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Oscillation Criteria of Second-Order Dynamic Equations on Time Scales

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Abstract: In this paper, we consider the oscillation behavior of the following second-order nonlinear dynamic equation. $\left(\lambda(s)\Psi\left(\frac{1}{\varphi^{\Delta}(s)}(y(\varphi(s)))^{\Delta}\right)\right)^{\Delta}+\eta(s)\Phi(y(\tau(s)))=0, s\in[s_0,\infty)_{\mathbb{T}}$. By employing generalized Riccati transformation and inequality scaling technique, we establish some oscillation criteria.

Keywords: second order dynamic equation; oscillation; nonlinear equation; Riccati technique; delta derivative

MSC: 26E70; 34K11; 39A10

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1. Introduction

The past decade has witnessed the tremendous development of time scale theory in many fields such as inequality and dynamic equation, which was established by Hilger [1] in 1988. The theory, which unified the representation of discrete and continuous, has received a large amount of attention and studies. For details, we refer to [2–8] .

In general, we cannot obtain analytical solutions of an arbitrarily high order dynamic equation, so the oscillation and asymptotic behavior of solutions is what we often focus on. Dynamic equation has many applications [3,9], and the research of its properties is significant. A great number of researches [10–22] have been done to explore the sufficient conditions which ensure every solution is oscillation in second-order dynamic equations on time scales.

To be specific, Erbe, Hassan and Perterson [13] explored the following equation in 2009,

$$(\lambda(s)(y^{\Delta}(s))^{\gamma})^{\Delta}+\eta(s)(y(\tau(s)))^{\gamma}=0, \qquad s\in[s_0,\infty)_{\mathbb{T}}. \tag{1}$$

Erbe, Perterson and Saker [14] considered the following equation in 2007,

$$(\lambda(s)y^{\Delta}(s))^{\Delta}+\eta(s)\Psi(y(\tau(s)))=0, \qquad s\in[s_0,\infty)_{\mathbb{T}}. \tag{2}$$

In 2008 and 2004, the authors investigated the following equations in [11,12], respectively.

$$(\lambda(s)(y^{\Delta}(s))^{\gamma})^{\Delta}+\eta(s)(y(s))^{\gamma}=0, \qquad s\in[s_0,\infty)_{\mathbb{T}}. \tag{3}$$

$$(\lambda(s)y^{\Delta}(s))^{\Delta}+\eta(s)\Phi(y(\sigma(s)))=0, \qquad s\in[s_0,\infty)_{\mathbb{T}}. \tag{4}$$

In 2017, Agwo, Khodier and Hassan [15] considered the following equation, which provides a general form of the above equations.

$$(\lambda(s)\Psi(y^{\Delta}(s)))^{\Delta}+\eta(s)\Phi(y(\tau(s)))=0, \qquad s\in[s_0,\infty)_{\mathbb{T}}. \tag{5}$$

In this paper, we focus on the following second-order nonlinear dynamic equation:

$$\Big(\lambda(s)\Psi\big(\frac{1}{\varphi^{\Delta}(s)}\big(y(\varphi(s))\big)^{\Delta}\big)\Big)^{\Delta}+\eta(s)\Phi(y(\tau(s)))=0,\qquad s\in[s_0,\infty)_{\mathbb{T}},\tag{6}$$

which gives a more unified form of the equations in the above reference, where functions λ, Ψ, φ, η, Φ and τ are defined in hypotheses in Section 2, and Δ means the delta derivative on time scales (see [2]).

Remark 1. *If $\varphi(s)=s$, then Equation (6) transforms into (5); if $\varphi(s)=s$, $\Psi(s)=\Phi(s)=s^{\gamma}$, where $\gamma=\frac{2m+1}{2n+1}, m,n\in\mathbb{N}_+$, then Equation (6) transforms into (1); if $\varphi(s)=s$, $\Psi(s)=s$, then Equation (6) transforms into (2); if $\varphi(s)=s$, $\Psi(s)=\Phi(s)=s^{\gamma}$ and $\tau(s)=s$, where $\gamma=\frac{2m+1}{2n+1}, m,n\in\mathbb{N}_+$, then Equation (6) transforms into (3); if $\varphi(s)=s$, $\Psi(s)=s$ and $\tau(s)=\sigma(s)$, then Equation (6) transforms into (4).*

We will establish two kinds of oscillation criteria via different methods, respectively.

Next section is organized as follows, some lemmas and propositions, which are helpful for the proof of Theorems 1, 2, and 3, are introduced firstly. Secondly, we will list some hypotheses to simplify our statement. Furthermore, then Theorem 1, which explored some qualities when Equation (6) has a positive solution, will be established. Finally, we obtain two kinds of oscillation criteria in Theorems 2 and 3 by employing Theorem 1, respectively.

2. Main Results

To complete the proof of the desired Theorems 1, 2 and 3, we need the following lemmas and propositions, which can be found in [2,23].

