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Article

Pointwise k-Pseudo Metric Space

Yu Zhong [1,*], Alexander Šostak [2] and Fu-Gui Shi [3]

1 College of Science, North China University of Technology, Beijing 100144, China
2 Department of Mathematics and Computer Science, University of Latvia, LV-1459 Riga, Latvia; sostaks@latnet.lv
3 School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China; fuguishi@bit.edu.cn
* Correspondence: zhongyu@ncut.edu.cn or zhongyu199055@126.com

Abstract: In this paper, the concept of a k-(quasi) pseudo metric is generalized to the L-fuzzy case, called a pointwise k-(quasi) pseudo metric, which is considered to be a map $d : J(L^X) \times J(L^X) \longrightarrow [0, \infty)$ satisfying some conditions. What is more, it is proved that the category of pointwise k-pseudo metric spaces is isomorphic to the category of symmetric pointwise k-remote neighborhood ball spaces. Besides, some L-topological structures induced by a pointwise k-quasi-pseudo metric are obtained, including an L-quasi neighborhood system, an L-topology, an L-closure operator, an L-interior operator, and a pointwise quasi-uniformity.

Keywords: pointwise k-(quasi) pseudo metric; pointwise k-remote neighborhood ball system; L-quasi neighborhood system; L-topology

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1. Introduction

Metric spaces play an important role in the research and applications of mathematics. Since Zadeh introduced fuzzy set theory, there have been many interesting and creative works in which different approaches to the concept of a fuzzy metric were introduced and corresponding theories were developed and used for various applications [1–18].

In 1979, Erceg [1] constructed the theory of fuzzy metrics by considering the Hausdorff distance function between L-subsets and studied their topological properties. Subsequently, Erceg's fuzzy metric was widely studied, in particular, Deng [19], Liang [20], and Peng [21] greatly contributed to its development. However, Erceg failed to build the distance function between L-fuzzy points and his approach does not directly reflect the relationships between a fuzzy point and its quasi-neighborhood. Besides, the topologies induced by Erceg's fuzzy metrics are not first countable that can be considered as a certain deficiency of this theory.

In order to solve these defects, Shi [11] introduced a new theory of pointwise metrics by treating a fuzzy metric as a mapping $d : J(L^X) \times J(L^X) \to [0, \infty)$, where the set $J(L^X)$ is the set of all L-fuzzy points on X. The theory of Shi's pointwise metrics is different from Erceg's fuzzy metric and has many advantages. Shi's pointwise metrics are well coordinated with the corresponding pointwise topology. Besides, his methods seem more simpler and more immediate. Moreover, Shi's pointwise metrics also solved the problem that the pointwise metric topology is first countable and showed that a Shi's pointwise metric can induce an Erceg's metric on L^X.

As a generalization of metric spaces, the notion of metric-type spaces was introduced by Bakhtin [22] in 1989, and later was rediscovered by Czerwik [23] under the name of b-metric space in 1993. In order to describe the concept more vividly, Šostak [24] used the name "k-metric space" to replace metric-type spaces and b-metric spaces, which makes the triangle inequality to a more general axiom: $d(x, z) \le k(d(x, y) + d(y, z))$, where $k \ge 1$ is a fixed constant.

Recently, Hussain [25] and Nădăban [26] introduced the concept of a fuzzy b-metric and discussed the corresponding fixed point theorem. A similar concept under the name of a fuzzy k-pseudo metric was independently introduced and some topological properties of fuzzy k-pseudo metric spaces were studied in [24]. Although definitions of (fuzzy) b-metric and (fuzzy) k-metric are very similar, there is a fundamental difference if we consider the categories of such spaces. For example, while countable products exist in the category of (fuzzy) b-metric spaces, the product of two (fuzzy) k-metric spaces may fail to be (fuzzy) k-metric.

By modifying the definition of a fuzzy k-pseudo metric in [24], Zhong and Šostak [27] proposed an alternative definition of a fuzzy k-pseudo metric, which is treated as a map $M : X \times X \times [0, \infty) \longrightarrow [0, 1]$ satisfying some modified conditions. Actually, a fuzzy k-pseudo metric can be viewed as a generalization of the fuzzifying case of a crisp k-pseudo metric. However, this approach prevents defining the distance function between L-fuzzy points and cannot induce any L-structures. Until now, researches about fuzzy k-pseudo metrics lack the L-fuzzy case of crisp k-pseudo metric; that is to say, there is no author that gives a definition of a pointwise k-pseudo metric and considers its induced L-topological structures. Therefore, these are our starting points for writing this paper.

The main aims of this paper are to introduce the concept of a pointwise k-(quasi) pseudo metric and to discuss its characterizations by a pointwise k-remote neighborhood ball system. Besides, we show that many L-topological structures can be induced by a pointwise k-quasi-pseudo metric.

This paper is organized as follows. In Section 2, some necessary definitions and results about k-pseudo metric spaces and L-topological spaces are recalled. In Section 3, the definitions of a pointwise k-(quasi) pseudo metric and a pointwise k-remote neighborhood ball system are introduced. Moreover, relationships between pointwise k-(quasi) pseudo metrics and pointwise k-remote neighborhood ball systems are discussed. In Section 4, some L-structures induced by a pointwise k-quasi-pseudo metric are constructed, including an L-quasi neighborhood system, an L-topology, an L-closure operator, an L-interior operator, and a pointwise quasi-uniformity.

2. Preliminaries

Throughout this paper, $(L, \vee, \wedge, \leq, ')$ denotes a complete, completely distributive De Morgan algebra, i.e., a completely distributive lattice with an order-reserving involution $'$. Moreover, $\perp_L$ and $\top_L$ be its smallest and largest elements, respectively. Let X be a non-empty set. L^X denotes the set of all L-fuzzy subsets on X and L^X is also a completely distributive De Morgan algebra when it inherits the structure of the lattice L in a natural way, by defining $\wedge, \vee, \leq$ and $'$ pointwisely. The smallest element and the largest element in L^X are denoted by $\perp_{L^X}$ and $\top_{L^X}$, respectively.

