $f_k = h_k + \overline{g_k}$ are of the form given by

$$h_k(z) = z - \sum_{n=2}^{\infty}|a_n|\varphi_n z^n \quad \text{and} \quad g_k(z) = \sum_{n=1}^{\infty}|b_n|\varphi_n z^n \quad (|b_1| < 1). \tag{10}$$

In this article, we obtain inclusion properties, sufficient conditions, and coefficient bounds for functions in the the class $\mathcal{MH}^{\sigma,s}_{k,q}(\lambda,\delta,\alpha)$. A representation theorem and distortion bounds for the class $\overline{\mathcal{MH}^{\sigma,s}_{k,q}(\lambda,\delta,\alpha)}$ are also established. We will examine the closure properties for the class $\overline{\mathcal{MH}^{\sigma,s}_{k,q}(\lambda,\delta,\alpha)}$ under the generalized q-Bernardi–Libera–Livingston integral operator $L^q_c(f)$.

2. A Set of Main Results

We begin by stating and proving Theorem 1 below.

Theorem 1. *Let $f \in \mathcal{MH}^{\sigma,s}_{k,q}(\lambda,\delta,\alpha)$, where f is given by (1). Then, f_k defined by (2) is in*

$$\mathcal{MH}^{\sigma,s}_{1,q}(\lambda,\delta,\alpha) =: \mathcal{MH}^{\sigma,s}_{q}(\lambda,\delta,\alpha).$$

Proof. Let $f \in \mathcal{MH}^{\sigma,s}_{k,q}(\lambda,\delta,\alpha)$. Then, upon replacing $re^{i\theta}$ by $\epsilon^v re^{i\theta}$, where $\epsilon^v = 1$ $(v = 0,1,2,\cdots,k-1)$ in (7), we have

$$\Im\left(\frac{\frac{\partial}{\partial\theta}\left\{D^{\sigma,s}_{\lambda,\delta,q}f\left(\epsilon^v re^{i\theta}\right)\right\}}{D^{\sigma,s}_{\lambda,\delta,q}f_k\left(\epsilon^v re^{i\theta}\right)}\right) \geqq \alpha.$$

According to the definition of f_k, and as $\epsilon^v = 1 \ \ (v = 0,1,2,\cdots,k-1)$, we know that

$$f_k\left(\epsilon^v re^{i\theta}\right) = \epsilon^v f_k\left(re^{i\theta}\right) \qquad (v = 0,1,2,\cdots,k-1).$$

Thus, by summing up, we get

$$\Im\left(\frac{1}{k}\sum_{v=0}^{k-1}\frac{\frac{\partial}{\partial\theta}\left\{D^{\sigma,s}_{\lambda,\delta,q}f\left(\epsilon^v re^{i\theta}\right)\right\}}{\epsilon^v D^{\sigma,s}_{\lambda,\delta,q}f_k\left(re^{i\theta}\right)}\right) = \Im\left(\frac{\frac{\partial}{\partial\theta}\left\{D^{\sigma,s}_{\lambda,\delta,q}f_k\left(re^{i\theta}\right)\right\}}{D^{\sigma,s}_{\lambda,\delta,q}f_k\left(re^{i\theta}\right)}\right) \geqq \alpha,$$

that is, $f_k \in \mathcal{MH}^{\sigma,s}_q(\lambda,\delta,\alpha)$. $\square$

If we let $q \to 1-$, in Theorem 1, we have the following result.

Corollary 1. *Let $f \in \mathcal{MH}^{\sigma,s}_k(\lambda,\delta,\alpha)$ where f is given by (1). Then, f_k defined by (2) is in the class*

$$\mathcal{MH}^{\sigma,s}_1(\lambda,\delta,\alpha) =: \mathcal{MH}^{\sigma,s}(\lambda,\delta,\alpha).$$