Definition 1. *For $s\in\mathbb{T}$, forward jump operator $\sigma\colon\mathbb{T}\to\mathbb{T}$ is defined by*

$$\sigma(s)=\inf\{l\in\mathbb{T}:l>s\}.$$

Definition 2. *If $\varphi\colon\mathbb{T}\to\mathbb{R}$ and for all given $\epsilon>0$ there exists a neighborhood U and $\varphi^{\Delta}(s)$ of s such that*

$$\Big|\varphi(\sigma(s))-\varphi(r)-\varphi^{\Delta}(s)(\sigma(s)-r)\Big|\le\epsilon|\sigma(s)-r|,$$

for all $r\in U$, then φ is called delta differentiable on $\mathbb{T}^k$, where $\mathbb{T}^k$ is defined as follows:

$$\mathbb{T}^k=\begin{cases}\mathbb{T}\setminus(\rho(\sup\mathbb{T}),\sup\mathbb{T}), & \text{if }\sup\mathbb{T}<\infty,\\ \mathbb{T}, & \text{if }\sup\mathbb{T}=\infty.\end{cases}$$

Moreover, for convenience, we denote $f^{\sigma}(s)$ as $f(\sigma(s))$.

Proposition 1 ([2] (Corollary 2.47))**.** *Suppose $\psi\in C([\alpha,\beta))$ is delta derivative, then ψ is increasing (decreasing) if and only if $\psi^{\Delta}(s)\ge 0(\le 0)$ for all $s\in[\alpha,\beta)$.*

Lemma 1 ([2] (Theorem 2.57))**.** *Suppose function f is continuous and function $g\colon\mathbb{T}\to\mathbb{R}$ is delta-differentiable. Then, $f(g(\cdot))$ is delta-differentiable with*

$$(f(g(s)))^{\Delta}=\Big(\int_0^1 f'(g(s)+t\mu(s)g^{\Delta}(s))\mathrm{d}t\Big)g^{\Delta}(s).$$

Lemma 2 ([2] (Theorem 2.62))**.** *Suppose $g\colon\mathbb{T}\to\mathbb{R}$ is strictly increasing, $f\colon\tilde{\mathbb{T}}\to\mathbb{R}$, where $\tilde{\mathbb{T}}=g(\mathbb{T})$ is a time scale. If $g^{\Delta}(s)$ and $f^{\tilde{\Delta}}(g(s))$ exist for all $s\in\mathbb{T}^k$, then*

$$(f(g(s)))^{\Delta}=f^{\tilde{\Delta}}(g(s))g^{\Delta}(s).$$

Some necessary hypotheses, in order to simplify the statement, are listed as follows before we give the theorems.

Hypothesis 1 (H1). *$\mathbb{T}$ is an unbounded time scale, $s_0 \in \mathbb{T}$, we write $[s_0, \infty) \cap \mathbb{T}$ as $[s_0, \infty)_{\mathbb{T}}$ and denote $\big(y(\varphi(s))\big)^{\Delta}$ as $y(\varphi(s))^{\Delta}$.*

Hypothesis 2 (H2). *Function Ψ defined on $\mathbb{R}$ is odd, continuous, increasing and has inverse function Ψ^{-1}. Exist positive constants K_1, K_2, function Ψ meets the conditions below for all $s, s_1, s_2 \in [s_0, \infty)_{\mathbb{T}}$*

(1) $s\Psi(s) > 0$;

(2) $s\Psi^{-1}(s) > 0$;

(3) $\Psi(s_1 s_2) \geq K_1 \Psi(s_1)\Psi(s_2)$;

(4) $\Psi^{-1}(s_1 s_2) \leq K_2 \Psi^{-1}(s_1)\Psi^{-1}(s_2)$.

Hypothesis 3 (H3). *Function Φ is continuous, increasing with $s\Phi(s) > 0$ and has $L > 0$ subject to $\frac{s\Phi^{\Delta}(s)}{\Phi(s)} \geq L$ for all $s \in [s_0, \infty)_{\mathbb{T}}$;*

Hypothesis 4 (H4). *$\lambda, \eta : \mathbb{T} \to \mathbb{R}^{+}$ is a monotonically increasing function, φ is positive increasing function with $\int_{s_0}^{\infty} \varphi^{\Delta}(t)\Phi^{-1}(\frac{1}{\lambda(t)})\Delta t = \infty$ and compound function $\varphi^{-1}(\tau(\cdot))$ exists.*

Hypothesis 5 (H5). *$\tau : \mathbb{T} \to \mathbb{T}$ is increasing function with $\lim\limits_{s\to\infty} \tau(s) = \infty$.*

Hypothesis 6 (H6). *$\int_{s_0}^{\infty} \eta(t)\Phi(\tau(t))\Delta t = \infty$ and inverse function $\varphi^{-1}(\tau(\cdot))$ is increasing.*

Hypothesis 7 (H7). *exists an $N > 0$ subject to $\Phi(s_1 s_2) \geq N\Phi(s_1)\Phi(s_2)$ for all $s_1, s_2 \in [s_0, \infty)_{\mathbb{T}}$.*

In fact, based on the Lemma 2, we have

$$\big(y(\varphi(s))\big)^{\Delta} = y^{\tilde{\Delta}}(\varphi(s))\varphi^{\Delta}(s),$$

where $\tilde{\Delta}$ is the Delta derivative on $\tilde{\mathbb{T}} = \varphi(\mathbb{T})$. Hence Equation (6) has another form.

$$\Big(\lambda(s)\Psi\big(y^{\tilde{\Delta}}(\varphi(s))\big)\Big)^{\Delta} + \eta(s)\Phi(y(\tau(s))) = 0, \qquad s \in [s_0, \infty)_{\mathbb{T}}.$$

The following theorem, which explored some qualities of Equation (6) under the assumption that has a positive solution on $[s_0, \infty)_{\mathbb{T}}$, is foundational in this paper.