We say that a is wedge-below b in L, in symbols, $a \prec b$, if for every subset $D \subseteq L$, $\bigvee D \geq b$ implies $a \leq d$ for some $d \in D$. The wedge below relation in a completely distributive lattice has the interpolation property, i.e., if $a \prec b$, then there exists $c \in L$ such that $a \prec c \prec b$. Moreover, it is easy to see that $a \prec \bigwedge_{i \in I} b_i$ implies $a \prec b_i$ for any $i \in I$, whereas $a \prec \bigvee_{i \in I} b_i$ implies $a \prec b_i$ for some $i \in I$ [28].

An element a in L is called a co-prime element if $b \vee c \geqslant a$ implies $b \geqslant a$ or $c \geqslant a$ for any $b, c \in L$ [28]. The set of all nonzero co-prime elements of L is denoted by $J(L)$, such as, if $L = [0, 1]$, then $J(L) = (0, 1]$. And the set of all nonzero co-prime elements of L^X is denoted by $J(L^X)$. It is easy to see that $J(L^X)$ is exactly the set of all L-fuzzy points x_λ, namely $J(L^X) = \{x_\lambda \in L^X \mid x \in X, \lambda \in J(L)\}$, where x_λ is an L-fuzzy set from X to L such that $x_\lambda(x) = \lambda$, and $= \perp_L$ otherwise . Let $f : X \longrightarrow Y$ be a map. Define $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ and $f_L^{\leftarrow} : L^Y \leftrightarrow L^X$ by $\forall A \in L^X$, $f_L^{\rightarrow}(A)(y) = \bigwedge_{f(x)=y} A(x)$ and $\forall B \in L^Y$, $f_L^{\leftarrow}(B) = B \circ f$.

First, we recall the definition of k-metric as it was introduced in [24].

Definition 1 ([24]). *Let $k \geq 1$ be a fixed constant and let $d : X \times X \longrightarrow [0, \infty)$ be a mapping such that $\forall x, y, z \in X$,*

(D1) $d(x, x) = 0$;
(D2) $d(x, z) \leq k(d(x, y) + d(y, z))$.

Then, d is called a k-pseudo-quasi metric. A k-quasi-pseudo metric is called a k-pseudo metric if it is symmetric,

(D3) $d(x,y) = d(y,x)$;

If the axiom (D1) is replaced by a stronger axiom:

*(D1)** $d(x,y) = 0 \Leftrightarrow x = y$;

then d is called a k-metric and the pair (X,d) *is called a k-metric space.*

Example 1. *Let* $\mathbb{R}$ *be the set of real numbers and let* $d : \mathbb{R} \times \mathbb{R} \longrightarrow [0,\infty)$ *be a mapping defined by* $d(x,y) = |x-y|^2$ *for all* $x,y \in \mathbb{R}$. *Then, d is a* 2*-metric. Similarly, let* $(X, \| \ \|)$ *be a normed space. There also exists* 2*-metric on X defined by* $d(x,y) = \|x-y\|^2$ *for all* $x,y \in X$.

Example 2. *Let X be the set of Lebesgue measurable functions on* $[a,b]$ *such that* $\int_a^b |f(x)|^2 dx < \infty$. *Define* $d : X \times X \longrightarrow [0,\infty)$ *by* $d(f,g) = \int_a^b |f(x) - g(x)|^2 dx$ *for any* $f,g \in X$. *Then, d is also a* 2*-metric.*

The concept of neighborhood systems is very important and fundamental in topology. However, the situation is more complicated when it is generalized to the L-fuzzy case. The important work of Pu and Liu [29], in which they generalized crisp neighborhood systems to quasi neighborhood systems, has drive to a great development of the theory of L-topological spaces.

Chang [30] first introduced fuzzy theory into topology. The notion of Chang's fuzzy topology was generalized to L-fuzzy setting by J.A. Goguen [31,32], which is now called an L-topology.

In what follows, the notions of an L-topology, an L-quasi neighborhood system, an L-closure operator and an L-interior operator are recalled.

Definition 2 ([31,32])**.** *An L-topology* $\mathcal{T}$ *on X is a subset of* L^X *satisfying:*

(LT1) $\perp_{L^X}, \top_{L^X} \in \mathcal{T}$;

(LT2) $\forall A, B \in \mathcal{T} \Rightarrow A \wedge B \in \mathcal{T}$;

(LT3) $\forall \{A_j\}_{j\in J} \subseteq \mathcal{T} \Rightarrow \bigvee_{j\in J} A_j \in \mathcal{T}$.

A continuous mapping from an L-topological space $(X, \mathcal{T}^X)$ to an L-topological space $(Y, \mathcal{T}^Y)$ is a mapping $f : X \longrightarrow Y$ such that $\forall B \in \mathcal{T}^Y$, $f_L^{\leftarrow}(B) \in \mathcal{T}^X$, where $f_L^{\leftarrow}(B)(x) = B(f(x))$ for any $x \in X$. The category of L-topological spaces and their continuous mappings is denoted by L-**Top**.

We say that a fuzzy point x_λ quasi-coincides with A if $\lambda \nleq A'(x)$ or equivalently $x_\lambda \nleq A'$ [29,33]. In case $L = [0,1]$, x_λ is quasi-coincident with A if and only if $A(x) > 1 - \lambda$. Then, we have the following definition.

Definition 3 ([33])**.** *An L-quasi neighborhood system on X is a family of* $\mathcal{Q} = \{\mathcal{Q}_{x_\lambda} \subseteq L^X \mid x_\lambda \in J(L^X)\}$ *satisfying the following conditions:*

(LQ1) $\top_{L^X} \in \mathcal{Q}_{x_\lambda}, \perp_{L^X} \notin \mathcal{Q}_{x_\lambda}$;

(LQ2) $\forall U \in \mathcal{Q}_{x_\lambda} \Rightarrow x_\lambda \nleq U'$;

(LQ3) $\forall U \in \mathcal{Q}_{x_\lambda}, U \leq V \Rightarrow V \in \mathcal{Q}_{x_\lambda}$;

(LQ4) $\forall U, V \in \mathcal{Q}_{x_\lambda} \Rightarrow U \wedge V \in \mathcal{Q}_{x_\lambda}$

(LQ5) $\forall U \in \mathcal{Q}_{x_\lambda}$, *there exists* $V \in L^X$ *such that* $x_\lambda \nleq V \geq U'$ *and* $V' \in \mathcal{Q}_{y_\mu}$. *for all* $y_\mu \nleq V$.

