mathematics

MDPI

Article

Multiple-Solution Tasks in Pre-Service Teachers Course on Combinatorics

Ingrid Semanišinová

Institute of Mathematics, Faculty of Science, Pavol Jozef Šafárik University, 040 01 Košice, Slovakia; ingrid.semanisinova@upjs.sk

Abstract: In the paper, we present a study devoted to the utilization of multiple-solution tasks (MSTs) in combinatorics as a part of a pre-service teachers course on didactics of mathematics from the view of the mathematics teachers' specialized knowledge (MTSK) theoretical framework. The study was carried out over the standard course of a summer semester in 2021. The course was attended by 13 pre-service teachers (PSTs). It was carried out online, due to COVID-19 restrictions. Ten combinatorial multiple-solution tasks were assigned to the PSTs. Analyzing pre-service teachers solutions to these tasks, we sought the description and better understanding of the combinatorial knowledge of the topic from the perspective of MSTK. The results revealed some critical aspects of mathematical knowledge in combinatorics that pre-service teachers education should focus on.

Keywords: mathematics teacher education; mathematics teachers' specialized knowledge; multiple-solution tasks; problem solving; combinatorics

MSC: 97B50; 97D50; 97K20

check for updates

Citation: Semanišinová, I. Multiple-Solution Tasks in Pre-Service Teachers Course on Combinatorics. *Mathematics* **2021**, *9*, 2286. https://doi.org/10.3390/math9182286

Academic Editors: Jarmila Novotná and Alena Hošpesová

Received: 7 August 2021
Accepted: 13 September 2021
Published: 16 September 2021

1. Introduction

The study examines the potential of multiple-solution tasks in a pre-service teachers course on combinatorics. Our study is motivated by the following two stimuli.

Firstly, one of the essential goals during pre-service teachers' (PSTs') preparation is to improve their ability to provide relevant feedback to students. Providing feedback to students' solutions to tasks chosen for the lesson is one of the most common activities that a mathematics teacher should do. During teaching, a teacher must be able to determine immediately whether the student's solution of the task is correct and, if not, they need to identify where and why the problem arose. The teacher should also know which representation of the task situation and/or which solution method is specific for the task as well as which one is more abstract. It is important for them to identify concepts and properties which are intertwined with the presented solution, and also how different students' solutions are connected with each other. Proper reaction to the students' solutions is influenced by such knowledge of the teacher.

The role of the teacher in creating high-quality conditions for the learning of their students is undeniable (see [1,2]). It is known that the teacher's content knowledge has a major impact on students' knowledge because it determines the teachers' teaching and, consequently, the learning of the students (see [3–5]). Therefore, the selection of proper mathematical tasks for PSTs focused on developing knowledge specific for teachers is decisive [6–8].

Secondly, combinatorics has quite a strong position in the Slovak mathematics curricula [9]. The main reason is its applicability in probability and computer science. Moreover, according to Lockwood [10], combinatorics is well established in mathematics for its rich potential in a problem-solving context. However, students face a great deal of difficulties as they meet nonstandard counting problems; these difficulties were documented by several authors (cf. [10–12]). That is why special attention to mathematics teachers' specialized knowledge related to combinatorics is of great importance.

Moreover, the study was inspired by experience from a final state exam in didactics of mathematics, which is an obligatory part of the study program for pre-service teachers of mathematics. During the exam, we assigned the following tasks to the pre-service teacher: *Student solves the following task: "The math teacher wants to choose a four-member team to represent the class in a mathematical competition. The teacher will choose the team from six good students. In how many different ways can he choose four students from six?" His solution is in Figure 1.*

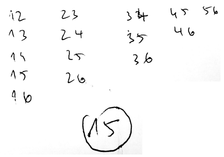


Figure 1. Exam task.

What Feedback Will You Provide to the Student?

The pre-service teacher claimed that the result is correct, but it was obtained by an incorrect method. The PST argued that the task requires finding four-elements subset of a six-element set. However, a few minutes before, the same PST formulated correctly the property for binomial coefficients $\binom{n}{k} = \binom{n}{n-k}$. Nonetheless, she did not observe utilization of this property in the interpretation of the correct student's solution of the task. This indicates a discrepancy between the knowledge of quite elementary mathematical concepts and methods and their application and integration into the teaching process in the classroom.

All of this was an important stimulus for further research in this area that aimed to find out whether the discovered problem during the exam is accidental or it shows a significant gap in the preparation of PSTs.

Mathematical knowledge for teaching combinatorics, particularly the differences between pedagogical content knowledge and content knowledge in combinatorics of PSTs, were explored in [13]. In this study, we reframe the research within the model of mathematics teachers' specialized knowledge (Carrillo et al., 2018). The model helps us better describe the specialized content knowledge needed for teachers.

In order to describe the content knowledge in combinatorics in more depth, we use the Lockwood model of students' combinatorial thinking (see [10]), which is helpful to understand students' learning. Lockwood analyzed the proposed model from the point of view of students' thinking. We use it to explore PSTs' combinatorial content knowledge.

In addition, we use MSTs to explore understanding of mathematical activity of PSTs when solving a combinatorial problem. In connection with PSTs education, MSTs are studied in relation to the development of PSTs' problem-solving competences (see [14]). The ways in which teachers implement MSTs in their lessons were explored in [15]. However, research on the use of MSTs in teacher training in combinatorial context has not been published yet.

2. Theoretical Background

In this section, we provide an overview of the sources concerning our motivation that was conducted in the introduction.

2.1. Mathematics Teacher's Specialized Knowledge

Several conceptualizations of teachers' knowledge were developed in the last two decades, such as the Knowledge Quartet [4], Mathematical Knowledge for Teaching [16], and Mathematics Teacher Specialized Knowledge [17]. All of them are aimed at a deeper understanding of different parts of a teacher's knowledge.

In the paper, we use the model Mathematics Teacher Specialized Knowledge (MTSK) since it considers specialization as a general feature of mathematics teachers' knowledge. This model focuses on mathematics teachers' specific knowledge with respect to teaching the subject and eliminates references to common content knowledge shared with other users of mathematics. According to Scheiner et al. [18] (p. 160), "mathematics teachers need to know subject matter in a qualitatively different way than other practitioners of mathematics (mathematicians, physicists, engineers, among others), and they need to hold a qualitatively different kind of knowledge than teachers of other subjects (physics teachers, biology teachers, history teachers, among others)".

The model MTSK has two parts: mathematical knowledge and pedagogical content knowledge (see Figure 2).

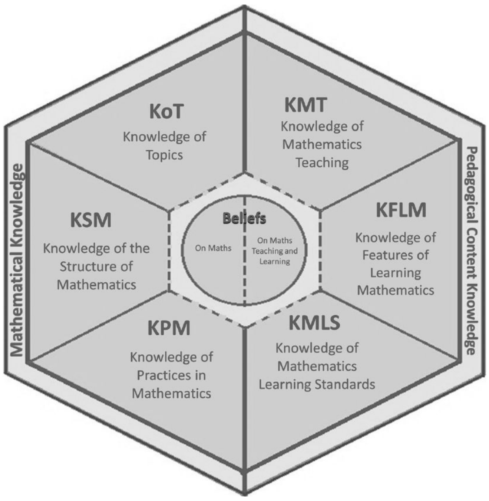


Figure 2. The mathematics teacher's specialized knowledge model (adapted from [17] (p. 241)).

