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A Characterization of GRW Spacetimes

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Abstract: We show presence a special torse-forming vector field (a particular form of torse-forming of a vector field) on generalized Robertson–Walker (GRW) spacetime, which is an eigenvector of the de Rham–Laplace operator. This paves the way to showing that the presence of a time-like special torse-forming vector field ξ with potential function ρ on a Lorentzian manifold (M, g), dim$M > 5$, which is an eigenvector of the de Rham Laplace operator, gives a characterization of a GRW-spacetime. We show that if, in addition, the function $\xi(\rho)$ is nowhere zero, then the fibers of the GRW-spacetime are compact. Finally, we show that on a simply connected Lorentzian manifold (M, g) that admits a time-like special torse-forming vector field ξ, there is a function f called the associated function of ξ. It is shown that if a connected Lorentzian manifold (M, g), dim$M > 4$, admits a time-like special torse-forming vector field ξ with associated function f nowhere zero and satisfies the Fischer–Marsden equation, then (M, g) is a quasi-Einstein manifold.

Keywords: generalized Robertson–Walker spacetime; special torse-forming vector fields; de Rham–Laplace operator; quasi-Einstein manifold

MSC: 83F05; 53C25

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1. Introduction

It is well known that through cosmological considerations the space being homogeneous and isotropic in the large scale, picks the Robertson–Walker metrics. It amounts to the fact that an n-dimensional spacetime, $n > 3$, acquires the form $I \times_\varphi N$, with metric $g = -dt^2 + \varphi^2 \bar{g}$, where I is an open interval, φ is a smooth positive function defined on I, and $(N, \bar{g})$ is an $(n-1)$-dimensional Riemannian manifold of constant curvature. An n-dimensional generalized Robertson–Walker spacetime (GRW-spacetime) is $I \times_\varphi N$, with metric $g = -dt^2 + \varphi^2 \bar{g}$, where $(N, \bar{g})$ is an $(n-1)$-dimensional Riemannian manifold (cf. [1,2]). An interesting characterization of GRW-spacetime was obtained by Chen (cf. [3]), by proving that a Lorentzian manifold (M, g) admits a non-trivial time-like concircular vector field, if, and only if, it is a GRW-spacetime. Additionally, for interesting characterizations of GRW-spacetimes using torse-forming vector fields and Weyl tensors, we refer to (cf. [4,5]).

A concircular vector field ξ on a semi-Riemannian manifold (M, g) satisfies:

$$\nabla_U \xi = \rho U, \quad U \in \mathfrak{X}(M),$$

where ρ is a scalar, ∇ is a Levi–Civita connection, and $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M (cf. [5–7]). For other characterizations of GRW-spacetimes, we refer to (cf. [2,3,8,9]).

Yano generalized concircular vector fields by introducing a torse-forming vector field on semi-Riemannian manifold (M, g) (cf. [10]), defined by:

$$\nabla_U \xi = \rho U + \alpha(U)\xi, \quad U \in \mathfrak{X}(M), \tag{1}$$

where α is a 1-form called the torsed 1-form. Naturally, if $\alpha = 0$, then a torse-forming vector field is a concircular vector field. These vector fields are also used in characterizing a GRW-spacetime (cf. [2,4]). In [11], Chen considered an interesting special class of torse-forming vector field, requiring ξ to be nowhere zero and satisfying $\alpha(\xi) = 0$, that is the torse-forming vector field is perpendicular to the dual-vector field to torsed form α, called torqued vector fields.

In the present paper, we introduce on a Lorentzian manifold a special type of torse-forming vector field. A unit time-like torse-forming vector field ξ on a Lorentzian manifold (M, g) is said to be a *special torse-forming vector field* if it satisfies:

$$\nabla_U \xi = \rho(U + \eta(U)\xi), \quad U \in \mathfrak{X}(M), \tag{2}$$

where ρ is a non-zero function and η is 1-form dual to ξ. We call ρ the potential function of the special torse-forming vector field ξ. Note that for a special torse-forming vector field, using Equation (1), we have $\alpha(U) = -\rho\eta(U)$, that is ξ is a torse-forming vector field, which is parallel to the vector field dual α as opposed to the torqued vector field where ξ is orthogonal to the vector field dual α. Moreover, from the definition of special torse-forming vector field ξ on a Lorentzian manifold, it follows that under no situation, it reduces to a concircular vector field.

