mathematics

Article

Covariance Principle for Capital Allocation: A Time-Varying Approach

Jilber Urbina [1], Miguel Santolino [2] and Montserrat Guillen [2,*]

[1] Banco de la Producción, Managua 13077, Nicaragua; JilberUrbina@banpro.com.ni
[2] Department of Econometrics, Riskcenter-IREA, University of Barcelona, 08034 Barcelona, Spain; msantolino@ub.edu
* Correspondence: mguillen@ub.edu; Tel.: +34-9340-20484

Abstract: The covariance allocation principle is one of the most widely used capital allocation principles in practice. Risks change over time, so capital risk allocations should be time-dependent. In this paper, we propose a dynamic covariance capital allocation principle based on the variance-covariance of risks that change over time. The conditional correlation of risks is modeled by means of a dynamic conditional correlation (DCC) model. Unlike the static approach, we show that in our dynamic capital allocation setting, the distribution of risk capital allocations can be estimated, and the expected future allocations of capital can be predicted, providing a deeper understanding of the stochastic multivariate behavior of risks. The methodology presented in the paper is illustrated with an example involving the investment risk in a stock portfolio.

Keywords: dynamic allocation; risk management; contribution; diversification

Citation: Urbina, J.; Santolino, M.; Guillen, M. Covariance Principle for Capital Allocation: A Time-Varying Approach. *Mathematics* **2021**, *9*, 2005. https://doi.org/10.3390/math9162005

Academic Editor: Miltiadis Chalikias

Received: 23 July 2021
Accepted: 18 August 2021
Published: 21 August 2021

1. Introduction

Risk in financial and actuarial applications is often defined as a random variable associated with losses [1,2]. Risk measures are functions aimed at describing and quantifying risks and are of great use in the field of capital allocation [3]. Capital allocation problems arise when a total amount associated with the aggregate risk has to be distributed across the multiple units of risk that make up this total risk [4].

Examples of capital allocation problems can be found in many areas of finance, where agents attempt to define investment risks [5–11], and in insurance, where they involve the allocation of the total solvency capital requirement across the business lines or the total costs or expenses across the coverage of an insurance policy, among other areas [12–14]. The guidelines on how the capital should be shared among the units are determined by capital allocation principles. Capital allocation principles are defined by a capital allocation criterion and a given risk measure. The combination of an allocation criterion and a risk measure will usually give different capital allocation solutions. Capital allocation principles have been motivated based on game theory [15–18] and optimization methods [3,4,19,20]. A general theoretical framework in which the majority of the capital allocation principles defined in the literature can be accommodated is provided by [3]. A recent review and comparison of techniques can be found in [21].

One of the most frequently used capital allocation principles in practice is the covariance allocation principle, which is based on the proportional allocation criterion and the variance of the individual risk and the covariance of the individual risk with the aggregate risk. The covariance principle for capital allocation as well as the allocation problem are described in [3,22]. Some of the reasons for the preference of this proportional covariance allocation criterion by practitioners include the fact that: (i) the principle takes into account dependence between risks, (ii) it defines allocations that can be computed easily and, (iii) allocation outcomes may effortlessly be explained by managers.

Evolution over time is a natural behavior for risks, as outlined in [23–28]. Risks change over time, so capital risk allocations should be time-dependent. This is a key issue in financial institutions, who are required to hold a risk capital in order to mitigate the effects of adverse events. The risk capital is derived from the risk-based assessment of the aggregate risk of all its divisions, and this aggregate risk changes over time. The time dimension of the risk evaluation is also relevant in the context of operational risk models, as outlined in [29], where miscalculation may not imply an immediate loss, but sooner or later economic consequences become evident. Therefore, it is imperative that managers have a tool that allows them to evaluate the evolution of risk as well as the allocations of capital to cover losses over time. In this paper, we propose a dynamic capital allocation principle based on the variance-covariance of risks that change over time. The conditional correlation of risks is modeled by means of a dynamic conditional correlation (DCC) model [30]. We show how the DCC model can be useful for predicting the expected future capital allocations in a dynamic capital allocation setting. The methodology presented in the paper is illustrated with an example involving the investment risk in a stock portfolio.

Most of the previous research deals with capital allocation principles where risk dynamics are not considered and value to share remains constant over time [3,22,31]. From the point of view of an asset allocation strategy of investors, different studies deal with risks that vary over time. Some examples are the studies of [32–34], among others. Ref. [32] analyzes the risk position of an investor at different points in time to prevent welfare losses. Ref. [33] considers the problem of optimally allocating funds in an investment portfolio, given that the returns covariance matrix varies over time. Similarly [34], considers a time-varying covariance matrix in the context of portfolio diversification based on the assumption that investors use the Markowitz's mean-variance portfolio theory to determine the optimal asset allocation. The approach of these studies is different from the present one, since they focus on the gains in diversification for selecting their optimal portfolios.

Dynamics in capital risk allocations have gained attention in recent years [23,24,27]. The study of [23] is based on [35], who showed the link between capital allocations and compositional data, where compositional data mean that relative frequency or percent information is more relevant than absolute values. Ref. [23] proposed a flexible capital allocation scheme based on multivariate compositional time series for modeling time-varying risk positions and how aggregate losses behave over time [36–38]. While [27] shows a comparison between static and dynamic allocations [24], highlight that time-dependence should be taken into account in capital allocation problems because shocks are not known a priori but, instead, they appear over time. An example of constant covariance matrix can be found in [39].