A continuous mapping from an L-quasi neighborhood space $(X, \mathcal{Q}^X)$ to an L-quasi neighborhood space $(Y, \mathcal{Q}^Y)$ is a mapping $f : X \longrightarrow Y$ such that $\forall x_\lambda \in J(L^X)$, $\forall U \in \mathcal{Q}^Y_{f(x)_\lambda}$, $f_L^{\leftarrow}(U) \in \mathcal{Q}^X_{x_\lambda}$. The category of L-quasi neighborhood spaces and their continuous mappings is denoted by L-**QNS**.

Remark 1. *In [33,34], it is shown that the category L-**Top** is isomorphic to the category of L-**QNS**. Specifically speaking, if $\mathcal{T}$ is an L-topology, then $\mathcal{Q}^{\mathcal{T}} = \{\mathcal{Q}^{\mathcal{T}}_{x_\lambda} \mid x_\lambda \in J(L^X)\}$ is an L-quasi neighborhood system, where $\mathcal{Q}^{\mathcal{T}}_{x_\lambda} = \{U \in L^X \mid \exists V \in \mathcal{T}, s.t., x_\lambda \not\leq V' \geq U'\}$. Conversely, if $\mathcal{Q} = \{\mathcal{Q}_{x_\lambda} \mid x_\lambda \in J(L^X)\}$ is an L-quasi neighborhood system, then $\mathcal{T}^{\mathcal{Q}} = \{U \in L^X \mid \forall x_\lambda \not\leq U'$, $U \in \mathcal{Q}_{x_\lambda}\}$ is an L-topology. In addition, $\mathcal{T}^{\mathcal{Q}^{\mathcal{T}}} = \mathcal{T}$, $\mathcal{Q}^{\mathcal{T}^{\mathcal{Q}}} = \mathcal{Q}$.*

It is well known that the closure operator and the interior operator are convenient alternative approaches to characterize a topology. In the following, we recall the definitions of an L-closure operator and an L-interior operator.

Definition 4 ([33,35])**.** *An L-closure operator on X is a mapping $cl : L^X \longrightarrow L^X$ satisfying the following conditions:*
(LC1) $cl(\perp_{L^X}) = \perp_{L^X}$;
(LC2) $A \leq cl(A)$;
(LC3) $cl(A \vee B) = cl(A) \vee cl(B)$;
(LC4) $cl(cl(A)) = cl(A)$.

Definition 5 ([35])**.** *An L-interior operator on X is a mapping $int : L^X \longrightarrow L^X$ satisfying the following conditions:*
(LI1) $int(\top_{L^X}) = \top_{L^X}$;
(LI2) $int(A) \leq A$;
(LI3) $int(A \wedge B) = int(A) \wedge int(B)$;
(LI4) $int(int(A)) = int(A)$.

Remark 2. *In [33–35], it is shown that there is a one-to-one correspondence between L-topologies and L-closure operators. That is, if $\mathcal{T}$ is an L-topology, then $cl^{\mathcal{T}}(A) = \bigvee\{x_\lambda \in J(L^X) \mid \forall x_\lambda \not\leq U \geq A, U' \notin \mathcal{T}\}$ is an L-closure operator. Conversely, if cl is an L-closure operator, then $\mathcal{T}^{cl} = \{A \in L^X \mid \forall x_\lambda \not\leq A', x_\lambda \not\leq cl(A')\}$ is an L-topology. In addition, $\forall A \in L^X$, $cl(A) = (int(A')')$, $int(A) = (cl(A'))'$.*

3. Pointwise k-Pseudo Metric Space

In this section, first, we will introduce the definition of a pointwise k-pseudo metric which is inspired by the idea of Shi's pointwise pseudo metric. Second, we will prove that there is a bijection between pointwise k-pseudo metrics and pointwise k-remote neighborhood ball systems.

Definition 6. *Let $k \geq 1$ be a fixed constant. A **pointwise k-quasi-pseudo metric** on L^X is a mapd $: J(L^X) \times J(L^X) \longrightarrow [0, \infty)$ satisfying the following conditions: $\forall x_\lambda, y_\mu, z_\nu \in J(L^X)$*
(LKD1) $d(x_\lambda, x_\lambda) = 0$;
(LKD2) $d(x_\lambda, z_\nu) \leq k(d(x_\lambda, y_\mu) + d(y_\mu, z_\nu))$;
(LKD3) $d(x_\lambda, y_\mu) = \bigwedge_{\nu \prec \mu} d(x_\lambda, y_\nu)$;
(LKD4) $\forall \gamma \leq \lambda$, $d(x_\gamma, y_\mu) \leq d(x_\lambda, y_\mu)$.
*A pointwise k-quasi-pseudo metric d is called a **pointwise k-pseudo metric** if it is symmetric, i.e., it satisfies*
(LKD5) $\bigwedge_{\gamma \not\leq \lambda'} d(x_\gamma, y_\mu) = \bigwedge_{\nu \not\leq \mu'} d(y_\nu, x_\lambda)$

Remark 3. *If $L = \{0, 1\}$, then each fuzzy condition reduces to the corresponding condition of a crisp k-pseudo metric. To be specific, (LKD1) and (LKD2) correspond to (D1) and (D2), respectively. (LKD3) and (LKD4) are naturally hold when d is a crisp k-metric, which are essential in the later research content of this paper. (LKD5) is a generalization of fuzzy symmetry, since it will be reduced to the symmetry of a crisp k-metric when $L = \{0, 1\}$. That is to say, the condition $\bigwedge_{\gamma \not\leq \lambda'} d(x_\gamma, y_\mu) = \bigwedge_{\nu \not\leq \mu'} d(y_\nu, x_\lambda)$ reduces to $d(x_1, y_1) = d(y_1, x_1)$.*

Example 3. *Let X be any set and $L = [0,1]$. Then, $J(L) = (0,1]$. Define $d : J(L^X) \times J(L^X) \longrightarrow [0,\infty)$ by $\forall x_\lambda, y_\mu \in J(L^X)$,*

$$d(x_\lambda, y_\mu) = \begin{cases} |\lambda - \mu|^2, & \lambda > \mu; \\ 0, & \lambda \le \mu. \end{cases}$$

Then, d is a pointwise 2-pseudo metric and d is not a pointwise pseudo metric.