Both parts of the model are divided into three subdomains. The mathematical knowledge part is divided into the following subdomains:

- Knowledge of topic (KoT)—includes mathematical content, such as knowledge of definitions, properties and their foundations, procedures, phenomenology and applications, and representation systems.
- Knowledge of the structure of mathematics (KSM)—integrates connections between mathematical topics, e.g., connections between definitions and properties from different topics that help to solve the same problem in different ways.
- Knowledge of practices in mathematics (KPM)—contains knowledge in mathematics, which enables the teacher to explore and create new knowledge [19], to emphasize characteristics of mathematics as a scientific discipline and the difference between mathematics per se and mathematics in the school.

According to Carrillo et al. [17] (p. 246), the pedagogical content knowledge part is related to mathematics itself. It is not the intersection between mathematical and general pedagogical knowledge, but it is a specific type of knowledge of pedagogy which derives chiefly from mathematics. The part of pedagogical content knowledge consists of the following three subdomains:

- Knowledge of mathematics teaching (KMT)—concerns the selection of tasks, activities and teaching methods for a lesson, for example, with regard to the emergence of possible misconceptions. It also includes knowledge of how to assess students and, according to the situation, deciding how to continue with the teaching process. Part of this knowledge comprises information on teaching materials, resources and digital tools that can be used for specific mathematical content to facilitate student learning.
- Knowledge of features of learning mathematics (KFLM)—includes understanding of the cognitive process that students must go through in order to understand mathemat-

ical concepts, their properties and problem solving methods. Moreover, it includes knowledge about the reasons for students' difficulties and knowledge about what and how could help them in taking over the subject matter.

- Knowledge of mathematics learning standards (KMLS)—emphasizes the importance of the teacher being aware of the country curriculum and other sources, which might include, for example, nonofficial curriculum documents and other materials, which enable the teacher to be critical and reflective in considering what the student should learn.

The central position in the MTSK model is reserved for teachers' beliefs on mathematics and on mathematics teaching and learning because they are assumed to interact with other parts of the model (e.g., [17,20]).

The MTSK model helps us to understand deeper the mathematical knowledge of PSTs. We want to characterize what PSTs know about the content that he/she will teach. Therefore, in this study we essentially deal with the subdomain, knowledge of topics (KoT). This subdomain comprises teachers' knowledge of mathematics content, which have to be deeper than the knowledge that students should attain. Teachers have to know not only the solutions for tasks that their students are supposed to solve, but they also should be able to support their students' understanding, for example, being able to quickly respond to various students' solutions (correct or not).

According to Carrillo et al. [17], the subdomain KoT of the MTSK comprises four subcategories: knowledge of definitions of concepts, their properties and foundations; procedures; phenomenology; and representation systems linked with a concrete topic.

Definitions, properties and their foundations

This subcategory includes knowledge of definitions, properties and their foundations together with connections between them. The teacher, who is aware of these connections, is able to solve the same problem in several ways, using different concepts and/or different properties of the concept from the topic. In the context of combinatorics in the Slovak curriculum, it comprises, for example, the concepts of permutations, arrangement with repetition and without repetitions, combinations, factorial, binomial coefficient, properties such as $\binom{n}{k} = \binom{n}{n-k}$, formulas for counting different combinatorial operations, connections between arrangement without repetition and combinations, etc.

Phenomenology and its applications

This subcategory involves a connection of the topic to the real context and to the mathematical content itself. In combinatorics, it includes, for example, different contexts in which particular combinatorial concept could emerge and different combinatorial interpretations of the same situation, e.g., $\frac{n!}{k!(n-k)!}$ could be interpreted as combinations without repetitions or permutations with repetitions.

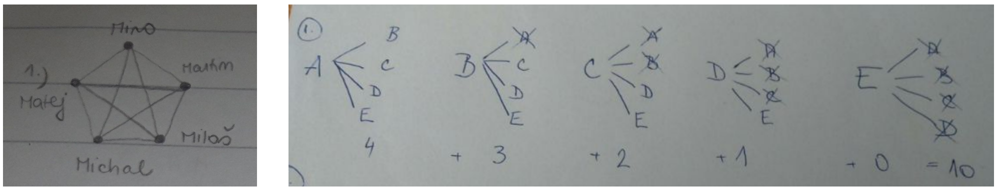
Mathematical procedures

This part of the KoT includes knowledge of how to do something, when the procedure could be used and why it works, e.g., which combinatorial principle is appropriate to solve the particular combinatorial problem. The subcategory contains appraisal of generality and/or effectivity of the used procedure too.

Representation systems

In connection with combinatorics, the chosen representation depends very often on the particular context of a task. The chosen representation may play a key role in discovering the same mathematical model for several combinatorial problems.

For example, in Figure 3a, there is representation using a specific combinatorial structure—complete graph for the solution to the task: "How many handshakes will take place at a meeting of five people if each person shakes hands with every other person once?" which shows that the task is analogous to the task: "There are five points on a sheet such that no three are collinear. What is the number of lines that one can obtain by joining these in pairs?" On the other hand, the representation in Figure 3b in terms of tree graphs can help to uncover the recurrence relation $h(n) = h(n-1) + n - 1$.



(**a**) Complete graph (**b**) Tree graph

Figure 3. Different representations for the handshaking task.

2.2. Problem Solving and Multiple-Solution Tasks

Problem solving is a necessary part of teaching mathematics. Several authors (e.g., [14,21,22]) wrote that solving challenging, uncommon problems requires mathematical knowledge that promote the development of abilities, such as understanding of the problem, creating connections between the new problem and problems solved in the past, use of multiple representations in solving the problems, recognizing similarities in the structure of different problems, awareness of the beauty of the problem and its solutions. Problem solving is also considered a means for the development of connected and robust mathematical knowledge. In this way, solving a problem develops deeper understanding of the studied mathematical concept or of the solving process, which leads to construction of content knowledge [23,24]. Consistent with [8,25,26] we consider mathematical problem solving to be an essential component of PST education.

Since the nature of mathematics teacher's knowledge is specific, mathematics teacher educators are constantly searching appropriate problems and tasks that are valuable in the professional development of mathematics teachers [8,15,27,28]. The tasks designed for teachers should encourage them to examine different situations in order to perceive the mathematical complexity on the one hand and sensitivity to students on the other hand [7,8,15], which means that they are valuable from a mathematical and pedagogical point of view.

Multiple sources, e.g., [14,25,29], consider that solving problems in different ways is an appropriate tool for developing mathematical knowledge and constructing the web of mathematical ideas and compatible ways of thinking, which are necessary for good mathematics teachers (see [8]). Multiple-solution tasks (MSTs) turned out to be an effective instructional tool to advance teachers' knowledge of the topic, to stimulate teachers' knowledge about students' learning and to provide opportunities to learn more about mathematics and its nature (see [15]). MSTs provide teachers with opportunities to extend and connect their mathematical and pedagogical understandings to create "new" knowledge [8]. According to Leikin and Lev [30], MSTs can be used for different research purposes, e.g., for the evaluation of students' (teachers') mathematical creativity (especially their fluency, flexibility and originality), for the analysis of their mathematical understanding. In our study, we use MSTs to find the level of PSTs' KoT and to explore possibilities for its development. Additionally, we search for more solutions to the task forces PSTs to think outside the box and to use concepts, properties and methods in which they may lack practice.

2.3. Combinatorics in PSTs Education

Combinatorics has rich potential in a problem solving context [11,29,31,32], and moreover, it has applications in various areas, e.g., probability and computer science. Lebowitz in [33] emphasize that problems from discrete mathematics (in the Slovak mathematics curriculum [9] represented mainly by combinatorial problems) do not require specific input knowledge and can be solved with students of several ages. Many problems have various accessible solution methods and have no exactly established technical terms and symbolism, which allow a solver to be free in the selection of the representation. As stated by Hart in [34], such problems have a significant pedagogical power, and they allow achieving valuable goals in teaching mathematics.