In this paper, we propose a methodology to model dynamics of capital allocations. Our methodology is flexible enough to capture the impact of shocks as they arrive over time. We are also interested in capital risk allocations along the same lines as [23], where relative capital risk allocations are more important than the absolute capital risk allocations. The dynamics must be consistent, since the fact that proportions must add up to one is an implicit constraint. Ref. [23] considers any capital allocation solution as a composition and applies a compositional Vector Autoregressive (VAR) model to predict future risk compositions. In this framework, compositions need to be isometric log-ratio transformed to be unconstrained, which makes the interpretation of results a complex task, particularly for vectors with a high dimension. Instead of considering capital allocation solutions as compositions and using composition techniques to model them, our analysis here is restricted to capital allocation principles that are determined by the variance and covariance of risks, such as the covariance capital allocation principle. In this paper, we focus on the dynamics of the covariance structure and apply a DCC model to deal with a time-varying covariance matrix. Our approach differs from [23]. The main difference is that, in our study, the risk dependence structure is defined by a time-varying covariance which is modeled by a DCC model, while [23] consider a VAR model with a time-invariance dependence structure (constant covariance) to model time-varying compositions. The main advantage of our approach is that the dynamics of the covariance serve as a simpler, more intuitive and more interpretable alternative to time-varying compositions.

The aim of the paper is to define the time-dependent covariance allocation principle and to describe the steps that need to be implemented in practice. The paper is structured as follows. The main concepts related to capital allocation problems and the covariance capital allocation principle are briefly described in Section 2. Section 3 is devoted to the description of the dynamics of the covariance capital allocation principle. A case study of capital allocation solutions using the DCC model is provided in Section 4. The main results are described in Section 5 and a discussion is provided in Section 6. Finally, concluding remarks are presented in Section 7.

2. The Allocation Problem

The allocation problem refers to the subdivision of aggregate capital held by a firm across its various constituents, which might be business lines, types of exposure, territories, or even individual products in a portfolio of insurance policies [22].

Following [1,3,22], we consider a portfolio of n individual losses (random variables) $X_1, X_2, \dots, X_n$ materializing at a fixed future date T. The aggregate loss, S, is defined as the sum of the individual losses:

$$S = \sum_{i=1}^{n} X_i. \tag{1}$$

The aggregate loss, S, in (1) can be interpreted, for instance, as [1,22]:

1. The total loss of an insurance company, with the individual losses being those arising from the business lines;
2. The loss from an insurance portfolio, with the individual losses corresponding to the losses of the different policies;
3. The loss of a financial conglomerate, with the individual losses being those losses suffered by its subsidiaries;
4. The loss of a bank due to fraud in current accounts and credit card, respectively.

Once the aggregate loss S is defined and assumed to be random, our concern is with the economic capital, K, for covering that loss, and we assume this amount of capital to be exogenously determined by a risk measure that is a function of the total amount $\rho(S)$ [40,41]. We also consider the company's intention to allocate K across its various business lines so that K is distributed among $K_1, K_2, \dots, K_n$. Thus, the full allocation requirement is [5]:

$$\rho(S) = K = \sum_{i=1}^{n} K_i. \tag{2}$$

The capital allocation problem is defined as:

$$S - K = \sum_{i=1}^{n} (X_i - K_i), \tag{3}$$

where the quantity $(X_i - K_i)$ expresses the loss minus the allocated capital for unit i. The risk capital allocation problem can be interpreted as on optimization problem [3,19]. The allocation problem based on the quadratic optimization criterion is defined as [3]:

$$\min_{K_1,\dots,K_n} \sum_{i=1}^{n} E\left[\zeta_i \frac{(X_i - K_i)^2}{v_i}\right], \; such that, \; \sum_{i=1}^{n} K_i = K, \tag{4}$$

where the non-negative real number v_i is a measure of exposure or business volume of the ith unit, such as revenue, insurance premium, etc. These scalar quantities are chosen to add up to 1 and ζ_i is a function that measures the deviations of the losses X_i from their respective allocated capital levels K_i. X_i and K_i were defined previously. If ζ_i is a non-negative random variable with expected value $E[\zeta_i] = 1$ for all i, then the following theorem is satisfied.

Theorem 1. *The optimal allocation problem in (4) has the following unique solution*:

$$K_i = E(\zeta_i X_i) + v_i \left(K - \sum_{i=1}^{n} E(\zeta_i X_i) \right), \; i = 1, \ldots, n. \tag{5}$$

A detailed proof of the solution for this minimization problem can be found in [3]. Optimization criteria alternative to the quadratic deviation can be found in [20]. Capital allocations for non-linear aggregated risks are discussed in [9]

Covariance Allocation Principle

If we consider a proportional allocation rule such as:

$$K_i = \frac{K}{E[Sh(S)]} E[X_i h(S)], \tag{6}$$

with h being a non-negative and non-decreasing function, such that $E[h(S)] = 1$, [3] showed that the general setting (5) includes (6) when $\zeta_i = h(S)$ and $v_i = \frac{E[X_i h(S)]}{E[Sh(S)]}$. The covariance principle stems naturally from this when choosing $h(S) = S - E(S)$ and using the philosophy of the plug-in principle. Setting $h(S) = S - E(S)$, the aim is to determine $E[X_i h(S)]$ and $E[Sh(S)]$.

For $E[X_i h(S)]$, we have:

$$\begin{aligned} E[X_i h(S)] &= E[X_i(S - E(S))] \\ &= E(X_i S) - E(X_i)E(S) \\ &= Cov(X_i, S). \end{aligned}$$

For $E[Sh(S)]$ to be explicitly found, we proceed as follows:

$$\begin{aligned} E[Sh(S)] &= E[S(S - E(S))] \\ &= E\left(S^2\right) - [E(S)]^2 \\ &= Var(S). \end{aligned}$$

These two expressions give rise to the covariance allocation principle:

$$K_i = \frac{K}{Var[S]} Cov(X_i, S), \quad i = 1, \ldots, n. \tag{7}$$

As can be deduced from (7), this is a proportional allocation principle based only on the variance of the i-th business and their covariance with the total loss [3,5,22]. A drawback of this principle is that it does not account for changes in risk exposure over time; it remains fixed, limiting the decision maker [3].