We study the role of a time-like special torse-forming vector field ξ on a Lorentzian manifold (M, g) in characterizing GRW-spacetimes. It is achieved by using the de Rham–Laplace operator $\boxdot$ (cf. [12]) and a time-like special torse-forming vector field ξ with potential function ρ on a connected Lorentzian manifold (M, g), $\dim M > 5$, through showing that $\boxdot\xi = \sigma\xi$ holds for a smooth function σ, if, and only if, (M, g) is a GRW-spacetime (see Theorem 1). We also show that if the function $\xi(\rho)$ is nowhere zero on M, then the fibers of GRW-spacetime $I \times_\varphi N$ are compact (see Theorem 2).

If ξ is a special torse-forming vector field on a simply connected Lorentzian manifold (M, g), then the dual-1-form η is closed (see Equation (15)), and, therefore, there is a function f such that $\eta = df$. Thus, the special torse-forming vector field ξ on a simply connected Lorentzian manifold (M, g) satisfies $\xi = \nabla f$, call this function f the associated function of ξ. Recall that a Lorentzian manifold (M, g) is said to be a quasi-Einstein manifold (cf. [13]) if its Ricci tensor has the following expression:

$$Ric = f_1 g + f_2 \beta \otimes \beta, \tag{3}$$

where f_1, f_2 are scalars and β is a 1-form on M. Exact solutions of the Einstein field equations can provide very important information about quasi-Einstein manifolds. For example, the Robertson–Walker spacetimes are quasi-Einstein manifolds. For this reason, the study of quasi-Einstein manifolds is important. It is shown that if the associated function f of the special torse-forming vector field ξ on a simply connected Lorentzian manifold (M, g), $\dim M > 4$, satisfies (i) f is nowhere zero and (ii) f is a solution of the Fischer–Marsden equation, then (M, g) is a quasi-Einstein manifold (see Theorem 3). Additionally, it is shown that if the scalar curvature τ of a simply connected Lorentzian manifold (M, g), $\dim M \geq 4$, is a constant and possesses a special torse-forming vector field ξ with potential function ρ and associated function f satisfying the above two conditions, then the potential function ρ is an eigenfunction of the Laplace operator Δ (see Corollary 1).

2. Preliminaries

Let φ be a smooth function on an n-dimensional connected Lorentzian (M, g). The Hessian operator H_φ is defined by:

$$H_\varphi(V) = \nabla_V \nabla\varphi, \quad V \in \mathfrak{X}(M), \tag{4}$$

where $\nabla\varphi$ is the gradient of φ and Hessian $Hess(\varphi)$ is defined by (cf. [14]):

$$Hess(\varphi)(U_1, U_2) = g\big(H_\varphi(U_1), U_2\big), \quad U_1, U_2 \in \mathfrak{X}(M). \tag{5}$$

The Laplacian $\Delta\varphi$ of the function φ is given by $\Delta\varphi = div(\nabla\varphi)$, and it satisfies:

$$\Delta\varphi = trH_\varphi. \tag{6}$$

Let ξ be a time-like special torse-forming vector field on a Lorentzian (M, g). Then, using the expression for the curvature tensor field

$$R(F_1, F_2)F_3 = \nabla_{F_1}\nabla_{F_2}F_3 - \nabla_{F_2}\nabla_{F_1}F_3 - \nabla_{[F_1,F_2]}F_3, \quad F_1, F_2, F_3 \in \mathfrak{X}(M)$$

and Equation (2), we compute:

$$R(F_1, F_2)\xi = F_1(\rho)F_2 - F_2(\rho)F_1 + (F_1(\rho)\eta(F_2) - F_2(\rho)\eta(F_1))\xi + \rho^2(\eta(F_2)F_1 - \eta(F_1)F_2).$$