Theorem 1. *Assume (H1)–(H5) hold and Equation (6) has a solution $y(s) > 0$ on $[s_0, \infty)_{\mathbb{T}}$. Then, there exists an S subject to the following hold for all $s \in [S, \infty)_{\mathbb{T}}$.*

(1) $\lambda(s)\Psi\big(\frac{1}{\varphi^{\Delta}(s)} y(\varphi(s))^{\Delta}\big)$ *is strictly decreasing, namely,* $\Big(\lambda(s)\Psi\big(\frac{1}{\varphi^{\Delta}(s)} y(\varphi(s))^{\Delta}\big)\Big)^{\Delta} < 0$;

(2) $y(\varphi(s))$ *is increasing, namely,* $y(\varphi(s))^{\Delta} \geq 0$;

(3) $\frac{y(\varphi(s))^{\Delta}}{\varphi^{\Delta}(s)} = y^{\tilde{\Delta}}(\varphi(s))$ *is decreasing, namely,* $\big(\frac{y(\varphi(s))^{\Delta}}{\varphi^{\Delta}(s)}\big)^{\Delta} < 0$ *and* $\big(y^{\tilde{\Delta}}(\varphi(s))\big)^{\Delta} < 0$; *and* $\frac{y(\varphi(s))^{\Delta\Delta}}{y(\varphi(s))^{\Delta}} < \frac{\varphi^{\Delta\Delta}(s)}{\varphi^{\Delta}(s)}$;

(4) $y(\varphi(s)) \geq R(s)y(\varphi(s))^{\Delta}$ *where*

$$R(s) = \frac{1}{K_2^2 \varphi^{\Delta}(s)\Psi^{-1}(\frac{1}{\lambda(s)})} \int_{s_0}^{s} \frac{\varphi^{\Delta}(t)}{\Psi^{-1}(\lambda(t))}\Delta t;$$

(5) *if (H6) and (H7) hold, then*

$$y(\varphi(s)) \geq \frac{\varphi(s)}{\varphi^{\Delta}(s)} y(\varphi(s))^{\Delta}.$$

Proof. (1). Since $y(s) > 0$ on $[s_0, \infty)_{\mathbb{T}}$, we have s_1 such that $y(\tau(s)) > 0$ on $[s_1, \infty)_{\mathbb{T}}$. Then,

$$\Big(\lambda(s)\Psi\big(\frac{1}{\varphi^{\Delta}(s)} y(\varphi(s))^{\Delta}\big)\Big)^{\Delta} = -\eta(s)\Phi(y(\tau(s))) < 0, \qquad s \in [s_1, \infty)_{\mathbb{T}},$$

from which we can deduce $\lambda(s)\Psi\big(\frac{1}{\varphi^{\Delta}(s)} y(\varphi(s))^{\Delta}\big)$ is strictly decreasing.

(2). To establish the desired conclusion, we assume it does not hold which means there exists an s_2 such that $y(\varphi(s))^{\Delta} < 0$ on $[s_2, \infty)_{\mathbb{T}}$. Base on the fact that

$$\Big(\lambda(s)\Psi\big(\frac{1}{\varphi^{\Delta}(s)} y(\varphi(s))^{\Delta}\big)\Big)^{\Delta} < 0,$$

we obtain

$$\lambda(s)\Psi\big(\frac{1}{\varphi^{\Delta}(s)} y(\varphi(s))^{\Delta}\big) < \lambda(s_2)\Psi\big(\frac{1}{\varphi^{\Delta}(s_2)} y(\varphi(s_2))^{\Delta}\big) := l_1 < 0, \qquad s \in (s_2, \infty)_{\mathbb{T}}.$$

According to the Hypothesis (H2), we yield

$$y(\varphi(s))^{\Delta} < \varphi^{\Delta}(s)\Psi^{-1}\big(\frac{l_1}{\lambda(s)}\big) \leq \varphi^{\Delta}(s) K_2 \Psi^{-1}(l_1)\Psi^{-1}\big(\frac{1}{\lambda(s)}\big), \qquad s \in (s_2, \infty)_{\mathbb{T}}.$$

Delta integrate from s_2 to s on both sides arrive at

$$y(\varphi(s)) < y(\varphi(s_2)) + K_2\Psi^{-1}(l_1) \int_{s_2}^{s} \varphi^{\Delta}(t)\Psi^{-1}\big(\frac{1}{\lambda(t)}\big)\Delta t, \qquad s \in (s_2, \infty)_{\mathbb{T}}.$$

Noting that $\Psi^{-1}(l_1) < 0$ and $\int_{s_0}^{\infty} \varphi^{\Delta}(t)\Psi^{-1}\big(\frac{1}{\lambda(t)}\big)\Delta t = \infty$, hence the contradiction can be concluded.