In this paper, we propose a strategy to overcome such an inconvenience by considering all elements in (7) as time-varying random variables, by introducing a subscript t in each variable, and keeping K constant. Without loss of generality, we assume that K is independent of time, but K could be considered to be time-dependent in this setting. Note that the capital to be allocated to K is assumed to be exogenously determined. We focus on the proportion (in relative terms) of the total capital to allocate to each risk, and that the capital size affects the scale but not the proportion.

3. Time-Varying Approach of the Covariance Allocation Principle

Considering a given point in time, t, we can estimate the proportion of capital, $K_{i,t}$, to be assigned to each individual business unit according to:

$$K_{i,t} = \frac{K}{Var(S_t)} Cov(X_{i,t}, S_t) \qquad i = 1, 2, \ldots, n. \tag{8}$$

where K is the total capital to be allocated into the i-th business unit, $X_{i,t}$ represents the loss reported by the i-th business unit at time t and S_t is the aggregate loss at time t.

The underlying assumption in covariance allocation principle so far has been that t remains fixed. However, in Equation (8), we allow $K_{i,t}$ to be time-varying and provide an estimation of $K_{i,t}$ for each point in time, that is $t = 1, 2, \ldots, T$.

Up until now, time-varying structure in the aggregate loss has not been given the attention it deserves. Under our setting, we provide the practitioner with a simple yet parsimonious principle to allocate capital among several business units when time becomes a crucially important variable.

Considering the time-varying covariance allocation principle as in Equation (8), some transformations can be completed as follows:

$$K_{i,t} = \frac{K}{Var(S_t)} \rho_{i,t} \sqrt{Var(X_{i,t}) \cdot Var(S_t)} \quad \text{follows from}$$

$$ov(X_{i,t}, S_t) = \rho_{i,t} \sqrt{Var(X_{i,t}), Var(S_t)}, \; then$$

$$K_{i,t} = \rho_{i,t} \sqrt{Var(X_{i,t})} \frac{\sqrt{Var(S_t)}}{Var(S_t)} K.$$

Finally, Equation (8) can be re-written as:

$$K_{i,t} = \rho_{i,t} \sqrt{\frac{Var(X_{i,t})}{Var(S_t)}} K \tag{9}$$

where $\rho_{i,t}$ is the Pearson correlation coefficient between $X_{i,t}$ and S_t.

Both $Var(X_{i,t})$ and $Var(S_t)$ are assumed to evolve as a generalized autoregressive conditional heteroskedasticity (GARCH) type process and $\rho_{i,t}$ follows a dynamic conditional correlation scheme, DCC(1,1). Considering that K in (9) is exogenously determined, we can then easily obtain an estimation for $K_{i,t}$ due to the two-step structure of the DCC likelihood (for more details, see [30]). To ensure that the capital allocations add up to K, we apply the following transformation:

$$\widetilde{K}_{i,t} = \frac{K_{i,t}}{\sum_{i=1}^{n} K_{i,t}} \tag{10}$$

3.1. The Dynamic Conditional Correlation model

Let $\mathrm{x}_t = [X_{1,t}, \ldots, X_{n,t}]'$ be a vector of losses that follows a multivariate gaussian process $\mathrm{x}_t \sim N(\mu, \mathrm{H}_t)$ where μ is the vector of unconditional means and H_t is the conditional covariance matrix. The conditional covariance matrix H_t is defined as:

$$\mathrm{H}_t = \mathrm{D}_t \mathrm{R}_t \mathrm{D}_t,$$

where R_t is the conditional correlation matrix and D_t is a diagonal matrix with conditional standard deviations on the diagonal, with:

$$\mathrm{R}_t = E[\xi_t \xi_t\prime | \Omega_{t-1}]$$

$$\xi_t = \mathrm{D}_t^{-1}(\mathrm{x}_t - \mu),$$

where ξ_t is a vector of standardized residuals and Ω_{t-1} is the information set available up to $t-1$. Therefore, we can write the vector of losses as $\mathrm{x}_t = \mu + \mathrm{H}_t^{1/2}\xi_t$ with $\xi_t \sim N(0, \mathrm{I_n})$ where $\mathrm{I_n}$ is the identity matrix. The estimator of the conditional correlation matrix is:

$$\mathrm{R}_t = (1-a-b)\overline{\mathrm{R}} + a\xi_{t-1}\xi_{t-1}\prime + b\mathrm{R}_{t-1}$$

where a and b are scalar parameters satisfying the stability constraint of the form $a+b<1$, and $\overline{\mathrm{R}}$ is a positive definite matrix with unit elements along the main diagonal consisting of the unconditional covariance matrix of the standardized errors. In order to obtain a proper correlation, the following transformation is required $\mathrm{R}_t{}^* = \left\{\mathrm{diag}(\mathrm{R}_t)^{-1}\mathrm{R}_t\mathrm{diag}(\mathrm{R}_t)^{-1}\right\}$, where $\rho^*_{i,j,t}$ is a typical element of $\mathrm{R}_t{}^*$ [30].

The DCC model has been widely used in finance [42,43]. Extensions of the DCC model can be found in [44,45], and their limitations are discussed in [46].

3.2. Estimation Procedure

As the estimation procedure is fully described in [30], here, we briefly point out the main aspects. In order to estimate the parameters of the DCC model, we follow the two-step procedure of [5]. Let ψ be the collection of parameters of the univariate GARCH model, where ψ could be a standard GARCH or an EGARCH or even a Beta-t-EGARCH, and let Ξ be the vector of DCC parameter (a,b), the quasi log-likelihood (QL) takes the following form:

$$\begin{aligned} QL(\psi,\Xi) &= QL_1(\psi) + QL_2(\psi,\Xi) \\ &= -\frac{1}{2}\sum_{t=1}^{T}(n\log(2\pi) + 2\log|\mathrm{D}_t| + \mathrm{x}'_t\mathrm{D}_t^{-2}\mathrm{x}_t) \\ &\quad -\frac{1}{2}\sum_{t=1}^{T}(\log|\mathrm{R}_t| + \xi_t'\mathrm{R}_t^{-1}\xi_t + \xi_t'\xi_t). \end{aligned}$$

The separation of $QL(\psi,\Xi)$ into $QL_1(\psi)$ and $QL_2(\psi,\Xi)$ indicates that we can first estimate the parameters of the univariate GARCH-type processes contained in ψ by maximizing $QL_1(\psi)$ to obtain $\hat{\psi}$, then we can plug $\hat{\psi}$ into $QL_2(\psi,\Xi)$ so that it becomes $QL_2(\hat{\psi},\Xi)$, where standardized residuals $\hat{\xi}_t = \hat{\mathrm{D}}_t^{-1}(\mathrm{x}_t - \hat{\mu})$ are used in the second stage.

