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Conditions for the Existence of Absolutely Optimal Portfolios

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Abstract: Let Δ_n be the n-dimensional simplex, $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$ be an n-dimensional random vector, and $\mathcal{U}$ be a set of utility functions. A vector $\mathbf{x}^* \in \Delta_n$ is a $\mathcal{U}$-absolutely optimal portfolio if $\mathrm{E}(u(\boldsymbol{\xi}^{\mathbf{T}}\mathbf{x}^*)) \geq \mathrm{E}(u(\boldsymbol{\xi}^{\mathbf{T}}\mathbf{x}))$ for every $\mathbf{x} \in \Delta_n$ and $u \in \mathcal{U}$. In this paper, we investigate the following problem: For what random vectors, $\boldsymbol{\xi}$, do $\mathcal{U}$-absolutely optimal portfolios exist? If $\mathcal{U}_2$ is the set of concave utility functions, we find necessary and sufficient conditions on the distribution of the random vector, $\boldsymbol{\xi}$, in order that it admits a $\mathcal{U}_2$-absolutely optimal portfolio. The main result is the following: If x^0 is a portfolio having all its entries positive, then x^0 is an absolutely optimal portfolio if and only if all the conditional expectations of ξ_i, given the return of portfolio x^0, are the same. We prove that if $\boldsymbol{\xi}$ is bounded below then CARA-absolutely optimal portfolios are also $\mathcal{U}_2$-absolutely optimal portfolios. The classical case when the random vector $\boldsymbol{\xi}$ is normal is analyzed. We make a complete investigation of the simplest case of a bi-dimensional random vector $\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$. We give a complete characterization and we build two dimensional distributions that are absolutely continuous and admit $\mathcal{U}_2$-absolutely optimal portfolios.

Keywords: random vector; utility function; absolutely optimal portfolio; CARA absolutely optimal portfolio

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1. Introduction. Defining the Problem

Let (Ω, K, P) be a fixed probability space and $\boldsymbol{\xi}$: $\Omega \rightarrow \mathbb{R}^n$ be an n–dimensional random vector. The random vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$ will be called the vector of return rates (or the financial market). Let S be a positive number. A portfolio of sum S is a vector $\mathbf{x} \in \mathbb{R}_+^n$ with the property that $x_1 + \ldots + x_n = S$. The set of all portfolios of sum S will be denoted by $\Delta_n(S)$. A standard portfolio is a portfolio of sum 1, that is, an element of standard simplex $\Delta_n(1)$.

We shall also use the set $P_{n-1}(S) = \{\mathbf{x} \in \mathbb{R}^{n-1} \mid x_i \geq 0\ \forall\ i = 1, \ldots, n-1$ and $x_1 + \ldots + x_{n-1} \leq S\}$. Instead of $P_{n-1}(1)$, we shall write P_{n-1}. We shall denote by $\mathbf{e}$ the n-dimensional vector with all entries equal to one, that is $\mathbf{e} = (1, 1, \ldots, 1)$. The return of portfolio $\boldsymbol{x}$ is the random variable

$$Y(\mathbf{x}) = \mathbf{x}^T\boldsymbol{\xi} = x_1\xi_1 + \ldots + x_n\xi_n \tag{1}$$

A utility function is a continuous non-decreasing function u: $I \rightarrow \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval. Usually, the interval is either $[0,\infty)$ or the whole real line. A utility function of the form $u(x) = e^{rx}$, $x \in I$ (with $r > 0$) or $u(x) = -e^{-rx}$, $x \in I$ $(r > 0)$ is called a CARA-utility.

The set of CARA-utilities will be denoted with $\boldsymbol{\mathcal{U}}_3$.

The main problem in the portfolio theory is to find "the best" portfolio. How do you compare two portfolios in order to decide which of them is better? In the framework of expected utility theory, any decision maker has a utility function U: $I \rightarrow \mathbb{R}$ and he tries to maximize the expected utility of a portfolio: that is, he tries to maximize

$$V(\mathbf{x}, \boldsymbol{\xi}; u) = \mathrm{E}u\left(\mathbf{x}^T\boldsymbol{\xi}\right) \tag{2}$$

for all portfolios $\mathbf{x} \in \Delta_n(S)$, provided that $\mathbf{x}^T\boldsymbol{\xi} \in I$.

From the point of view of expected utility theory, a decision maker with utility u prefers the portfolio $\mathbf{y}$ to portfolio $\mathbf{x}$ if

$$V(\mathbf{x}, \boldsymbol{\xi}; u) \leq V(\mathbf{y}, \boldsymbol{\xi}; u) \Leftrightarrow \mathrm{E}u\left(\mathbf{x}^T\boldsymbol{\xi}\right) \leq \mathrm{E}u\left(\mathbf{y}^T\boldsymbol{\xi}\right)$$

To avoid trivialities, we shall also suppose that the relation (1.2) makes sense; more precisely, we shall always assume that we deal with utilities having the property $\boldsymbol{u}(\mathbf{x}^T\boldsymbol{\xi}) \in \mathbf{L}^1(\Omega, K, P)$ for any $\mathbf{x} \in \mathbb{R}_+^n$.

A decision maker is called risk-avoiding if the utility is concave. The reason for this is that if u is concave then, by Jensen's inequality, $\mathrm{E}u(Y) \leq u(\mathrm{E}Y)$ for any random variable Y. The meaning of this is that such a decision maker always prefers a sure amount of money to any random one having the same expected value.

We shall assume in the sequel that all the decision makers are risk-avoiding—after all, they will not be interested in portfolio theory otherwise.

Let $\boldsymbol{\mathcal{U}}_2(S, \boldsymbol{\xi})$ denote the set of all concave utilities, u having the property that $u\ (\mathbf{x}^T\boldsymbol{\xi})$ makes sense and, moreover, $u\ (\mathbf{x}^T\boldsymbol{\xi}) \in \mathrm{L}^1(\Omega, K, P)$ for all $\mathbf{x} \in \Delta_n(S)$.

Let $\boldsymbol{\xi}$ be a fixed random vector from $\mathbb{R}^n$ and $u \in \boldsymbol{\mathcal{U}}_2(S, \boldsymbol{\xi})$ be a fixed utility. Consider the problem:

$$\mathrm{P(S)}: \max\{\mathrm{E}u(\mathbf{x}^T\boldsymbol{\xi}): \mathbf{x} \in \Delta_n(S)\}$$

The meaning of the above optimization problem is that that any rational decision maker wants to maximize his expected utility. Notice that the mapping $\mathbf{x} \to \mathrm{E}u(\mathbf{x}^T\boldsymbol{\xi})$, $\mathbf{x} \in \Delta_n(S)$ is concave and continuous. The concavity is obvious and the continuity comes from Lebesgue's domination principle: if $\mathbf{x}_n \to \mathbf{x}$, the sequence of random variables $(u(\mathbf{x}_n{}^T\boldsymbol{\xi}))_n$ converges to $u(\mathbf{x}^T\boldsymbol{\xi})$ and is dominated by the random variable

$$\max_i(\,|\,u(S\ \xi_i)\,|\,) \in \mathrm{L}^1\ (\Omega\ K,P)$$

As $\Delta_n(S)$ is a compact set and the mapping h is continuous, the problem $\mathrm{P}(S)$ always has solutions. Moreover, the set of solutions is a closed convex subset from $\Delta_n\ (S)$. This is called the set of optimal portfolios and is denoted by $\mathrm{Opt}(\xi,S;u)$. A portfolio $\mathbf{x} \in \Delta_n\ (S)$ is called an equal weight portfolio if all its entries are equal, that is $\mathbf{x} = \left(\frac{S}{n}, \ldots, \frac{S}{n}\right)$.

Our study was motivated by the following well known fact:

Theorem 1. *If ζ_i, $I = 1, 2, \ldots, n$ are independent and identically distributed (i.i.d.) random variables, $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \ldots, \zeta_n)$ then the equal weight portfolio is optimal for the problem P(S):*

$$\mathbf{x^o} = \left(\frac{S}{n}, \ldots, \frac{S}{n}\right) \in\ \mathbf{Opt}(\zeta, S; u) \text{ for every } u\ \in\ \boldsymbol{\mathcal{U}}_2(S,\ \boldsymbol{\xi}) \tag{3}$$

In other words, any risk-avoiding decision maker agrees that the equal weight portfolio is optimal, provided that the random variables $(\xi_i)_{1 \leq i \leq n}$ are i.i.d. Put another way, if the return of the assets has the same distribution and they are independent, then the investor's choice to put the same amount of money in each asset is optimal from the point of view of the expected utility of the return.

In [1], Samuelson generalized the above result. He showed that the hypothesis "ξ_i, $I = 1, 2, \ldots, n$ are i.i.d. random variables" may be replaced by a more general condition:

Denote by $F_{\boldsymbol{\xi}}$ the distribution of the random vector $\boldsymbol{\xi}$, that is $F_{\boldsymbol{\xi}}\ (B) = P(\boldsymbol{\xi} \in B)$. Call $F_{\boldsymbol{\xi}}$ symmetrical if $F_{\boldsymbol{\xi}_{\mathbf{o}}\boldsymbol{\sigma}} = F_{\boldsymbol{\xi}}$ for any permutation σ of the set $\{1, 2, \ldots, n\}$. Here, $\boldsymbol{\xi}\ \boldsymbol{\sigma}$ means the vector $(\xi_{\sigma(1)}, \xi_{\sigma(2)}, \ldots, \xi_{\sigma(n)})$.

Samuelson [1] showed that ***if the distribution of $\boldsymbol{\xi}$ is symmetrical, then the equal weight portfolio is optimal*** for the problem P(S).

Hadar and Russell [2] gave an alternative proof of Samuelson's result based on stochastic dominance. In [3], Tamir found a more general condition than that belonging to Samuelson. He showed that, if $F_{\xi_o \sigma} = F_{\xi}$ for σ = (2, 3, ... , n, 1), then the equal weight portfolio is optimal for the problem P(S). Thus, it is enough that $F_{(\xi_1,\xi_2,\ldots,\xi_n)} = F_{(\xi_2,\ldots,\xi_n,\xi_1)}$ in order that the equal weight portfolio be optimal no matter the concave utility, u. In other words, the equal weight portfolio is absolutely optimal.

Another direction of research was initiated in Deng and Li [4]. They found necessary and sufficient conditions for the optimality of equal weight portfolios for the minimum variance problem.