Moreover, based on the Lemma 2, we obtain $y(\varphi(s))^{\Delta} = y^{\tilde{\Delta}}(\varphi(s))\varphi^{\Delta}(s)$. Therefore, we can also find that $y^{\tilde{\Delta}}(\varphi(s)) > 0$ according to the fact that $\varphi^{\Delta}(s) > 0$.

(3). Noting that

$$\begin{aligned} &\Big(\lambda(s)\Psi\big(\frac{1}{\varphi^{\Delta}(s)} y(\varphi(s))^{\Delta}\big)\Big)^{\Delta} \\ = \ &\lambda^{\Delta}(s)\Psi\big(\frac{1}{\varphi^{\Delta}(s)} y(\varphi(s))^{\Delta}\big) + \lambda^{\sigma}(s)\Big(\Psi\big(\frac{1}{\varphi^{\Delta}(s)} y(\varphi(s))^{\Delta}\big)\Big)^{\Delta} < 0, \end{aligned}$$

thus

$$\lambda^{\sigma}(s)\Big(\Psi\big(\frac{1}{\varphi^{\Delta}(s)} y(\varphi(s))^{\Delta}\big)\Big)^{\Delta} < -\lambda^{\Delta}(s)\Psi\big(\frac{1}{\varphi^{\Delta}(s)} y(\varphi(s))^{\Delta}\big) \leq 0.$$

The conditions in Lemma 1 hold for $\Psi\big(\frac{1}{\varphi^{\Delta}(s)} y(\varphi(s))^{\Delta}\big)$, thus

$$\begin{aligned} &\Big(\Psi\big(\frac{1}{\varphi^{\Delta}(s)} y(\varphi(s))^{\Delta}\big)\Big)^{\Delta} \\ = \ &\Big(\int_0^1 \Psi'\big(\frac{1}{\varphi^{\Delta}(s)} y(\varphi(s))^{\Delta} + t\mu(s)\big(\frac{y(\varphi(s))^{\Delta}}{\varphi^{\Delta}(s)}\big)^{\Delta}\big)\mathrm{d}t\Big)\big(\frac{y(\varphi(s))^{\Delta}}{\varphi^{\Delta}(s)}\big)^{\Delta}. \end{aligned}$$

Since function Ψ is increasing, namely, $\int_0^1 \Psi'(\frac{1}{\varphi^\Delta(s)}y(\varphi(s))^\Delta + t\mu(s)(\frac{y(\varphi(s))^\Delta}{\varphi^\Delta(s)})^\Delta)\mathrm{d}t > 0$, we have

$$\lambda^\sigma(s)\Big(\int_0^1 \Psi'(\frac{1}{\varphi^\Delta(s)}y(\varphi(s))^\Delta + t\mu(s)(\frac{y(\varphi(s))^\Delta}{\varphi^\Delta(s)})^\Delta)\mathrm{d}t\Big)(\frac{y(\varphi(s))^\Delta}{\varphi^\Delta(s)})^\Delta < 0,$$

which can deduce that $(\frac{y(\varphi(s))^\Delta}{\varphi^\Delta(s)})^\Delta < 0$ for all $s \in [s_0, \infty)_{\mathbb{T}}$. Using the delta quotient rule, we have

$$(\frac{y(\varphi(s))^\Delta}{\varphi^\Delta(s)})^\Delta = \frac{y(\varphi(s))^{\Delta\Delta}\varphi^\Delta(s) - y(\varphi(s))^\Delta\varphi^{\Delta\Delta}(s)}{\varphi^\Delta(s)\varphi^\Delta(\sigma(s))} < 0,$$

so we also have $\frac{y(\varphi(s))^{\Delta\Delta}}{y(\varphi(s))^\Delta} < \frac{\varphi^{\Delta\Delta}(s)}{\varphi^\Delta(s)}$.

(4). We want to determine a lower bound of $y(\varphi(s))/y(\varphi(s))^\Delta$. By employing the Hypothesis (H2), we have

$$\begin{aligned} & \Psi^{-1}\Big(\lambda(s)\Psi(\frac{1}{\varphi^\Delta(s)}y(\varphi(s))^\Delta)\Big) \\ \leq\ & K_2\Psi^{-1}(\lambda(s))\Psi^{-1}\Big(\Psi(\frac{1}{\varphi^\Delta(s)}y(\varphi(s))^\Delta)\Big) = \frac{K_2\Psi^{-1}(\lambda(s))}{\varphi^\Delta(s)}y(\varphi(s))^\Delta, \end{aligned}$$

namely,

$$y(\varphi(s))^\Delta \geq \frac{\varphi^\Delta(s)\Psi^{-1}\Big(\lambda(s)\Psi(\frac{1}{\varphi^\Delta(s)}y(\varphi(s))^\Delta)\Big)}{K_2\Psi^{-1}(\lambda(s))}.$$

Delta integrates both sides from s_0 to s, along with the fact that $\lambda(s)\Psi(\frac{1}{\varphi^\Delta(s)}y(\varphi(s))^\Delta)$ decreases and $\Psi^{-1}(s)$ increases, yields

$$\begin{aligned} y(\varphi(s)) - y(\varphi(s_0)) &\geq \int_{s_0}^s \frac{\varphi^\Delta(t)\Psi^{-1}\Big(\lambda(t)\Psi(\frac{1}{\varphi^\Delta(t)}y(\varphi(t))^\Delta)\Big)}{K_2\Psi^{-1}(\lambda(t))}\Delta t \\ &\geq \frac{\Psi^{-1}\Big(\lambda(s)\Psi(\frac{1}{\varphi^\Delta(s)}y(\varphi(s))^\Delta)\Big)}{K_2}\int_{s_0}^s \frac{\varphi^\Delta(t)}{\Psi^{-1}(\lambda(t))}\Delta t. \end{aligned} \tag{7}$$