For the first part of the likelihood, $QL_1(\psi)$ consists of estimating $Var(X_{i,t}|\Omega_{t-1})$ for all i and $QL_2(\hat{\psi},\Xi)$ allows for estimation of the conditional correlations.

4. Case Study Definition

In the next two sections, we develop the steps to apply the time-dependent covariance allocation principle defined in Section 3 in an investment portfolio. This allows the manager to have:

- A time-varying allocation scheme for covering loss i in time t.
- A historical evolution of exposure as well as the distribution of the economic capital.
- A time series model for capital allocation, with which the manager can forecast both variance and covariance in order to predict the future K_i.

In this section, we describe the data that were previously used by [23]. The same dataset is used here to connect our findings with results in [23]. All calculations were carried out in R software [47].

Data

We assume that n = 3, and that an investor selects the S&P 500 (SP500) index, the DAX index, and the CAC 40 index, respectively. We use these indexes from 1 January 2000 to 31 December 2016 and monitor their daily value for every working day. We consider a buy-and-hold portfolio with an initial equal investment to each index on 1 January 2000. The original aggregate investment in the portfolio is 100,000 dollars.

Figure 1 shows the investment evolution according to each portfolio component, with the DAX and the SP500 showing a similar pattern within the overall sample. Until 2009, CAC40 was synchronized with the other two components, but afterward it evolved at lower levels.

Figure 1. Investment evolution of the aggregate daily value of a portfolio that invests 100,000 units in the S&P 500 index, the DAX index, and the CAC 40. The investment is equally split into three components on 1 January 2000 and the investment goes on until 31 December 2016.

The daily value of the investment in each index is considered a source of risk, and the aggregated daily value of the portfolio has an overall risk. In this illustration, we focus on daily losses (returns with negative sign). That is, we compute the difference in log scale of the daily value of the investment in each index, where a positive value indicates a loss (the daily value of the index is lower than that of the previous daily value) and a negative value indicates a gain (the daily value of the index is higher than the previous daily value). In terms of the daily losses (the differences in the logs), a similar behavior is observed in Figure 2 among the investment components, with the SP500 having the lowest variance.

A summary of main statistics of daily losses is shown in Table 1.

Table 1. Descriptive statistics for daily losses.

	Mean	Variance	Cov (Index, Total)
SP500	-4.4×10^{-5}	3.0×10^{-5}	7.3×10^{-5}
DAX	-5.5×10^{-5}	4.6×10^{-5}	1.1×10^{-4}
CAC40	2.0×10^{-5}	4.3×10^{-5}	1.0×10^{-4}

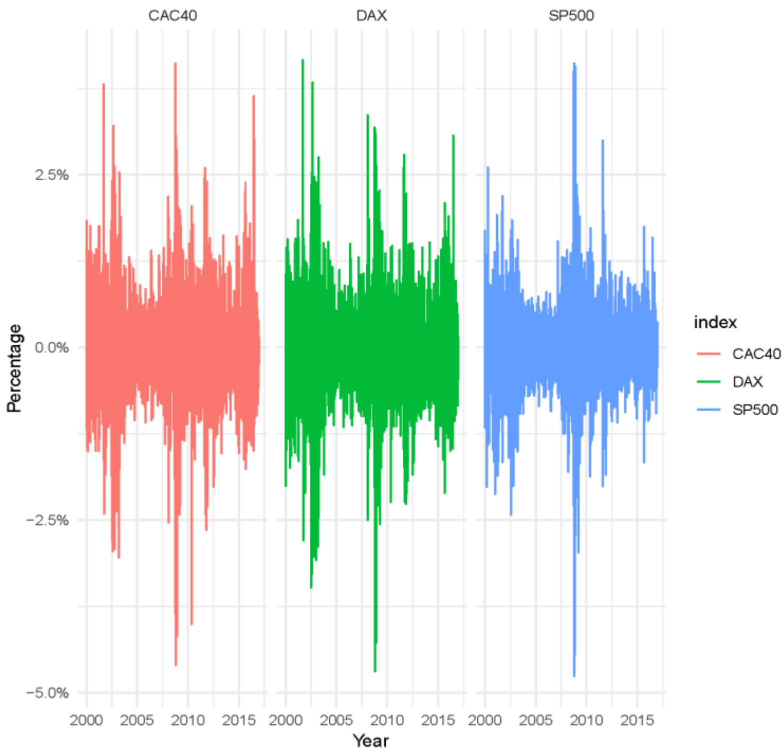

Figure 2. Daily losses (in percentage) of a portfolio that invests 100,000 units in the S&P 500 index, the DAX index, and the CAC 40 on 1 January 2000. The investment is equally split across the three components.

5. Results

Taking the capital allocation from Equation (7), i.e., daily losses are considered individual observations and no time-varying approach is followed, and using statistics shown in Table 1, the following capital assignment scheme based on covariance allocation principle is obtained: SP500 25.68%, DAX 37.93% and CAC40 36.39%.

The time-varying approach of the covariance capital allocation principle is now considered. GARCH models are fitted to data to model $Var(X_{i,t})$ and $Var(S_t)$, where i = 1,2,3 for daily losses (on log scale) of SP500, DAX and CAC40 indices, respectively, and $\rho_{i,t}$ follows a dynamic conditional correlation scheme, DCC(1,1).