On the other hand, consistent with [10–13,35], combinatorial problems are difficult for students at various levels. In particular, if the context of a combinatorial situation is nonstandard, it is quite common that high achievers in mathematics do not solve a problem correctly (see [10]).

Lockwood [10] created a model of students' combinatorial thinking to conceptualize the solving of combinatorial tasks and, consequently, help students to develop combinatorial thinking so as to achieve success when solving combinatorial tasks (see Figure 4). The Lockwood model highlights the importance of developing the relationships between sets of outcomes (the collection of objects being counted), counting processes (enumeration process or series of processes in which a counter engages as they solve a problem), and formulas/expressions (see [10] for more detailed explanation of the notions). She states that "for a given counting problem, a student may work with one or more of these components and may explicitly or implicitly coordinate them" [10] (p. 253).

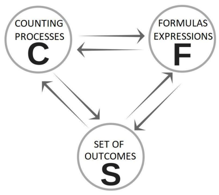


Figure 4. Model of students' combinatorial thinking (adapted from [10]).

The components of the Lockwood model and connections between the components describe intra-conceptual connections within the topic. With respect to the model MTSK, they belong to the subdomain KoT part of the subcategory definitions, properties and their foundations (see [17]).

3. Procedure, Participants, Tasks and Research Question

The study was carried out during the summer semester of 2021 in a course for pre-service mathematics teachers in their third year of training. It was aimed at solving school problems from a particular part of the Slovak mathematical curriculum content called combinatorics, probability and statistics. The course is a part of the bachelor program for pre-service mathematics teachers at the Faculty of Science of Pavol Jozef Šafárik University in Košice, Slovakia. The program has a standard study period of three years and is focused on mathematical knowledge for teachers. Pedagogical content knowledge for mathematical teachers and general pedagogy and psychology mostly comprise the content of the corresponding master program.

The participants of the course were 13 pre-service teachers (10 females and 3 males). All participants had completed several mathematics courses (e.g., geometry, algebra, calculus). Moreover, students had taken the discrete mathematics course, part of which is aimed at combinatorics. During this course, they studied combinatorial concepts, methods and principles, counting techniques, e.g., the inclusion–exclusion principle and basic combinatorial theorems, such as binomial theorem. PSTs also met a few different contexts in which combinatorial problems can emerge.

The course was carried out online due to COVID-19 restrictions. We used the MS Teams platform for our online sessions. PSTs handed in their homework, using the Assignments feature. The solutions of the tasks given in the session were shared with us using Class Notebook implemented in MS Teams. This allowed us to see the PSTs' work during the session. PSTs in most of the cases made a photo of their solution and shared it with us. Afterwards, we chose a representative sample of different PSTs' solutions, which created a collective solution space. We shared the sample with the PSTs, and a group discussion followed. In order to better organize the whole group online discussion and create conditions for involving as many PSTs as possible, we decided to use Class Notebook implemented in MS Teams and its Collaboration Space during the discussion. Using this feature allowed every PST to pose his/her question, write a comment, opinion, etc., anonymously, and all participants saw each other's contributions. After these contributions were written, we started to discuss verbally. Part of the Collaboration Space with contributions of four PSTs is shown in Figure 5.

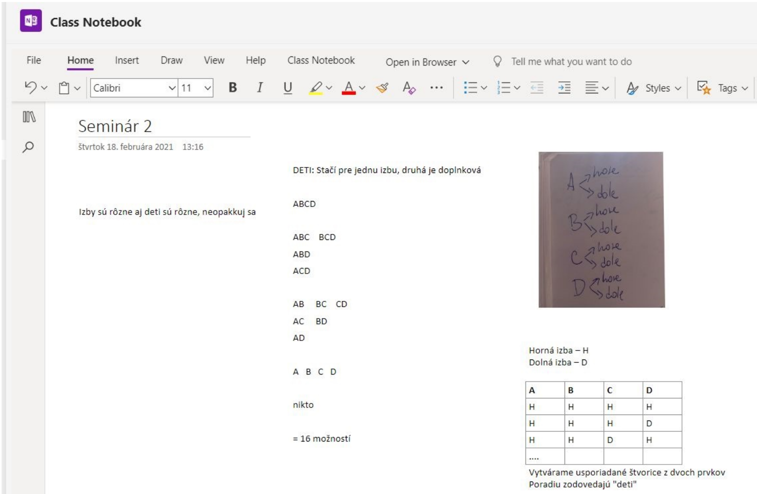


Figure 5. Part of Collaboration Space with contributions of four PSTs.

In this study, we analyze written PSTs' solutions of MSTs, which they solved throughout four problem-solving sessions (2 h each) over four weeks. Our goal is to describe mathematical knowledge of PSTs from the MTSK perspective in the context of combinatorics. The PSTs agreed to their participation in the research. They were informed that their solution would be anonymized before analysis for research purposes.

We assigned 10 MSTs (see Tables 1 and 2) to the PSTs. Two MSTs (1.1 Handshaking and 1.2 Glasses) were solved during the first session, while the other three tasks (2.1 Balls, 2.2 Children, 2.3 Cars) were assigned at the end of the first session as homework. The solutions to these tasks were discussed during the second session.

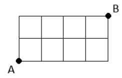
Table 1. Combinatorial MSTs for Sessions 1 and 2.

1.1—Handshaking Mike meets four colleagues: Martin, Milo, Michael and Matthew. Each of them shakes hands with every other colleague once. How many handshakes take place? **1.2—Glasses** John places 6 cups in a row on the table. Four cups are from one set and two are from another (cups from one set look the same). How many different possibilities to arrange the cups in a row does John have?
2.1—Balls [11] In an urn, there are three marbles each numbered with a digit: 2, 4 or 7. We extract a marble from the urn and note down its number. Without replacing the first marble, we extract another one and note down its number. Finally, we extract the last marble from the urn. How many three-digit numbers can we obtain with this method? For example, we could obtain the number 724. **2.2—Children [11]** Four children—Alice, Bert, Carol and Diana—go to spend the night at their grandmother's home. She has two different rooms available (one on the ground floor and another upstairs) in which she could place all or some of the children to sleep. In how many different ways can the grandmother place the children in the two different rooms? For example, she could use only one room to place the children, or she could place Alice, Bert and Carol in the ground floor room and Diana in the upstairs room. **2.3—Cars [11]** Adam has four different colored cars (black, orange, white and grey). He decides to distribute the cars to his friends, Kyle, Luke and Noah. In how many different ways can he distribute the cars? For example, he could give all the cars to Luke.

Table 2. Combinatorial MSTs for Sessions 3 and 4.

3.1—Team
The math teacher wants to choose a four-member team to represent the class in the mathematical competition. He will choose the team from six good students. In how many different ways he can choose four students from six?

3.2—Paths
Determine by how many ways we can get from point A to point B in the square grid in the figure if we follow only the lines of the grid, and we are allowed to go only to the right and up (i.e., we are not allowed to go to the left or down).



3.3—Hockey
Hockey match between Finland and Germany finished 5:2. How many different ways can the course of the match have?

4.1—Staircase [36]
Suppose that a staircase comprises 10 steps and that you can climb the steps one or two steps at a time. In how many different ways can you climb these 10 steps?

4.2—Blocks [36]
You have 2-by-10 rectangular frame (see the figure below) as well as 10 rectangular blocks, each having the dimensions 2 by 1. Your task is to fill the frame with the 10 blocks so that no blocks overlap and the frame is entirely filled. In how many different ways can you arrange the 10 blocks?