Noting that

$$\begin{aligned} \frac{y(\varphi(s))^\Delta}{\varphi^\Delta(s)} &= \Psi^{-1}\Big(\frac{1}{\lambda(s)}\lambda(s)\Psi(\frac{1}{\varphi^\Delta(s)}y(\varphi(s))^\Delta)\Big) \\ &\leq K_2\Psi^{-1}(\frac{1}{\lambda(s)})\Psi^{-1}(\lambda(s)\Psi(\frac{1}{\varphi^\Delta(s)}y(\varphi(s))^\Delta)), \end{aligned}$$

namely,

$$\Psi^{-1}(\lambda(s)\Psi(\frac{1}{\varphi^\Delta(s)}y(\varphi(s))^\Delta)) \geq \frac{y(\varphi(s))^\Delta}{K_2\varphi^\Delta(s)\Psi^{-1}(\frac{1}{\lambda(s)})}. \tag{8}$$

Substitute (8) into (7) and note that $y(s_0) > 0$, immediately we get

$$y(\varphi(s)) \geq y(\varphi(s))^\Delta \frac{1}{K_2^2\varphi^\Delta(s)\Psi^{-1}(\frac{1}{\lambda(s)})}\int_{s_0}^s \frac{\varphi^\Delta(t)}{\Psi^{-1}(\lambda(t))}\Delta t = y(\varphi(s))^\Delta R(s).$$

(5). We can add some conditions to get a more concise lower bound of the function $y(\varphi(s))/y(\varphi(s))^\Delta$. Set

$$F(s) = y(\varphi(s)) - \frac{\varphi(s)}{\varphi^\Delta(s)}y(\varphi(s))^\Delta.$$

We claim that there exists an s_3 such that $F(s) \geq 0$ on $[s_3, \infty)$, if not, exists an s_4 such that $F(s) < 0$ on $[s_4, \infty)$. We investigate the monotony of $y(\varphi(s))/\varphi(s)$,

$$\Big(\frac{y(\varphi(s))}{\varphi(s)}\Big)^{\Delta} = \frac{y(\varphi(s))^{\Delta}\varphi(s) - y(\varphi(s))\varphi^{\Delta}(s)}{\varphi(s)\varphi^{\sigma}(s)} = -\frac{F(s)\varphi^{\Delta}(s)}{\varphi(s)\varphi^{\sigma}(s)} > 0,$$

which means $y(\varphi(s))/\varphi(s)$ is increasing on $[s_4, \infty)$. Since $\tau(s)$ is increasing, there exists an s_5 such that $\tau(s_5) > s_4$ and $y(\tau(s)) \geq l_2\tau(s)$ for $s \in [s_5, \infty)$, where $l_2 = \frac{y(\tau(s_4))}{\tau(s_4)}$.

Delta integrates Equation (6) from s_5 to ∞ on both sides, we yield

$$\lim_{s\to\infty} \lambda(s)\Psi\big(\frac{1}{\varphi^{\Delta}(s)}y(\varphi(s))^{\Delta}\big) - \lambda(s_5)\Psi\big(\frac{1}{\varphi^{\Delta}(s_5)}y(\varphi(s_5))^{\Delta}\big) + \int_{s_5}^{\infty} \eta(t)\Phi(y(\tau(t)))\Delta t = 0,$$

hence we have

$$\begin{aligned}
\lambda(s_5)\Psi\big(\frac{1}{\varphi^{\Delta}(s_5)}y(\varphi(s_5))^{\Delta}\big) &> \int_{s_5}^{\infty} \eta(t)\Phi(y(\tau(t)))\Delta t \geq \int_{s_5}^{\infty} \eta(t)\Phi(l_2\tau(t))\Delta t \\
&\geq \int_{s_5}^{\infty} \eta(t)N\Phi(l_2)\Phi(\tau(t))\Delta t \\
&= N\Phi(l_2)\int_{s_5}^{\infty} \eta(t)\Phi(\tau(t))\Delta t = \infty,
\end{aligned}$$

and a contradiction is obtained. Hence, $F(s) \geq 0$, which means $y(\varphi(s)) \geq \frac{\varphi(s)}{\varphi^{\Delta}(s)}y(\varphi(s))^{\Delta}$ and $y(\tau(s)) \leq \frac{y(\tau(s_3))}{\tau(s_3)}\tau(s)$ on $[s_3, \infty)_{\mathbb{T}}$. □

Theorems 2 and 3 give an oscillation criteria of Equation (6) by employing some conclusions in Theorem 1, respectively.