An interesting problem is to find larger classes of matrices **A** with the following properties:

(i) $F_{\xi}(\mathbf{Ax}) = F_{\xi}(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^n$;
(ii) The equal weight portfolio is optimal for the problem P(S);

Tamir's result [3] for cyclically symmetry of F_{ξ} is the most general condition we know so far on F_{ξ}.

Definition 1. *We state that $\mathbf{x}^o$ is an absolutely optimal portfolio with sum S if $\mathbf{x}^o \in Opt(\boldsymbol{\xi},S;u)$ for every utility $u \in \mathcal{U}_2(S, \boldsymbol{\xi})$. We state that $\boldsymbol{\xi}$ (or better F_{ξ}) has the absolute optimal portfolio property (AOP) if an absolutely optimal portfolio does exist.*

In the following, we shall write 'simply absolute portfolio' instead of 'absolutely optimal portfolio'.

We investigate the following problem: For what random vectors, ξ, do $\mathcal{U}$ -absolute portfolios exist?

In the second section, we recall some facts about stochastic dominance relations and we prove some auxiliary results.

In Section 3, we solve the problem, if $\mathcal{U} = \mathcal{U}_2$, what is the set of concave utility functions; we find necessary and sufficient conditions in order that F_{ξ} has the AOP property, i.e., F_{ξ} admits an absolute portfolio. The main result is the following: If x^0 is a portfolio having all its entries positive, then x^0 is an absolute portfolio if and only if all the conditional expectations of ξ_i, given the return of portfolio x^0, are the same.

In Section 4, we prove that, if $\boldsymbol{\xi}$ is bounded below, then CARA-absolute portfolios are also U_2-absolute portfolios. In the case when ξ_i are rates of returns of assets in a financial market, then all ξ_i are bounded below, hence the above assertion holds.

In Section 5, the classical case when the random vector $\boldsymbol{\xi}$ is normal is analyzed.

We make a complete investigation of the simplest case of a bi-dimensional random vector $\boldsymbol{\xi} = (\xi_1, \xi_2)$ in Section 6. We give a complete characterization and we build two dimensional distributions that are absolutely continuous and admit $\mathcal{U}_2$-absolute portfolios.

The AOP property is connected with the stochastic ordering; actually, this means that the set of probability distributions of the random variables $\mathbf{x}^T\boldsymbol{\xi}$, $\mathbf{x} \in P_n(S)$ has a maximum in the increasing concave order.

2. Stochastic Dominance

The increasing concave order, denoted by "icv", is extensively used in economics, where it is called second-order stochastic dominance: see Belzunce [5], Levy [6], Sriboonchita [7], or Shaked and Shantikumar [8]. Firstly, let us recall the definition of the increasing concave order.

Definition 2. *Let X,Y be two integrable random variables. We state that X is dominated by Y in the increasing concave order if:*

$$\mathrm{E}u(X) \leq \mathrm{E}u(Y) \text{ for every concave utility } u \text{ such that } u(X), u(Y) \in L^1 \tag{4}$$

We shall denote this domination relation by "$X \prec_{\text{icv}} Y$". If the inequality (4) holds for any concave function, u, not necessarily non-decreasing, we say that X is concavely dominated by Y and write "$X \prec_{\text{cv}} Y$". It is known that $X \prec_{\text{cv}} Y$ if, and only if, $X \prec_{\text{icv}} Y$ and $\mathrm{E}X = \mathrm{E}Y$. Notice that the relation "$\prec_{\text{icv}}$" is an order relation between the probability distributions F and G of X and Y and can be better stated as:

$$F \prec_{\text{icv}} G \Leftrightarrow \int u dF \le \int u dG \text{ for every } u \in \mathcal{U}_2 \text{ such that } \int |u| dF, \int |u| dG < \infty$$

In this case, we can restate (3) as:

Proposition 1. *Let* $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$ *be a random vector with i.i.d. integrable entries and* $\mathbf{x^o} = \left(\frac{S}{n}, \ldots, \frac{S}{n}\right)$ *with* $S > 0$. *Then* $\mathbf{x}^T\boldsymbol{\xi} \prec_{\boldsymbol{icv}} (\mathbf{x^o})^T\boldsymbol{\xi}$ *for every* $\mathbf{x} \in \Delta_n(S)$.

Proof. Let $X = (\mathbf{x^o})^T\boldsymbol{\xi} = S\frac{\xi_1+\xi_2+\ldots+\xi_n}{n}$. We confirm that $\mathrm{E}(\xi_i \mid X) = \mathrm{E}(\xi_1 \mid X)$. Indeed, $\mathrm{E}(\xi_i \mid X)$ is a random variable of the form $g_i(X)$ having the property that $\mathrm{E}(\xi_i h(X)) = \mathrm{E}(g_i(X)h(X))$ for every bounded measurable function $h{:}\mathbb{R}\to\mathbb{R}$. As ξ_i are i.i.d.,

$$\begin{aligned} \mathrm{E}(\xi_i h(X)) &= \int x_i h(\tfrac{S}{n}(x_1 + \ldots + x_n)) dF^{\otimes n}(x_1, x_2, \ldots, x_n) \\ &= \int x_1 h(\tfrac{S}{n}(x_1 + \ldots + x_n)) dF^{\otimes n}(x_1, x_2, \ldots, x_n) = \mathrm{E}(\xi_1 h(X)) \end{aligned}$$

The last equality holds because of the equality $\mathrm{d}F^{\otimes n}(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \mathrm{d}F^{\otimes n}(x_1, \ldots, x_n)$. It follows that $\mathrm{E}(\xi_i \mid X) = \mathrm{E}(\xi_1 \mid X)\ \forall\, i = 2, \ldots, n$.

Consequently $X = \mathrm{E}(X \mid X) = \mathrm{E}(S\frac{\xi_1+\xi_2+\ldots+\xi_n}{n} \mid X) = \frac{S}{n}\,(\mathrm{E}(\xi_1 \mid X) + \mathrm{E}(\xi_2 \mid X) + \ldots + \mathrm{E}(\xi_n \mid X)) = \frac{S}{n}\cdot n\cdot \mathrm{E}(\xi_1 \mid X) = S\mathrm{E}(\xi_1 \mid X)$. It follows that $\mathrm{E}(\xi_1 \mid X) = X/S$.

Let $\mathbf{x} \in \Delta_n(S)$ be arbitrary and $v(x) = u(x/S)$. Then, by Jensen's inequality, $\mathrm{E}u(\mathbf{x}^T\boldsymbol{\xi}) = \mathrm{E}(\mathrm{E}(u(\mathbf{x}^T\boldsymbol{\xi}) \mid X)) = \mathrm{E}\left(\mathrm{E}u\left(S\left(\frac{x_1\xi_1+\ldots+x_n\xi_n}{S}\right)|X\right)\right) = \mathrm{E}\left(\mathrm{E}v\left(\frac{x_1\xi_1+\ldots+x_n\xi_n}{S}\right)|X\right) \le \mathrm{E}\left(v\left(\mathrm{E}\left(\frac{x_1\xi_1+\ldots+x_n\xi_n}{S}\right)|X\right)\right) = \mathrm{E}(v(\mathrm{E}(\xi_1 \mid X))) = \mathrm{E}(v(X/S)) = \mathrm{E}u(X)$.

Therefore, $\mathrm{E}u(\mathbf{x}^T\boldsymbol{\xi}) \le \mathrm{E}u(X) = \mathrm{E}u((\mathbf{x^o})^T\boldsymbol{\xi})$; hence, $\mathbf{x^o}$ is indeed optimal. □

Remark. *Taking into account the fact that* ξ_i *have the same expectation (recall that they are identically distributed) we could say even more:* $\mathbf{x}^T\boldsymbol{\xi} \prec_{\boldsymbol{cv}} (\mathbf{x^o})^T\boldsymbol{\xi}$ *for every* $\mathbf{x} \in \Delta_n(S)$. *Moreover, from the above proof, one can easily see that the assumption that the entries* ξ_i *were i.i.d. was not essential. What we really used was the fact that the distribution* $F^{\otimes n}$ *is symmetrical (A random vector* $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$ *is said to be symmetrically distributed if all its permutations* $(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(n)})$ *have the same distribution as* $\boldsymbol{\xi}$*). Thus, the result from Proposition 1 holds for every symmetrically distributed random vector.*

It is important to keep in mind the following simple properties:

Proposition 2.

(i) *If* $X \prec_{\mathbf{icv}} Y$ *then* $\mathrm{E}X \le \mathrm{E}Y$;
(ii) *If* $X \prec_{\mathbf{cv}} Y$ *then* $\mathrm{Var}(X) \ge \mathrm{Var}(Y)$;
(iii) *If* $f{:}\mathbb{R} \to \mathbb{R}$ *is non-decreasing and concave, then* $X \prec_{\mathbf{icv}} Y \Rightarrow f(X) \prec_{\mathbf{icv}} f(Y)$;
(iv) *(Invariance with respect to convolution): If* $X_j \prec_{\mathbf{icv}} Y_j$, $j = 1,2$ *and* (X_1, X_2) *are independent and* (Y_1, Y_2) *are independent, then* $X_1 + X_2 \prec_{\mathbf{icv}} Y_1 + Y_2$;
(v) *(Invariance with respect to mixture): If* $F_j \prec_{\mathbf{icv}} G_j$, $j = 1,2$ *and* $0 \le \lambda \le 1$, *then* $(1-\lambda)F_1 + \lambda F_2 \prec_{\mathbf{icv}} (1-\lambda)G_1 + \lambda G_2$;
(vi) *(Invariance with respect to weak convergence): If* $F_n \prec_{\mathbf{icv}} G_n\ \forall\, n$, $\mathrm{Supp}(F_n) \subseteq [0, \infty)$, $\forall\, n$. $\mathrm{Supp}(G_n) \subseteq [0, \infty)\ \forall\, n$ and $F_n \Rightarrow F$, $G_n \Rightarrow G$ then $F \prec_{\mathbf{icv}} G$.

Proof.