During the second session, the PSTs started solving the next three MSTs (3.1 Team, 3.2 Paths, 3.3 Hockey), but due to lack of time, only the solution to the first task (3.1 Team) out of these three MSTs was discussed. During the third session, we discussed solutions to the remaining two tasks (3.2 Paths, 3.3 Hockey), which the PSTs finished as homework. At the end of the third session, we assigned the last two MSTs (4.1 Staircase, 4.2 Blocks). The solutions to these tasks were discussed during the fourth session. Numbers assigned to the tasks indicate during which session we discussed the PSTs' solutions and the order of the task at the session.

The tasks were formulated in different real contexts. Five tasks (1.1 Handshaking, 1.2 Glasses, 3.1 Team, 3.2 Paths, and 3.3 Hockey) can be solved using binomial coefficients (using either combination without repetition or permutation with repetition), and three tasks (2.1 Balls, 2.2 Children, 2.3 Cars) can be solved using arrangement with repetition. The choice of tasks (2.1 Balls, 2.2 Children and 2.3 Cars) was motivated by Dubois' concept of the implicit combinatorial model (see [37]), which emphasizes that combinatorial problems may be classified by different models (selection, partition, distribution) according to the context in which the problem is formulated. Batanero et al. in [11] (p. 196) wrote that "implicit combinatorial model was identified as a didactic variable which shows strong effect on both the problem difficulty and the type of error, e.g., they noticed that some pupils who could apply the definition of the combinatorial operation for the selection model were not able to transfer this definition when changing the problem to a different combinatorial model (partition, distribution)".

Out of 10 listed MSTs, 4 were standard combinatorial tasks (1.1 Handshaking, 1.2 Glasses, 2.1 Balls and 3.1 Team). The other six combinatorial tasks were nonstandard (2.2 Children, 2.3 Cars, 3.2 Paths, 3.3 Cars, 4.1 Staircase and 4.2 Blocks). We consider a standard combinatorial task to be a task that is possible to solve using one combinatorial operation. This combinatorial operation can be easily identified directly from the context of the task without any reformulation. According to Dubois' concept of the implicit combinatorial model (see [37]), it usually belongs to the selection model (a set of m objects, from which a sample of n elements is drawn). Such tasks are most common in Slovak textbooks, and teachers prefer them in their lessons (see [38]). Other tasks, we consider to be nonstandard.

Each of first eight tasks can be solved involving only one combinatorial operation. The solution of last two tasks (4.1 Staircase, 4.2 Blocks) are more complex combinatorial tasks involving more combinatorial operations. Both tasks lead to the same mathematical model and the same numerical solution.

The written PSTs' work from the sessions was collected and analyzed with respect to the results of the whole group's discussion. We created collective solution space (CSS) for each MST. According to Leikin and Lev [30], CSS is a set of different solutions produced by a group of individuals—in our case, it was group of 13 PSTs. Consistent with [30] (p. 185), the solutions are considered different "if they are based on: different representations of some mathematical concepts involved in the task, different properties (definitions or theorems) of mathematical objects within a particular field, or different properties of a mathematical object in different fields".

The analysis of PSTs' written solutions is aimed at the subdomain KoT of the MTSK model to address the following research question: what KoT in the area of combinatorics is exhibited by PSTs when solving MSTs?

4. Data Analysis

Our research has a qualitative design. It presents the case study about the knowledge in combinatorics that PSTs demonstrated during solving 10 MSTs. We obtained two different types of data: PSTs' solution of the 10 MSTs and notes written (inserted) in the Class Notebook Collaboration Space before the whole group discussion. We used semantic analysis of problem solving strategies used by the PSTs' solutions. Firstly, the solution strategies were categorized according to the components of the Lockwood model as a set of outcomes, counting principle and formula. The Table 3 contains the outcome of this categorization.

Secondly, each of the categories—set of outcomes, counting process and formula—was divided into subcategories. The division was created according to differences between solutions, following [15] (the used representation and/or properties of the mathematical object). In such a way, we created CSS for each task. Table 4 shows the size of CSS for each MSTs used in the study.

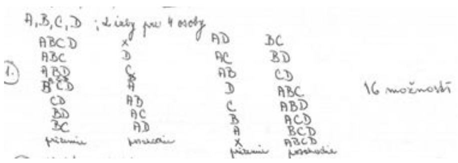
Table 3. Categorization of written solution according to the Lockwood model.

	Set of Outcomes	Counting Process	Formula
1.1 Handshaking	4	8	6
1.2 Glasses	3	3	6
2.1 Balls	6	15	4
2.2 Children	8	16	0
2.3 Cars	6	20	1
3.1 Team	2	1	12
3.2 Paths	12	5	7
3.3 Hockey	7	2	9
4.1 Staircase	7	18	1
4.2 Blocks	7	17	1

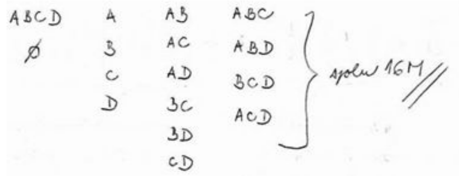
Table 4. Size of CSS.

Number of Solutions	0	1	2	3	4	|CSS|
1.1 Handshaking	4	2	6	0	1	**6**
1.2 Glasses	5	5	2	1	0	**4**
2.1 Balls	0	2	10	1	0	**6**
2.2 Children	1	0	12	0	0	**8 + 2**
2.3 Cars	0	1	11	0	1	**7**
3.1 Team	1	10	1	1	0	**3**
3.2 Paths	1	3	6	3	0	**9**
3.3 Hockey	3	4	4	2	0	**8**
4.1 Staircase	0	3	7	3	0	**9**
4.2 Blocks	0	4	6	3	0	**9**

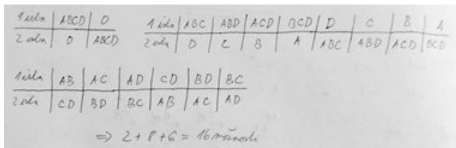
We can see from the Table 4 that the CSS have a bigger size if the task is nonstandard (from 7 to 10 different solutions). For the task 2.2 Children, 10 different solutions were created, although 2 of them were produced after the discussion. Figures 6–8 demonstrate these 10 solutions. Four solutions in Figure 6 used a set of outcomes to derive these solutions.



(**a**) Set of outcomes—systematic listing



(**b**) Set of outcomes—systematic listing—coding only one room



(**c**) Set of outcomes—systematic listing—table representation

Horna ízba-H
Dolná ízba-D

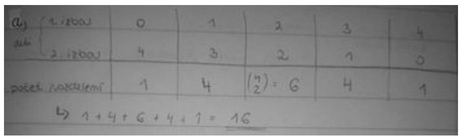
A	B	C	D
H	H	H	H
H	H	H	D
H	H	D	H
……			

Vytvárame usporíadané štvoríce z dvoch prvkov
Poradlu zodovedajú"detí"

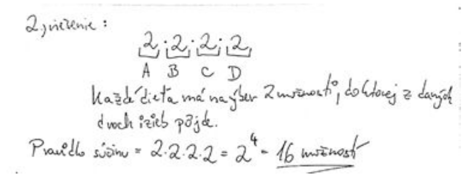
(**d**) Set of outcomes—systematic listing—assigning rooms to children

Figure 6. CSS for the MSTs 2.2 Children—Set of outcomes.

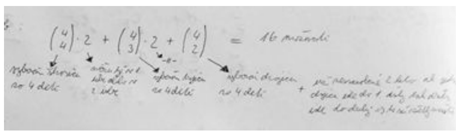
Another four solutions used counting process (see Figure 7).