Time-varying covariance capital allocations obtained by applying Equation (8) are shown in Figure 3.

The time-varying capital allocation principle succeeds in capturing the dynamics of losses of each portfolio component, especially of that related to US and periods of turmoil, such as the 2008 subprime crisis. Periods of growth in capital allocations are preceded by periods of turmoil, such as dotcom and the subprime crisis, as can be seen in Figures 1 and 2. Before 2009, all components behave quite similarly but, from 2009, CAC40 slows down its risk exposure and, hence, its need for capital. In contrast, DAX and SP500 speed up their level of exposure, demanding more assigned capital to cover the possible risks.

Figure 3 shows two episodes where SP500 is above CAC40 and DAX; these periods correspond to the dotcom crisis and the subprime crises, as mentioned earlier. Outside these time windows, SP500 is the index with the lowest volatility and the least exposure to risk and, therefore, the one with the lowest required capital. Note how the dynamic set-up of our new approach to capital allocation is able to capture the stable periods and fluctuations of turmoil in each component of the portfolio.

According to the evolution in capital allocations, the average value for the SP500 is 24.53% with 25% of the episodes below 20.84% and 75% of them below 27.71%. However, there were episodes coinciding with crises where capital allocation exceeded 40%. In

January 2001, it averaged 40.92%; November 2007, 43.62%; October 2008, 41.87%; and March 2009, 40.57%.

Main statistics of the estimated time-varying covariance capital allocations from 1 January to 31 December 2016 are shown in Table 2.

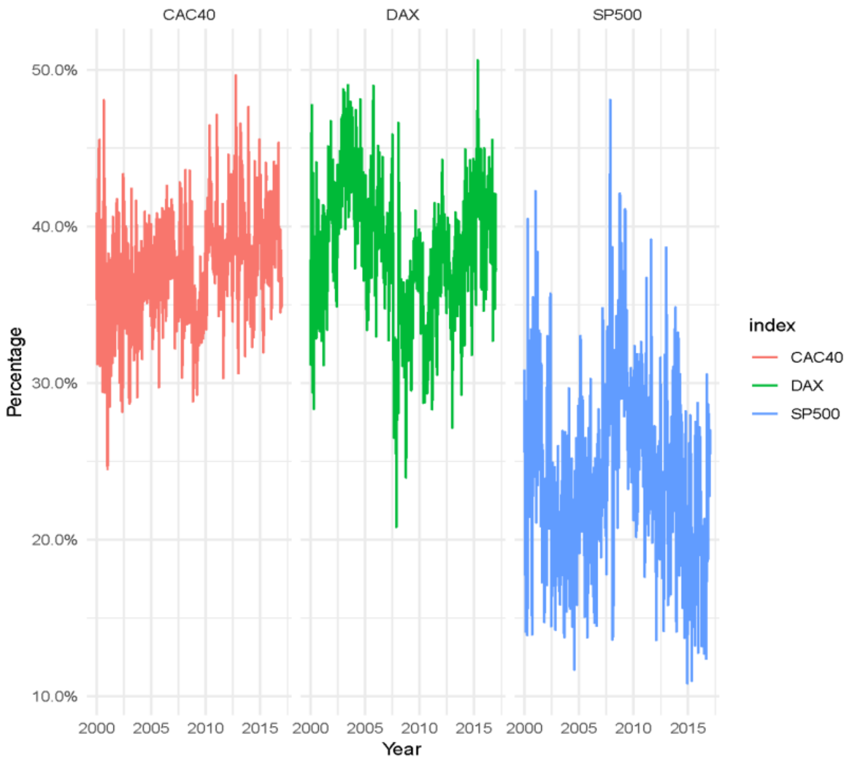

Figure 3. Time-varying covariance capital allocations.

Table 2. Descriptive statistics for dynamic allocation principle over the entire sample.

	Mean	St.Dev.	Min	Quartile 1	Quartile 3	Max
SP500	24.53	5.06	11.19	20.84	27.71	47.89
DAX	38.39	4.03	21.10	35.98	41.20	51.01
CAC40	37.10	3.23	24.91	34.87	39.19	48.78

It is worth noting that the allocation obtained from Equation (7) based on the static covariance capital allocation principle without time dependence structure is close (but not equal) to the average of the time-dependent covariance capital allocations based on Equation (8) shown in Table 2. The dynamic structure proposed in this paper provides richer information to decision makers than the static covariance capital allocation outcome, such as the proportion of cases of the capital allocations in the different sources of risks which were below or above a certain threshold.

In addition, our dynamic framework enables the prediction of future covariance capital allocations. Figure 4 includes the 10-days-ahead forecast for dynamic covariance capital allocations. The fitted dynamic capital allocation for the last observed day (30 December 2016) was 27.4% (SP500), 37.6% (DAX) and 35.0% (CAC40), and the predicted covariance capital allocation for the first projected day was 27.7% (SP500), 37.7% (DAX) and 34.6% (CAC40) with no significant fluctuations over the following nine days.