(i) Take the utility $u(x) = x$;

(ii) Take the concave function $f(x) = -x^2$;
(iii) Notice that if u,f are non-decreasing and concave, then $u \circ f$ is also non-decreasing concave;
(iv) See, for instance [5–8];
(v) Obvious;
(vi) We know that $\int udF_n \leq \int udG_n$ for any concave utility u:$[0,\infty) \to \mathbb{R}$ and we want to check that $\int udF \leq \int udG$. We claim that $\int udF_n \to \int udF$, $\int udG_n \to \int udG$. The reason for this is that:

$$\left|\int udF - \int udF_n\right| \leq \left|\int udF - \int \min(u, M)dF\right| + \left|\int \min(u, M)dF - \int \min(u, M)dF_n\right| = I_1(M) + I_2(M, n)$$

Let $\varepsilon > 0$. Then there exists $M_\varepsilon > 0$, such that $I_1(M_\varepsilon) < \varepsilon/2$. The function $f = \min(u,M_\varepsilon)$ is continuous and bounded; hence, $I_2(M_\varepsilon,n) \to 0$ as $n \to \infty$. Thus, for n that is great enough, $\left|\int udF - \int udF_n\right| < \varepsilon$. Therefore, $\int udF_n \to \int udF$, $\int udG_n \to \int udG$.

Thus, our problem becomes:

Let $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$ be a random vector, $S > 0$, $F_{\mathbf{x}} = P \circ (\mathbf{x}^T \boldsymbol{\xi})^{-1}$ be the distribution of the random variable $\mathbf{x}^T \boldsymbol{\xi}$, and let $D(\boldsymbol{\xi}) = \{P \circ (\mathbf{x}^T \boldsymbol{\xi})^{-1} \mid \mathbf{x} \in \Delta_n(S)\}$. When does the family $D(\boldsymbol{\xi})$ have the greatest element with respect to the order "$\prec_{\mathbf{icv}}$"?

Or, even better.

Let F be a probability distribution on $\mathbb{R}^n$, $L_{\mathbf{x}}$: $\mathbb{R}^n \to \mathbb{R}$ be the linear mappings $L_{\mathbf{x}}(\mathbf{y}) = \mathbf{x}^T\mathbf{y}$, and $F_{\mathbf{x}} = F \mathrm{o} (L_{\mathbf{x}})^{-1}$ and $D(F) = \{F_{\mathbf{x}} \mid \mathbf{x} \in \Delta_n(S)\}$. When does $D(F)$ admit the greatest element with respect to the order "$\prec_{\mathbf{icv}}$"?

Then Proposition 1 states that, when $F = G^{\otimes n}$ with G a probability distribution on the real line, then $D(F)$ admits the greatest element. Or, more generally, that if F is symmetric, it also has the greatest element. □

A portfolio $\mathbf{x^o} \in \Delta_n(S)$, with the property that $\boldsymbol{F_{x^o}}$ is the greatest element of $D(F)$, will be called an absolutely optimal portfolio of sum S associated to F. We chose this name because all risk-avoiding decision makers will agree that this maximal element is the best among all portfolios of sum S.

We restate the Definition 1. in stochastic order terms:

Definition 3. *We state that the random vector $\boldsymbol{\xi}$ has the absolute optimum property (AOP) for sum S if the family $D(\boldsymbol{\xi})$ has greatest element with respect to the order "$\prec_{\mathrm{icv}}$". If an absolute optimal portfolio corresponding to this greatest element is $\boldsymbol{x^o} \in \Delta_n(S)$, we will write that $\boldsymbol{\xi}$ has the property AO (x^o) or that $\xi \in AO$ (x^o). In terms of distributions, we state that $F \in$ **AO** ($\mathbf{x^o}$). Precisely:*

$$\xi \in \boldsymbol{AO}\ (\mathbf{x^o}) \Leftrightarrow \mathbf{x}^T \boldsymbol{\xi} \prec_{\mathbf{icv}} (\mathbf{x^o})^T \boldsymbol{\xi} \text{ for every } \mathbf{x} \in \Delta_n(S)$$

Or, in terms of distributions:

$$F \in \boldsymbol{AO}\ (\mathbf{x}^o) \Leftrightarrow \boldsymbol{F} \circ (\boldsymbol{L_{\mathbf{x}}})^{-\mathbf{1}} \prec_{\mathrm{icv}} \boldsymbol{F} \circ (\boldsymbol{L_{\mathbf{x^o}}})^{-\mathbf{1}} \text{ for every } \mathrm{x} \in \Delta_n(S)$$

where $L_{\mathbf{x}}$ is the linear form $L_{\mathbf{x}}(\mathbf{y}) = \mathbf{x}^T \mathbf{y}$, $\mathbf{y} \in \mathbb{R}^n$.

Then, for a given $S > 0$, the class $\boldsymbol{AO}$ of distributions on $\mathbb{R}^n$ is defined by:

$$\boldsymbol{AO} = \bigcup_{\boldsymbol{x^o} \in \Delta_n(S)} \boldsymbol{AO}(\boldsymbol{x^o})$$

3. Properties of Class *AO*

At this point, we do not know if the $\boldsymbol{AO}$ class contains other distributions on $\mathbb{R}^n$ beside the symmetric ones. However, the following properties of the class $\boldsymbol{AO}$ are immediate. We shall state them both in terms of random vectors and in terms of distributions.

Proposition 3. *Invariance Properties. Let $S > 0$ be fixed.*

(*i*) *If* $\boldsymbol{\xi} \in \boldsymbol{AO}$ ($\mathbf{x^o}$), $a \in \mathbb{R}$, $\lambda > 0$ *then* $a{\cdot}\mathbf{e} + \lambda\boldsymbol{\xi} \in \boldsymbol{AO}$ ($\mathbf{x^o}$). *Here* $\mathbf{e} = (1, 1, \ldots, 1) \in \mathbb{R}^n$ (***invariance with respect to scaling***);

(*ii*) *If* $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2$ *are independent and both belong to* $\boldsymbol{AO}$ ($\mathbf{x^o}$), *then* $\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 \in \boldsymbol{AO}$ ($\mathbf{x^o}$) (***invariance with respect to convolutions***);

(*iii*) *If* $\boldsymbol{\xi_1}$, $\boldsymbol{\xi_2}$ *are independent,* $A \in K$ *is independent on both of them and* $\boldsymbol{\xi_1}$, $\boldsymbol{\xi_2} \in \boldsymbol{AO}$ ($\mathbf{x^o}$), *then the random vector* $\xi = \xi_1 1_A + \xi_2 1_{A^c}$ *also belongs to* $\boldsymbol{AO}$ ($\mathbf{x^o}$) *(invariance with respect to mixtures);*

(*iv*) *If* $(\boldsymbol{\xi}_n)_n$ *is a sequence of non-negative random vectors and* $\boldsymbol{\xi_n} \to \boldsymbol{\xi}$ (*in distribution*), then $\boldsymbol{\xi_n} \in \boldsymbol{AO}(\mathbf{x^o})$ $\forall$ n implies $\boldsymbol{\xi} \in \boldsymbol{AO}(\mathbf{x^o})$ (*invariance with respect to weak convergence of nonnegative distributions*);

(*v*) Let $\boldsymbol{\xi} \in \boldsymbol{AO}(\mathbf{x^o})$. *Then* $\mathbf{x^o}/S$ *is absolutely optimal among all the portfolios of sum* 1, *or the same thing in terms of distributions.*

(*i*) $F \in \boldsymbol{AO}(\mathbf{x^o})$, $a \in \mathbb{R}$, $\lambda > 0 \Rightarrow \delta_{a\cdot\mathbf{1}} * F\mathbf{o}(h_\lambda)^{-1} \in \boldsymbol{AO}(\mathbf{x^o})$; here, h_λ is the homothety $h_\lambda(x) = \lambda x$;

(*ii*) $F_1, F_2 \in \boldsymbol{AO}(\mathbf{x^o}) \Rightarrow F_1{*}F_2 \in \boldsymbol{AO}(\mathbf{x^o})$;

(*iii*) $F_1, F_2 \in \boldsymbol{AO}(\mathbf{x^o})$, $0 \le \lambda \le 1 \Rightarrow (1 - \lambda)F_1 + \lambda F_2 \in \boldsymbol{AO}(\mathbf{x^o})$;

(*iv*) $\mathrm{Supp}(F_n) \subseteq \mathbb{R}^n_+$ $\forall$ n, $F_n \Rightarrow F$, $F_n \in \boldsymbol{AO}(\mathbf{x^o})$ $\forall$ $n \Rightarrow F \in \boldsymbol{AO}(\mathbf{x^o})$;

(*v*) *Suppose that* $\mathbf{x^o} \in \Delta_n(S)$ and $F \in \boldsymbol{AO}(\mathbf{x^o})$, *then* $F \in \boldsymbol{AO}(\mathbf{x^o}/S)$ *as well. This means that when dealing with distributions of class* $\boldsymbol{AO}$ *we always can assume that* $S = 1$.

Proof.

(i) The assumption is that $\mathbf{x}^T\boldsymbol{\xi} \prec_{\mathbf{icv}} (\mathbf{x^o})^T\boldsymbol{\xi}$ $\forall$ $\mathbf{x} \in \Delta_n(S)$. Notice that $\mathbf{x}^T(a{\cdot}\mathbf{e} + \lambda\boldsymbol{\xi}) = aS + \lambda\mathbf{x}^T\boldsymbol{\xi}$ and $(\mathbf{x^o})^T(a{\cdot}\mathbf{e} + \lambda\boldsymbol{\xi}) = aS + \lambda(\mathbf{x^o})^T\boldsymbol{\xi}$. Let $X = \mathbf{x}^T\boldsymbol{\xi}$ and $X^o = (\mathbf{x^o})^T\boldsymbol{\xi}$. We know that $X \prec_{\mathbf{icv}} X^o$. Then $aS + \lambda X \prec_{\mathbf{icv}} aS + \lambda X^o$, because of Proposition 2., the function $x \mapsto aS + \lambda x$ is increasingly concave.

(ii) Now we know that $\mathbf{x}^T\boldsymbol{\xi}_1 \prec_{\mathbf{icv}} (\mathbf{x^o})^T\boldsymbol{\xi}_1$ $\forall$ $\mathbf{x} \in \Delta_n(S)$ and that $\mathbf{x}^T\boldsymbol{\xi}_2 \prec_{\mathbf{icv}} (\mathbf{x^o})^T\boldsymbol{\xi}_2$. By Proposition 2. (iv), we see that $\mathbf{x}^T(\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2) \prec_{\mathbf{icv}} (\mathbf{x^o})^T(\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2)$.

