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Composition Vector Spaces as a New Type of Tri-Operational Algebras

Omid Reza Dehghan [1,†], Morteza Norouzi [1,†] and Irina Cristea [2,*,†]

[1] Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, Bojnord 94531, Iran; dehghan@ub.ac.ir (O.R.D.); m.norouzi@ub.ac.ir (M.N.)
[2] Centre for Information Technologies and Applied Mathematics, University of Nova Gorica, 5000 Nova Gorica, Slovenia
* Correspondence: irina.cristea@ung.si or irinacri@yahoo.co.uk; Tel.: +386-0533-15-395
† These authors contributed equally to this work.

Abstract: The aim of this paper is to define and study the composition vector spaces as a type of tri-operational algebras. In this regard, by presenting nontrivial examples, it is emphasized that they are a proper generalization of vector spaces and their structure can be characterized by using linear operators. Additionally, some related properties about foundations, composition subspaces and residual elements are investigated. Moreover, it is shown how to endow a vector space with a composition structure by using bijective linear operators. Finally, more properties of the composition vector spaces are presented in connection with linear transformations.

Keywords: composition vector space; foundation; strong composition subspace; composition linear transformation

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1. Introduction

Menger [1] in 1944 initiated the studies on the theory of the tri-operational algebras, based on his interest to algebra of functions, that led him to investigate the behavior of the functions under various operations: addition, multiplication, or composition. He observed that these operations on functions have an important role in the algebra of functions. The composition of functions is associative, non-commutative and does not depend on addition and multiplication, but it is connected to them by the one-sided distributive laws: $(f+g)\circ h = f\circ h + g\circ h$ and $(f\cdot g)\circ h = (f\circ h)\cdot(g\circ h)$. This was a strong motivation for Menger to define a new algebraic structure, namely a *tri-operational algebra* [2,3]. Moreover, a tri-operational algebra is a special type of commutative ring, such as the ring of polynomials, the ring of infinitely differentiable functions on $\mathbb{R}$, and so on. This idea was clearly presented by Adler [4] in 1961, when he defined the *composition rings* as commutative rings endowed with an additional operation, called *composition*, that is related to the two operations of the rings. Twenty years later, Kaiser and Nöbauer [5,6] studied more in depth the composition of polynomials and polynomial functions, especially k-dimensional V-composition algebras, where V is a variety containing subgroups, or near rings, composition rings, and composition lattices, investigated also in [7]. While in classical algebra these ideas were only sporadically deepened in the 20th century—we may refer here to the papers of Veldsman [8], or Gallina and Morini [9]—they opened a new line of research in hypercompositional algebra. This is the theory of algebraic hypercompositional structures (called also hyperstructures), i.e., algebraic structures with one or more hyperoperations, which synthesize elements of the support set of the structure and have the result of a subset of the support set instead of one element only, as the classical operations have. The study in this direction was opened by Cristea and Jancic-Rasovic [10] by defining the composition hyperrings, which better describe the structure of the hyperrings of polynomials [11]. This work was then continued in two directions: the study of the (m,n)-hyperrings endowed with a composition operation [12] and the study of the hyperrings endowed with a composition, i.e., an n-ary hyperoperation [13]. Combining these two directions, one may study the composition (m, n, k)-hyperrings [14], that are (m, n)-hyperrings endowed with a k-ary composition hyperoperation. The same initial idea of [10] was applied also for EL-hyperstructures [15], which are hyperstructures constructed from quasi-ordered semigroups.

Inspired by all of these researches, in this manuscript we define composition vector spaces as a proper generalization of the vector spaces, by introducing a new operation, called *composition*, connected with the operations of the vector spaces. Additionally, we define and study some properties of the left or right foundations, composition subspaces, and residual elements in composition vector spaces. We prove that every subspace of a composition vector space is not generally a composition subspace, while the set V_W of all residual elements of a composition vector space V modulo a subspace W is the largest strong composition subspace of V such that W can be written as the intersection between V_W and the left foundation of V. We also show that the intersection of any maximal composition subspace of V with the left foundation $LF(V)$ is a maximal subspace in $LF(V)$. Moreover, in every composition vector space V, there is no nontrivial left foundation such that V is a group under the composition operation. In the last part of the paper we study properties of the composition vector spaces related to linear operators. We define a family of linear operators associated with a composition vector space. If any operator of this family is a bijection, then the composition vector space is called automorphic. We prove that any vector space may have a composition structure obtained by using bijective linear operators, which lead us to conclude that the image of an automorphic composition vector space under an onto composition linear transformation is automorphic, too. The paper ends with a conclusive section.