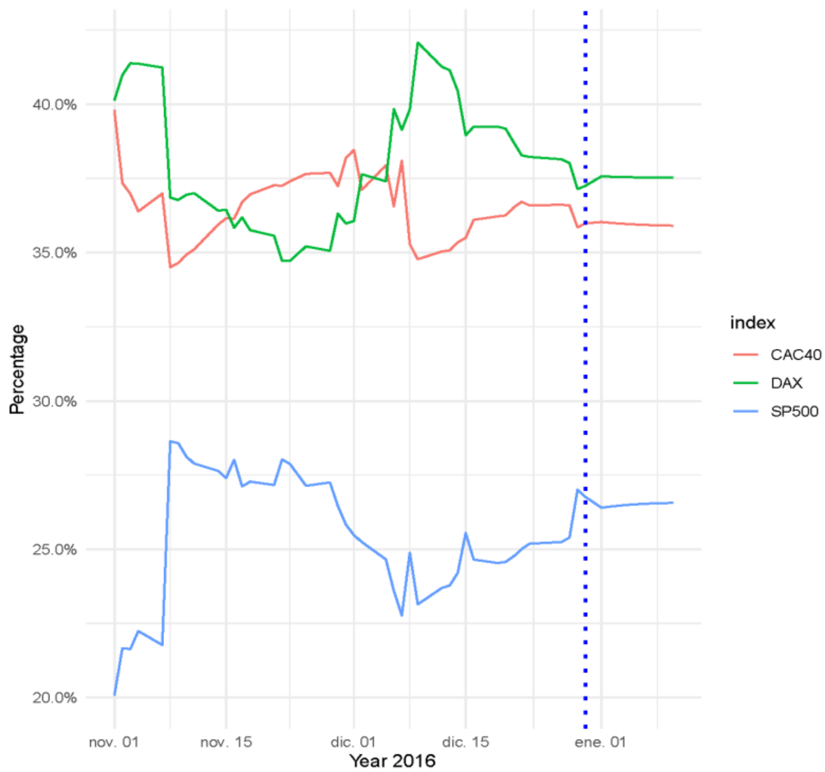

Figure 4. 10-days-ahead covariance capital allocation predictions: observed period from 1 November to 31 December 2016.

In conclusion, the behavior of indices (and their log returns) is continuously changing over the observation period in which the initial configuration is no longer observed. The dynamic capital allocation framework proposed in this paper is flexible enough to adapt to the variations in the performance of the stock indices and to properly reflect the underlying risk involving the investment on these indices. By contrast, the static capital allocation scheme given by Equation (7) is qualitative rather than quantitative in the sense that it effectively provides an aggregate ordering of the financial indices on the demand of capital to cover potential losses, but it is not an indicator of the dynamic behavior of these financial indices.

6. Discussion

Risks change over time, but most capital allocation principles suggested in the literature are analyzed in a static setting [3–5,9,14–20,22]. Therefore, managers cannot directly apply these capital allocation principles to have a time-varying allocation scheme for covering losses that can be used to forecast future allocations. Here, we provide to managers with a dynamic capital allocation scheme, so that managers can forecast capital risk allocations.

Most previous studies dealing with risks that vary over time are in the field of asset allocation strategies in the portfolio selection [32–34,41]. Although optimal asset allocation and optimal capital risk allocation are strongly interconnected, their goals and, in consequence, their optimal allocation strategies are also different. Optimal asset allocations are focused on the trade-off between return and risk such that, given assets with equal returns, more risky assets receive lower allocations [41]. Capital risk allocation principles are focused on the assets' contribution to the aggregate risk, so that riskier assets receive higher allocations [5,13], meaning that they contribute more than others to total risk. Even the concept of risk differs. The risk of assets in the context of portfolio selection is often evaluated by the variance of returns [41]. In our setting, the risk of the assets is evaluated by the covariance between the individual losses and the aggregate losses of a given portfolio.

In the context of capital risk allocations, an interesting contribution incorporating risk dynamics is provided by [23]. They proposed a time-dependent capital allocation scheme based on multivariate compositional time series for modeling time-varying risk positions. Our research differs on the term basis of the analysis, which is either monthly or daily. They apply a Euler allocation rule with a quantile risk measure to compute capital allocations with a monthly basis. We apply a covariance allocation principle which for a time-dependent capital allocation scheme on a daily basis. With the same dataset, the length of the compositional time series in [23] is 210 observations and, in our case, the time series has 4212 observations. Having a higher frequency and, accordingly, a larger sample size can be relevant for inference. The number of (unknown) parameters grows with the number of risks much faster in [23] than in our setting, so the framework proposed in [23] could require larger samples for efficient parameter estimation. In terms of risk management, we can predict the daily variations in capital risk allocations and, therefore, provide a tool to managers who care about daily monitoring of risks. We showed that the time-dependent covariance principle properly reflects the daily dynamics of risks with asymmetric impacts of dotcom and the subprime crises on the SP500, DAX and CAC40 indices.

Another difference is that capital risk allocations in [23] must always be strictly positive, since the compositional modeling is based on log transformations. Therefore, [23] cannot apply their methodology, for instance, in a context of expected negative losses (i.e., profits) of the asset and null covariance with the aggregate loss of the portfolio. Our methodology is not restricted to positive values, so we can deal with negative or null capital risk allocations. Finally, in terms of interpretability, the time-varying relative capital allocation scheme proposed in this study is determined by the time-varying covariance matrix of assets' losses. Coefficient estimates of the dynamic conditional correlation model can be directly interpreted in terms of their effect on the conditional covariance matrix of assets' losses and, therefore, also on the relative capital risk allocation outcomes. The multivariate time-series modeling with constant covariance matrix in [23] is based on isometric log transformations of losses, which makes interpretation of the estimated coefficients of the vector autoregressive model in terms of their effect on the capital allocation outcomes very difficult, particularly for high-dimensional problems [23,36].

This study is not exempt of limitations. The dependence structure of risks is determined by the covariance matrix of losses. A future line of research is to investigate alternatives to the DCC for modeling the conditional correlation of losses, such as the generalized autoregressive conditional correlation model [48]. Another limitation is that the methodology does not distinguish between long run correlations and short run correlations. Long run correlations are related to the natural behavior of risks, as business evolves with the economy, and short run correlations represent the risk behavior with -negative/positive- shocks such as news arriving, as speculations, moral hazard problems or temporal market distortions, among others.

A future line of research is to include these different correlations in the dynamic capital allocation setting. Finally, managers are particularly interested in the contributions of individual risks on the tail of the distribution of aggregate losses [1,14,49]. Based on [49–51] incorporates the Tail Covariance Premium Adjusted (TCPA) as a component for capital allocations for risks with heavy tails, but this approach remains in a static setup. The analysis of time-varying tail dependence structures and their application in capital allocations problems is a promising line for future work.