(iii) Let $F_j = P\mathrm{o}(\boldsymbol{\xi}_j)^{-1}$ with j = 1,2 and $1 - \lambda = P(A)$. Then the distribution of $\boldsymbol{\xi}$ is $(1 - \lambda)F_1 + \lambda F_2$. We assumed that $F_j \in AO(\mathbf{x^o}) \Leftrightarrow F_j \circ (L_{\mathbf{x}})^{-1} \prec_{\mathrm{icv}} F_j \circ (L_{\mathbf{x^o}})^{-1} \forall$ $\mathbf{x} \in \Delta_n(S)$, j = 1,2. Then, by Proposition 2. (v), we have $(1-\lambda)\left(F_1 \circ (L_{\mathbf{x}})^{-1}\right) + \lambda\left(F_2 \circ (L_{\mathbf{x}})^{-1}\right) \prec_{\mathrm{icv}} (1-\lambda)\left(F_1 \circ (L_{\mathbf{x^o}})^{-1}\right) + \lambda\left(F_2 \circ (L_{\mathbf{x^o}})^{-1}\right)$ or $((1-\lambda)F_1 + \lambda F_2) \circ (L_{\mathbf{x}})^{-1} \prec_{\mathrm{icv}} ((1-\lambda)F_1 + \lambda F_2) \circ (L_{\mathbf{x^o}})^{-1}$ for every $\mathbf{x} \in \Delta_n(S)$, hence $(1 - \lambda)F_1 + \lambda F_2 \in AO(\mathbf{x^o})$.

(iv) A consequence of Proposition 2 (v).

(v) Obvious: $\mathbf{x}^T\boldsymbol{\xi} \prec_{\mathbf{icv}} (\mathbf{x^o})^T\boldsymbol{\xi}$ $\forall$ $\mathbf{x} \in \Delta_n(S) \Rightarrow \mathbf{x}^T\boldsymbol{\xi}/S \prec_{\mathbf{icv}} (\mathbf{x^o})^T\boldsymbol{\xi}/S$ $\forall$ $\mathbf{x} \in \Delta_n(S) \Leftrightarrow \mathbf{y}^T\boldsymbol{\xi} \prec_{\mathbf{icv}} (\mathbf{x^o}/S)^T\boldsymbol{\xi}$ $\forall$ $\mathbf{x} \in \Delta_n(1)$. □

Important Remark *Based on the above proposition, namely on point (v), it suffices to assume that* $S = 1$. ***In the sequel we shall agree that always*** $\boldsymbol{S = 1}$***. Thus,*** $\boldsymbol{\Delta_n}$ ***will mean*** $\boldsymbol{\Delta_n(1)}$. *Now we give a necessary and sufficient condition in order that a distribution on* $\mathbb{R}^n$, *F be in the class* $AO(\boldsymbol{x^o})$, *provided that* $x_i^0 > 0$ *for all* $i = 1, \ldots, n$.

Proposition 4. *Let* $\boldsymbol{x^o} \in \Delta_n$ *be such that* $x_i^0 > 0$ $\forall$ $i = 1, \ldots, n$. *Let also* $X^0 = (\boldsymbol{x^o})^T\boldsymbol{\xi}$. *Then*

$$\boldsymbol{\xi} \in AO\ (\mathbf{x}^o) \Leftrightarrow \mathrm{E}(\xi_1 \middle| X^0) = \mathrm{E}(\xi_2 \middle| X^0) = \ldots = \mathrm{E}(\xi_n \middle| X^0) = X^0 \quad (5)$$

Proof. The easy part is "$\Leftarrow$". Indeed, let $\mathbf{x} \in \Delta_n$ and let $u \in \boldsymbol{\mathcal{U}}_2(1, \xi)$. Then

$$\mathrm{E}u(\mathbf{x}^T\boldsymbol{\xi}) = \mathrm{E}[\mathrm{E}u(\mathbf{x}^T\boldsymbol{\xi}|X^0)] \le \mathbf{E}[u(\mathrm{E}(\mathbf{x}^T\xi|X^0)] = \mathrm{E}[u(x_1\mathrm{E}(\xi_1|X^0) + x_2\mathrm{E}(\xi_2|X^0) + \ldots + x_n\mathrm{E}(\xi_n|X^0))] = \mathrm{E}u(x_1X^0 + x_2X^0 + \ldots + x_nX^0) = \mathrm{E}u(X^0) = \mathrm{E}u((\mathrm{x}^o)^{\boldsymbol{T}}\xi);$$

hence, $\mathbf{x^o}$ is an absolute optimum. We prove now "$\Rightarrow$". Let $u \in \mathcal{U}_2(\mathbf{1}, \boldsymbol{\xi})$. Consider the function $h_u{:}P_{n-1} \to \mathbb{R}$ defined by:

$$h_u(x_1, \ldots, x_{n-1}) = \mathrm{E}u(x_1\xi_1 + \ldots + x_{n-1}\xi_{n-1} + (1 - x_1 - \ldots - x_{n-1})\xi_n) \tag{6}$$

□

Suppose that u is differentiable, then h_u is also differentiable. Let $\mathbf{g}$ = Grad(h_u). Thus,

$$g_i(x_1, \ldots, x_{n-1}) = \frac{\partial h_u}{\partial x_i}(x_1, \ldots, x_{n-1}) = E[(\xi_1 - \xi_n)u\prime(x_1\xi_1 + \ldots + x_{n-1}\xi_{n-1} + (1 - x_1 - \ldots - x_{n-1})\xi_n] \tag{7}$$

Let $\mathbf{y^o} = \left(x_1^0, x_2^0, \ldots, x_{n-1}^0\right) \in P_{n-1}$. We know that h_u attains its maximum at $\mathbf{y^o}$ and $\mathbf{y^o}$ is an interior point in P_{n-1}. Then:

$$\mathbf{g}(\mathbf{y^o}) = 0 \Leftrightarrow \mathrm{E}\left[(\xi_i - \xi_n)u\prime(X^0\ )\right] = 0\ \forall\ i = 1, \ldots, n-1\ \forall\ u\ \in\ \mathcal{U}_2(\mathbf{1}, \boldsymbol{\xi}) \tag{8}$$

Recall that u is non-decreasing and concave; hence, u' is decreasing, continuous, and non-negative. Conversely, any function $g{:}\mathbb{R} \to [0,\infty)$ continuous and decreasing is the derivative of some concave utility. Taking a sequence of this kind of functions that monotonically converges to $1_{(-\infty,a]}$, we arrive at the conclusion that:

$$\mathrm{E}\left[(\xi_i - \xi_n);\ X^0\ \le\ a\right] = 0 \text{ for every } i = 1, \ldots, n-1 \text{ and } a\ \in\ \mathbb{R} \tag{9}$$

Denote by $\boldsymbol{\mathcal{B}}$ ($\mathbb{R}$) the σ-algebra of the Borel sets on the real line.

Let $\boldsymbol{C} = \{A \in \boldsymbol{\mathcal{B}}(\mathbb{R})\text{: } \mathrm{E}((\xi_i - \xi_n)1_A(X^0) = 0\}$. According to (9), $\boldsymbol{C}$ contains all the intervals $(-\infty, a]$. Moreover, it fulfills the following conditions:

(i) $\mathbb{R} \in \boldsymbol{C}$;
(ii) $A \in \boldsymbol{C} \Rightarrow A^c \in \boldsymbol{C}$;
(iii) $A_n \in \boldsymbol{C}$ and $(A_n)_n$ are disjoint $\Rightarrow \underset{n}{\cup} A_n \in \boldsymbol{C}$.

Property (i) follows if we let $a \to \infty$. Property (ii) results from the following series of equalities: $0 = \mathrm{E}(\xi_i - \xi_n) - \mathrm{E}((\xi_i - \xi_n)1_A(X^0) = \mathrm{E}((\xi_i - \xi_n)(1 - 1_A)(X^0)$. Property (iii) follows from the Lebesgue's domination principle.

Such a family of sets is called a Dynkin − system or a *U-system*. So$\boldsymbol{C}$ is a *U-system that contains the intervals* $(-\infty, a]$. It then contains the *U-system generated by these intervals*, but it is standard knowledge that this *U-system is the Borel* σ-algebra *on the real line*, $\boldsymbol{\mathcal{B}}$ ($\mathbb{R}$). Thus, $\boldsymbol{C}$ contains all the Borelian sets. The conclusion is that:

$$\mathrm{E}\left[(\xi_i - \xi_n)\ 1_A(X^0)\right] = 0\ \forall\ I = 1\ , \ldots,\ n - 1\ \forall\ A \in\ \boldsymbol{\mathcal{B}}\ (\mathbb{R}) \tag{10}$$

By a standard argument (if (10) holds for indicators, then it holds for simple functions and it holds for positive measurable functions). It follows that:

$$\mathrm{E}\left[\xi_i\ f(X^0\ )\right] = \mathrm{E}\left[\xi_n\ f(X^0)\right]\ \forall\ i = 1, \ldots, n-1\ \forall\ f : \mathbb{R}\ \to\ \mathbb{R} \text{ measurable bounded} \tag{11}$$

The conclusion is that $\mathrm{E}(\xi_i \mid X^0) = \mathrm{E}(\xi_n \mid X^0)$. Indeed, $\mathrm{E}(\xi_i \mid X^0)$ is a random variable of the form $\varphi_i(X^0)$ with the property that $\mathrm{E}(\varphi_i(X^0)f(X^0)) = \mathrm{E}(\xi_i f(X^0))$ for any f that is measurable and bounded. Hence, $\mathrm{E}(\varphi_i(X^0)f(X^0)) = \mathrm{E}(\varphi_n(X^0)f(X^0))$ for any f that is measurable and bounded; therefore, $\varphi_i(X^0) = \varphi_n(X^0)$. We claim that, in this case, $\mathrm{E}(\xi_i \mid X^0) = X^0$. Indeed,

$$X^0 = \mathrm{E}(X^0 \middle| X^0\ ) = \mathrm{E}(x^o{}_1\xi_1 + \ldots + x^o{}_n\xi_n \middle| X^0\ ) = x_1^0\,\mathrm{E}(\xi_1 \middle| X^0\ ) + \ldots + x_n^0\,\mathrm{E}(\xi_n \middle| X^0\ ) =$$
$$(x_1^0 + \ldots + x_n^0\ )\mathrm{E}(\xi_n \middle| X^0\ ) = \mathrm{E}(\xi_n \middle| X^0)$$

Hence:

$$\mathrm{E}(\xi_i \Big| X^0\ ) = \mathrm{E}(X_n \Big| X^0\ ) = X^0$$

As a consequence, we find a necessary condition in order that $\boldsymbol{\xi}$ may have the property ***AO***.