2. Foundations of Composition Vector Spaces

In this section we will first introduce the concept of composition vector space and support it using several non-trivial examples.

Definition 1. *Let F be a field and $(V, +, \cdot, F)$ a vector space over F, so $\cdot$ is an operation defined as $\cdot : F \times V \longrightarrow V$. An algebraic structure $(V, +, *, \cdot, F)$ is said to be a left composition vector space over F, if $(V, *)$ is a semigroup and the following conditions hold, for all $x, y, z \in V$ and $a \in F$:*

(1) $(x + y) * z = (x * z) + (y * z)$;
(2) $(a \cdot x) * y = a \cdot (x * y)$.

If $x * (y + z) = (x * y) + (x * z)$ and $x * (a \cdot y) = a \cdot (x * y)$ hold, then $(V, +, *, \cdot, F)$ is called a right composition vector space over F.

Moreover, we say that V is left (right) unitary, if there exists $I \in V$ such that $I * x = x$ $(x * I = x)$, for all $x \in V$. In this case I is called a left (right) identity. As usual, V is called unitary if it is left and right unitary.

We denote the unit in V by I in order to not confuse it with the unit 1 of the field F and the zero vector 0.

In the following, we will illustrate this concept using several examples, including the most well-known vector spaces.

Example 1. *Consider a vector space $(V, +, \cdot, F)$ over a field F.*
(1) *V is a left and right composition vector space, whenever the operation $*$ is defined by $x * y = 0$, for all $x, y \in V$. In this case, V is called a null composition vector space. It is clear that it has no identity.*
(2) *For all $x, y \in V$, define $x * y = x$ (respectively, $x * y = y$). Then V is a left (respectively, right) composition vector space over F. Note that V is not left (respectively right) unitary, unless $V = \{0\}$, while every vector of V is a right (left) identity.*

(3) *Consider* $(L(V), +, \cdot, F)$ *to be the vector space of all linear operators on* V *and define* $*$ *as the composition of linear operators. Then* $L(V)$ *is a unitary composition vector space.*

Example 2. *Define the operation* $*$ *on the space* $V = \{f : S \longrightarrow F, f \text{ is a function}\}$ *of all functions from a set* S *to a field* F *as* $(f * g)(s) = f(s)g(s)$*. Then* V *is a unitary composition vector space over* F*, where the function* $1 : S \longrightarrow F$ *defined by* $1(s) = 1_F$*, for any* $s \in S$*, is the identity.*

Example 3. $(F^n, +, \cdot, *, F)$ *is a unitary composition vector space for any field* F*, where the composition is defined as* $(a_1, \dots, a_n) * (b_1, \dots, b_n) = (a_1b_1, \dots, a_nb_n)$*, having the unit* $(1, 1, \dots, 1)$*.*

Example 4. (1) *The vector space* $M_n(F)$ *of all* $n \times n$ *matrices over a field* F *is a unitary composition vector space with the composition being the ordinary multiplication of matrices.*
(2) *The vector space* $M_{m\times n}(F)$ *of all* $m \times n$ *matrices over a field* F *is a unitary composition vector space, where the composition is defined by* $[a_{ij}] * [b_{ij}] = [a_{ij}b_{ij}]$*.*

Example 5. *Let* P *be the space of polynomial functions over a field* F*.*
Define the composition of polynomials as $\sum_{i=0}^{\infty} a_i x^i * \sum_{j=0}^{\infty} b_j x^j = \sum_{k=0}^{\infty} c_k x^k$*, where* $c_k = \sum_{t=0}^{k} a_t b_{k-t}$ *and* $k = 0, 1, 2, \dots$*. Then* P *is a unitary composition vector space over* F*, where* 1 *and* x *are their identities, respectively.*

Example 6. *The set* $\mathrm{C}(\mathbb{R})$ *of all real continuous functions over* $\mathbb{R}$ *is a unitary composition vector space with the sum, composition and scalar product of functions.*

It is clear that any property of left composition vector spaces holds also for right composition vector spaces. Thus, we will consider only the left ones, unless otherwise specified, and "composition vector space" means "left composition vector space".