Proof. Step 1: we shall check d satisfies (LKD1)-(LKD5).

(LKD1) $d(x_\lambda, x_\lambda) = 0$ is trivial.

(LKD2) It suffices to consider the case when $d(x_\lambda, z_\nu) > 0$, i.e., $d(x_\lambda, z_\nu) = |\lambda - \nu|^2$ and $\lambda > \nu$. If one of $d(x_\lambda, y_\mu)$ and $d(y_\mu, z_\nu)$ equals 0, say $d(x_\lambda, y_\mu) = 0$, then $\lambda \le \mu$. Thus, $\nu < \mu$. Therefore, $d(y_\mu, z_\nu) = |\mu - \nu|^2$. As $|\lambda - \nu|^2 \le |\mu - \nu|^2 \le 2|\mu - \nu|^2$, it follows that $d(x_\lambda, z_\nu) \le 2(d(x_\lambda, y_\mu) + d(y_\mu, z_\nu))$. If $d(x_\lambda, y_\mu) = |\lambda - \mu|^2$, $\lambda > \mu$ and $d(y_\mu, z_\mu) = |\mu - \nu|^2$, $\mu > \nu$, then $|\lambda - \nu|^2 \le |(\lambda - \mu) + (\mu - \nu)|^2 \le (|\lambda - \mu| + |\mu - \nu|)^2 \le 2(|\lambda - \mu|^2 + |\mu - \nu|^2)$. This shows $d(x_\lambda, z_\nu) \le 2(d(x_\lambda, y_\mu) + d(y_\mu, z_\nu))$.

(LKD3) Suppose $d(x_\lambda, y_\mu) = 0$, i.e., $\lambda \le \mu$. If $\lambda = \mu$, then $\bigwedge_{\nu<\mu} d(x_\lambda, y_\nu) = \bigwedge_{\nu<\mu} |\lambda - \nu|^2 = 0$. If $\lambda < \mu$, then there exists ν such that $\lambda < \nu < \mu$. Therefore, $d(x_\lambda, y_\nu) = 0$. Thus, $\bigwedge_{\nu<\mu} d(x_\lambda, y_\nu) = 0$. This shows $d(x_\lambda, y_\mu) = \bigwedge_{\nu<\mu} d(x_\lambda, y_\nu)$.

Suppose $d(x_\lambda, y_\mu) > 0$, i.e., $d(x_\lambda, y_\mu) = |\lambda - \mu|^2$ and $\lambda > \mu$. For any $\nu < \mu$, we have $\lambda > \nu$ and $\bigwedge_{\nu<\mu} d(x_\lambda, y_\nu) = \bigwedge_{\nu<\mu} |\lambda - \nu|^2 = |\lambda - \mu|^2$. This shows $d(x_\lambda, y_\mu) = \bigwedge_{\nu<\mu} d(x_\lambda, y_\nu)$.

(LKD4) The proof is similar to that of (LKD3) and omitted here.

(LKD5) It need to prove that $\bigwedge_{s>1-\lambda} d(x_s, y_\mu) = \bigwedge_{\nu>1-\mu} d(y_\nu, x_\lambda)$. If $1 - \lambda - \mu \ge 0$, then $s > 1 - \lambda > \mu$, $\nu > 1 - \mu > \lambda$. Therefore, $\bigwedge_{s>1-\lambda} d(x_s, y_\mu) = \bigwedge_{s>1-\lambda} |s - \mu|^2 = |1 - \lambda - \mu|^2$
and $\bigwedge_{\nu>1-\mu} d(y_\nu, x_\lambda) = \bigwedge_{\nu>1-\mu} |\nu - \lambda|^2 = |1 - \mu - \lambda|^2$. Thus, $\bigwedge_{s>1-\lambda} d(x_s, y_\mu) = \bigwedge_{\nu>1-\mu} d(y_\nu, x_\lambda)$. If $1 - \lambda - \mu < 0$, then there exist $s > 1 - \lambda$ and $v > 1 - \mu$ such that $1 - \lambda < s < \mu$ and $1 - \mu < \nu < \lambda$. Therefore, $\bigwedge_{s>1-\lambda} d(x_s, y_\mu) = 0$ and $\bigwedge_{\nu>1-\mu} d(y_\nu, x_\lambda) = 0$. Thus, $\bigwedge_{s>1-\lambda} d(x_s, y_\mu) = \bigwedge_{\nu>1-\mu} d(y_\nu, x_\lambda)$.

Step 2: we shall show that d is not a pointwise pseudo metric.

Let $\lambda = \frac{5}{8}$, $\mu = \frac{3}{8}$ and $\nu = \frac{1}{8}$. Then,

$$d(x_\lambda, z_\nu) = |\frac{5}{8} - \frac{1}{8}|^2 = \frac{1}{4}, \;\; d(x_\lambda, y_\mu) = |\frac{5}{8} - \frac{3}{8}|^2 = \frac{1}{16}, \;\; d(y_\mu, z_\nu) = |\frac{3}{8} - \frac{1}{8}|^2 = \frac{1}{16}.$$

Therefore, $d(x_\lambda, z_\nu) \le 2(d(x_\lambda, y_\mu) + d(y_\mu, z_\nu))$ and $d(x_\lambda, z_\nu) \nleq d(x_\lambda, y_\mu) + d(y_\mu, z_\nu)$. □

Definition 7. *A mapping $f : X \longrightarrow Y$ between pointwise k-quasi-pseudo metric spaces (X, d_X) and (Y, d_Y) is called non-expansive if $\forall x_\lambda, y_\mu \in J(L^X)$,*

$$d_Y\big(f(x)_\lambda, f(y)_\mu\big) \le d_X(x_\lambda, y_\mu).$$

It is easy to check that pointwise k-quasi-pseudo metric spaces and their non-expansive mappings form a category, denoted by L-**KPQMS**.

By Definition 6, it is not hard to get the following properties.

Proposition 1. *Let d be a pointwise k-quasi-pseudo metric on X. Then, the following statements hold.*

(LKD1)* $\forall \lambda \le \mu$, $d(x_\lambda, x_\mu) = 0$.