Corollary 1. *If* $\boldsymbol{\xi} \in \boldsymbol{AO}\ (\mathbf{x^o})$ *and* $x_i^0 > 0\ \forall\ I = 1, \ldots, n$, *then* $E\xi_1 = E\xi_2 = \ldots = E\xi_n$.

Proof. According to Proposition 3.2, $\mathrm{E}(\xi_1 \mid X^0) = \mathrm{E}(\xi_2 \mid X^0) = \ldots = \mathrm{E}(\xi_n \mid X^0)$. If we average the preceding sequence of equalities, we obtain the conclusion of the Corollary. □

What if $\boldsymbol{\xi} \in AO\ (\mathbf{x^o})$ but some of the components of $\mathbf{x^o}$ are equal to 0? Is that possible?

In order to answer this question, let us suppose that the components of $\boldsymbol{\xi}$ are ordered in such a way that $\mathrm{E}X_1 \le \mathrm{E}X_2 \le \ldots \le \mathrm{E}X_n$. It is possible that some of the expectations coincide. In general, we shall use the convention:

$$\mu_1 \le \mu_2 \le \ldots \le \mu_k < \mu_{k+1} = \mu_{k+2} = \ldots = \mu_n$$

Denote by INC(k), the set of all random vectors $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$ with the above property.

For $k = 0$, we imply that all the expectations coincide and for $k = n - 1$ imply that $\mu_{n-1} < \mu_n$. Then Corollary 1. states that, if $\boldsymbol{\xi}$ has an absolute optimal portfolio with all the components positive, then $\boldsymbol{\xi} \in$ INC(0).

Proposition 5. *Let* $\xi \in$ *INC(k) with* $k \ge 1$.

If $\boldsymbol{\xi}$ has an absolute optimal portfolio $\mathbf{x^o}$, then $x_1^0 = \ldots = x_k^0 = 0$. Moreover, $\mathrm{Var}((\mathbf{x^o})^T\boldsymbol{\xi}) \le \mathrm{Var}(\mathbf{x}^T\boldsymbol{\xi})\ \forall\ \mathbf{x} \in \Delta_n$.

In words: If an absolute optimal portfolio does exist, it should both maximize the expectation and minimize the variance.

Proof. Let $X^0 = (\mathbf{x^o})^T\boldsymbol{\xi}$. Suppose, ad absurdum, that ${x^o}_i > 0$ for some $i \le k$. Then $\mathrm{E}X^0 < \mathrm{E}X_n$; hence, X^0 cannot be optimal, according to Proposition 2. (i), which states that $\mathrm{E}X_n \le \mathrm{E}X^0$. Therefore, the possible absolute optima must be of the form $\mathbf{x^o} = \left(0, 0, \ldots, x_{k+1}^0, \ldots, x_n^0\right)$. All the portfolios $\mathbf{x}$ with $x_j = 0\ \forall\ j \le k$ have the property that $\mathrm{E}(\mathbf{x}^T\boldsymbol{\xi}) = \mu_n = \max_{1 \le j \le n} \mu_j$. The absolute optimal portfolio $\mathbf{x^o}$, provided that it does exist at all, should dominate them in the "$\prec_{\mathbf{icv}}$" order. As it has the same expectation, μ_n, it should dominate them in the concave order "$\prec_{\mathbf{cv}}$"; according to Proposition 2 (ii), $\mathrm{Var}((\mathbf{x^o})^T\boldsymbol{\xi}) \le \mathrm{Var}(\mathbf{x}^T\boldsymbol{\xi})$ for all such $\mathbf{x}$. If another portfolio, say $\mathbf{y} \in \boldsymbol{\Delta_n}$, would exist such that $\mathrm{Var}((\mathbf{x^o})^T\boldsymbol{\xi}) > \mathrm{Var}(\mathbf{y}^T\boldsymbol{\xi})$, then $(\mathbf{x^o})^T\boldsymbol{\xi}$ could not dominate $\mathbf{y}^T\boldsymbol{\xi}$ in the "$\prec_{\mathbf{icv}}$" order. □

Let us denote $P_{n-1}(k) = \{\mathbf{y} \in P_{n-1}\colon y_1 = \ldots = y_k = 0, y_j > 0\ \forall\ j > k\ \}$. Note that $P_{n-1}(0)$ is the interior of P_{n-1}. Let $\Delta_{n,k} = \{\mathbf{x} \in \Delta_n\colon x_1 = \ldots = x_k = 0, x_j > 0\ \forall\ j > k\ \}$. Note that $\Delta_{n,0} = \{\mathbf{x} \in \Delta_n\colon x_j > 0\ \forall\ j\}$. We want to find necessary and sufficient conditions in order that $\boldsymbol{\xi} \in \boldsymbol{AO}\ (\mathbf{x^o})$, $\mathbf{x^o} \in \Delta_{n,k}$. We shall use the following particular case of Kuhn–Tucker conditions, which probably is well known:

Lemma 1. *Let* $f\colon P_{n-1} \to \mathbb{R}$ *be concave and differentiable and let* $\boldsymbol{g}$ = *Grad(f). Let* $\boldsymbol{a} \in \mathrm{P_{n-1}}\ (k)$. Then:

$$f(\mathbf{a}) \ge f(\mathbf{y})\ \forall\ \mathbf{y} \in P_{n-1} \Leftrightarrow g_j(\mathbf{a}) \le 0\ \forall\ j \le k,\ g_j(\mathbf{a}) = 0\ \forall\ j > k \tag{12}$$

Proof. "⇒": Let **a**,**y** $\in P_{n-1}$ and $\varphi_{\mathbf{a},\mathbf{y}}$: $[0,1] \to \mathbb{R}$ be defined as $\varphi_{\mathbf{a},\mathbf{y}}(t) = f((1-t)\mathbf{a} + t\mathbf{y})$, $t \in [0,1]$. Then, $\varphi_{\mathbf{a},\mathbf{y}}$ is differentiable, concave, and its maximum is at $t = 0$. Therefore, it is non-increasing, hence $(\varphi_{\mathbf{a},\mathbf{y}})'(0) \le 0$. As $\mathbf{a} \in P_{n-1}(k)$, and we see that:

$$(\varphi_{\mathbf{a},\mathbf{y}})'(0) = \sum_{j=1}^{n-1}(y_j - a_j)g_j(\boldsymbol{a}) = \sum_{j=1}^{k} y_j g_j(\boldsymbol{a}) + \sum_{j=k+1}^{n-1}(y_j - a_j)g_j(\boldsymbol{a}) \le 0\ \forall \mathbf{y} \in P_{n-1}$$

If we choose $\mathbf{y} \in P_{n-1}(k)$, the condition becomes $\sum_{j=k+1}^{n-1}(y_j - a_j)g_j(\boldsymbol{a}) \le 0$, which of course implies $g_j(\mathbf{a}) = 0\ \forall j > k$ (we can choose $\mathbf{y} = \mathbf{a} \pm \varepsilon \mathbf{e_j}$ with $\varepsilon > 0$ small enough). Thus, we get:

$$\sum_{j=1}^{k} y_j g_j(\mathbf{a}) \le 0\ \forall\, \mathbf{y} \in P_{n-1} \Rightarrow g_j(\mathbf{a}) \le 0\ \forall\, j \le k$$

Conversely, if $g_j(\mathbf{a}) \le 0\ \forall j \le k$, $g_j(\mathbf{a}) = 0\ \forall j > k$, it is obvious that $(\varphi_{\mathbf{a},\mathbf{y}})'(0) \le 0$; as $\varphi_{\mathbf{a},\mathbf{y}}$ is concave, this fact implies that $\varphi_{\mathbf{a},\mathbf{y}}$ is non-increasing, hence $\varphi_{\mathbf{a},\mathbf{y}}(0) \ge \varphi_{\mathbf{a},\mathbf{y}}(1) \Leftrightarrow f(\mathbf{a}) \ge f(\mathbf{y})$. □

Proposition 6. *Let $x^o \in \Delta_{n,k}$ and $X^o = (x^o)^T\xi$. Then $\boldsymbol{\xi} \in \boldsymbol{AO}\ (\mathbf{x^o}) \Leftrightarrow E(\xi_j/X^0 \le a) \le E(\xi_n/X^0 \le a)\ \forall j \le k$, $a \in \mathbb{R}$ and:*

$$\mathrm{E}(\xi_j \big| X^0) = X^0\ \forall\, j > k \tag{13}$$

Proof. Let u be a concave differentiable utility, $\mathbf{z} \in P_{n-1}(k)$, defined as:

$$\mathbf{z} = \left(0, 0, \ldots, x^0_{k+1}, \ldots, x^0_{n-1}\right) \text{ and } h_u(\mathbf{y}) = \mathrm{E}u(y_1\xi_1 + \ldots + y_{n-1}\xi_{n-1} + (1 - y_1 - \ldots - y_{n-1})\xi_n)$$

If $\boldsymbol{\xi} \in \boldsymbol{AO}\ (\mathbf{x^o})$, then the concave differentiable function, h_u, attains its maximum at $\mathbf{z}$. Let $\mathbf{g} = \mathrm{Grad}(h_u)$. According to Lemma 1, $g_j(\mathbf{z}) \le 0\ \forall j \le k$, $g_j(\mathbf{z}) = 0\ \forall j > k$ or:

$$\mathrm{E}\Big[(\xi_j - \xi_n)u'(X^0)\Big] \le 0\ \forall\, j \le k, \mathrm{E}\Big[(\xi_j - \xi_n)u'(X^0)\Big] = 0\ \forall\, j > k\ \forall\, u \in \boldsymbol{\mathcal{U}}_2 \text{ differentiable} \tag{14}$$

□

In the same way as we did in the proof of Proposition 4., we can take u' to be $1_{(-\infty,\, a]}$. Thus, (14) becomes:

$$\mathrm{E}\Big[(\xi_j - \xi_n)1_{(-\infty,a]}(X^0)\Big] \le 0\ \forall\, j \le k, \mathrm{E}\Big[(\xi_j - \xi_n)\, 1_{(-\infty,a]}\ (X^0)\Big] = 0\ \forall\, j > k\ \forall\, a \in \mathbb{R} \tag{15}$$

which implies (13), by the same argument used in the proof of Proposition 4.:

$$\mathrm{E}(\xi_j \big| X^0) = \mathrm{E}(\xi_n \big| X^0)\ \forall\, j > k$$

which implies that:

$$X^o = E(X^o|X^o) = \sum_{j=k+1}^{n} x_j^o E(\xi_j|X^o) = \left(\sum_{j=k+1}^{n} x_j^o\right) E(\xi_n|X^o) = E(\xi_n|X^o)$$

Hence, $\mathrm{E}(\xi_j \mid X^o) = X^o$ for every $j > k$.