Theorem 2. *Let $f = h + \overline{g}$ given by (1) and $f_k = h_k + \overline{g_k}$ with h_k and g_k given by (2). Suppose also that*

$$\begin{aligned}\sum_{n=2}^{\infty}\psi_n(\lambda,\sigma,\delta,s,q)&\left(\frac{[n]_q-\alpha\varphi_n}{1-\alpha}|a_n| + \frac{[n]_q+\alpha\varphi_n}{1-\alpha}|b_n|\right)\\ &\leqq 1-\left(\frac{1+\alpha\varphi_1}{1-\alpha}\right)\psi_1|b_1|,\end{aligned} \tag{11}$$

where φ_n and $\psi_n(\lambda,\sigma,\delta,s,q)$ given by (3) and (6) with

$$a_1 = 1,\ \ l \geqq 1,\ \ \lambda \geqq 0,\ \ (k \geqq 1) \qquad \text{and} \qquad \delta,\sigma,s \in N_0.$$

Then, the function f is sense-preserving harmonic univalent in $\mathbb{U}$ and $f \in \mathcal{MH}^{\sigma,s}_{k,q}(\lambda,\delta,\alpha)$.

Proof. To prove that $f \in \mathcal{MH}^{\sigma,s}_{k,q}(\lambda,\delta,\alpha)$, we only need to show that if (11) holds true, then the required condition (7) is satisfied. From (7), we can write

$$\Re\left(\frac{z\mathfrak{D}_q D^{\sigma,s}_{\lambda,\delta,q}h(z) - (-1)^s\overline{z\mathfrak{D}_q D^{\sigma,s}_{\lambda,\delta,q}g(z)}}{D^{\sigma,s}_{\lambda,\delta,q}h_k(z) + (-1)^s\overline{D^{\sigma,s}_{\lambda,\delta,q}g_k(z)}}\right) = \Re\left(\frac{\mathcal{T}(z)}{\mathcal{R}(z)}\right),$$

where

$$\mathcal{T}(z) = z\mathfrak{D}_q D^{\sigma,s}_{\lambda,\delta,q}h(z) - (-1)^s\overline{z\mathfrak{D}_q D^{\sigma,s}_{\lambda,\delta,q}g(z)}$$

and

$$\mathcal{R}(z) = D^{\sigma,s}_{\lambda,\delta,q}h_k(z) + (-1)^s\overline{D^{\sigma,s}_{\lambda,\delta,q}g_k(z)}.$$

Now, using the fact that

$$\Re(w) \geqq \alpha \iff |1-\alpha+w| \geqq |1+\alpha-w|,$$

it suffices to show that

$$|\mathcal{T}(z) + (1-\alpha)\mathcal{R}(z)| - |\mathcal{T}(z) - (1+\alpha)\mathcal{R}(z)| \geqq 0.$$

Upon substituting for $\mathcal{T}(z)$ and $\mathcal{R}(z)$ into (11), we find that

$$\begin{aligned}
&|\mathcal{T}(z)+(1-\alpha)\mathcal{R}(z)|-|\mathcal{T}(z)-(1+\alpha)\mathcal{R}(z)| \\
&\geqq (2-\alpha)|z|-\sum_{n=2}^{\infty}\psi_n(\lambda,\sigma,\delta,s,q)\Big([n]_q+(1-\alpha)\varphi_n\Big)|a_n||z|^n \\
&\quad -\sum_{n=1}^{\infty}\psi_n(\lambda,\sigma,\delta,s,q)\Big([n]_q+(1-\alpha)\varphi_n\Big)|b_n||z|^n-\alpha|z| \\
&\quad -\sum_{n=2}^{\infty}\psi_n(\lambda,\sigma,\delta,s,q)\Big([n]_q-(1+\alpha)\varphi_n\Big)|a_n||z|^n \\
&\quad -\sum_{n=1}^{\infty}\psi_n(\lambda,\sigma,\delta,s,q)\Big([n]_q+(1+\alpha)\varphi_n\Big)|b_n||z|^n \\
&=2(1-\alpha)|z|\left[1-\sum_{n=2}^{\infty}\psi_n(\lambda,\sigma,\delta,s,q)\left(\frac{[n]_q-\alpha\varphi_n}{1-\alpha}\right)|a_n||z|^{n-1}\right. \\
&\quad \left.-\sum_{n=1}^{\infty}\psi_n(\lambda,\sigma,\delta,s,q)\left(\frac{[n]_q+\alpha\varphi_n}{1-\alpha}\right)|b_n||z|^{n-1}\right]
\end{aligned}$$