Theorem 2. *Assume (H1)–(H5) hold and exists a Δ-differentiable function $\delta\colon \mathbb{R} \to \mathbb{R}^+$ satisfies*

$$\int_{s_0}^{\infty} \delta(t)\eta(t)e_{-C(t)}(t, s_0)\Delta t = \infty, \tag{9}$$

where

$$C(s) = \frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)} - \frac{L\delta(s)}{\delta^{\sigma}(s)B(\varphi^{-1}(\tau(s)))},$$

and

$$B(s) = y(\varphi(s_0)) + K_2\Psi^{-1}(l_1)\int_{s_0}^{s} \varphi^{\Delta}(t)\Psi^{-1}\big(\frac{1}{\lambda(t)}\big)\Delta t.$$

Then, Equation (6) is oscillatory.

Proof. We can assume, without loss of generality, that exists a positive solution $y(s)$ of Equation (6). If $y(s)$ is negative, we can take $y^*(s) = -y(s)$, where $y^*(s)$ is a positive solution of Equation (6). Take

$$\omega(s) = \delta(s)\frac{\lambda(s)\Psi\big(\frac{1}{\varphi^{\Delta}(s)}y(\varphi(s))^{\Delta}\big)}{\Phi(y(\tau(s)))}.$$

Then, we obtain

$$
\begin{aligned}
\omega^{\Delta}(s) &= \Big(\frac{\delta(s)}{\Phi(y(\tau(s)))}\Big)^{\Delta}\lambda(\sigma(s))\Psi(\frac{1}{\varphi^{\Delta}(\sigma(s))}y(\varphi(\sigma(s)))^{\Delta}) \\
&+ \frac{\delta(s)}{\Phi(y(\tau(s)))}(\lambda(s)\Psi(\frac{1}{\varphi^{\Delta}(s)}y(\varphi(s))^{\Delta}))^{\Delta} \\
&= -\delta(s)\eta(s) + \lambda(\sigma(s))\Psi(\frac{1}{\varphi^{\Delta}(\sigma(s))}y(\varphi(\sigma(s)))^{\Delta}) \\
&\quad \Big(\frac{\delta^{\Delta}(s)\Phi(y(\tau(s))) - \delta(s)\Big(\Phi(y(\tau(s)))\Big)^{\Delta}}{\Phi(y(\tau(s)))\Phi(y(\tau(\sigma(s))))}\Big) \\
&= -\delta(s)\eta(s) + \omega^{\sigma}(s)\Big(\frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)} - \frac{\delta(s)\big(\Phi(y(\tau(s)))\big)^{\Delta}}{\delta^{\sigma}(s)\Phi(y(\tau(s)))}\Big) \\
&\leq -\delta(s)\eta(s) + \omega^{\sigma}(s)\Big(\frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)} - \frac{L\delta(s)}{\delta^{\sigma}(s)y(\tau(s))}\Big).
\end{aligned}
$$

As we have proved in Theorem 1 (2), we have

$$
y(\varphi(s))^{\Delta} < \varphi^{\Delta}(s)\Psi^{-1}(\frac{l_1}{\lambda(s)}) \leq K_2\varphi^{\Delta}(s)\Psi^{-1}(l_1)\Psi^{-1}(\frac{1}{\lambda(s)}), \qquad s \in (s_0,\infty)_{\mathbb{T}},
$$

Delta integrate from s_0 to s on both sides, and we get

$$
y(\varphi(s)) - y(\varphi(s_0)) < K_2\Psi^{-1}(l_1)\int_{s_0}^{s}\varphi^{\Delta}(t)\Psi^{-1}(\frac{1}{\lambda(t)})\Delta t,
$$

then $y(\varphi(s))$ has an upper bound $B(s)$ and $y(\tau(s)) \leq B(\varphi^{-1}(\tau(s)))$.
Hence, we get

$$
\omega^{\Delta}(s) \leq -\delta(s)\eta(s) + \omega^{\sigma}(s)\Big(\frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)} - \frac{L\delta(s)}{\delta^{\sigma}(s)B(\varphi^{-1}(\tau(s)))}\Big) = -\delta(s)\eta(s) + \omega^{\sigma}(s)C(s).
$$

Noting that

$$
\begin{aligned}
\Big(\omega(s)e_{-C(s)}(s,s_0)\Big)^{\Delta} &= \omega^{\Delta}(s)e_{-C(s)}(s,s_0) - C(s)\omega^{\sigma}(s)e_{-C(s)}(s,s_0) \\
&= \Big(\omega^{\Delta}(s) - C(s)\omega^{\sigma}(s)\Big)e_{-C(s)}(s,s_0) \leq -\delta(s)\eta(s)e_{-C(s)}(s,s_0),
\end{aligned}
$$

Delta integrate from s_0 to s and letting $s \to \infty$, and we have

$$
\lim_{s\to\infty}\omega(s)e_{-C(s)}(s,s_0) - \omega(s_0)e_{-C(s)}(s_0,s_0) + \int_{s_0}^{\infty}\delta(t)\eta(t)e_{-C(t)}(t,s_0)\Delta t \leq 0,
$$

which is a contradiction based on the condition (9), where exponential function $e_f(t,s)$ is defined by

$$
e_f(t,s) = \exp\Big(\int_s^t \frac{1}{\mu(\eta)}\ln\big(1 + f(\eta)\mu(\eta)\big)\Delta\eta\Big).
$$

Hence, we complete the proof. □

By changing the method we deal with $\omega^{\Delta}(s)$, we can get another oscillation criteria.