Above equation gives expression for the Ricci tensor Ric of the Lorentzian manifold (M, g):

$$Ric(V, \xi) = -(n-4)V(\rho) + \Big(\xi(\rho) + (n-3)\rho^2\Big)\eta(V), \quad V \in \mathfrak{X}(M). \tag{7}$$

Note that the Ricci operator Q of the Lorentzian manifold (M, g) is given by $Ric(U, V) = g(QU, V)$, $U \in \mathfrak{X}(M)$, and, therefore, Equation (7) implies:

$$Q\xi = -(n-4)\nabla\rho + \Big(\xi(\rho) + (n-3)\rho^2\Big)\xi \tag{8}$$

and:

$$Ric(\xi, \xi) = -(n-3)\Big(\xi(\rho) + \rho^2\Big). \tag{9}$$

The Laplace operator Δ acting on vector fields on the Lorentzian manifold (M, g) is defined by:

$$\Delta U = \sum_{i=1}^{n}\Big(\nabla_{v_i}\nabla_{v_i}U - \nabla_{\nabla_{v_i}v_i}U\Big), \quad U \in \mathfrak{X}(M), \tag{10}$$

where $\{v_1, \dots, v_n\}$ is a local orthonormal frame on M. The de Rham–Laplace operator $\boxdot$ on the Lorentzian manifold (M, g) is $\boxdot : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by (cf. [12]):

$$\boxdot U = \Delta U + QU, \quad U \in \mathfrak{X}(M). \tag{11}$$

Lemma 1. *Let ξ be a time-like special torse-forming vector on an n-dimensional Lorentzian manifold (M, g) with potential function ρ. Then:*

$$\boxdot\xi = -(n-5)\nabla\rho + 2(n-2)\rho^2\xi.$$

Proof. Using Equation (2), for $U \in \mathfrak{X}(M)$, we have:

$$\nabla_U\nabla_U\xi - \nabla_{\nabla_U U}\xi = U(\rho)U + U(\rho)\eta(U)\xi + \rho^2\|U\|^2\xi + 2\rho^2\eta(U)^2\xi + \rho^2\eta(U)U.$$

Since ξ is a time-like unit vector field, choosing a local frame $\{v_1, \ldots, v_{n-1}, \xi\}$ on M, where v_i, $i = 1, \ldots, n-1$ are spacelike unit vector fields in the above equation, to conclude:

$$\Delta\xi = \nabla\rho - \left(\xi(\rho) - (n-1)\rho^2\right)\xi.$$

Thus, using Equations (8) and (11) with the above equation, we conclude:

$$\boxdot\xi = -(n-5)\nabla\rho + 2(n-2)\rho^2\xi.$$

□

3. Characterizing GRW Spacetimes

Consider an n-dimensional GRW-spacetime $M = I \times_\varphi N$ with metric $g = -dt^2 + \varphi^2\overline{g}$. Then, $\xi = \frac{d}{dt}$ is a time-like unit vector field on (M, g). Let ∇ be the Levi–Civita connection on (M, g). Then, for a $U \in \mathfrak{X}(M)$, we have $U = h\xi + E$, $E \in \mathfrak{X}(N)$. If we denote by $\eta = dt$, then $\eta(U) = g(U, \xi) = -h$, where $\eta(\xi) = g(\xi, \xi) = -1$. Using fundamental equations for the warped product (cf. [8]), we have:

$$\nabla_U\xi = \nabla_{h\xi+E}\xi = \nabla_E\xi = \frac{\xi(\varphi)}{\varphi}E = \frac{\xi(\varphi)}{\varphi}(U - h\xi) = \frac{\xi(\varphi)}{\varphi}(U + \eta(U)\xi).$$

Thus,

$$\nabla_U\xi = \rho(U + \eta(U)\xi), \quad U \in \mathfrak{X}(M), \quad \rho = \frac{\xi(\varphi)}{\varphi}, \tag{12}$$

this proves, ξ is a special torse-forming vector field on the GRW-spacetime (M, g). Now, using the expression for the Ricci tensor for the warped product $I \times_\varphi N$ (cf. [8]), we have:

$$Ric(\xi, E) = 0, \quad E \in \mathfrak{X}(N),$$

which implies $Q(\xi) = \lambda\xi$ for a smooth function λ on I. Furthermore, choosing a local frame $\{v_1, \ldots, v_{n-1}\}$ on N, we have a local orthonormal frame $\{\xi, v_1, \ldots, v_{n-1}\}$ on M. Then, using Equation (12), we have $\nabla_\xi\xi = 0$, $\nabla_{v_i}\xi = \rho v_i$, $v_i(\rho) = 0$, and:

$$\nabla_{v_i}\nabla_{v_i}\xi = \rho\nabla_{v_i}v_i, \ \nabla_\xi\nabla_\xi\xi = 0.$$