$$\begin{aligned}
&=2(1-\alpha)|z|\left[1-\psi_1(\lambda,\sigma,\delta,s,q)\left(\frac{[1]_q+\alpha\varphi_n}{1-\alpha}\right)|b_1|\right. \\
&\quad \left.-\sum_{n=2}^{\infty}\psi_n(\lambda,\sigma,\delta,s,q)\left(\frac{[n]_q-\alpha\varphi_n}{1-\alpha}|a_n|+\frac{[n]_q+\alpha\varphi_n}{1-\alpha}|b_n|\right)\right].
\end{aligned}$$

The last expression is non-negative by (11), and therefore $f\in\mathcal{MH}_{k,q}^{\sigma,s}(\lambda,\delta,\alpha)$. □

The next theorem gives a coefficient bound for functions in the class $\mathcal{MH}_{k,q}^{\sigma,s}(\lambda,\delta,\alpha)$.

Theorem 3. *The function $f\in\mathcal{MH}_{k,q}^{\sigma,s}(\lambda,\delta,\alpha)$ if and only if*

$$\begin{aligned}
&\left(D_{\lambda,\delta,q}^{\sigma,s}h(z)*\frac{(\xi+1)z}{(1-z)(1-qz)}-D_{\lambda,\delta,q}^{\sigma,s}h_k(z)*\frac{(\xi-1+2\alpha)z}{(1-z)}\right) \\
&-(-1)^s\left(\overline{D_{\lambda,\delta,q}^{\sigma,s}g(z)}*\frac{(\xi+1)z}{(1-z)(1-qz)}+\overline{D_{\lambda,\delta,q}^{\sigma,s}g_k(z)}*\frac{(\xi-1+2\alpha)z}{(1-z)}\right)\neq 0,
\end{aligned}$$

where

$$|\xi|=1 \quad (\xi\neq -1) \qquad \textit{and} \qquad z\in\mathbb{U}.$$

Proof. From (7), $f\in\mathcal{MH}_{k,q}^{\sigma,s}(\lambda,\delta,\alpha)$ if and only if $z=re^{i\theta}$ in $\mathbb{U}$, we have

$$\Re\left(\frac{z\mathfrak{D}_qD_{\lambda,\delta,q}^{\sigma,s}h(z)-(-1)^s\overline{z\mathfrak{D}_qD_{\lambda,\delta,q}^{\sigma,s}g(z)}}{D_{\lambda,\delta,q}^{\sigma,s}h_k(z)+(-1)^s\overline{D_{\lambda,\delta,q}^{\sigma,s}g_k(z)}}\right)\geqq\alpha,$$

which readily yields

$$\Re\left(\frac{1}{1-\alpha}\left[\frac{z\mathfrak{D}_qD_{\lambda,\delta,q}^{\sigma,s}h(z)-(-1)^s\overline{z\mathfrak{D}_qD_{\lambda,\delta,q}^{\sigma,s}g(z)}}{D_{\lambda,\delta,q}^{\sigma,s}h_k(z)+(-1)^s\overline{D_{\lambda,\delta,q}^{\sigma,s}g_k(z)}}-\alpha\right]\right)\geqq 0.$$

Now, as

$$\frac{1}{1-\alpha}\left(\frac{z\mathfrak{D}_q D^{\sigma,s}_{\lambda,\delta,q}h(z)-(-1)^s\overline{z\mathfrak{D}_q D^{\sigma,s}_{\lambda,\delta,q}g(z)}}{D^{\sigma,s}_{\lambda,\delta,q}h_k(z)+(-1)^s\overline{D^{\sigma,s}_{\lambda,\delta,q}g_k(z)}}-\alpha\right)=1 \text{ at } z=0,$$

the above-required condition is equivalent to

$$\frac{1}{1-\alpha}\left(\frac{z\mathfrak{D}_q D^{\sigma,s}_{\lambda,\delta,q}h(z)-(-1)^s\overline{z\partial_q D^{\sigma,s}_{\lambda,\delta,q}g(z)}}{D^{\sigma,s}_{\lambda,\delta,q}h_k(z)+(-1)^s\overline{D^{\sigma,s}_{\lambda,\delta,q}g_k(z)}}-\alpha\right)\neq\frac{\xi-1}{\xi+1}, \tag{12}$$