Proposition 1. *If* $(V, +, \cdot, *, F)$ *is a composition vector space, then the space* $(L(V), +_L, \cdot_L, *_L, F)$ *of all operators on* V *is a composition vector space, too, where* $f *_L g : V \longrightarrow V$ *is defined by* $(f *_L g)(x) = f(x) * g(x)$*, for all* $f, g \in L(V)$*. Moreover, if* I *is a left (right) identity of* V*, then* $I_L : V \longrightarrow V$*, defined by* $I_L(x) = I$*, is a left (right) identity of* $L(V)$*.*

Proof. The conditions in Definition 1 are all fulfilled for the space $L(V)$. $\square$

Definition 2. *Let* V *be a composition vector space. Then an element* $x \in V$ *is called a left (right) constant, if* $x * y = x$ *(respectively* $y * x = x$*), for all* $y \in V$*. If* $W \subseteq V$*, then the set of all left (right) constants in* W *is called the left (right) foundation of* W *and is denoted by* $LF(W)$ *(respectively* $RF(W)$*). The element* x *is said to be a constant if it is both left and right constant, while* $F(W)$ *is the set of all constants of* W*.*

Example 7. (1) *The zero vector is the only constant in any null composition vector space.*
(2) *In Example 1(2), if* V *is a left composition vector space, then* $LF(V) = V$ *and* $RF(V) = \emptyset$*. Similarly, if* V *is a right composition vector space, then* $LF(V) = \emptyset$ *and* $RF(V) = V$*. It is clear that there is no any constant in both of them.*
(3) *Continuing with Example 1(3), any constant operator of* V *is a constant of* $L(V)$*, given in Example 1(3), as well as constant functions are constants of the composition vector space in Example 2.*

Example 8. *If* x *is a constant of a composition vector space* V*, then the function* $f : V \longrightarrow V$*, defined by* $f(t) = x$ *for all* $t \in V$*, is a constant of the composition vector space* $L(V)$*.*

In what follows, it is assumed that V is a composition vector space over F, unless otherwise stated. A subspace W of V is called a *composition subspace* if it is closed under composition. In this case, we write $W \leqslant^c V$.

Example 9. *Every subspace of a null composition vector space is a composition subspace.*

However, generally, not every subspace of a composition vector space is a composition subspace, as we can see below.

Example 10. *The set $\mathbb{R}^{\mathbb{R}}$ of all functions from $\mathbb{R}$ to $\mathbb{R}$ is a composition vector space under the sum, scalar product (i.e., $(f \cdot g)(t) = f(t) \cdot g(t)$) and composition of functions. Consider $W = \{f \in \mathbb{R}^{\mathbb{R}} \mid f(-1) = 0\}$. Then W is a subspace of $\mathbb{R}^{\mathbb{R}}$, but it is not a composition subspace, because $(f + cg)(-1) = 0$, while $(f * g)(-1) = f(g(-1)) = f(0)$, which is not always 0, for all $f, g \in \mathbb{R}^{\mathbb{R}}$ and $c \in \mathbb{R}$.*

The next proposition summarizes some elementary properties of a composition vector space. They follow immediately from the definition.

Proposition 2. *Let V be a composition vector space. Then:*

(1) $0 \in LF(V)$.
(2) $x \in LF(V)$ *implies that* $y * x \in LF(V)$ *for all* $y \in V$.
(3) $x \in LF(V)$ *if and only if* $x * 0 = x$.
(4) $LF(V) \leqslant^c V$.
(5) *If* $W \leqslant^c V$, *then* $LF(W) \leqslant^c LF(V)$.
(6) *For fixed* $x, y \in V$, *if* $x * z = y$ *for all* $z \in V$, *then* $y \in LF(V)$.