Furthermore, using Equation (12), we have:

$$\nabla_{\nabla_{v_i}v_i}\xi = \rho(\nabla_{v_i}v_i + g(\nabla_{v_i}v_i, \xi)\xi) = \rho\nabla_{v_i}v_i - \rho g(v_i, \nabla_{v_i}\xi)\xi = \rho\nabla_{v_i}v_i - \rho^2\xi.$$

Thus, the rough Laplace operator Δ acting on ξ is given by:

$$\Delta\xi = \left(\nabla_\xi\nabla_\xi\xi - \nabla_{\nabla_\xi\xi}\xi\right) + \sum_{i=1}^{n-1}\left(\nabla_{v_i}\nabla_{v_i}\xi - \nabla_{\nabla_{v_i}u_i}\xi\right) = (n-1)\rho^2\xi.$$

Now, we see that the de Rham–Laplace operator $\boxdot$ acting on ξ is given by:

$$\boxdot\xi = \left((n-1)\rho^2 + \lambda\right)\xi.$$

Hence, GRW-spacetime (M, g) admits a special torse-forming vector field ξ, which is an eigenvector of the de Rham–Laplace operator $\boxdot$.

Theorem 1. *An n-dimensional connected Lorentzian manifold (M, g), $n > 5$, is a GRW-spacetime $I \times_\varphi N$, if, and only if, it admits a time-like special torse-forming vector field ξ, which is an eigenvector of the de Rham–Laplace operator on (M, g).*

Proof. Let (M,g) be a connected Lorentzian manifold, $n > 5$, ξ be a time-like special torse-forming vector field on (M,g) with $\boxdot\xi = \lambda\xi$, λ being a scalar. We denote by ∇ the Levi–Civita connection on (M,g); using Equation (2), we have:

$$\nabla_\xi \xi = 0. \tag{13}$$

Define a smooth distribution $\mathcal{D}$ on M by:

$$\mathcal{D} = \{U \in \mathfrak{X}(M) : \eta(U) = 0\}. \tag{14}$$

Note that Equation (2) gives:

$$d\eta(U,V) = g(\nabla_U \xi, V) - g(\nabla_V \xi, U) = 0, \quad U, V \in \mathfrak{X}(M), \tag{15}$$

that is the dual-1-form η to ξ is closed. Thus, for $E, F \in \mathcal{D}$, we have $\eta([E,F]) = -d\eta(E,F) = 0$, that is $[E,F] \in \mathcal{D}$, proving that the distribution $\mathcal{D}$ is integrable. Let N be a leaf of $\mathcal{D}$. Then, N is a hypersurface of M with unit normal ξ. Using Equation (2), we observe that for $E \in \mathfrak{X}(N)$,

$$\nabla_E \xi = \rho E, \tag{16}$$

that is the shape operator S of N is given by:

$$S(E) = -\rho E, \quad E \in \mathfrak{X}(N). \tag{17}$$

Now, as $\boxdot\xi = \lambda\xi$, where λ is a scalar on M, using Lemma 1, we get:

$$-(n-5)\nabla\rho + 2(n-2)\rho^2\xi = \lambda\xi. \tag{18}$$

On taking the inner product in above equation with ξ yields

$$\lambda = (n-5)\xi(\rho) + 2(n-2)\rho^2$$

and substituting this value of λ in Equation (18), we have:

$$-(n-5)\nabla\rho = (n-5)\xi(\rho)\xi. \tag{19}$$

Above equation on taking the inner product with $E \in \mathfrak{X}(N)$, gives $(n-5)E(\rho) = 0$, and the assumption $n > 5$ implies $E(\rho) = 0$, that is ρ is a constant on the hypersurface N. Therefore, Equation (17) implies that N is a totally umbilical hypersurface of M. Moreover, the orthogonal complementary distribution $\mathcal{D}^\perp$ to $\mathcal{D}$ is one-dimensional spanned by ξ, and by Equation (13), the integral curves of the distribution $\mathcal{D}^\perp$ are geodesics on M. Thus, (M,g) is the warped product $I \times_\varphi N$ (cf. [15]), that is (M,g) is a GRW-spacetime.