where

$$|\xi|=1 \quad (\xi\neq -1) \qquad \text{and} \qquad 0<|z|<1.$$

Thus, by a simple algebraic manipulation, the inequality (12) yields

$$\begin{aligned}0\neq{}&(\xi+1)\left(z\mathfrak{D}_q D^{\sigma,s}_{\lambda,\delta,q}h(z)-(-1)^s\overline{z\mathfrak{D}_q D^{\sigma,s}_{\lambda,\delta,q}g(z)}\right)\\&-(\xi-1+2\alpha)\left(D^{\sigma,s}_{\lambda,\delta,q}h_k(z)+(-1)^s\overline{D^{\sigma,s}_{\lambda,\delta,q}g_k(z)}\right)\\={}&\left(D^{\sigma,s}_{\lambda,\delta,q}h(z)*\frac{(\xi+1)z}{(1-z)(1-qz)}-D^{\sigma,s}_{\lambda,\delta,q}h_k(z)*\frac{(\xi-1+2\alpha)z}{(1-z)}\right)\\&-(-1)^s\left(\overline{D^{\sigma,s}_{\lambda,\delta,q}g(z)}*\frac{(\xi+1)z}{(1-z)(1-qz)}+\overline{D^{\sigma,s}_{\lambda,\delta,q}g_k(z)}*\frac{(\xi-1+2\alpha)z}{(1-z)}\right),\end{aligned}$$

which is the condition asserted in Theorem 3. □

Next, the condition (11) is also necessary for functions in the class $\overline{\mathcal{MH}^{\sigma,s}_{k,q}(\lambda,\delta,\alpha)}$, which is clarified in Theorem 4 below.

Theorem 4. *Let $f=h+\overline{g}$ with h and g given by (9) and $f_k=h_k+\overline{g_k}$ with h_k and g_k given by (10). Then, $f\in\overline{\mathcal{MH}^{\sigma,s}_{k,q}(\lambda,\delta,\alpha)}$ if and only if*

$$\begin{aligned}&\sum_{n=2}^{\infty}\psi_n(\lambda,\sigma,\delta,s,q)\left(\frac{[n]_q-\alpha\varphi_n}{1-\alpha}|a_n|+\frac{[n]_q+\alpha\varphi_n}{1-\alpha}|b_n|\right)\\&\qquad\leqq 1-\left(\frac{1+\alpha\varphi_1}{1-\alpha}\right)\psi_1|b_1|,\end{aligned} \tag{13}$$

where φ_n and $\psi_n(\lambda,\sigma,\delta,s,q)$ are given by (3) and (6) with

$$a_1=1,\ l\geqq 1,\ \lambda\geqq 0,\ (k\geqq 1) \qquad \text{and} \qquad \delta,\sigma,s\in N_0.$$

Proof. The direct part of the proof follows from Theorem 2 by noting that if the analytic and co-analytic parts of $f=h+\overline{g}\in\mathcal{MH}^{\sigma,s}_{k,q}(\lambda,\delta,\alpha)$ are given in (9), then $f\in\overline{\mathcal{MH}^{\sigma,s}_{k,q}(\lambda,\delta,\alpha)}$.

Let us prove the converse part by contradiction. We show that $f\notin\overline{\mathcal{MH}^{\sigma,s}_{k,q}(\lambda,\delta,\alpha)}$ if the condition (13) holds true. Thus, we can write

$$\Re\left(\frac{z\mathfrak{D}_q D^{\sigma,s}_{\lambda,\delta,q}h(z)-(-1)^s\overline{z\mathfrak{D}_q D^{\sigma,s}_{\lambda,\delta,q}g(z)}}{D^{\sigma,s}_{\lambda,\delta,q}h_k(z)+(-1)^s\overline{D^{\sigma,s}_{\lambda,\delta,q}g_k(z)}}\right)\geqq\alpha,$$