(LKD3)* $\forall \nu \le \mu$, $d(x_\lambda, y_\nu) \ge d(x_\lambda, y_\mu)$.

In order to discuss some L-topological type structures induced by a pointwise k-pseudo metric, we need to introduce the concept of a pointwise k-remote neighborhood ball system, which is a generalization of the opposite of the crisp spherical neighborhood system $R(x, r) = (B(x, r))' = \{y \in X \mid d(x, y) \ge r\}$.

Definition 8. *Let $k \geq 1$ be a fixed constant. A* **pointwise k-remote neighborhood ball system** *on X is defined to be a set $\mathcal{R} = \{R_r \mid r \in (0,\infty)\}$ of maps $\{R_r : J(L^X) \longrightarrow L^X\}$ satisfying $\forall x_\lambda, y_\mu, z_\nu \in J(L^X)$, $\forall r, s > 0$,*

(LKR1) $\bigwedge_{r>0} R_r(x_\lambda) = \perp_{L^X}$;
(LKR2) $x_\lambda \not\leq R_r(x_\lambda)$;
(LKR3) $R_s \odot R_r \geq R_{k(r+s)}$, *where* $(R_s \odot R_r)(x_\lambda) = \bigwedge\{R_s(y_\mu) \,|\, y_\mu \not\leq R_r(x_\lambda)\}$;
(LKR3)$R_r(x_\lambda) = \bigwedge_{s<r} R_s(x_\lambda)$;
(LKR4)$\forall \gamma \leq \lambda$, $R_r(x_\gamma) \leq R_r(x_\lambda)$.

The pair $(X, \mathcal{R})$ is called a pointwise k-remote neighborhood ball space. $\mathcal{R}$ is called **symmetric***, if it satisfies*

(LKR6) $y_\mu \not\leq \bigwedge_{\gamma \not\leq \lambda'} R_r(x_\gamma) \Leftrightarrow x_\lambda \not\leq \bigwedge_{\nu \not\leq \mu'} R_r(y_\nu)$.

Definition 9. *A mapping $f : X \longrightarrow Y$ between pointwise k-remote neighborhood ball spaces $(X, \mathcal{R}^X)$ and $(Y, \mathcal{R}^Y)$ is called continuous if $\forall r > 0$, $\forall x_\lambda \in J(L^X)$,*

$$f_L^{\leftarrow}\Big(R_r^Y(f(x)_\lambda)\Big) \leq R_r^X(x_\lambda)).$$

It is easy to check that pointwise k-remote neighborhood ball spaces and their continuous mappings form a category, denoted by L-**KRNBS**.

Proposition 2. *Let $(X, \mathcal{R})$ be a pointwise k-remote neighborhood ball space. Then, for any $x_\lambda \in J(L^X)$ and for all $r, s \in (0,\infty)$,*

(LKR4)* $s \leq r \Rightarrow R_r(x_\lambda) \leq R_s(x_\lambda)$.

In the following, the relationships between pointwise k-pseudo metrics and pointwise k-remote neighborhood ball systems are discussed.

Let d be a pointwise k-quasi-pseudo metric on X. For any $r \in (0,\infty)$, define a mapping $R_r^d : J(L^X) \longrightarrow L^X$ by $\forall x_\lambda \in J(L^X)$,

$$R_r^d(x_\lambda) = \bigvee\{y_\mu \in J(L^X) \mid d(x_\lambda, y_\mu) \geq r\}.$$

Before proving that $\mathcal{R}^d = \{R_r^d \mid r \in (0,\infty)\}$ is a pointwise k-remote neighborhood ball system, we need the following useful lemma.

Lemma 1. *Let d be a pointwise k-quasi-pseudo metric on X. For any $r \in (0,\infty)$ and for all $x_\lambda, y_\mu \in J(L^X)$,*

$$y_\mu \leq R_r^d(x_\lambda) \Leftrightarrow d(x_\lambda, y_\mu) \geq r, \;\; i.e., \;\; y_\mu \not\leq R_r^d(x_\lambda) \Leftrightarrow d(x_\lambda, y_\mu) < r.$$

Proof. From the definition of R_r^d, it is obvious that $d(x_\lambda, y_\mu) \geq r$ implies $y_\mu \leq R_r^d(x_\lambda)$. On the other hand, suppose that $y_\mu \leq R_r^d(x_\lambda)$. For any $y_\nu \prec y_\mu$, as

$$y_\nu \prec R_r^d(x_\lambda) = \bigvee\{y_\mu \in J(L^X) \mid d(x_\lambda, y_\mu) \geq r\},$$

there exists $y_t \in J(L^X)$ such that $d(x_\lambda, y_t) \geq r$ and $y_\nu \prec y_t$. By (LKD3)*, we know $d(x_\lambda, y_\nu) \geq d(x_\lambda, y_t) \geq r$. Thus, $d(x_\lambda, y_\mu) = \bigwedge_{\nu \prec \mu} d(x_\lambda, y_\nu) \geq r$. □

Theorem 1. *Let d be a pointwise k-quasi-pseudo metric on X. Then, $\mathcal{R}^d = \{R_r^d \mid r \in (0,\infty)\}$ is a pointwise k-remote neighborhood ball system, where $R_r^d(x_\lambda) = \bigvee\{y_\mu \in J(L^X) \mid d(x_\lambda, y_\mu) \geq r\}$.*

Proof. We need to check (LKR1)-(LKR5) in Definition 8.

(LKR1) Assume that $\bigwedge_{r>0} R_r^d(x_\lambda) = B \neq \bot_{L^X}$. For each $y_\mu \leq B$, we have $y_\mu \leq R_r^d(x_\lambda)$ for all $r > 0$. By Lemma 1, we get $d(x_\lambda, y_\mu) \geq r$ for any $r > 0$, which contradicts with the fact that $d(x_\lambda, y_\mu) \in [0, \infty)$. Therefore, $\bigwedge_{r>0} R_r^d(x_\lambda) = \bot_{L^X}$.
(LKR2) It follows from Lemma 1 and (LKD1).
(LKR3) Let $x_\lambda \in J(L^X)$.