(**a**) Counting process—addition principle



(**b**) Counting process—multiplication principle

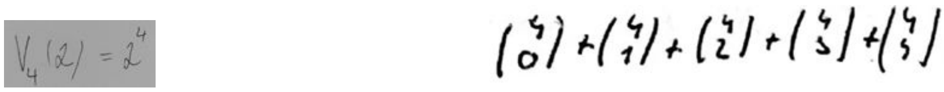


(**c**) Counting process—multiplication and addition principle

(**d**) Counting process—pictorial representation

Figure 7. CSS for the MSTs 2.2 Children—Counting process.

The last two examples of solutions from CSS apply formula (see Figure 8).



(**a**) Formula—arrangement with repetition (**b**) Formula—combinations and addition principle

Figure 8. CSS for the MSTs 2.2 Children—Formula.

The solutions created by the PSTs after the discussion are highlighted in blue.

Aiming to answer the research question, we suggested indicators for subcategories of combinatorial KoT. The indicators (see Table 5) were proposed, consistent with [39], according to experience from previous semesters, which was led by the author.

Table 5. Categories and indicators of combinatorial knowledge of topics.

KoT Subcategory	Indicators
Definitions, properties and their foundations	KoT-C-DP-1 Knowledge of definitions (permutations, permutations with repetition, arrangements with and without repetition, combinations, factorial, binomial coefficient) KoT-C-DP-2: Knowledge of combinatorial principles (addition and multiplication principle) KoT-C-DP-3: Knowledge of notation for arrangements, permutations and combinations KoT-C-DP-4: Knowledge of properties of binomial coefficients and factorials, e.g., $\binom{n}{k} = \binom{n}{n-k}$ KoT-C-DP-5: Knowledge of connections between set of outcomes, counting principles and formulas (components of Lockwood model)
Phenomenology and its applications	KoT-C-PH-1: Knowledge of different meanings of combinatorial expression, e.g., expression $\binom{n}{k}$ can be interpreted as binomial coefficient, number of k-elements subset of n-element set, number which equals to $\frac{n!}{(n-k)!k!}$, etc. KoT-C-PH-2: Knowledge about different contexts in which combinatorial operation can arise
Mathematical procedures	KoT-C-MP-1: Knowledge of method for creating set of outcomes systematically, e.g., using tree graphs KoT-C-MP-2: Knowledge of method for calculation of combinatorial expressions (with binomial coefficients, factorials, etc.)
Representation systems	KoT-C-RS-1: Knowledge of different types of representations (pictorial, verbal, graphical, symbolic) KoT-C-RS-2: Knowledge of connections between different types of representations and combinatorial operations

Thirdly, we analyzed the PSTs' solutions and notes written (inserted) in the Class Notebook Collaboration Space, according to the KoT subcategories and indicators in Table 5.

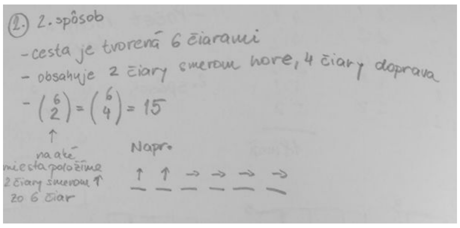
5. Results

In this section, the presence of evidence of PSTs' knowledge of topic is discussed. PSTs' written solutions are analyzed, following the indicators of subcategories of KoT described in Table 5.

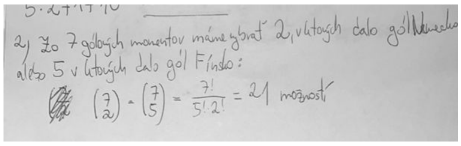
Definitions, properties and their foundations

Each of the 13 PSTs have revealed KoT-C-DP-1, 2 and 3 in their written solution. They correctly utilized concepts, such as permutations, permutations with repetition, arrangements with and without repetition, combinations, factorial and binomial coefficients, and used correct formulas to calculate them. Combinatorial principles and notation were used properly in the solutions too.

On the other hand, the data revealed difficulties in the indicator knowledge of properties of binomial coefficients (KoT-C-DP-4). The written solution of the task 2.2 Children and the following discussion showed that no PST perceived the relation $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$, and in the task 3.1 Team, nobody used the property $\binom{n}{k} = \binom{n}{n-k}$. The PSTs have already encountered such formulas several times during the discrete mathematics course. After the whole group discussion about the solutions of the task 3.1 Team, the property $\binom{n}{k} = \binom{n}{n-k}$ appeared in the solutions of task 3.2 in the work of three PSTs and in the solutions of task 3.3 Hockey in the work of one PST (see Figure 9). This experience shows that, although PSTs know the formulas for binomial coefficients, which result from the Pascal triangle, it is not natural for them to use these formulas to solve a combinatorial task.



(**a**) 3.2 Paths



(**b**) 3.3 Hockey

Figure 9. Using the relation $\binom{n}{k} = \binom{n}{n-k}$ in the solutions of tasks.

The PSTs have also demonstrated gaps in the indicator KoT-C-DP-5—knowledge of connections between the set of outcomes, counting principles and formulas (components of the Lockwood model). No PST identified that the combinatorial operation suitable for the solution of task 2.2 Children is arrangement with repetition. During discussion, they argued in the way that "Children are different and rooms are different. In this situation no repetition is present, and therefore in such a case it is not possible to apply the known formula for arrangements with repetitions". Consequently, the PSTs were asked to choose the set of outcomes from the examples presented on the whiteboard or to create new ones, which corresponds to their counting process (see Figure 7b. After approximately 10 minutes, four new sets of outcomes emerged; one of them is shown in Figure 6d. Afterwards, the formula $V_4(2) = 2^4$ (Slovak notation for arrangements with repetition) appeared with the explanation that we create a 4-tuple from 2 elements—arrangement with repetition (room on the ground floor and upstairs)—in the notes of one PST.

The gaps in the knowledge of connections between components of the Lockwood model were also shown in the solution of task 2.3 Cars. Five PSTs wrote the correct expression 3^4; however, nobody explained how we can obtain this expression from the set of outcomes. Only one PST mentioned in the solution that the result of the task is an arrangement with repetition.

As for task 3.2 Paths, six PSTs solved it in one way—using the set of outcomes. Five PSTs identified the correct combinatorial operation intertwined with the set of outcomes and/or with the counting process. These PSTs also explained how this operation is connected with the set of outcomes and counting process. Two PSTs used other methods of solution. They filled the number of possible paths to each vertex of square grid (see Figure 10). Unfortunately, they did not recognize that this representation is intertwined with the Pascal triangle and its property: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Seven PSTs used a correct combinatorial operation to solve the task 3.3 Hockey. Nevertheless, only two of them described the set of outcomes which leads to this combinatorial operation. One of them created six possibilities for the first goal of Germany and then for each of these six possibilities, found the position for the second goal of Germany. This representation leads to the expression $6 + 5 + 4 + 3 + 2 + 1$, which was correctly identified as $\binom{7}{2}$. Another PST created 6-tuples from two letters F (Finland) and N (Germany, in Slovak language "Nemecko") and consequently obtained possibilities FFFFFNN, FFFFNFN, etc. In this representation, each 6-tuple represents who scored which goal.

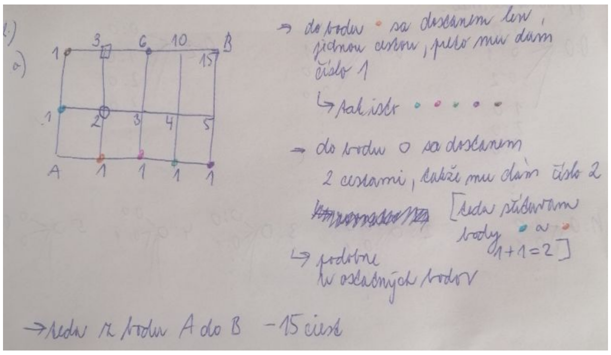


Figure 10. Part of Pascal triangle in the solution of task 3.2 Paths.