which is equivalent to

$$\Re\left(\frac{z\mathfrak{D}_q D^{\sigma,s}_{\lambda,\delta,q}h(z)-(-1)^s\overline{z\mathfrak{D}_q D^{\sigma,s}_{\lambda,\delta,q}g(z)}}{D^{\sigma,s}_{\lambda,\delta,q}h_k(z)+(-1)^s\overline{D^{\sigma,s}_{\lambda,\delta,q}g_k(z)}}\right)-\alpha\geqq 0,$$

that is,

$$\Re\left(\left[(1-\alpha)z-\sum_{n=2}^{\infty}\psi_n(\lambda,\sigma,\delta,s,q)\left([n]_q-\alpha\varphi_n\right)|a_n|z^n \right.\right.$$
$$\left.-(-1)^s\overline{\sum_{n=1}^{\infty}\psi_n(\lambda,\sigma,\delta,s,q)\left([n]_q+\alpha\varphi_n\right)|b_n|z^n}\right]$$
$$\left.\cdot\left[z-\sum_{n=2}^{\infty}\psi_n(\lambda,\sigma,\delta,s,q)\varphi_n|a_n|z^n+(-1)^s\overline{\sum_{n=1}^{\infty}\psi_n(\lambda,\sigma,\delta,s,q)\varphi_n|b_n|z^n}\right]^{-1}\right).$$
$$\geqq 0.$$

Thus, clearly, the above-required condition holds true for all values of z ($|z| = r < 1$). Upon choosing the values of z on the non-negative real axis such that $0 \leqq z = r < 1$, we find that

$$\Re\left(\left[(1-\alpha)-\sum_{n=2}^{\infty}\psi_n(\lambda,\sigma,\delta,s,q)\left([n]_q-\alpha\varphi_n\right)|a_n|r^{r-1}\right.\right.$$
$$\left.-\sum_{n=1}^{\infty}\psi_n(\lambda,\sigma,\delta,s,q)\left([n]_q+\alpha\varphi_n\right)|b_n|r^{n-1}\right]$$
$$\left.\cdot\left[1-\sum_{n=2}^{\infty}\psi_n(\lambda,\sigma,\delta,s,q)\varphi_n|a_n|r^{n-1}+\sum_{n=1}^{\infty}\psi_n(\lambda,\sigma,\delta,s,q)\varphi_n|b_n|r^{n-1}\right]^{-1}\right)$$
$$\geqq 0,$$

which can be written as follows:

$$\frac{\mathcal{Q}(q)-\left[\sum_{n=2}^{\infty}\psi_n(\lambda,\sigma,\delta,s,q)\left(\left([n]_q-\alpha\varphi_n\right)|a_n|+\left([n]_q+\alpha\varphi_n\right)|b_n|\right)\right]r^{r-1}}{1+|b_1|-\left(\sum_{n=2}^{\infty}\psi_n(\lambda,\sigma,\delta,s,q)\varphi_n(|a_n|+|b_n|)\right)r^{r-1}}\geqq 0, \tag{14}$$

where

$$\mathcal{Q}(q)=(1-\alpha)-\psi_1(\lambda,\sigma,\delta,s,q)\left([1]_q+\alpha\varphi_n\right)|b_1|.$$

If the condition (13) does not hold true, then the numerator in (14) is negative for r sufficiently close to 1. Therefore, there exists a $z_0 = r_0$ in $(0,1)$ for which the quotient in (14) is negative. This contradicts the required condition for $f \in \overline{\mathcal{MH}^{\sigma,s}_{k,q}(\lambda,\delta,\alpha)}$. Our proof of the converse part Theorem 4 by contradiction is thus completed. □

The following theorem gives the distortion bounds for functions in the class $\overline{\mathcal{MH}^{\sigma,s}_{k,q}(\lambda,\delta,\alpha)}$.