Conversely, if the conditions from the right hand of (13) hold, then $\mathrm{E}(\xi_i u'(X^o)) \le \mathrm{E}(\xi_n u'(X^o))\ \forall j \le k$ and $\mathrm{E}(\xi_i u'(X^0)) = \mathrm{E}(\xi_n u'(X^0))\ \forall j > k$ for any u concave that is differentiable and non-decreasing. By Lemma 1., the function h_u attains its maximum at $\mathbf{x^o}$ for any differentiable concave utility. Thus $\mathrm{E}u((\mathbf{x^o})^T\boldsymbol{\xi}) \ge \mathrm{E}u((\mathbf{x})^T\boldsymbol{\xi})$ for any $\mathbf{x} \in \Delta_n$, u differentiable concave that is non-decreasing. However, the restriction that u be differentiable is not a serious one: any utility is the uniform limit of differentiable ones.

Corollary 2. *Let $\boldsymbol{x^o} \in \Delta_{n,k}$. A sufficient condition in order that $\boldsymbol{\xi} \in AO$ ($\boldsymbol{x^o}$) is*

$$\mathrm{E}\left[(\xi_j - \xi_n)\middle|X^0\right] \leq 0\,\forall\, j \leq k, \mathrm{E}\left[(\xi_j - \xi_n)\middle|X^0)\right] = 0\,\forall\, j > k \tag{16}$$

Proof. Obviously, $\mathrm{E}[(\xi_j - \xi_n) \mid X^0] \leq 0$ implies $\mathrm{E}(\xi_i u'(X^0)) \leq \mathrm{E}(\xi_n u'(X^0))\ \forall\, u$ differentiable, concave, and non-decreasing, such that the integral makes sense. □

4. The Class *CARA*AO

In this section, we shall deal only with short tailed random vectors. A random vector $\xi = (\xi_1,\ldots,\xi_n)$ is ***short tailed*** if the moment generating functions, $\boldsymbol{m_i}(t) = \mathrm{E}[\exp(t\xi_i)]$, are finite in a neighborhood of 0. If we consider only CARA utilities of the form $u(x) = -\,e^{-rx}$, we can give the following.

Definition 4. *Let $\boldsymbol{x^o} \in \Delta_n$ and $\boldsymbol{\xi}$ be a random vector. We say that $\boldsymbol{\xi}$* ***has the property CARAAO($\mathbf{x^o}$)*** *(and write $\xi \in$* ***CARAAO($\mathbf{x^o}$)****) if:*

$$Ee^{-r(\mathbf{x}^o)^T\xi} \leq Ee^{-r\mathbf{x}^T\xi} \tag{17}$$

for any $r > 0$ such that both the expectations are finite. The union of all the classes ***CARAAO($\mathbf{x^o}$)*** *is defined to be the class* ***CARAAO****.*

It is obvious that, if $\xi \in \mathbf{AO}(\mathbf{x^o})$, then $\xi \in$ ***CARA*AO($\mathbf{x^o}$)**

The fact is that, in some cases, the classes ***CARA*AO** and **AO** coincide. Some conditions under which the equality of classes holds are presented in the following proposition.

Proposition 7. *Suppose that $\boldsymbol{\xi}$ is bounded below, in the sense that there exists $m \in \mathbb{R}$ such that $\xi_i \geq m$ a.s. $\forall\, i = 1,\ldots,n$. Then,* ***CARAAO($\boldsymbol{x^o}$) = AO($\boldsymbol{x^o}$)*** *for every $\boldsymbol{x^o} \in \Delta_{n,0}$. Or, in words: if all the components of $\boldsymbol{x^o}$ are positive, then the classes* ***CARAAO($\boldsymbol{x^o}$)*** *and* ***AO($\boldsymbol{x^o}$)*** *coincide.*

Proof. Let $\mathbf{x^o} \in \Delta_{n,0}$ and $\mathbf{y^o} = \left(x_1^0, x_2^0, \ldots, x_{n-1}^0\right) \in \mathrm{P}_{n-1}$. Let h: $\mathrm{P}_{n-1} \to \mathbb{R}$ defined by

$$h(\mathbf{x}) = e^{-r(x_1\xi_1+\ldots+x_{n-1}\xi_{n-1}+(1-x_1-\ldots-x_{n-1})\xi_n)},\ \mathbf{x} \in P_{n-1}$$

The function h is concave and attains its maximum at $\mathbf{y^o}$, which is an interior point of the compact $P_{\boldsymbol{n-1}}$. Let $X^0 = (\mathbf{x^o})^T\boldsymbol{\xi}$. Then:

$$(\mathrm{Grad\ h})(\mathrm{y^o}) = 0 \Leftrightarrow \mathrm{E}[(\mathrm{X_i} - \mathrm{X_n})\mathrm{e}^{-\mathrm{r}\xi}] = 0\ \forall\ \mathrm{r} \geq 0,\ \mathrm{i} = 1,..,\mathrm{n} - 1$$

Let $\mathrm{Z} = e^{-X^0} \leq e^{-m}$ a.s. The condition (Grad h)($\mathbf{y^o}$) = 0 becomes:

$$\mathrm{E}(\xi_j Z^r) = \mathrm{E}(\xi_n Z^r)\ \forall\, r > 0\ \forall\, j = 1,\ldots,n-1 \tag{18}$$

Which further implies:

$$\mathrm{E}(\xi_j f(\mathrm{Z})) = \mathrm{E}(\xi_n f(\mathrm{Z})) \text{ for any polynomial function } f,\ \forall\, j = 1,\ldots,n-1 \tag{19}$$

As the polynomials are dense in $C([0.e^{-m}])$, equality (19) holds for any bounded continuous function, f. The standard procedure is: Approximate the indicators $1_{(a,b]}$ with continuous functions and check that (19) implies the equality $\mathrm{E}(\xi_j\, 1_{(a,b]}(\mathrm{Z})) = \mathrm{E}(\xi_n\, 1_{(a,b]}(\mathrm{Z}))$ $\forall\, j = 1,\ldots,n\ \forall\, a < b \in \mathbb{R}$. The conclusion is that $\mathrm{E}(\xi_j \mid \mathrm{Z}) = \mathrm{E}(\xi_n \mid \mathrm{Z})$ for any $j = 1,\ldots,n-1$. As Z and X^0 generate the same σ-algebra; we see that condition (5) is satisfied. The proof is finalized by applying Proposition 4. □

Remark. *In the case when ξ_I are rates of returns of assets in a financial market, we have* $\xi_i \geq -1$ *for all* i. *The hypothesis that* ξ_i *are bounded below from the preceding proposition is verified.*

Example. *Suppose that ξ_j ~ Gamma(a_j,a_j), j = 1,...,n with a_j > 0 and $(\xi_j)_{1\leq j\leq n}$ are independent. Then* $\boldsymbol{\xi}$ *∈ CARAAO ($\boldsymbol{x^o}$) with:*

$$x^o = \left(\frac{a_1}{S}, \frac{a_2}{S}, \ldots, \frac{a_n}{S}\right), S = \sum_{j=1}^{n} a_j$$

Indeed, we have to minimize the function:

$$h(x_1, \ldots, x_{n-1}) = \ln\left[\mathrm{E}e^{-r(x_1\xi_1+\ldots+x_{n-1}\xi_{n-1}+(1-x_1-\ldots-x_{n-1})\xi_n)}\right] \text{ with } \mathbf{x} \in P_{n-1}$$

As $(\xi_j)_j$ are independent,

$$h(\mathbf{x}) = a_1 ln\frac{a_1}{a_1+rx_1} + a_2 ln\frac{a_2}{a_2+rx_2} + \ldots + a_n ln\frac{a_n}{a_n+sx_n} \text{ with } x_n = 1 - x_1 - \ldots - x_{n-1}$$

After some calculus, one obtains the gradient $\mathbf{g}$ = Grad(h). Its components are:

$$g_i(\mathbf{x}) = \frac{r^2(a_i x_n - a_n x_i)}{(a_i + rx_i)(a_n + rx_n)} \text{ thusGrad}(h) = \mathbf{0} \Leftrightarrow x_i = \frac{a_1}{a_n} x_n$$

Therefore, $\mathbf{x^o} = \left(\frac{a_1}{S}, \frac{a_2}{S}, \ldots, \frac{a_n}{S}\right)$. From the above proposition, $\mathbf{x^o}$ is an absolute optimal portfolio.

Remark. *Even for n = 2, it is **not true** that if ξ_1 and ξ_2 are independent and Eξ_1 = Eξ_2, then (ξ_1,ξ_2) has the property **AO**. Suppose that ξ_1 ~ Uniform(−a,a) and ξ_2 ~ Uniform(−b,b) for some a,b > 0, a ≠ b. Then the minimum point of the function*

$$h(x) = Ee^{-rx\xi_1 - r(1-x)\xi_2} = \frac{sinh(arx)sinh(br(1-x)}{abr^2 x(1-x)}, \ 0 \leq x \leq 1$$

depends strongly on r; hence, $\boldsymbol{\xi}$ *cannot belong to **CARAAO**.*

Counterexample. *Proposition 7 says that if the CARA-absolute optimal portfolio is in the interior of the simplex Δ_n, then in many cases it is an absolutely optimal portfolio as well. However, if $\boldsymbol{x^o}$ is not an interior point, if it belongs to the face $\Delta_{n,k}$, k ≥ 1 of the simplex, that fails to be true. Let $\boldsymbol{x}_1$ = (0;1), $\boldsymbol{x}_2$ = (0;3), $\boldsymbol{x}_3$ = (1;3), and $\boldsymbol{x}_4$ = (3;2) four points in the plane, and suppose that (ξ_1,ξ_2) ~ $\begin{pmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 & \boldsymbol{x}_4 \\ \alpha & \gamma & \delta & \beta \end{pmatrix}$. Then ξ_1 ~ $\begin{pmatrix} 0 & 1 & 3 \\ \alpha+\gamma & \delta & \beta \end{pmatrix}$, ξ_2 ~ $\begin{pmatrix} 1 & 2 & 3 \\ \alpha & \beta & \gamma+\delta \end{pmatrix}$.*

The condition that $\mathbf{x^o}$ = (0;1) is the absolute optimal portfolio is that:

$$\mathrm{E}(\xi_1 \mid \xi_2 \leq a) \leq \mathrm{E}(\xi_1 \mid \xi_2 \leq a) \ \forall \ a$$

One can easily check that $\mathbf{x^o}$ = (0;1) is the absolute optimal portfolio if and only if $\beta \leq \alpha$; $\mathbf{x^o}$ = (0;1) is an absolute CARA-optimal portfolio if and only if [$\frac{\beta}{2\alpha} \leq 1$ and $3\gamma + 2\delta \geq \beta - \alpha$] or [$\frac{\beta}{2\alpha} > 1$ and $3\gamma + 2\delta \geq \left(\frac{\beta}{2\alpha}\right)^2$]. $\xi_1 \prec_{\mathbf{icv}} \xi_2 \Leftrightarrow \mathrm{E}\xi_1 \leq \mathrm{E}\xi_2 \Leftrightarrow 3\gamma + 2\delta \geq \beta - \alpha$.