7. Conclusions

This study proposes a new method for dynamically allocating capital among business lines. In this paper, we develop the covariance capital allocation principle in a dynamic setting, in which the conditional variance and the conditional correlation of risks is modeled by means of GARCH-type processes and a dynamic conditional correlation scheme, respectively. The dynamic framework provided in the paper allows for the expected evolution of risks and their future demand of capital to be foreseen. We illustrate an example of all the steps to compute time-dependent capital allocations based on the covariance allocation principle for investment losses in a stock portfolio.

We show that our strategy provides a better understanding of the time-dependent behavior of risks and their variations on the demand of capital to cover potential losses over time. The static covariance allocation principle provides aggregate information about the behavior of risks in a particular timespan, but it does not capture the dynamics of the individual risks in this period of time. However, time-dependent capital allocations properly reflect the dynamic nature of risks with asymmetric impacts on the underlying individual risks, such as the impact of dotcom and the subprime crises on the SP500, DAX and CAC40 indices and, consequently, the variations in capital allocations on these indices.

Author Contributions: Conceptualization, M.G.; methodology, M.G. and J.U.; software, J.U.; validation, M.G., J.U. and M.S.; formal analysis, M.S.; investigation, J.U.; resources, M.G.; data curation, M.S.; writing—original draft preparation, J.U.; writing—review and editing, M.G. and M.S.; visualization, M.S.; supervision, M.G.; project administration, M.G.; funding acquisition, M.G. and M.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Spanish Ministry of Science and Innovation under Grant PID2019-105986GB-C21 and by the Catalan Government under Grant 2020-PANDE-00074.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The data presented in this study are available in Yahoo Finance, https://finance.yahoo.com/, accessed on 21 August 2021.

Acknowledgments: We thank the editor and anonymous referees for their helpful comments. M.G. and M.S. acknowledge the support of the Spanish Ministry of Science and Innovation under Grant PID2019-105986GB-C21. M.G thanks the support of the ICREA Academia and Fundación BBVA support for research on Big Data. M.S thanks the support of the Catalan Government under Grant 'Pandemies 2020'.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. McNeil, A.J.; Frey, R.; Embrechts, P. *Quantitative Risk Management: Concepts, Techniques and Tools*; Princeton University Press: Princeton, NJ, USA, 2005.
2. Belles-Sampera, J.; Guillen, M.; Santolino, M. *Risk Quantification and Allocation Methods for Practitioners*; Amsterdam University Press: Amsterdam, The Netherlands, 2017.
3. Dhaene, J.; Tsanakas, A.; Valdez, E.A.; Vanduffel, S. Optimal Capital Allocation Principles. *J. Risk Insur.* **2012**, *79*, 1–28. [CrossRef]
4. Dhaene, J.; Goovaerts, M.; Kaas, R. Economic capital allocation derived from risk measures. *N. Am. Actuar. J.* **2003**, *7*, 44–59. [CrossRef]
5. Tasche, D. *Risk Contributions and Performance Measurement*; Research Paper Zentrum Mathematik SCA; Zentrum Mathematik SCA: Munich, Germany, 2000.
6. Haugh, M.; Iyengar, G.; Song, I. A generalized risk budgeting approach to portfolio construction. *J. Comput. Financ.* **2017**, *21*, 29–60. [CrossRef]
7. Costa, G.; Kwon, R. Generalized Risk Parity Portfolio Optimization: An ADMM Approach. *J. Glob. Optim.* **2020**, *78*, 207–238. [CrossRef]
8. Asimit, V.; Peng, L.; Wang, R.; Yu, A. An efficient approach to quantile capital allocation and sensitivity analysis. *Math. Financ.* **2019**, *29*, 1131–1156. [CrossRef]
9. Boonen, T.J.; Tsanakas, A.; Wüthrich, M.V. Capital allocation for portfolios with non-linear risk aggregation. *Insur. Math. Econ.* **2017**, *72*, 95–106. [CrossRef]
10. Buch, A.; Dorfleitner, G.; Wimmer, M. Risk capital allocation for RORAC optimization. *J. Bank Financ.* **2011**, *35*, 3001–3009. [CrossRef]
11. Rahl, L. *Risk Budgeting: Risk Appetite and Governance in the Wake of the Financial Crisis*, 2nd ed.; Risk Books: London, UK, 2012.
12. Myers, S.C.; Read, J.A., Jr. Capital allocation for insurance companies. *J. Risk Insur.* **2001**, *68*, 545–580. [CrossRef]
13. Belles-Sampera, J.; Guillen, M.; Santolino, M. GlueVaR risk measures in capital allocation applications. *Insur. Math. Econ.* **2014**, *58*, 132–137. [CrossRef]
14. Van Gulick, G.; de Waegenaere, A.; Norde, H. Excess based allocation of risk capital. *Insur. Math. Econ.* **2012**, *50*, 26–42. [CrossRef]
15. Kalkbrener, M. An axiomatic approach to capital allocation. *Math. Financ.* **2005**, *15*, 425–437. [CrossRef]
16. Denault, M. Coherent allocation of risk capital. *J. Risk* **2001**, *4*, 1–34. [CrossRef]
17. Boonen, T.J. τ-value for risk capital allocation problems. *Oper. Res. Lett.* **2020**, *48*, 752–757. [CrossRef]