Theorem 5. *If $f \in \overline{\mathcal{MH}^{\sigma,s}_{k,q}(\lambda,\delta,\alpha)}$, then*

$$|f(z)| \geqq (1-|b_1|)r-\frac{1}{\psi_2(\lambda,\sigma,\delta,s,q)}\left(\frac{1-\alpha}{[2]_q-\alpha\varphi_2}-\frac{1+\alpha}{[2]_q-\alpha\varphi_2}|b_1|\right)r^2 \tag{15}$$

and

$$|f(z)| \leqq (1+|b_1|)r+\frac{1}{\psi_2(\lambda,\sigma,\delta,s,q)}\left(\frac{1-\alpha}{[2]_q-\alpha\varphi_2}-\frac{1+\alpha}{[2]_q-\alpha\varphi_2}|b_1|\right)r^2, \tag{16}$$

where φ_n *and* $\psi_n(\lambda, \sigma, \delta, s, q)$ *are given by* (3) *and* (6) *with*

$$a_1 = 1, \ \ l \geqq 1, \ \ \lambda \geqq 0, \ \ (k \geqq 1) \qquad \textit{and} \qquad \delta, \sigma, s \in N_0.$$

Proof. We will only prove the left-hand inequality of Theorem 5. The arguments for proving the right-hand inequality are similar and so we omit the details involved.

Let $f \in \overline{\mathcal{MH}^{\sigma,s}_{k,q}(\lambda, \delta, \alpha)}$. Then, by taking the modulus of $f(z)$, we obtain

$$\begin{aligned}
|f(z)| &\geqq (1 - |b_1|)r - \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^n \\
&\geqq (1 - |b_1|)r - \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^2 \\
&\geqq (1 - |b_1|)r - \frac{1-\alpha}{\psi_2(\lambda, \sigma, \delta, s, q)\Big([2]_q - \alpha\varphi_2\Big)} \\
&\qquad \cdot \sum_{n=2}^{\infty} \frac{\psi_2(\lambda, \sigma, \delta, s, q)\Big([2]_q - \alpha\varphi_2\Big)}{1-\alpha}(|a_n| + |b_n|)r^2 \\
&\geqq (1 - |b_1|)r - \frac{1-\alpha}{\psi_2(\lambda, \sigma, \delta, s, q)\Big([2]_q - \alpha\varphi_2\Big)}\left(1 - \frac{1+\alpha}{1-\alpha}|b_1|\right)r^2 \\
&= (1 - |b_1|)r - \frac{1}{\psi_2(\lambda, \sigma, \delta, s, q)}\left(\frac{1-\alpha}{[2]_q - \alpha\varphi_2} - \frac{1+\alpha}{[2]_q - \alpha\varphi_2}|b_1|\right)r^2,
\end{aligned}$$

which proves the inequality (15). □

The following covering result follows from the left-hand inequality in Theorem 5.

Corollary 2. *If* $f \in \overline{\mathcal{MH}^{\sigma,s}_{k,q}(\lambda, \delta, \alpha)}$, *then*

$$\{w : |w| < Q_1(\lambda, \sigma, \delta, s, q) - Q_2(\lambda, \sigma, \delta, s, q)|b_1|\} \subset f(\mathbb{U}),$$

where

$$Q_1(\lambda, \sigma, \delta, s, q) = \frac{2\psi_2(\lambda, \sigma, \delta, s, q) - 1 - (\psi_2(\lambda, \sigma, \delta, s, q) - 1)\alpha}{\psi_2(\lambda, \sigma, \delta, s, q)\Big([2]_q - \alpha\varphi_2\Big)}$$

and

$$Q_2(\lambda, \sigma, \delta, s, q) = \frac{2\psi_2(\lambda, \sigma, \delta, s, q) - 1 - (\psi_2(\lambda, \sigma, \delta, s, q) + 1)\alpha}{\psi_2(\lambda, \sigma, \delta, s, q)\Big([2]_q - \alpha\varphi_2\Big)}.$$

Finally, we will examine the closure properties of the class $\overline{\mathcal{MH}^{\sigma,s}_{k,q}(\lambda, \delta, \alpha)}$ under the generalized q-Bernardi–Libera–Livingston integral operator $L^q_c(f)$ which is defined by

$$L^q_c(f(z)) = \frac{[c+1]_q}{z^c}\int_0^z t^{c-1}\, f(t)\, d_q t \qquad (c > -1).$$

Theorem 6. *Let* $f \in \overline{\mathcal{MH}^{\sigma,s}_{k,q}(\lambda, \delta, \alpha)}$. *Then,* $L^q_c(f(z)) \in \overline{\mathcal{MH}^{\sigma,s}_{k,q}(\lambda, \delta, \alpha)}$.