After three sessions, we assigned tasks 4.1 and 4.2 to the PSTs. Seven PSTs out of thirteen connected the set of outcomes, counting process and formula each other (4 of them explained only the idea of how to obtain the complete set of outcomes).

PHENOMENOLOGY AND ITS APPLICATIONS

In the context of task 3.3 Hockey, we describe the knowledge of different meanings of combinatorial expression (KoT-C-PH-2) demonstrated by the PSTs. The solution of this task leads to the expression $\binom{7}{2}$. The PSTs identified the following four different ways to derive this solution:

1. Having 7 places for goals, we choose 2 places out of 7 to be the goal of Finland (3 PSTs);
2. Having 7 places for goals, we choose 5 places out of 7 to be the goal of Germany and then realize that the expression $\binom{7}{5} = \binom{7}{2}$ (1 PST);
3. Creating 7-letter arrangements from 5 letters F and 2 letters N, which leads to the expression $\frac{7!}{2!\cdot 5!}$, which equals to $\binom{7}{2}$ (2 PSTs, the connection with the binomial coefficient was formulated during the discussion);
4. deriving the expression $6+5+4+3+2+1$ which is equal to $\binom{7}{2}$ (1 PST).

Unfortunately, only 1 PST worked with two different meanings (items 1 and 2). Others who arrived to the combinatorial expression $\binom{7}{2}$, worked with one isolated meaning, which they discovered in their solutions.

The ability to identify different contexts in which combinatorial operation can arise (KoT-C-PH-2) was not demonstrated in the solutions of first eight MSTs, except for two cases. In the written solution of task 2.3 Cars, one PST wrote that: "the task is analogical to the task how many n-digit numbers we can create from m digits, which can repeat". No other references to the same combinatorial operation were identified among the tasks 2.1 Balls, 2.2 Children and 2.3 Cars. As for the tasks 3.1 Team, 3.2 Paths and 3.3 Hockey, 12 PSTs used different representation system for the solution of each of these tasks. This showed they did not see the analogy between the tasks. In the discussion, they realized that "the tasks 3.1, 3.2 and 3.3 use the same combinatorial operation". A solution of one PST that showed the analogy of task 3.3 with task 3.2 using the same representation is in Figure 11.

The PSTs were asked to find as many solutions as they could. However, except for one PST, they did not identify analogies between the tasks during the first three sessions before we started to discuss the solutions in the CSS. This was improved when solving tasks 4.1 and 4.2. Nine PSTs identified that the tasks are basically the same. Moreover, they associated the result of these tasks with the Fibonacci sequence. This can be interpreted as progress compared to the solutions of the previous tasks, but of course, further research needs to be carried out on that.

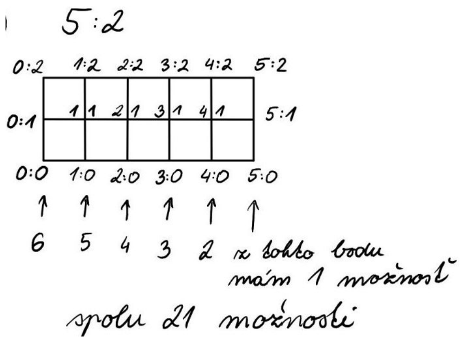


Figure 11. Analogy between task 3.2 and 3.3.

Mathematical procedures

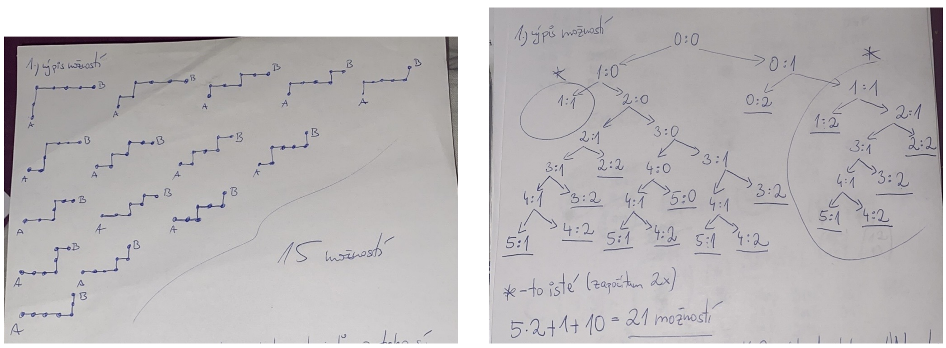
Mathematical procedures tied with combinatorics do not need special input knowledge or handling complex concepts. Each PST in our group exhibited knowledge of method for creating set of outcomes systematically (KoT-C-MP-1). In order to create the set of outcomes systematically, they organized the outcomes using tables, tree-diagrams or dividing the set of all outcomes into disjoint sets. The PSTs also showed knowledge of method for calculation of combinatorial expressions (KoT-C-MP-2).

Representation systems

For task 2.2 Children, 12 PSTs used at least two different representations. This shows that the PSTs were aware of different possibilities for representing a situation in combinatorial tasks (KoT-C-RS-1). The most common representation used by the PSTs in solutions of different tasks was tree diagram (used by 10 PSTs at least once). This representation helped them to organize set of outcomes or to divide all possibilities into disjoint subsets.

Representations which the PSTs used for standard combinatorial tasks (1.1 Handshaking, 1.2 Glasses, 2.1 Balls and 3.1 Team) were mostly common. In most cases, the PSTs described verbally why they chose the combinatorial operation, sometimes supplemented by the set of outcomes, the rest just wrote the formula without an explanation (three PSTs for task 1.2 and four PSTs for task 3.1). It seems that standard tasks were not challenging for the PSTs to look for more solutions, which is in agreement with [14].

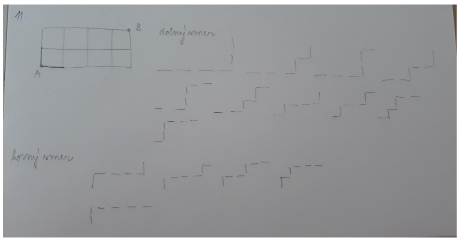
The choice of representation for the nonstandard tasks (2.2 Children, 2.3 Cars, 3.2 Paths, 3.3 Hockey, 4.1 Staircase and 4.2 Blocks) showed strong dependency on the context of the task. In Figure 12, we can see how context influenced the choice of the PST's representation used for the tasks 3.2 Paths and 3.3 Hockey. Although both tasks can be solved using one combinatorial operation (combinations or permutations with repetition), the chosen representation prevents the PST from discovering this analogy. Without the whole-group discussion on CSS, this analogy would stay undiscovered and the intra-conceptual connection would not be achieved.



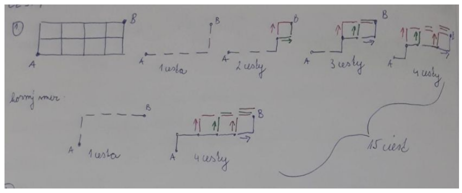
(**a**) 3.2 Paths (**b**) 3.3 Hockey

Figure 12. Representation depends on the context of the task.

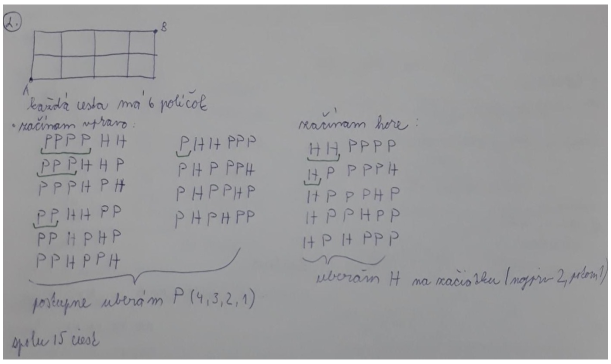
Knowledge of connections between different types of representations and combinatorial operations (KoT-C-RS-2) was explicitly demonstrated in one PST's solution of task 3.2 Paths. Figure 13 shows three solutions with the gradual refinement of representation. From the third set of outcomes, the connection with the combinatorial operation (permutations with repetition or combinations) is easy to see. Such gradual switching from one representation to more abstract one has a significant pedagogical power.