Theorem 3. *Suppose (H1)–(H7), $\tau(\sigma(s)) = \sigma(\tau(s))$, $\varphi(\sigma(s)) = \sigma(\varphi(s))$ hold, there exists a Δ-differentiable function $\delta\colon \mathbb{R} \to \mathbb{R}^+$ satisfies*

$$\int_{s_0}^{\infty} \frac{E(t)}{1 + D(t)\mu(t)} e_{-\frac{D(t)}{1+D(t)\mu(t)}}(t, s_0)\Delta t = -\infty, \tag{10}$$

and $D(s) > 0$, where

$$D(s) = \frac{\delta^\Delta(s)}{\delta(s)} - \frac{L\delta^\sigma(s)}{l_2\delta(s)\tau(s)},$$

$$E(s) = -N\delta^\sigma(s)\eta(s)\Phi\big(H(s)\big),$$

and

$$H(s) = \frac{\tau(s)}{\varphi^\Delta(\varphi^{-1}(\tau(s)))\mu(\varphi^{-1}(\tau(s))) + \tau(s)}.$$

Then Equation (6) is oscillatory.

Proof. We assume that there exists a solution $y(s)$ of Equation (6) is eventually positive, and take

$$\omega(s) = \delta(s)\frac{\lambda(s)\Psi(\frac{1}{\varphi^\Delta(s)}y(\varphi(s))^\Delta)}{\Phi(y(\tau(s)))},$$

then

$$\begin{aligned}
\omega^\Delta(s) &= \delta^\Delta(s)\frac{\lambda(s)\Psi(\frac{1}{\varphi^\Delta(s)}y(\varphi(s))^\Delta)}{\Phi(y(\tau(s)))} + \delta^\sigma(s)\Big(\frac{\lambda(s)\Psi(\frac{1}{\varphi^\Delta(s)}y(\varphi(s))^\Delta)}{\Phi(y(\tau(s)))}\Big)^\Delta \\
&= \frac{\delta^\Delta(s)}{\delta(s)}\omega(s) + \delta^\sigma(s)\Big(\frac{(\lambda(s)\Psi(\frac{1}{\varphi^\Delta(s)}y(\varphi(s))^\Delta))^\Delta\Phi(y(\tau(s)))}{\Phi(y(\tau(s)))\Phi(y(\tau(\sigma(s))))} \\
&\quad - \frac{\lambda(s)\Psi(\frac{1}{\varphi^\Delta(s)}y(\varphi(s))^\Delta)(\Phi(y(\tau(s))))^\Delta}{\Phi(y(\tau(s)))\Phi(y(\tau(\sigma(s))))}\Big) \\
&= \frac{\delta^\Delta(s)}{\delta(s)}\omega(s) - \delta^\sigma(s) \times \\
&\quad \Big(\frac{\eta(s)\Phi(y(\tau(s)))}{\Phi(y(\tau(\sigma(s))))} + \frac{\lambda(s)\Psi(\frac{1}{\varphi^\Delta(s)}y(\varphi(s))^\Delta)(\Phi(y(\tau(s))))^\Delta}{\Phi(y(\tau(s)))\Phi(y(\tau(\sigma(s))))}\Big).
\end{aligned}$$

Based on conclusion (5) in Theorem 1, we have

$$\frac{y(\varphi(s))}{y(\varphi(s))^\Delta} = \frac{y(\varphi(s))}{\frac{y(\varphi(\sigma(s))) - y(\varphi(s))}{\mu(s)}} \geq \frac{\varphi(s)}{\varphi^\Delta(s)},$$

namely,

$$\frac{y(\varphi(s)))}{y(\varphi(\sigma(s)))} \geq \frac{\varphi(s)}{\varphi^\Delta(s)\mu(s) + \varphi(s)}.$$

Since $\tau(\sigma(s)) = \sigma(\tau(s))$ and $\varphi(\sigma(s)) = \sigma(\varphi(s))$, we have

$$\begin{aligned}
\frac{y(\tau(s))}{y(\tau(\sigma(s)))} &= \frac{y(\tau(s))}{y(\sigma(\tau(s)))} \\
&= \frac{y(\varphi(\varphi^{-1}(\tau(s))))}{y(\sigma(\varphi(\varphi^{-1}(\tau(s)))))} \\
&= \frac{y(\varphi(\varphi^{-1}(\tau(s))))}{y(\varphi(\sigma(\varphi^{-1}(\tau(s)))))} \\
&\geq \frac{\varphi(\varphi^{-1}(\tau(s)))}{\varphi^{\Delta}(\varphi^{-1}(\tau(s)))\mu(\varphi^{-1}(\tau(s))) + \varphi(\varphi^{-1}(\tau(s)))} \\
&= \frac{\tau(s)}{\varphi^{\Delta}(\varphi^{-1}(\tau(s)))\mu(\varphi^{-1}(\tau(s))) + \tau(s)} := H(s).
\end{aligned}$$

Whereupon, we have

$$\frac{\Phi(y(\tau(s)))}{\Phi(y(\tau^{\sigma}(s)))} \geq \frac{\Phi(H(s)y(\tau^{\sigma}(s)))}{\Phi(y(\tau^{\sigma}(s)))} \geq N\Phi(H(s)). \tag{11}$$