The next result provides a characterization of the constants of a composition vector space.

Proposition 3. *Let V be a unitary composition vector space with the identity I. Then $I \in F(V)$ if and only if there exists $x \in F(V)$, $x \neq 0$, such that $x * y \neq 0$ and $y * x \neq 0$, for all $y \in V$, $y \neq 0$.*

Proof. Let $I \in F(V)$. Then $I * y = I = y * I$, for all $y \in V$. Hence $I - I * y = 0$ and so $x * (I - I * y) = 0$, for all $x \in V, x \neq 0$. Hence $x = x * I = x * (I * y) = x * y \neq 0$, that is $x \in LF(V)$. Similarly, we have $x \in RF(V)$ and therefore $x \in F(V)$.

Conversely, for $x \in F(V), x \neq 0$ and $y \in V$ we have $x * I = x = x * y = x * (I * y)$. Hence $x * (I - I * y) = 0$, which implies that $I = I * y$. Similarly, $I = y * I$ and thus $I \in F(V)$. □

Definition 3. *Let V be a composition vector space and W be a subspace of $LF(V)$. We say that $x \in V$ is a residual element modulo W if $x * LF(V) \subseteq W$. The set of all residual elements modulo W is denoted by V_W, i.e., $V_W = \{x \in V \mid x * y \in W, \ \forall y \in LF(V)\}$.*

For instance, consider $(\mathbb{R}^2, +, \cdot, *, \mathbb{R})$ under the composition defined in Example 1 (2) and the subspace $W = \{(x, 0) \mid x \in \mathbb{R}\}$. Then $LF(\mathbb{R}^2) = \mathbb{R}^2$ and $\mathbb{R}^2_W = W$.

Proposition 4. *Let V be a composition vector space and W a subspace of $LF(V)$, i.e., $W \leqslant LF(V)$. Then the following assertions hold:*

(1) $V_W \leqslant V$.
(2) *If* $W \leqslant^c V$, *then* $V_W \leqslant^c V$.
(3) $x * y \in V_W$, *for all* $x \in V_W$ *and* $y \in V$.
(4) $V_W \cap LF(V) = W$.
(5) V_W *is the largest subspace of* V *satisfying Conditions* (3) *and* (4).

Proof. (1), (2) It is easy to see that V_W is closed under the operations of V.
(3) For all $z \in LF(V)$ we have $y * z \in LF(V)$, since $(y * z) * t = y * (z * t) = y * z$, for all $t \in V$. Hence $(x * y) * z = x * (y * z) \in W$. Then $x * y \in V_W$.
(4) If $x \in V_W \cap LF(V)$, then $x = x * 0 \in W$, by Proposition 2 (1) and (3). Now if $x \in W \subseteq$

$LF(V)$, then $x * y = x \in W$ for all $y \in LF(V)$, that is, $x \in V_W \cap LF(V)$.
(5) Let U be a subspace of V such that $U \cap LF(V) = W$ and $x * y \in U$ for all $x \in U$ and $y \in V$. Then for any $u \in U$ and $t \in LF(V)$, $u * t \in U \cap LF(V) = W$, which implies that $u \in V_W$. □

If W is a composition subspace of $LF(V)$, then there exists a composition subspace U of V with the left foundation W, where $U \neq W$, if $LF(V) \neq V$. In this way, we can construct a new composition subspace.

Corollary 1. *Let V be a composition vector space and $W \leqslant^c LF(V)$. Then*

$$LF(V_W) = W.$$

Proof. If $t \in LF(V_W)$, then $t = t * 0$ because $0 \in V_W$. Additionally, $t * 0 \in W$ since $t \in V_W$ and $0 \in LF(V)$. Thus $t \in W$ and so $LF(V_W) \subseteq W$. On the other hand, if $t \in W$, then $t \in V_W \cap LF(V)$, by Proposition 4(4). Thus $t * y = t$ for all $y \in V_W$. Hence $t \in LF(V_W)$. □

Notice that $LF(W) = W$ for any composition subspace W of $LF(V)$, i.e., W is the smallest composition subspace U of V containing W such that $LF(U) = W$ (see Figure 1).