Thus, there exists no implications that "$\mathbf{x}^0$ is an absolute CARA-optimal portfolio, ⇒ $\mathbf{x}^0$ is an absolute optimal portfolio" or even "$\xi_1 \prec_{\mathrm{icv}} \xi_2 \Rightarrow \mathbf{x^o}$ = (0;1) is an absolute CARA-optimal portfolio", as we thought.

5. The Normal Case

This is the classic case. The literature concerning it is extensive. The unidimensional normal distributions are very convenient because the "$\prec_{\mathbf{icv}}$" order is easy to establish:

Normal $(\mu,\sigma^2) \prec_{\mathbf{icv}}$ Normal $(\mu',\sigma'^2) \Leftrightarrow \mu \leq \mu'$, $\sigma \geq \sigma'$.

Moreover, if ξ ~ Normal $(\boldsymbol{\mu},\mathbf{C})$ is a n-dimensional random vector and $\mathbf{x} \in \Delta_n(S)$ is a portfolio of sum S, then $\mathbf{x}^T\xi$ ~ Normal$(\mathbf{x}^T\boldsymbol{\mu}, \mathbf{x}^T\mathbf{C}\mathbf{x})$. Therefore, it is very easy to compare two portfolios $\mathbf{x}$ and $\mathbf{y}$:

$$\mathbf{x}^T\xi \prec_{\mathbf{icv}} \mathbf{y}^T\xi \Leftrightarrow \mathbf{x}^T\boldsymbol{\mu} \leq \mathbf{y}^T\boldsymbol{\mu} \text{ and } \mathbf{x}^T\mathbf{C}\mathbf{x} \geq \mathbf{y}^T\mathbf{C}\mathbf{y} \tag{20}$$

Because these relations are homogeneous, the sum S does not matter; hence, we always can restrict ourselves to consider only portfolios from Δ_n.

We can now answer the question: when does $\xi \in \mathbf{AO}$ provided that ξ ~ Normal $(\boldsymbol{\mu},\mathbf{C})$?

Assuming the convention INC(k), we agree that $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_k < \mu_{k+1} = \mu_{k+2} = \ldots = \mu_n$.

Proposition 8. *Let $\boldsymbol{\xi}$ ~ Normal $(\boldsymbol{\mu},\boldsymbol{C})$. Then $\boldsymbol{\xi} \in \boldsymbol{AO}$ if the Pareto problem*

$$\begin{cases} \text{maximize } \left(\mathbf{x}^T\mu\right) \\ \text{minimize } \left(\mathbf{x}^T\boldsymbol{C}\boldsymbol{x}\right) \\ \boldsymbol{x}^T\cdot\boldsymbol{e} = 1,\ \boldsymbol{x} \geq 0, \end{cases} \tag{21}$$

admits at least solution.

Proof. Obvious from (20). □

Proposition 9. *Let $\boldsymbol{\xi}$ ~ Normal $(\boldsymbol{\mu},\boldsymbol{C}) \cap$ INC(k). Then*

(i) *If $k = 0$, then $\xi \in \mathbf{AO}$;*

1. *If $k \geq 1$ then $\xi \in \mathbf{AO}$* if *at least one solution $\mathbf{x}^{\mathbf{o}}$ of the problem*

$$\begin{cases} \textit{minimize } \left(\mathbf{x}^T\mathbf{C}\mathbf{x}\right) \\ \mathbf{x}^T\cdot\mathbf{e} = 1,\ \mathbf{x} \geq 0, \end{cases} \tag{22}$$

*has the property that $x_j^0 = 0\ \forall\, j \leq k$. A **sufficient** condition for that to happen is that*

$$c_{i,j} \geq c_{n,j}\ \forall\, j = k+1,\ldots,n\ \forall\, i = 1,\ldots,k \tag{23}$$

Proof.

(i) There is nothing to prove: all the portfolios have the same expectation.

(ii) (Replace x_n with $1 - x_1 - x_2 - \ldots - x_{n-1}$ and let $f(\mathbf{x}) = \mathrm{Var}(\mathbf{x}^T\mathbf{C}\mathbf{x})$. Then:

$$f(\mathbf{x}) = \sum_{1\leq i,j\leq n-1} c_{i,j}x_ix_j + 2\sum_{1\leq i\leq n-1} c_{i,n}x_i(1 - x_1 - \ldots - x_{n-1}) + c_{n,n}(1 - x_1 - \ldots - x_{n-1})^2 \tag{24}$$

Its gradient $\mathbf{g} = \mathrm{Grad}(f)$ has the components:

$$g_i(\mathbf{x}) = \sum_{j=1}^{n}\left(c_{i,j} - c_{n,j}\right)x_j, 1 \leq i \leq n-1 \tag{25}$$

We claim that the inequalities (23) imply that f attains a minimum at points of the form $\mathbf{x}^{\mathbf{o}} = (0,\ldots, 0, x_{k+1},\ldots, x_{n-1})$. Notice first that (23) implies that $g_i(\mathbf{x}) \geq 0\ \forall\, 1 \leq i \leq k$ for points of form $\mathbf{x} = (0,\ldots, 0, x_{k+1},\ldots, x_{n-1})$, because $g_i(\mathbf{x}) = \sum_{j=k+1}^{n}\left(c_{i,j} - c_{n,j}\right)x_j$. In order to prove our claim, let us consider the convex function h: $[0,\infty) \to \mathbb{R}$, $h(s) = f(sx_1,\ldots,sx_k, x_{n\text{-}k+1},\ldots,x_{n-1})$.

Note that h′(s) = $\sum_{i=1}^{k} x_i g_i(sx_1, \ldots, sx_k, x_{k+1}, \ldots x_{n-1})$; hence:

$$h'(0) = \sum_{i=1}^{k} x_i g_i(0, \ldots, 0, x_{k+1}, \ldots x_{n-1}) \geq 0$$

However, h is convex, hence its derivative is non-decreasing, hence it is non-negative, therefore h is non-decreasing itself. It follows that:

$$h(0) \leq h(1) \Leftrightarrow f(0,\ldots, 0, x_{k+1},\ldots, x_{n-1}) \leq f(x_1,\ldots, x_k, x_{k+1},\ldots, x_{n-1})$$

In conclusion, f can attain its minimum only in points of the form $(0,\ldots,0,x_{k+1},\ldots,x_{n-1})$. $g_i(\mathbf{x}) = \sum_{j=1}^{n} (c_{i,j} - c_{n,j}) x_j, 1 \leq i \leq n-1$. □

Corollary 9. *Let* $\boldsymbol{C}$ *be a non-negative defined matric. Let* $\boldsymbol{c}_{i,\cdot}$ *be its i'th row of* $\boldsymbol{C}$*. A sufficient condition that* $\boldsymbol{\xi} \sim N(\boldsymbol{\mu},\boldsymbol{C})$*,* $\boldsymbol{\xi} \in INC(k)$ *belong to* **AO** *is that* $\boldsymbol{c}_{i,\cdot} \geq \boldsymbol{c}_{n,\cdot}$ $\forall$ *i = 1,2,...,k.*

Example. *Take* $\boldsymbol{\mu} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix} 6 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$

Note that $\xi \sim N(\mu,\mathbf{C}) \in \mathbf{AO}(\mathbf{x^o})$ for some $\mathbf{x^o}$ from $\Delta_{3,1}$ because its first row is greater than the third one (each component of the first row is greater than the corresponding component of the third row).

One sees that $\mathbf{x^o} = (0, \frac{1}{2}, \frac{1}{2})$ implies $X^0 \sim N(2, \sqrt{\frac{3}{2}}^{\,2})$. This is the absolute optimum.

Remark. *The conditions from Corollary 3. are only sufficient, not necessary. For instance, the vector* $\boldsymbol{\xi} \sim N(\boldsymbol{\mu},\boldsymbol{C})$ *where* $\boldsymbol{\mu} = \begin{pmatrix} 1 \\ a \\ b \end{pmatrix}$ *and* $\boldsymbol{C} = \begin{pmatrix} 9 & 0 & 1 \\ 0 & 9 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, $1 \leq a \leq b$ *admits* $\mathbf{x^o} = (0, 0, 1)$*. Hence,* $X^0 \sim Normal(b,1)$*, in spite of the fact that neither the first row nor the second one are greater than the third row of the matrix.*

Remark. *If we want the absolute optimum porfolio to be of the form* $\mathbf{x^o} = (0,x_2,\ldots,x_n)$*, then the necessary and sufficient condition is that* $c_{1,2} \geq c_{n,2},\ldots, c_{1,n} \geq c_{n,n}$*. This is the same as the condition* $\boldsymbol{c}_{1,\cdot} \geq \boldsymbol{b}_{n,\cdot}$*. Indeed, because* $c_{i,j} = \sigma_i\sigma_j\rho_{i,j}$ *with* $Var(\xi_i) = \sigma_i^{\,2}$ *and* $\rho_{i,j}$ *is the correlation coefficient between* ξ_i *and* ξ_j*, then the inequality* $c_{1,i} \geq c_{n,i}$ *is equivalent to* $\sigma_1\rho_{1,i} \geq \sigma_n\rho_{n,i}$ $\forall$ *i = 2,...,n; hence,* $\sigma_1 \geq \sigma_n \max\left(\frac{\rho_{n,2}}{\rho_{1,2}}, \frac{\rho_{n,3}}{\rho_{1,3}}, \ldots, \frac{\rho_{n,n}}{\rho_{1,n}}\right)$ *and this clearly implies* $c_{1,1} \geq c_{1,n} \Leftrightarrow \sigma_1 \geq \sigma_n\rho_{1,n}$*. However, for k* $\geq$ *2, this is not true anymore. For instance, for k = 2, the conditions "*$c_{i,j} \geq c_{n,j}$ $\forall$ *j = 3,...,n* $\forall$ *i = 1,2" and "*$\boldsymbol{c}_{1,\cdot} \geq \boldsymbol{c}_{n,\cdot}$*,* $\boldsymbol{c}_{2,\cdot} \geq \boldsymbol{c}_{n,\cdot}$*" are not the same; the second one is stronger.*

For $n = 4$, an example could be $\mathbf{C} = \begin{pmatrix} 16 & 0 & 4 & 2 \\ 0 & 9 & 1 & 2 \\ 4 & 1 & 4 & 1 \\ 2 & 2 & 1 & 1 \end{pmatrix}$.