Consequently, using (11) and the Hypothesis (H6), we have

$$\begin{aligned}
\omega^{\Delta}(s) &\leq \frac{\delta^{\Delta}(s)}{\delta(s)}\omega(s) - \delta^{\sigma}(s)\Big(N\eta(s)\Phi(H(s)) \\
&\quad + \frac{\lambda(s)\Psi(\frac{1}{\varphi^{\Delta}(s)}y(\varphi(s))^{\Delta})(\Phi(y(\tau(s))))^{\Delta}}{\Phi(y(\tau(s)))\Phi(y(\tau(\sigma(s))))}\Big) \\
&\leq \frac{\delta^{\Delta}(s)}{\delta(s)}\omega(s) - \delta^{\sigma}(s)\Big(N\eta(s)\Phi(H(s)) + \frac{L\lambda(s)\Psi(\frac{1}{\varphi^{\Delta}(s)}y(\varphi(s))^{\Delta})}{y(\tau(s))\Phi(y(\tau(\sigma(s))))}\Big) \\
&= \frac{\delta^{\Delta}(s)}{\delta(s)}\omega(s) - \delta^{\sigma}(s)\Big(N\eta(s)\Phi(H(s)) + \frac{L\omega(s)\Phi(y(\tau(s)))}{\delta(s)y(\tau(s))\Phi(y(\tau(\sigma(s))))}\Big) \\
&\leq \frac{\delta^{\Delta}(s)}{\delta(s)}\omega(s) - \delta^{\sigma}(s)\Big(N\Phi(H(s))(\eta(s) + \frac{L\omega(s)}{\delta(s)y(\tau(s))})\Big).
\end{aligned}$$

Based on Theorem 1 (5), there exists s_3 such that the function

$$F(s) = y(\varphi(s)) - \frac{\varphi(s)}{\varphi^{\Delta}(s)}y(\varphi(s))^{\Delta},$$

is non-negative on $[s_3, \infty)$. Then direct calculations show that

$$\Big(\frac{y(\varphi(s))}{\varphi(s)}\Big)^{\Delta} = \frac{y(\varphi(s))^{\Delta}\varphi(s) - y(\varphi(s))\varphi^{\Delta}(s)}{\varphi(s)\varphi^{\sigma}(s)} = -\frac{F(s)}{\varphi(s)\varphi^{\Delta}(s)\varphi^{\sigma}(s)} \leq 0,$$

namely, $\frac{y(\varphi(s))}{\varphi(s)}$ is decreasing on $[s_3, \infty)$. Thus, $y(\tau(s)) \leq l_2\tau(s)$ holds on $[s_3, \infty)$, where $l_2 = \frac{y(\tau(s_3))}{\tau(s_3)}$.

We can arrive at

$$\omega^{\Delta}(s) \leq \frac{\delta^{\Delta}(s)}{\delta(s)}\omega(s) - \delta^{\sigma}(s)\Big(N\Phi(H(s))(\eta(s) + \frac{L\omega(s)}{\delta(s)l_2\tau(s)})\Big) = D(s)\omega(s) + E(s),$$

namely,

$$\omega^{\Delta}(s) \leq \frac{D(s)}{1 + D(s)\mu(s)}\omega^{\sigma}(s) + \frac{E(s)}{1 + D(s)\mu(s)}, \qquad s \in [s_3, \infty)_{\mathbb{T}}.$$

Noting that

$$\begin{aligned}
& \left(\omega(s)e_{-\frac{D(s)}{1+D(s)\mu(s)}}(s,s_0)\right)^{\Delta} \\
= \;& \omega^{\Delta}(s)e_{-\frac{D(s)}{1+D(s)\mu(s)}}(s,s_0) - \frac{D(s)}{1+D(s)\mu(s)}\omega^{\sigma}(s)e_{-\frac{D(s)}{1+D(s)\mu(s)}}(s,s_0) \\
= \;& \left(\omega^{\Delta}(s) - \frac{D(s)}{1+D(s)\mu(s)}\omega^{\sigma}(s)\right)e_{-\frac{D(s)}{1+D(s)\mu(s)}}(s,s_0) \\
\leq \;& \frac{E(s)}{1+D(s)\mu(s)}e_{-\frac{D(s)}{1+D(s)\mu(s)}}(s,s_0),
\end{aligned}$$

integrating from s_3 to s and letting $s \to \infty$, it yields

$$\begin{aligned}
& \lim_{s\to\infty} \omega(s)e_{-\frac{D(s)}{1+D(s)\mu(s)}}(s,s_3) - \omega(s_3)e_{-\frac{D(s)}{1+D(s)\mu(s)}}(s_3,s_3) \\
- & \int_{s_3}^{\infty} \frac{E(t)}{1+D(t)\mu(t)}e_{-\frac{D(t)}{1+D(t)\mu(t)}}(t,s_3)\Delta t \leq 0,
\end{aligned}$$

which is a contradiction according to (10). □

3. Conclusions

In this paper, we explore a more general second-order nonlinear dynamic equation. Some oscillation criteria are established by using generalized Riccati transformation. Our work has greatly promoted the development of dynamic equations on time scales.

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