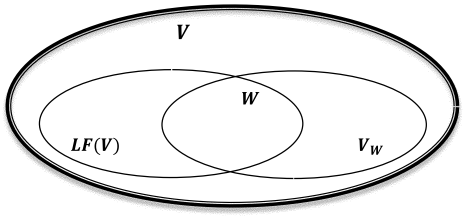

Figure 1. The relationship between composition subspaces.

Corollary 2. *If V is a composition vector space, then $V_{\{0\}}$ is a composition subspace of V such that $V_{\{0\}} = V$ if and only if $LF(V) = \{0\}$.*

Proof. By Proposition 2(1), $0 * 0 = 0$, so $\{0\}$ is a composition subspace of V. Thus by Proposition 4(1), $V_{\{0\}}$ is a composition subspace of V. Now if $V_{\{0\}} = V$, then for all $y \in LF(V)$, by Proposition 2(1),(3) and $y \in V_{\{0\}}$, we have $y = y * 0 = 0$. Moreover for all $x \in V$ and $y \in LF(V) = \{0\}$, $x * y \in LF(V) = \{0\}$ by Proposition 2(2). Hence $x \in V_{\{0\}}$. □

Proposition 5. *Let V be a unitary composition vector space and W be a maximal composition subspace of V. Then $W \cap LF(V)$ is a maximal subspace of $LF(V)$.*

Proof. Let $U = W \cap LF(V) \subsetneq X \subseteq LF(V)$ for some subspace X of $LF(V)$. Then there exists $x \in X \setminus U$ such that $x \notin W$. Hence $x^* + W = V$, where $x^* = \{x^n = \underbrace{x * \ldots * x}_{n} \mid n \in \mathbb{N}\} \leqslant^c V$. Thus, for any $t \in LF(V)$ there exist $x^n \in x^*$, for some $n \in \mathbb{N}$, and $w \in W$ such that $t = x^n + w$. It follows that $w = t - x^n \in LF(V) \cap W = U \subseteq X$. Then $t = x^n + w \in X$ and so $X = LF(V)$, i.e., U is a maximal subspace of $LF(V)$. □

Definition 4. *A composition subspace $W = (W, +, \cdot, *, F)$ is called a strong composition subspace of V, denoted $W \preccurlyeq V$, if $x * y \in W$ for all $x \in W$ and $y \in V$.*

Note that V_W is a strong composition subspace of V by Proposition 4(3). In addition, $LF(V) \preccurlyeq V$, since $(x * y) * z = x = x * y$ for all $x \in LF(V)$ and $y, z \in V$.

Lemma 1. *Let V be a unitary composition vector space with the identity I.*

(i) *If $W \preccurlyeq V$ and $I \in W$, then $W = V$.*
(ii) *If any nonzero element $x \in V$ has an inverse with respect to $*$, then V has no nontrivial strong composition subspace.*

Proof. (i) If $x \in V$, then $x = I * x \in W$ and clearly $W = V$.
(ii) Let $W \preccurlyeq V$ such that $W \neq \{0\}$. Then there exists at least one element $x \neq 0, x \in W$. Hence there exists $y \in V$ such that $I = x * y \in W$, which implies that $W = V$ by (i). □

Corollary 3. *Let $(V, +, \cdot, *, F)$ be a unitary composition vector space such that any nonzero $x \in V$ has an inverse respect to $*$ and $LF(V) \neq \{0\}$. Then $V = LF(V)$.*

Proof. It follows from Lemma 1(ii), since $LF(V) \preccurlyeq V$. □

In other words, in every composition vector space V, with the property that V is a group under the composition operation, there is no nontrivial left foundation.

Moreover, it is worth noticing that we cannot generally define a quotient structure on composition vector spaces in a natural way. For doing this, consider W as a strong composition subspace of a left and right composition vector space V. Then the quotient of V by W, i.e., $V/W = \{x + W \mid x \in V\}$, is constructed together with the natural operations and $(x + W) * (y + W) = (x * y) + W$.