Conversely, we have already seen that a GRW-spacetime $I \times_\varphi N$ admits a special torse-forming vector field ξ, which is an eigenvector of $\boxdot$. □

In the above result we have seen that the presence of a time-like special torse-forming vector field ξ on a Lorentzian manifold (M,g) satisfying $\boxdot\xi = \lambda\xi$ for scalar λ is a GRW-spacetime $I \times_\varphi N$. It is interesting to observe if in addition $\xi(\rho)$ is nowhere zero, then this condition has effect on the topology of N.

Theorem 2. *Let ξ be a time-like special torse-forming vector field with potential function ρ on an n-dimensional complete and connected Lorentzian manifold (M,g), $n > 5$. If ξ is an eigenvector of the de Rham–Laplace operator on (M,g) and the function $\xi(\rho)$ is nowhere zero, then (M,g) is GRW-spacetime $I \times_\varphi N$, with N compact.*

Proof. Let ξ be a time-like special torse-forming vector field on a Lorentzian manifold (M, g), $n > 5$, with ξ being an eigenvector of the de Rham Laplace operator on (M, g) and the function $\xi(\rho) \neq 0$ everywhere on M. Since $n > 5$, Equation (19) implies:

$$\nabla\rho = -\xi(\rho)\xi. \tag{20}$$

As ξ is a time-like unit vector field and $\xi(\rho)$ is nowhere zero, the above equation implies that $\nabla\rho$ is nowhere zero on M. Therefore, the potential function $\rho : M \to \mathbf{E}$ is a submersion, and each fiber $F_x = \rho^{-1}\{\rho(x)\}$, $x \in M$, is an $(n-1)$-dimensional smooth manifold; as $\{\rho(x)\}$ is compact in $\mathbf{E}$, we obtain that F_x is compact. Consider a smooth vector field:

$$\mathbf{u} = -\frac{\xi}{\xi(\rho)}$$

that has no zeros on M. Then, it follows that $\mathbf{u}(\rho) = -1$ and $\mathbf{u}$ has a local flow $\{\phi_s\}$ that satisfies:

$$\rho(\phi_s(x)) = \sigma(x) - s. \tag{21}$$

Recall the escape Lemma (cf. [16]), which states that if γ is a integral curve of $\mathbf{u}$ whose maximal domain is not all of $\mathbf{E}$, then the image of γ cannot lie in any compact subset of M. Using the escape lemma and Equation (21) on a complete and connected M, we obtain that $\mathbf{u}$ is complete and has global flow $\{\phi_s\}$. Now, define $f : \mathbf{E} \times F_x \to M$ by:

$$f(s, u) = \phi_s(u), \quad u \in F_x.$$

Then, f is smooth, and for each $u \in M$, we find $s \in \mathbf{E}$ such that $\phi_s(u) = y \in F_x$, satisfying $u = \phi_{-s}(y)$. Thus, $f(-s, y) = u$, that is f is an on-to map. We observe that, on taking (s_1, u_1), (s_2, v_2) in $\mathbf{E} \times F_x$ satisfying $f(s_1, u_1) = f(s_2, u_2)$, we have $\phi_{s_1}(u_1) = \phi_{s_2}(u_2)$, and using Equation (21), we obtain $\rho(u_1) - s_1 = \rho(u_2) - s_2$. As $u_1, u_2 \in F_x$, $\rho(u_1) = \rho(u_2)$, and we obtain $s_1 = s_2$. Thus, using $\phi_{s_1}(u_1) = \phi_{s_2}(u_2)$, we arrive at $u_1 = u_2$, that is f is one-to-one. Furthermore, we have:

$$f^{-1}(u) = (-s, y) = (-s, \phi_s(u)),$$

which is smooth. Hence, $f : \mathbf{E} \times F_x \to M$ is a diffeomorphism, where F_x is a compact subset of M. Using Theorem 3.1, we see that $I \times N$ is diffeomorphic to $\mathbf{E} \times F_x$, and as the open interval I is diffeomorphic to $\mathbf{E}$, the fiber N must be diffeomorphic to F_x. As F_x is compact, we obtain that N is compact. □