Proof. From the representation of $L_c^q\big(f(z)\big)$, it follows that

$$\begin{aligned} L_c^q\big(f(z)\big) &= \frac{[c+1]_q}{z^c} \int_0^z t^{c-1}[h(t)+\overline{g}(t)]\, d_q t \\ &= \frac{[c+1]_q}{z^c}\left[\int_0^z t^{c-1}\left(t+\sum_{n=2}^{\infty} a_n t^n\right) d_q t + \overline{\int_0^z t^{c-1}\left(\sum_{n=1}^{\infty} b_n t^n\right) d_q t}\right] \\ &= z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n z^n}, \end{aligned}$$

where

$$A_n = \frac{[c+1]_q}{[c+n]_q}\, a_n \qquad \text{and} \qquad B_n = \frac{[c+1]_q}{[c+n]_q}\, b_n.$$

Therefore, we get

$$\begin{aligned} &\sum_{n=2}^{\infty} \psi_n(\lambda,\sigma,\delta,s,q)\left(\frac{\big([n]_q-\alpha\varphi_n\big)[c+1]_q}{(1-\alpha)[c+n]_q}|a_n|\right) \\ &\qquad + \sum_{n=2}^{\infty} \psi_n(\lambda,\sigma,\delta,s,q)\left(\frac{\big([n]_q+\alpha\varphi_n\big)[c+1]_q}{(1-\alpha)[c+n]_q}|b_n|\right) \\ &\leqq \sum_{n=2}^{\infty} \psi_n(\lambda,\sigma,\delta,s,q)\left(\frac{\big([n]_q-\alpha\varphi_n\big)}{(1-\alpha)}|a_n| + \frac{\big([n]_q+\alpha\varphi_n\big)}{(1-\alpha)}|b_n|\right) \\ &< 1 - \left(\frac{1+\alpha\varphi_1}{1-\alpha}\right)\psi_1|b_1|. \end{aligned}$$

As $f \in \overline{\mathcal{MH}_{k,q}^{\sigma,s}(\lambda,\delta,\alpha)}$, by Theorem (4), we have $L_c^q\big(f(z)\big) \in \overline{\mathcal{MH}_{k,q}^{\sigma,s}(\lambda,\delta,\alpha)}$, as asserted by Theorem 6. □

3. Concluding Remarks and Observations

The theory of the basic (or q-) calculus has been applicable in many areas of mathematics and physics such as fractional calculus and quantum physics as described in Srivastava's recently-published survey-cum-expository review article [12]. However, researches on the q-calculus in connection with geometric function theory and, especially, harmonic univalent functions are fairly recent and not much has been published on this topic. Motivated by the recent works [12,38,39], we have made use of the quantum or basic (or q-) calculus to define and investigate new classes of harmonic univalent functions with respect to k-symmetric points, which are associated with a q-analog of the ordinary derivative operator. We have studied here such results as sufficient conditions, representation theorems, distortion theorems, integral operators, and sufficient coefficient bounds. Furthermore, we have highlighted some known consequences of our main results.

Basic (or q-) series and basic (or q-) polynomials, especially the basic (or q-) hypergeometric functions and basic (or q-) hypergeometric polynomials are applicable particularly in several diverse areas of mathematical and physical sciences (see, for example, [41] (pp. 350–351); see also [42–48]). Moreover, as we remarked above and in the introductory Section 1, in Srivastava's recently-published survey-cum-expository review article [12], the triviality of the so-called (p,q)-calculus was exposed and it also mentioned about the trivial and inconsequential variation of the classical q-calculus to the so-called (p,q)-calculus, the additional parameter p being redundant or superfluous $\big($see, for details, [12] (p. 340)$\big)$. Indeed one can apply Srivastava's observation in [12] to any attempt to produce the rather inconsequential and straightforward (p,q)-variations of the q-results which we have presented in this paper.

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