3. Linear Operators on Composition Vector Spaces

The aim of this section is to endow vector spaces V with a nontrivial composition structure by using linear operators. Additionally, some properties of the left foundation $LF(V)$ and nullifier N_V of V are investigated under linear operators. Moreover, automorphic composition vector spaces are defined and studied.

Consider the composition vector space $L(V)$, i.e., the set of all linear operators on V, defined in Example 1(3). The bijective elements of $L(V)$ form a composition subspace of $L(V)$, denoted by $BL(V)$. Let Γ be a subgroup of $(BL(V), \circ)$ and consider the sets $\Gamma_x = \{T(x) \mid T \in \Gamma\}$ for all $x \in V$, that are called *orbits*. Clearly, the family $\{\Gamma_x\}_{x \in V}$ is a partition for V and one can study the properties of the composition vector space with equivalence relations under this partition. For more details, in the following we give a characterization for it.

Any orbit Γ_x is called *principal* if $T(y) \neq y$ for all $y \in \Gamma_x$ and $Id \neq T \in \Gamma$. Note that $\Gamma_0 = \{0\}$ is not principal, while if $\Gamma = \{Id\}$, then every orbit $\Gamma_x = \{x\}$ is principal.

Proposition 6. *Let Γ be a subgroup of $(BL(V), \circ)$ and $x \in V$. Then the orbit Γ_x is principal if and only if, for all $y \in \Gamma_x$, the mapping $f : \Gamma \to \Gamma_x$ defined by $f(T) = T(y)$ is a bijection.*

Proof. Let Γ_x be principal. If $y \in \Gamma_x$ and $T_1, T_2 \in \Gamma_x$ such that $f(T_1) = f(T_2)$, then $(T_2^{-1} \circ T_1)(y) = y$ and so $T_2^{-1} \circ T_1 = Id$, i.e., $T_1 = T_2$, by principality of Γ_x. Thus f is injective. Clearly f is also surjective.

Conversely, if $y \in \Gamma_x$, $Id \neq T \in \Gamma$ and $T(y) = y$, then $f(T) = T(y) = y = I(y) = f(I)$. By injectivity of f, we have $T = I$, which is a contradiction. Hence Γ_x is principal. □

The following results highlight the correspondence between composition vector spaces and linear operators defined on them.

Proposition 7. *Let V be a composition vector space over the field F and the mappings $T_y : V \longrightarrow V$ be defined by $T_y(x) = x * y$, for all $x, y \in V$. Then T_y is a linear operator on V and $T_{T_x(y)} = T_x \circ T_y$.*

Proof. Consider $x_1, x_2, y \in V$ and $a \in F$. Then $T_y(x_1 + ax_2) = (x_1 + ax_2) * y = (x_1 * y) + (ax_2 * y) = (x_1 * y) + a(x_2 * y) = T_y(x_1) + aT_y(x_2)$, which shows the linearity of T. Moreover, for all $x, y, t \in V$, it follows that $T_{T_x(y)}(t) = t * T_x(y) = t * (y * x) = (t * y) * x = T_x(t * y) = T_x(T_y(t)) = (T_x \circ T_y)(t)$. □

Proposition 8. *Let V be a vector space and $\{T_y\}_{y \in V}$ be a family of linear operators on V such that $T_{T_x(y)} = T_x \circ T_y$, for all $x, y \in V$. Then V is a composition vector space by defining the composition by $x * y = T_y(x)$, for all $x, y \in V$.*

Proof. Let $x, y, z \in V$. Then $(x * y) * z = T_z(x * y) = T_z(T_y(x)) = T_{T_z(y)}(x) = x * T_z(y) = x * (y * z)$. Thus $(V, *)$ is a semigroup. Additionally, $(x + y) * z = T_z(x + y) = T_z(x) = T_z(y) = x * z + y * z$ and $(a \cdot x) * y = T_y(ax) = a \cdot T_y(x) = a \cdot (x * y)$, for all $a \in F$. Hence V is a composition vector space. □

Based on these last two results, we may conclude that the structure of any composition vector space V with identity I is characterized by the monoid $\{T_y \mid y \in V\}$ of linear operators, where T_I is its identity. The mentioned monoid is called the *family of the linear operators associated with V*.