18. Tsanakas, A.; Barnett, C. Risk capital allocation and cooperative pricing of insurance liabilities. *Insur. Math. Econ.* **2003**, *33*, 239–254. [CrossRef]
19. Zaks, Y.; Tsanakas, A. Optimal capital allocation in a hierarchical corporate structure. *Insur. Math. Econ.* **2014**, *56*, 48–55. [CrossRef]
20. Cai, J.; Wang, Y. Optimal capital allocation principles considering capital shortfall and surplus risks in a hierarchical corporate structure. *Insur. Math. Econ.* **2021**, *100*, 329–349. [CrossRef]
21. Guo, Q.; Bauer, D.; Zanjani, G. Capital Allocation Techniques: Review and Comparison. *Variance* **2021**, in press.
22. Urbina, J.; Guillen, M. An application of capital allocation principles to operational risk and the cost of fraud. *Expert Syst. Appl.* **2014**, *41*, 7023–7031. [CrossRef]
23. Boonen, T.J.; Guillen, M.; Santolino, M. Forecasting compositional risk allocations. *Insur. Math. Econ.* **2019**, *84*, 79–86. [CrossRef]
24. Bauer, D.; Kamiya, S.; Ping, X.; Zanjani, G. Dynamic capital allocation with irreversible investments. *Insur. Math. Econ.* **2019**, *85*, 138–152. [CrossRef]
25. Tsanakas, A. Dynamic capital allocation with distortion risk measures. *Insur. Math. Econ.* **2004**, *35*, 223–243. [CrossRef]
26. Laeven, R.J.; Goovaerts, M.J. An optimization approach to the dynamic allocation of economic capital. *Insur. Math. Econ.* **2004**, *35*, 299–319. [CrossRef]
27. Boonen, T.J. Static and dynamic risk capital allocations with the Euler rule. *J. Risk* **2019**, *22*, 1–15. [CrossRef]
28. Crowner, C. Dynamic capital allocation: Exploiting persistent patterns in currency performance. *Quant. Financ.* **2006**, *6*, 185–191. [CrossRef]
29. Bolancé, C.; Guillen, M.; Gustafsson, J.; Nielsen, J.P. *Quantitative Operational Risk Models*; Chapman & Hall/CRC: Boca Raton, FL, USA, 2012.
30. Engle, R. Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models. *J. Bus. Econ. Stat.* **2002**, *20*, 339–350. [CrossRef]
31. Tsanakas, A. To Split Or Not To Split: Capital Allocation with Convex Risk Measures. *Insur. Math. Econ.* **2009**, *44*, 268–277. [CrossRef]
32. Guidolin, M.; Timmermann, A. Economic implications of bull and bear regimes in uk stock and bond returns. *Econ. J.* **2005**, *115*, 111–143. [CrossRef]
33. Cumby, R.; Figlewski, S.; Hasbrouck, J. International asset allocation with time varying risk: An analysis and implementation. *Jpn. World Econ.* **1994**, *6*, 1–25. [CrossRef]
34. Flavin, T.; Wickens, M. Optimal international asset allocation with time-varying risk. *Scott. J. Polit. Econ.* **2006**, *53*, 543–564. [CrossRef]
35. Belles-Sampera, J.; Guillen, M.; Santolino, M. Compositional methods applied to capital allocation problems. *J. Risk* **2016**, *19*, 15–30. [CrossRef]
36. Kynčlová, P.; Filzmoser, P.; Hron, K. Modeling compositional time series with vector autoregressive models. *J. Forecast.* **2015**, *35*, 303–314. [CrossRef]
37. Snyder, R.; Ord, J.K.; Koehler, A.B.; McLaren, K.R.; Beaumont, A. Forecasting compositional time series: A state space approach. *Int. J. Forecast.* **2017**, *32*, 502–512. [CrossRef]
38. Wei, Y.; Wang, Z.; Wang, H.M.; Lie, Y. Compositional data techniques for forecasting dynamic change in china's energy consumption structure by 2020 and 2030. *J. Clean. Prod.* **2020**, *284*, 124702. [CrossRef]
39. Chalikias, M.S. Optimal Repeated Measurements for Two Treatment Designs with Dependent Observations: The Case of Compound Symmetry. *Mathematics* **2019**, *7*, 378. [CrossRef]
40. Belles-Sampera, J.; Guillen, M.; Santolino, M. What attitudes to risk underlie distortion risk measure choices? *Insur. Math. Econ.* **2016**, *68*, 101–109. [CrossRef]
41. Belles-Sampera, J.; Guillen, M.; Santolino, M. Beyond Value-at-Risk: GlueVaR Distortion Risk Measures. *Risk Anal.* **2014**, *34*, 121–134. [CrossRef] [PubMed]
42. Olmo, J. Optimal portfolio allocation and asset centrality revisited. *Quant. Financ.* **2021**, in press. [CrossRef]
43. Asanga, S.; Asimit, A.; Badescu, A.; Haberman, S. Portfolio Optimization under Solvency Constraints: A Dynamical Approach. *N. Am. Actuar. J.* **2014**, *18*, 394–416. [CrossRef]
44. Engle, R.F.; Ledoit, O.; Wolf, M. Large dynamic covariance matrices. *J. Bus. Econ. Stat.* **2019**, *37*, 363–375. [CrossRef]
45. Aielli, G.P. Dynamic conditional correlation: On properties and estimation. *J. Bus. Econ. Stat.* **2013**, *31*, 282–299. [CrossRef]
46. Caporin, M.; McAleer, M. Ten Things You Should Know about the Dynamic Conditional Correlation Representation. *Econometrics* **2013**, *1*, 115–126. [CrossRef]
47. R Core Team. *R: A Language and Environment for Statistical Computing*; R Foundation for Statistical Computing: Vienna, Austria, 2020; Available online: https://www.R-project.org/ (accessed on 21 August 2021).
48. McAleer, M.; Chan, F.; Hoti, S.; Lieberman, O. Generalized autoregressive conditional correlation. *Econ. Theory* **2008**, *24*, 1554–1583. [CrossRef]
49. Wang, M. Capital allocation based on the Tail Covariance Premium Adjusted. *Insur. Math. Econ.* **2014**, *57*, 125–131. [CrossRef]
50. Furman, E.; Landsman, Z. Tail variance premium with application for elliptical portfolio of risks. *ASTIN Bull.* **2006**, *36*, 433–462. [CrossRef]
51. Overbeck, L. Allocation of economic capital in loan portfolios. In *Measuring Risk in Complex. Stochastic Systems*; Franke, J., Haerdle, W., Stahl, G., Eds.; Lecture Notes in Statistics; Springer: New York, NY, USA, 2000; Volume 147, pp. 1–17.