Take any $y_\mu \in J(L^X)$ with

$$y_\mu \not\leq (R_s^d \odot R_r^d)(x_\lambda) = \bigwedge\{R_s^d(z_w) \mid z_w \not\leq R_r^d(x_\lambda)\}.$$

Then, there exists some $z_w \in J(L^X)$ such that $z_w \not\leq R_r^d(x_\lambda)$ and $y_\mu \not\leq R_s^d(z_w)$. By Lemma 1, we have $d(x_\lambda, z_w) < r$ and $d(z_w, y_\mu) < s$. It follows that

$$d(x_\lambda, y_\mu) \leq k(d(x_\lambda, z_w) + d(z_w, y_\mu)) < k(r+s).$$

Therefore, $y_\mu \not\leq R_{k(r+s)}^d(x_\lambda)$. By the arbitrariness of y_μ, we obtain $(R_s^d \odot R_r^d)(x_\lambda) \geq R_{k(r+s)}^d(x_\lambda)$, i.e., $R_s^d \odot R_r^d \geq R_{k(r+s)}^d$.
(LKR4) It can be obtained from the following equivalences:

$$\begin{aligned} & y_\mu \leq R_r^d(x_\lambda) \Leftrightarrow d(x_\lambda, y_\mu) \geq r \Leftrightarrow \forall s < r, d(x_\lambda, y_\mu) \geq s \\ \Leftrightarrow \quad & \forall s < r, y_\mu \leq R_s^d(x_\lambda) \Leftrightarrow y_\mu \leq \bigwedge_{s<r} R_s^d(x_\lambda). \end{aligned}$$

(LKR5) For any $\gamma \leq \lambda$, we have $d(x_\gamma, y_\mu) \leq d(x_\lambda, y_\mu)$. Thus,

$$R_r^d(x_\gamma) = \bigvee\{y_\mu \mid d(x_\gamma, y_\mu) \geq r\} \leq \bigvee\{y_\mu \mid d(x_\lambda, y_\mu) \geq r\} = R_r^d(x_\lambda).$$

□

Theorem 2. *If $f : (X, d_X) \longrightarrow (Y, d_Y)$ is non-expansive between pointwise k-quasi-pseudo metric spaces, then $f : (X, \mathcal{R}^{d_X}) \longrightarrow (Y, \mathcal{R}^{d_Y})$ is continuous between pointwise k-remote neighborhood ball spaces.*

Proof. It needs to check that $f_L^{\leftarrow}(R_r^{d_Y}(f(x)_\lambda)) \leq R_r^{d_X}(x_\lambda)$ for all $x_\lambda \in J(L^X)$ and for any $r > 0$.

By the definition of R_r^d, the inequality can be proved from the following:

$$\begin{aligned} & f_L^{\leftarrow}(R_r^{d_Y}(f(x)_\lambda)) = f_L^{\leftarrow}\left(\bigvee\{z_\nu \in J(L^Y) \mid d_Y(f(x)_\lambda, z_\nu) \geq r\}\right) \\ = \quad & \bigvee\left\{f_L^{\leftarrow}(z_\nu) \in J(L^X) \mid d_Y(f(x)_\lambda, z_\nu) \geq r\right\} \\ \leq \quad & \bigvee\left\{f^{-1}(z)_\nu \in J(L^X) \mid d_X(x_\lambda, f^{-1}(z)_\nu) \geq r\right\} \\ \leq \quad & \bigvee\left\{y_\mu \in J(L^X) \mid d_X(x_\lambda, y_\mu) \geq r\right\} = R_r^{d_X}(x_\lambda). \end{aligned}$$

□

Now, we shall consider the opposite problem: whether a pointwise k-quasi-pseudo metric can be induced by a pointwise k-remote neighborhood ball system? The answer is positive and its construction is defined as follows.

Let $\mathcal{R} = \{R_r \mid r \in (0, \infty)\}$ be a pointwise k-remote neighborhood ball system. Define a map $d^{\mathcal{R}} : J(L^X) \times J(L^X) \longrightarrow [0, \infty)$ by $\forall x_\lambda, y_\mu \in J(L^X)$,

$$d^{\mathcal{R}}(x_\lambda, y_\mu) = \bigwedge\{r \in (0, \infty) \mid y_\mu \not\leq R_r(x_\lambda)\}.$$

Before proving that $d^{\mathcal{R}}$ is a pointwise k-quasi-pseudo metric, we need the following meaningful lemma.

Lemma 2. *Let $\mathcal{R}$ be a pointwise k-remote neighborhood ball system. For any $r \in (0,\infty)$ and for all $x_\lambda, y_\mu \in J(L^X)$,*

$$d^{\mathcal{R}}(x_\lambda, y_\mu) < r \Leftrightarrow y_\mu \nleq R_r(x_\lambda) \text{ i.e., } d^{\mathcal{R}}(x_\lambda, y_\mu) \geq r \Leftrightarrow y_\mu \leq R_r(x_\lambda).$$

Proof. It can be obtained by the following implication:

$$d^{\mathcal{R}}(x_\lambda, y_\mu) < r \Leftrightarrow \exists s < r \text{ such that } y_\mu \nleq R_s(x_\lambda) \Leftrightarrow y_\mu \nleq \bigwedge_{s<r} R_s(x_\lambda) = R_r(x_\lambda).$$

□

Theorem 3. *Let $\mathcal{R} = \{R_r \mid r \in (0,\infty)\}$ be a pointwise k-remote neighborhood ball system. Then, $d^{\mathcal{R}}$ is a pointwise k-quasi-pseudo metric.*

Proof. Step 1: We show $d^{\mathcal{R}}$ is well-defined, namely, $d^{\mathcal{R}}(x_\lambda, y_\mu) \in [0,\infty)$. If $x_\lambda \neq y_\mu$, then there exists some $r > 0$ such that $y_\mu \nleq R_r(x_\lambda)$. By Lemma 2, we have $d^{\mathcal{R}}(x_\lambda, y_\mu) < r$. If $x_\lambda = y_\mu$, then $d^{\mathcal{R}}(x_\lambda, x_\lambda) = \bigwedge_{r>0} r = 0$. Thus, $d^{\mathcal{R}}(x_\lambda, y_\mu) \in [0,\infty)$.
Step 2: we check $d^{\mathcal{R}}$ satisfies (LKD1)-(LKD4).
(LKD1) $d^{\mathcal{R}}(x_\lambda, x_\lambda) = \bigwedge\{r \in (0,\infty) \mid x_\lambda \nleq R_r(x_\lambda)\} = \bigwedge_{r>0} r = 0$.
(LKD2) Let $s, t \in (0,\infty)$ such that $d^{\mathcal{R}}(x_\lambda, y_\mu) < s$ and $d^{\mathcal{R}}(y_\mu, z_\nu) < t$. By Lemma 2, we know $y_\mu \nleq R_s(x_\lambda)$ and $z_\nu \nleq R_t(y_\mu)$, which implies

$$z_\nu \nleq \bigvee\{R_t(y_\mu) \mid y_\mu \nleq R_s(x_\lambda)\} = (R_t \odot R_s)(x_\lambda).$$