Definition 5. *A mapping T between two composition vector spaces V and W is said to be a composition linear transformation if $T(x + y) = T(x) + T(y)$, $T(a \cdot x) = a \cdot T(x)$ and $T(x * y) = T(x) * T(y)$, for all $x, y \in V$ and $a \in F$.*

In the following we will investigate some properties of the left foundation $LF(V)$ of a composition vector space, by the help of linear operators.

Proposition 9. *Let $T : V \longrightarrow W$ be an onto composition linear transformation. Then $T^{-1}(LF(W)) = \ker T + LF(V)$.*

Proof. Suppose $x \in T^{-1}(LF(W))$. Then $T(x) \in LF(W)$ and so $T(x) * 0 = T(x)$. Thus $x = (x - (x * 0)) + (x * 0) \in \ker T + LF(V)$, by Proposition 2(2). Now, if $x \in \ker T + LF(V)$, then $x = v + t$, for some $v \in \ker T$ and $t \in LF(V)$. Thus, $T(x) = T(t)$. Hence, for all $y \in W$, $T(x) * y = T(t) * T(x') = T(t * x') = T(t) = T(x)$, where $y = T(x')$ for $x' \in V$. Therefore, $T(x) \in LF(W)$, and this completes the proof. □

Proposition 10. *The mapping $T : V \longrightarrow LF(V)^{LF(V)}$ defined by $T(x) = T_x$, where $T_x(y) = x * y$, is a linear transformation such that* $\ker T = V_{\{0\}}$.

Proof. For arbitrary $x, x' \in V$, $a \in F$ and $y \in LF(V)$, we have

$$T_{x+x'}(y) = (x + x') * y = (x * y) + (x' * y) = T_x(y) + T_{x'}(y),$$
$$T_{a \cdot x}(y) = (a \cdot x') * y = a \cdot (x * y) = a \cdot T_x(y),$$
$$T_{x * x'}(y) = (x * x') * y = x * (x' * y) = x * T_{x'}(y) = T_x(T_{x'}(y)) = (T_x \circ T_{x'})(y),$$

and $\ker T = \{x \in V \mid T(x) = 0\} = \{x \in V \mid x * y = 0,\ \forall y \in LF(V)\} = V_{\{0\}}$. □

Definition 6. *An element $x \in V$ is called a nullifier of V if $x * V = \{x * t \mid t \in V\} = \{0\}$. The set of nullifiers of V is denoted by N_V.*

Proposition 11. *Let V be a composition vector space. Then:*

(1) $N_V \leqslant^c V$.
(2) If $N_V = \{0\}$ and $x_1, x_2 \in V$ such that $x_1 * y = x_2 * y$, for all $y \in V$, then $x_1 = x_2$.
(3) If $N_V = \{0\}$ and $x * y = z$, for all $y \in V$, then $x = z \in LF(V)$.

Proof. (1) It is straightforward.
(2) By assumption, $(x_1 - x_2) * y = x_1 * y - x_2 * y = 0$, so $x_1 - x_2 \in N_V$. Thus $x_1 = x_2$.
(3) Since $z * y = (x * z) * y = x * (z * y) = z$ and $x * y = z$, then $x = z$ by (2). Hence $x * y = x$ and $z * y = z$ for all $y \in V$, which means that $x, z \in LF(V)$. □

Proposition 12. *The mapping $T : V \longrightarrow V^V$ defined by $T(x)(y) = T_x(y) = x * y$, is a linear transformation such that* $\ker T = N_V$.

Proof. Similar to the proof of Proposition 10. □

Definition 7. *A composition vector space V is said to be automorphic if every nonzero linear operator T_y associated with V is a bijection.*

Note that if V is an automorphic composition vector space and $y \in \cup\Gamma_x$ for the principal orbits Γ_x, then T_y is nonzero.

Theorem 1. *Let $f : V \longrightarrow W$ be an onto composition linear transformation such that $W \neq \{0\}$. If V is automorphic, then W is automorphic, too.*

Proof. Let $T_y : W \longrightarrow W$ be a nonzero linear operator associated with W, meaning that $T_y = T_{f(x)}$, for some $x \in V$. Note that T_x is a nonzero linear operator associated with V, because if $T_x = 0$, then $z * x = T_x(z) = 0$ for all $z \in V$. Hence $0 = f(0) = f(z * x) = T_y(f(z))$, that is, $T_y(W) = 0$. Hence $T_y = 0$, which is a contradiction.