(**a**) Dependency on the context



(**b**) Less dependency on the context



(**c**) Independence on the context

Figure 13. Refinement of the representation system for task 3.2 Paths.

Seven PSTs' solution of the tasks 4.1 and 4.2 exemplified in their solutions the transition from the representation related to the context (staircase and blocks) to arrangement of numbers 1 and 2 that sum to 10. This transition is valuable from the pedagogical point of view because it is one possible way for teachers to help students discover more abstract methods to solve the problem (counting process, formula) and, therefore, shift their combinatorial thinking.

6. Conclusions

The study presents the utilization of multiple-solution tasks in combinatorics as a part of the pre-service teachers course on didactics of mathematics in the view of the Mathematics Teachers' Specialized Knowledge (MTSK) theoretical framework.

Our findings showed a considerable gap in the development of PSTs' KoT in combinatorics. Although the PSTs have already completed their preparation regarding fundamental mathematical knowledge in combinatorics, we uncovered gaps in the ability to apply elementary properties of binomial coefficients during solution of combinatorial tasks and the ability to identify connections between components of the Lockwood model (set of outcomes, counting principles and formulas). This knowledge belongs to the subcategory of KoT—knowledge of definitions, properties and their foundations. Our findings show that expected intra-conceptual connections were not established in the awaiting way and level.

The ability to interpret a combinatorial expression in different ways was also poorly demonstrated as well as the capability to see and apply the same combinatorial operation in different contexts (subcategory phenomenology and its applications).

We have demonstrated that the PSTs' problem-solving ability was partially improved during the four sessions. The PSTs started connecting combinatorial tasks with various concepts and properties and demonstrated that their knowledge had become more interconnected. The use of MSTs stimulated PSTs for seeing relationships between different representations, different contexts and concepts and their properties. The whole group analysis of the solutions from CSS induced PSTs to perceive the level of their own understanding of notions, concepts, various representations and interpretations, which motivated them to work on themselves. These findings have led us to the conclusion that using MSTs, followed by a discussion concerning CSS created by the PSTs, forms settings for development (at least partly) of PSTs' combinatorial knowledge and skills, which are essential for their subsequent work in the classroom. The PSTs began to be interested in solving combinatorial tasks in different ways and in this way, they developed their flexibility, which is very important for improving their ability to supervise a mathematical discussion during their lessons aimed to combinatorics. Additionally, MSTs were a useful tool to find out whether PSTs have a conceptual understanding of the combinatorial concepts and their properties and consequently to find out which subcategory of combinatorial KoT was not sufficiently developed.

The Lockwood model of students' combinatorial thinking provided us with the language to characterize useful components and relationships involved in the solving of combinatorial tasks, and helped us to identify missing intra-conceptual connections and particular reasons behind PSTs' struggles. Consequently, it helped us take the whole group discussions to problematic points and to improve PSTs' understanding, particular combinatorial knowledge and skills. Moreover, our research showed the importance of relationships between two components of Lockwood model—formulas and the set of outcomes. Lockwood in [10] stated that the relationship between these components is possibly only a theoretical aspect of the model. Our study showed that from the pedagogical point of view, awareness of this relationship is important for PSTs to see connections between abstract solution (using formula) and a solution using set of outcomes (which is popular among most students).

MTSK conceptualization and identification of PSTs' KoT indicators helped us to gain a deeper understanding of the teacher content knowledge in combinatorics. Our outcomes emphasize the necessity of a modification of focus in teacher training with stress on building connections as it is claimed by Thompson et al. in [8] (p. 417): "If a teacher's conceptual structures comprise disconnected facts and procedures, their instruction is likely to focus on disconnected facts and procedures. In contrast, if a teacher's conceptual structures comprise a web of mathematical ideas and compatible ways of thinking, it will at least be possible that she attempts to develop these same conceptual structures in her students. We believe that it is mathematical understandings of the latter type that serve as a necessary condition for teachers to teach for students' high-quality understanding."

The study can be seen also as a contribution to developing a deeper and broader understanding of teachers' knowledge taught to PSTs, as it is required in [39]. This study contributes to this within the topic of combinatorics.

The results of our study provide us with helpful facts for further discussion concerning a reform of the bachelor program course aimed at the development of fundamental mathematical knowledge in combinatorics. Moreover, they provide us with arguments for the implementation of new approaches in courses focusing on the improvement of classroom competencies of PSTs.

The limitation of the study is a result of the relatively small sample size. The results in the other group of pre-service teachers may differ. Therefore, future research should include a larger sample size to conclude more generalizable data. Regardless of this limitation, we believe that this study will encourage other researchers in further research of KoT in combinatorics. The research also generated stimuli for focusing on other subdomains of the model MTSK in a combinatorial context too. MSTs seems to be a useful tool for developing knowledge in the subdomains KFLM and KMT of the model MTSK.

Funding: This work was supported by the Slovak grant KEGA 020UPJŠ-4/2020.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