Here, $\sigma_1 = 4$, $\sigma_2 = 3$, $\sigma_3 = 2$, $\sigma_1 = 1$. The minimum point is of form $\mathbf{x^o} = (0,0,\alpha,1-\alpha)$, in spite of the fact that neither the first row nor the second one are greater than the fourth one. Indeed, the components of the gradient are:

$$g_1(\mathbf{x}) = 14x_1 - 2x_2 + 3x_3 + x_4,\ g_2(\mathbf{x}) = -2x_1 + 7x_2 + x_4$$

and has the property that $g_1(\mathbf{x}) \geq 0$, $g_2(\mathbf{x}) \geq 0$ for $\mathbf{x} \in \Delta_{4,2}$. The reader may check that $\mathbf{x^o} = \mathbf{e_4}$. The optimal portfolio is $\mathbf{e_4} = (0,0,0,1)$ whenever $\xi \sim N(\mu,C)$ with $\mu_1 \leq \mu_2 < \mu_3 = \mu_4$.

6. The Case n = 2

In this section, we consider the case when the random vector $\boldsymbol{\xi}$ is bi-dimensional. This is the next simple case. Now $\boldsymbol{\xi}$ = (X,Y) and instead of writing $\boldsymbol{OA}(s,t)$ with $s,t \geq 0$, $s + t = 1$ we write $\boldsymbol{OA}(s)$. We shall give the method of how to construct probability distributions on $\mathbb{R}^2$ with property $\boldsymbol{OA}(s)$, $s \in [0,1)$. We shall assume that ξ either has a density f or has a discrete distribution on the set of integers. The same letter f will denote the density in the first case and the probability law in the second one: $f(x,y)$ will denote P($X = x$,$Y = y$). We shall state the conditions in both cases.

The case s > 0. We want the absolute optimum return to be Z = sX + (1 − s)Y. According to Proposition 4, the condition is that E(X | sX + (1 − s)Y) = E(Y | sX + (1 − s)Y).

In terms of densities, the condition means that:

$$\int_{\{sx+(1-s)y=a\}} (x-y)f(x,y)dx = 0\forall a \text{ (Absolutely continuous case)} \tag{26}$$

$$\sum_{\{sx+(1-s)y=a\}} (x-y)f(x,y) = 0\forall a \text{ (Discrete case)}$$

Or, explicitly written:

$$\int \left(x - \frac{a-sx}{1-s}\right) f\left(x, \frac{a-sx}{1-s}\right) dx = 0\forall a \text{ (Absolutely continuous case)} \tag{27}$$

$$\sum_{\{x\}} \left(x - \frac{a-sx}{1-s}\right) f\left(x, \frac{a-sx}{1-s}\right) \forall a \text{ (Discrete case)}$$

We shall focus on the absolute continuous case. After the substitution $u = x - a$ in (27) one obtains:

$$\int uf(a+u, a-\lambda u)du = 0\forall a \left(\text{with } \lambda = \frac{s}{1-\text{s}}\right) \tag{28}$$

Let us put ρ(a) = $\frac{1}{1-s}\int f(a+u, a-\lambda u)du$. Then ρ is a probability density. Let also

$$p_a(u) = \frac{f(a+u, a-\lambda u)}{(1-s)\rho(a)} \tag{29}$$

Then p_a is also a probability density and (28) becomes:

$$\int up_a(u)du = 0 \ \forall a \tag{30}$$

The meaning of (30) is that, if Z_a are random variables with density p_a, then EZ_a = 0. This is very easy to construct; take any random variables and center them. The conclusion is that we may construct as many distributions $F \in \boldsymbol{OA}(s)$ as follows:

(i) Take a probability density on [0,∞), denoted by ρ;
(ii) Take a family of densities on the real line, p_a, with property (30);
(iii) Set $f(a+u, a-\lambda u) = (1-s)p_a(u)\rho(a)$ or substituting $a + u$ by x and $a - \lambda u$ by y, we obtain:

$$f(xy) = (1-\text{s})p_{sx+(1-s)y}((1-s)(xy))\text{ρ}(sx+(1-s)y) \text{ and } F = f\cdot\lambda^2 \tag{31}$$

where λ^2 is the Lebesgue measure in the plane.

Then $F \in \boldsymbol{OA}(s)$.

Example 1. *Take* $p_a = \frac{1}{2a}1_{[-a,a]}$ *and* $\rho = 1_{[0,\,1]}$*. Then:*

$$f(x,y) = (1-s)p_{sx+(1-s)y}((1-s)(y-x))\rho(sx+(1-s)y) = \frac{1-s}{2(y-s(y-x))}1_{\Delta}(x,y)$$

where Δ *is the interior of the triangle ABC with A(0,0), B(0,*$\frac{1}{1-s}$*), C(*$\frac{1+2s}{2s}$*,* $\frac{1-4s^2}{4s(1-s)}$*).*

Notice, as a particular case, that if $s = \frac{1}{2}$, then $f(x,y) = \frac{1}{2(x+y)}1_\Delta(x,y)$ with A(0,0), B(0,2), C(2,0); and now the distribution F is symmetric. The reason for this is that all the densities, p_a, are symmetrical. The general form of a density of a distribution $F \in \mathbf{OA}(\frac{1}{2})$ is:

$$f(x,y) = \frac{1}{2}p_{\frac{x+y}{2}}\left(\frac{x-y}{2}\right)\rho\left(\frac{x+y}{2}\right)$$

which, of course, may not be symmetric.

A sufficient condition that the equality (28) holds is that the density, f, satisfies the condition:

$$f(a + x, a - \lambda x) = f(a - x, a + \lambda x) \text{ for any } a,x \tag{32}$$

This condition is similar to the symmetry. We can construct many such densities, starting with a symmetric density. It is enough to take a symmetric density, call it g, (meaning that $g(x,y) = g(y,x)\ \forall\ x,y$!). The reader can check that:

$$f(x,y) = \frac{2}{1-s}g\left(\frac{2s-1}{1-s}x + 2y, \frac{x}{1-s}\right) \tag{33}$$

is a density that satisfies (32). For $s = \frac{1}{2}$ we discover again the symmetric densities because (33) becomes $f(x,y) = 4g(2y,2x)$.

Example 2. *$g(x,y) = 1_{[0,1]\times[0,1]}(x,y) \Rightarrow f(x,y) = \frac{2}{1-s}1_D(x,y)$ where $D = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1-s, 0 \le (2s-1)x + (2-2s)y \le 1-s\}$.*

This is the uniform distribution in the interior of the parallelogram ABCD with A(0,0), B(t, $(s-t)/2$), C(t,s), D(0, $\frac{1}{2}$). Here $t = 1 - s$. If g would be the uniform density in the interior of the unity circle, then f would be the density for the uniform density in the interior of some ellipse, etc.

The case $s = 0$. The condition is that $E(X; Y \le a) \le E(Y; Y \le a)\ \forall\ a$. A sufficient condition is that $E(X \mid Y) \le Y$. Otherwise written, the pair (Y,X) should be a sub-martingale (or rather the first two terms of a sub-martingale). It is very easy to construct such distributions.

7. Conclusions

Our work was motivated by a result from mathematical folklore, which states that in the case of a financial market where the asset rates of return are i.i.d., the equal weight portfolio was an optimal portfolio for all risk-averse investors. Our attention was focused on finding necessary and sufficient conditions for the distribution of a financial market so that the equal weight portfolio is the optimal portfolio for all risk-averse investors. We generalized the previous problem by replacing the equal weight portfolio with a given portfolio. The necessary and sufficient conditions found were formulated using the conditional mean. In case the financial market has two assets, we have provided an algorithm for construction of the financial market that admits a given optimal portfolio. It is a challenge to find algorithms for the construction of probability distributions with property AOP in the case $n > 2$. The study in this paper could be developed considering sets of utility functions whose derivatives have a constant sign. For example, one can study existence conditions for absolute portfolios when the set of utility functions is composed from $\mathcal{U}_2$ utility functions that have a non-negative third derivative. More generally, one can investigate existence conditions for absolute portfolios in the case where the set $\mathcal{U}$ of utility functions is composed of functions u with the property that $(-1)^i u^{(i)} \le 0$, $i = 1, 2, \ldots, k$.

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References

1. Samuelson, P.A. General proof that diversification pays. *J. Financ. Quant. Anal.* **1967**, *2*, 1–13. [CrossRef]
2. Hadar, J.; Russell, W.R. Diversification of interdependent prospects. *J. Econ. Theory* **1974**, *7*, 231–240. [CrossRef]
3. Tamir, A. Symmetry and Diversification of Interdependent Prospects, Discussion Paper 139, 1975, Preprint Repec. Available online: http://www.kellogg.northwestern.edu/research/math/papers/139.pdf (accessed on 24 August 2021).
4. Deng, X.; Li, R. Some research on value range of equal weight portfolio risk. In Proceedings of the International Seminar on Future BioMedical Information Engineering (FBIE 2008), Wuhan, China, 18 December 2008; pp. 164–167.
5. Belzunce, F.; Riquelme, C.M.; Mulero, J. *An Introduction to Stochastic Orders*; Academic Press: New York, NY, USA, 2015.
6. Levy, H. *Stochastic Dominance: Investment Decision Making under Uncertainty*; Springer: New York, NY, USA, 2015.
7. Sriboonchita, S.; Wing-Keung, W.; Sompong, D.; Hung, T.N. *Stochastic Dominance and Applications to Finance, Risk and Economics*; CRC Press: Boca Raton, FL, USA, 2009.
8. Shaked, M.; Shanthikumar, J. *Stochastic Orders*; Springer Science & Business Media: New York, NY, USA, 2007.