It follows from $R_t \odot R_s \geq R_{k(s+t)}$ that

$$z_\nu \nleq R_{k(s+t)}(x_\lambda), \text{i.e.,} \, d^{\mathcal{R}}(x_\lambda, z_\nu) < k(s+t).$$

Thus, $d^{\mathcal{R}}(x_\lambda, z_\nu) \leq k(d^{\mathcal{R}}(x_\lambda, y_\mu) + d^{\mathcal{R}}(y_\mu, z_\nu))$ by the arbitrariness of s and t.
(LKD3) Take any $\nu \prec \mu$ with $y_\nu \nleq R_r(x_\lambda)$. Then, $y_\mu \nleq R_r(x_\lambda)$ and $d^{\mathcal{R}}(x_\lambda, y_\nu) \geq d^{\mathcal{R}}(x_\lambda, y_\mu)$. This shows $\bigwedge_{\nu \prec \mu} d^{\mathcal{R}}(x_\lambda, y_\nu) \geq d^{\mathcal{R}}(x_\lambda, y_\mu)$. On the other hand, suppose that $d^{\mathcal{R}}(x_\lambda, y_\mu) < r$. Then $y_\mu \nleq R_r(x_\lambda)$, which implies there exists some $\nu \prec \mu$ such that $y_\nu \nleq R_r(x_\lambda)$. This means $d^{\mathcal{R}}(x_\lambda, y_\nu) < r$. Further $\bigwedge_{\nu \prec \mu} d^{\mathcal{R}}(x_\lambda, y_\nu) < r$. By the arbitrariness of r, we deduce $\bigwedge_{\nu \prec \mu} d^{\mathcal{R}}(x_\lambda, y_\nu) \leq d^{\mathcal{R}}(x_\lambda, y_\mu)$.
(LKD4) It is easy to be proved from (LKR5) and the Definition of $d^{\mathcal{R}}$.
□

Theorem 4. *If $f : (X, \mathcal{R}^X) \longrightarrow (Y, \mathcal{R}^Y)$ is continuous between pointwise k-remote neighborhood ball spaces, then $f : (X, d^{\mathcal{R}^X}) \longrightarrow (Y, d^{\mathcal{R}^Y})$ is non-expansive between pointwise k-quasi-pseudo metric spaces.*

Proof. It needs to prove that $\forall x_\lambda, y_\mu \in J(L^X)$, $d^{\mathcal{R}^Y}(f(x)_\lambda, f(y)_\mu) \leq d^{\mathcal{R}^X}(x_\lambda, y_\mu)$. By the definition of $d^{\mathcal{R}}$ and the continuity of pointwise k-remote neighborhood ball systems, the inequality can be proved from the following:

$$\begin{aligned} d^{\mathcal{R}^X}(x_\lambda, y_\mu) &= \bigwedge\left\{r > 0 \mid y_\mu \nleq R_r^X(x_\lambda)\right\} \\ &\geq \bigwedge\left\{r > 0 \mid y_\mu \nleq f_L^{\leftarrow}\left(R_r^Y(f(x)_\lambda)\right)\right\} \\ &\geq \bigwedge\left\{r > 0 \mid f(y)_\mu \nleq R_r^Y(f(x)_\lambda)\right\} = d^{\mathcal{R}^Y}(f(x)_\lambda, f(y)_\mu) \end{aligned}$$

□

By Lemmas 1 and 2, it is easy to see that $\mathcal{R}^{d^{\mathcal{R}}} = \mathcal{R}$ and $d^{\mathcal{R}^d} = d$. Therefore, we can get the following theorem.

Theorem 5. *The category L-**KPQMS** is isomorphic to the category L-**KRNBS**.*

Finally, we shall study the relationship between symmetric versions of pointwise k-quasi-pseudo metric spaces and pointwise k-remote neighborhood ball spaces.

Theorem 6. *Let (X, d) be a pointwise k-pseudo metric space. Then, $\mathcal{R}^d$ is symmetric.*

Proof. The symmetry of $\mathcal{R}^d$ can be derived from the following.

$$\begin{aligned} & y_\mu \not\leq \bigwedge_{\gamma \not\leq \lambda'} R^d_r(x_\gamma) \Leftrightarrow \exists \gamma \not\leq \lambda', y_\mu \not\leq R^d_r(x_\gamma) \Leftrightarrow \exists \gamma \not\leq \lambda', d(x_\gamma, y_\mu) < r \\ \Leftrightarrow \; & \bigwedge_{\gamma \not\leq \lambda'} d(x_\gamma, y_\mu) = \bigwedge_{\nu \not\leq \mu'} d(y_\nu, x_\lambda) < r \Leftrightarrow \exists \nu \not\leq \mu', d(y_\nu, x_\lambda) < r \\ \Leftrightarrow \; & \exists \nu \not\leq \mu', x_\lambda \not\leq R^d_r(y_\nu) \Leftrightarrow x_\lambda \not\leq \bigwedge_{\nu \not\leq \mu'} R^d_r(y_\nu). \end{aligned}$$

□

Theorem 7. *Let $(X, \mathcal{R})$ be a pointwise k-remote neighborhood ball space. If $\mathcal{R}$ is symmetric, then $d^{\mathcal{R}}$ is symmetric.*

Proof. The symmetry of $d^{\mathcal{R}}$ can be deduced by the following implications.

$$\begin{aligned} \bigwedge_{\gamma \not\leq \lambda'} d^{\mathcal{R}}(x_\gamma, y_\mu) &= \bigwedge_{\gamma \not\leq \lambda'} \bigwedge_{y_\mu \not\leq R_r(x_\lambda)} r = \bigwedge_{y_\mu \not\leq \bigwedge_{\gamma \not\leq \lambda'} R_r(x_\lambda)} r \\ &= \bigwedge_{x_\lambda \not\leq \bigwedge_{\nu \not\leq \mu'} R_r(y_\nu)} r = \bigwedge_{\nu \not\leq \mu'} \bigwedge_{x_\lambda \not\leq R_r(y_\nu)} r = \bigwedge_{\nu \not\leq \mu'} d(y_\nu, x_\lambda) \end{aligned}$$

□

In Figure 1, we present a diagram visualizing the obtained relations between the concepts considered here.