References

1. König, J.; Blömeke, S.; Jentsch, A.; Schlesinger, L.; née Nehls, C.F.; Musekamp, F.; Kaiser, G. The links between pedagogical competence, instructional quality, and mathematics achievement in the lower secondary classroom. *Educ. Stud. Math.* **2021**, *107*, 189–212. [CrossRef]
2. Hošpesová, A.; Novotná, J.; Vondrová, N.; Moraová, H.; Tichá, M. From Teacher of Nations to Teacher of Mathematics. *Mathematics* **2021**, *9*, 1583. [CrossRef]
3. Grossman, P.L. *Learning to Practice: The Design of Clinical Experience in Teacher Preparation*; American Association of Colleges for Teacher Education and National Education Association: Washington, DC, USA, 2010; pp. 1–8.
4. Rowland, T.; Martyn, S.; Barber, P.; Heal, C. Primary teacher trainees' mathematics subject knowledge and classroom performance. *Res. Math. Educ.* **2000**, *2*, 3–18. [CrossRef]
5. Silverman, J.; Thompson, P. Toward a framework for the development of mathematical knowledge for teaching. *J. Math. Teach. Educ.* **2008**, *11*, 499–511. [CrossRef]
6. Krainer, K. Powerful tasks: A contribution to a high level of acting and reflecting in mathematics instruction. *Educ. Stud. Math.* **1993**, *24*, 65–93. [CrossRef]
7. Zaslavsky, O.; Leikin, R. Professional Development of Mathematics Teacher Educators: Growth Through Practice. *J. Math. Teach. Educ.* **2004**, *7*, 5–32. [CrossRef]
8. Thompson, P.W.; Carlson, M.P.; Silverman, J. The design of tasks in support of teachers' development of coherent mathematical meanings. *J. Math. Teach. Educ.* **2007**, *10*, 415–432. [CrossRef]
9. ŠPÚ. Inovovaný Štátny VzdelávacíProgram. The National Institute for Education in Slovak Republic. 2014. Available online: https://www.statpedu.sk/sk/svp/inovovany-statny-vzdelavaci-program/ (accessed on 2 September 2021).
10. Lockwood, E. A model of students' combinatorial thinking. *J. Math. Behav.* **2013**, *32*, 251–265. [CrossRef]
11. Batanero, C.; Navarro-Pelayo, V.; Godino, J.D. Effect of the implicit combinatorial model on combinatorial reasoning in secondary school pupils. *Educ. Stud. Math.* **1997**, *32*, 181–199. [CrossRef]
12. English, L.D. Combinatorics and the development of children's combinatorial reasoning. In *Exploring Probability in School. Mathematics Education Library*; Jones, G.A., Ed.; Springer: Boston, MA, USA, 2005; Volume 40, pp. 121–141. [CrossRef]
13. Hubeňáková, V.; Semanišinová, I. Preservice teachers' mathematical knowledge for teaching combinatorial thinking. In Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education (CERME11), Utrecht, The Netherlands, 5–10 February 2019; pp. 3921–3928. Available online: https://hal.archives-ouvertes.fr/hal-02430472/document (accessed on 6 August 2021).
14. Guberman, R.; Leikin, R. Interesting and difficult mathematical problems: changing teachers' views by employing multiple-solution tasks. *J. Math. Teach. Educ.* **2013**, *16*, 33–56. [CrossRef]
15. Leikin, R. Multiple-solution tasks: from a teacher education course to teacher practice. *Zdm Math. Educ.* **2011**, *43*, 993–1006. [CrossRef]
16. Ball, D.L.; Thames, M.H.; Phelps, G. Content knowledge for teaching: what makes it special? *J. Teach. Educ.* **2008**, *59*, 389–407. [CrossRef]
17. Carrillo-Yañez, J.; Climent, N.; Montes, M.; Contreras, L.C.; Flores-Medrano, E.; Escudero-Ávila, D.; Vasco, D.; Rojas, N.; Flores, P.; Aguilar-González, Á.; et al. The mathematics teacher's specialised knowledge (MTSK) model. *Res. Math. Educ.* **2018**, *20*, 236–253. [CrossRef]
18. Scheiner, T.; Montes, M.A.; Godino, J.D.; Carrillo, J.; Pino-Fan, L.R. What Makes Mathematics Teacher Knowledge Specialized? Offering Alternative Views. *Int. J. Sci. Educ.* **2019**, *17*, 153–172. [CrossRef]
19. Delgado-Rebolledo, R.; Zakaryan, D. Relationships Between the Knowledge of Practices in Mathematics and the Pedagogical Content Knowledge of a Mathematics Lecturer. *Int. J. Sci. Math. Educ.* **2020**, *18*, 567–587. [CrossRef]
20. Ribeiro, C.M.; Carrillo, J. Discussing a teacher MKT and its role on teacher practice when exploring data analysis. In Proceedings of the PME 35, Ankara, Turkey, 10–15 July 2011; pp. 41–48.
21. Schoenfeld, A.H. *Mathematical Problem Solving*; Academic Press: Orlando, FL, USA, 1985; ISBN 978-0-12-628870-4.
22. Silver, E. Fostering creativity through instruction rich in mathematical problem solving and problem posing. *ZDM* **1997**, *29*, 75–80. [CrossRef]
23. Lesh, R.; Zawojewski, J. Problem solving and modeling. In *Second Handbook of Research on Mathematics Teaching and Learning*; Lester, F., Ed.; Information Age: Charlotte, NC, USA, 2007; Chapter 17, pp. 763–804.
24. English, L.; Sriraman, B. Problem Solving for the 21st Century. In *Theories of Mathematics Education*; Advances in Mathematics Education; Sriraman, B., English, L., Eds.; Springer: Berlin/Heidelberg, Germany, 2010. [CrossRef]

25. Polya, G. *How to Solve It: A New Aspect of Mathematics Method*; Princeton University Press: Princeton, NJ, USA, 1973; ISBN 9780691164076.
26. Silver, E.A.; Ghousseini, H.; Gosen, D.; Charalambous, C.; Strawhun, B.T.F. Moving from rhetoric to praxis: Issues faced by teachers in having students consider multiple solutions for problems in the mathematics classroom. *J. Math. Behav.* **2005**, *24*, 287–301. [CrossRef]
27. Montes, M.; Climent, N.; Carrillo, J.; Contreras, L. Constructing tasks for primary teacher education from the perspective of Mathematics Teachers' Specialised Knowledge. In Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education (CERME11), Utrecht, The Netherlands, 5–10 February 2019; pp. 3978–3985. Available online: https://hal.archives-ouvertes.fr/hal-02430479/document (accessed on 6 August 2021).
28. Policastro, M.; Mellone, M.; Ribeiro, M.; Fiorentini, D. Conceptualising tasks for teacher education: From a research methodology to teachers' knowledge development. In *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education (CERME11)*; Jankvist, U., van den Heuvel-Panhuizen, M., Veldhuis, M., Eds.; Freudenthal Group, Utrecht University: Utrecht, The Netherlands, 2019; Volume TWG20. Available online: https://hal.archives-ouvertes.fr/hal-02430487/document (accessed on 6 August 2021).
29. NCTM. *Principles and Standards for School Mathematics*; National Council of Teachers of Mathematics: Reston, VA, USA, 2000.
30. Leikin, R.; Lev, M. Mathematical creativity in generally gifted and mathematically excelling adolescents: What makes the difference? *Zdm Math. Educ.* **2013**, *45*, 183–197. [CrossRef]
31. English, L.D. Young children's combinatoric strategies. *Educ. Stud. Math.* **1991**, *22*, 451–474. [CrossRef]
32. Kapur, J.N. Combinatorial Analysis and School Mathematics. *Educ. Stud. Math.* **1970**, *3*, 111–127. [CrossRef]
33. Lebowitz, R. Writing Discrete(ly). In *Discrete Mathematics in the Schools*; Rosenstein, J.G., Franzblau, D.S., Roberts, F.S., Eds.; DIMACS Series in Discrete Mathematics and Theoretical Computer Science; AMS: Providence, RL, USA, 1997; Volume 36, pp. 85–88, ISBN 978-0-8218-1137-5 (print)/978-1-4704-3994-1 (online).
34. Hart, E.W. Discrete Mathematical Modeling In The Secondary Curriculum: Rationale and Examples from The Core-Plus Mathematics Project. In *Discrete Mathematics in the Schools*; Rosenstein, J.G., Franzblau, D.S., Roberts, F.S., Eds.; DIMACS Series in Discrete Mathematics and Theoretical Computer Science; AMS: Providence, RL, USA, 1997; Volume 36, pp. 265–280, ISBN 978-0-8218-1137-5 (print)/978-1-4704-3994-1 (online).
35. Medová, J.; Ovary Bulková, K.; Čeretková, S. Relations between Generalization, Reasoning and Combinatorial Thinking in Solving Mathematical Open-Ended Problems within Mathematical Contest. *Mathematics* **2020**, *8*, 2257. [CrossRef]
36. Rubel, L.; Zolkower, B. On Blocks, Stairs, and beyond: Learning about the Significance of Representations. *Math. Teach. Learn. Teach. Pk–12* **2007**, *101*, 340–344. [CrossRef]
37. Dubois, J. Une systématique des configurations combinatoires simple. *Educ. Stud. Math.* **1984**, *15*, 37–57. [CrossRef]
38. Palenčárová, D. Riešenie Kombinatorických úloh na Základnej Škole. Ph.D. Thesis, Pavol Jozef Šafárik University, Košice, Slovakia, 2013. Available online: https://opac.crzp.sk/?fn=detailBiblioForm&sid=96D6D14AE6883BC5DB045F2A43F0&seo=CRZP-detail-kniha (accessed on 2 September 2021).
39. Zakaryan, D.; Ribeiro, M. Mathematics teachers' specialized knowledge: A secondary teacher's knowledge of rational numbers. *Res. Math. Educ.* **2019**, *21*, 25–42. [CrossRef]