Now, consider $w \in W$ and $f(v) = w$ for an arbitrary $v \in V$. Then there exists $t \in V$ such that $t * x = T_x(t) = v$. Thus $w = f(v) = f(t * x) = f(t) * f(x) = T_{f(x)}(f(t)) = T_y(f(t))$, which means that T_y is onto.

Moreover, if $T_y(w_1) = T_y(w_2)$ for $w_1 = f(v_1)$ and $w_2 = f(v_2)$ with $v_1, v_2 \in V$, then $f(T_x(v_1 - v_2)) = f(v_1 * x) - f(v_2 * x) = T_y(w_1) - T_y(w_2) = 0$. Hence $T_x(v_1 - v_2) \in \ker f$ and so $v_1 - v_2 \in \ker f$, since $T_x \in BL(V)$ has an inverse and $\ker f \preccurlyeq V$. It follows that $w_1 = w_2$, and thus T_y is injective, thefore W is automorphic. □

The following corollary is an immediate consequence of Theorem 1.

Corollary 4. *Let $f : V \longrightarrow W$ be an onto composition linear transformation such that $W \neq \{0\}$. If $T_x \in BL(V)$, then $T_{f(x)} \in BL(W)$, for all $x \in V$.*

Finally, the behavior of linear operators associated with composition vector spaces is investigated under composition linear transformations.

Theorem 2. *Let $f : V \longrightarrow W$ be an onto composition linear transformation, $\mathcal{A}$ and $\mathcal{B}$ be the set of all bijective linear transformations T_x and T_y associated with V and W, respectively. Then the function $\bar{f} : \mathcal{A} \longrightarrow \mathcal{B}$ defined by $\bar{f}(T_x) = T_{f(x)}$ is an onto group homomorphism such that* $\ker \bar{f} = \{T_x \in \mathcal{A} \mid T_x(t) - t \in \ker f,\ \forall t \in V\}$.

Proof. Consider $T_{x_1} = T_{x_2}$ for some $x_1, x_2 \in V$ and $w \in W$. Then $w = f(v)$, for some $v \in V$. Thus $T_{x_1}(v) = T_{x_2}(v)$ and hence $f(v * x_1) = f(v * x_2)$, which implies that $T_{f(x_1)}(w) = T_{f(x_2)}(w)$, that is $\bar{f}(T_{x_1}) = \bar{f}(T_{x_2})$. Additionally $\bar{f}(T_{x_1} \circ T_{x_2}) = \bar{f}(T_{T_{x_1}(x_2)}) = T_{f(x_2 * x_1)} = T_{T_{f(x_1)}(f(x_2))} = T_{f(x_1)} \circ T_{f(x_2)} = \bar{f}(T_{x_1}) \circ \bar{f}(T_{x_2})$, by Proposition 7. Thus, $\bar{f}$ is a homomorphism. Clearly, it is onto. Moreover, $T_x \in \ker \bar{f}$ if and only if $T_{f(x)} = Id$, equivalently with $T_{f(x)}(w) = w$ for all $w \in W$ iff $f(t) * f(x) = f(t)$ for all $t \in V$, i.e., $T_x(t) - t \in \ker f$ for all $t \in V$. This completes the proof. □

4. Conclusions

Defining a composition structure on a vector space, similarly as for rings, or hyperrings or ordered hyperstructures, permits us to study the new obtained tri-operational algebra, called in this case a composition vector space, also from the perspective of linear operators. It is interesting to notice that not all subspaces of a composition vector space are composition subspaces, as we could see in Example 10. Additionally, we have concluded that the structure of any composition vector space $(V, +, \cdot, *)$ with identity I may be characterized by a monoid $\{T_y \mid y \in V\}$, with the identity T_I, of linear operators on V, where $T_y(x) = x * y$, for any $x \in V$. Studying properties of the linear operators permits us to study new properties of the associated composition vector space.

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