4. Lorentzian Manifolds as Quasi-Einstein Manifolds

Fischer–Marsden considered the following differential equation on a semi-Riemannian manifold (M, g) (cf. [17]):

$$(\Delta f)g + fRic = Hess(f), \tag{22}$$

where f is a smooth function on M. We call the above differential equation the Fischer–Marsden equation. This differential equation is closely associated with Einstein spaces. A generalization of Einstein manifolds was considered in [13], where the authors defined quasi-Einstein manifolds. A semi-Riemannian manifold (M, g) is said to be a quasi-Einstein manifold if its Ricci tensor satisfies Equation (3). In this section, we use a unit time-like special torse-forming vector field ξ on a Lorentzian manifold (M, g) to find conditions under which (M, g) is a quasi-Einstein manifold.

Let ξ be a time-like special torse-forming vector field on a simply connected Lorentzian manifold (M, g). On using Equations (2) and (15), we have $d\eta = 0$, that is η is a closed 1-form and M is simply connected $\eta = df$ (exact) for a smooth function f on M. Thus, for a time-like special torse-forming ξ on a simply connected Lorentzian manifold (M, g), we have:

$$\xi = \nabla f \tag{23}$$

and we call the smooth function f in Equation (23) the *associated function* of ξ.

Theorem 3. *Let ξ be a time-like special torse-forming vector field on an n-dimensional simply connected Lorentzian manifold (M, g), $n > 4$, with potential function ρ and associated function f. If f is a nowhere zero solution of the Fischer–Marsden equation, then (M, g) is a quasi-Einstein manifold.*

Proof. Using Equations (2) and (23), we have:

$$H_f(U) = \rho(U + \eta(U)\xi),$$

which implies:

$$Hess(f) = \rho g + \rho\eta \otimes \eta, \quad \Delta f = (n-3)\rho. \tag{24}$$

Since f satisfies Fischer–Marsden equation, using Equations (22) and (24), we have:

$$f Ric = -(n-4)\rho g + \rho\eta \otimes \eta. \tag{25}$$

As f is nowhere zero, we have:

$$Ric = -(n-4)\left(\rho f^{-1}\right)g + \left(\rho f^{-1}\right)\eta \otimes \eta.$$

Hence, (M, g) is a quasi-Einstein manifold. □

If simply connected Lorentzian manifold (M, g) has scalar curvature $\tau = trQ$, using above result we have the following result that gives a relation between ρ and f of the time-like special torse-forming vector field ξ on (M, g).

Corollary 1. *Let ξ be a time-like special torse-forming vector field on an n-dimensional simply connected Lorentzian manifold (M, g), $n \geq 4$, with potential function ρ and associated function f. If f is a solution of the Fischer–Marsden equation, then:*

$$\rho = -\frac{\tau}{(n-3)^2} f.$$

In particular, if the scalar curvature τ of (M, g) is a constant, then the potential function ρ is an eigenfunction of the Laplace operator Δ.

Proof. Let ξ be a time-like special torse-forming vector field on a simply connected Lorentzian manifold (M, g), $n \geq 4$, with potential function ρ and associated function f. Suppose f satisfies Equation (22). Then, Equation (25), gives

$$f\tau = -(n-4)(n-2)\rho - \rho = -(n-3)^2\rho.$$

Hence,

$$\rho = -\frac{\tau}{(n-3)^2} f.$$

Now, if τ is a constant, then the above equation in view of Equation (24) implies:

$$\Delta\rho = -\frac{\tau}{(n-3)}\rho,$$

that is the potential function ρ is an eigenfunction of Δ. □

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