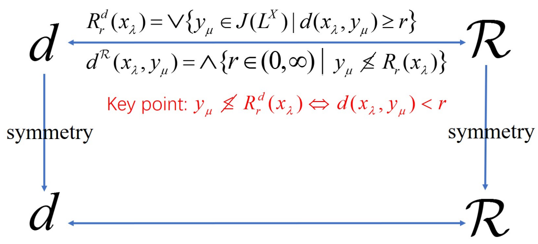

Figure 1. The relationship between d and $\mathcal{R}$.

4. *L*-Structures Induced by a Pointwise *k*-Quasi-Pseudo Metric

In this section, we shall give some L-structures induced by a pointwise k-quasi-pseudo metric.

At first, let us recall some facts about crisp k-metric spaces. Let (X, d) be a k-metric space. Define $B(x, r) = \{y \in X \mid d(x, y) < r\}$. Then, the set $\mathcal{N}^d = \{\mathcal{N}^d_x \mid x \in X\}$ is a neighborhood system, where $\mathcal{N}^d_x = \{A \subseteq X \mid \exists r > 0, \; B(x, r) \subseteq A\}$. Moreover, $\mathcal{T}^d = \{A \subseteq X \mid \forall x \in A, \; \exists r > 0, \; B(x, r) \subseteq A\}$ is a topology.

However, $\mathcal{S}^d = \{A \subseteq X \mid A = \bigcup_{i \in I} B(x_i, \varepsilon_i)\}$ is not a topology, is only a supratopology (or called a pre-topology) and $\mathcal{T}^d \subsetneq \mathcal{S}^d$. The reason is that every open ball $B(x, r)$ need not to be an open set in $\mathcal{T}^d$ because of the violation of triangle inequality in a k-metric space. Readers can refer to the following counterexample.

Example 4 ([24])**.** *Let $X = \{a\} \cup [b, c]$ and the length of $[b, c]$ is s. Let $d_t \in [b, c]$ with $d_t - b = t$ for any $t \in (0, s)$. The distance on $[b, c]$ is the usual Euclidean metric and define $d(a, b) = s$, $d(a, c) = 2s$, $d(a, d_t) = 2s - t$. Then, d is a 2-metric. However, $B(b, \delta) \nsubseteq B(a, s + \varepsilon)$ for any $\varepsilon > 0$ and $\delta > 0$.*

Through the relationships between pointwise k-quasi-pseudo metrics and pointwise k-remote neighborhood ball systems (see Figure 1), we would like to generalize crisp conclusions to L-fuzzy cases.

First, we introduce an L-quasi neighborhood system induced by a pointwise k-remote neighborhood ball system in the following theorem.

Theorem 8. *Let $(X, \mathcal{R})$ be a pointwise k-remote neighborhood ball space. For any $x_\lambda \in J(L^X)$, define $\mathcal{Q}^{\mathcal{R}}_{x_\lambda} \subseteq L^X$ as follows:*

$$\mathcal{Q}^{\mathcal{R}}_{x_\lambda} = \{A \in L^X \mid \exists r \in (0, \infty),\ A' \leq R_r(x_\lambda)\}.$$

Then, $\mathcal{Q}^{\mathcal{R}} = \{\mathcal{Q}^{\mathcal{R}}_{x_\lambda} \mid x_\lambda \in J(L^X)\}$ is an L-quasi neighborhood system.

Proof. We need to check that $\mathcal{Q}^{\mathcal{R}}$ satisfies (LQ1)-(LQ5) in Definition 3.

(LQ1)–(LQ3) hold obviously.

(LQ4) For any $A, B \in \mathcal{Q}^{\mathcal{R}}_{x_\lambda}$, there exist r and s such that $A' \leq \mathcal{R}_r(x_\lambda)$ and $B' \leq \mathcal{R}_s(x_\lambda)$. Let $t = r \wedge s$. Then $\mathcal{R}_r(x_\lambda) \leq \mathcal{R}_t(x_\lambda)$ and $\mathcal{R}_s(x_\lambda) \leq \mathcal{R}_t(x_\lambda)$. It follows that $(A \wedge B)' = A' \vee B' \leq \mathcal{R}_t(x_\lambda)$. This shows $A \wedge B \in \mathcal{Q}^{\mathcal{R}}_{x_\lambda}$.

(LQ5) For any $A \in \mathcal{Q}^{\mathcal{R}}_{x_\lambda}$, there exist $r > 0$ such that $A' \leq \mathcal{R}_r(x_\lambda)$. Let

$$B = \bigwedge\{R_{\frac{s}{2k}}(z_\nu) \mid R_r(x_\lambda) \leq R_s(z_\nu)\}.$$

Then, it is not difficult to get $y_\mu \leq B \Leftrightarrow \forall R_r(x_\lambda) \leq R_s(z_\nu),\ y_\mu \leq R_{\frac{s}{2k}}(z_\nu)$.

Next, we shall show $x_\lambda \nleq B \geq A'$ and $\forall y_\mu \nleq B$, $B' \in \mathcal{Q}_{y_\mu}$.

(i) As $x_\lambda \nleq R_{\frac{r}{2k}}(x_\lambda)$, it follows that $x_\lambda \nleq B$. Take any $y_\mu \in J(L^X)$ with $y_\mu \leq A'$, we have $y_\mu \leq R_r(x_\lambda) \leq R_{\frac{r}{2k}}(x_\lambda)$. Therefore, $y_\mu \leq B$. This implies $A' \leq B$. Thus, $x_\lambda \nleq B \geq A'$.

(ii) For any $y_\mu \nleq B$, there exists $\frac{s}{2k} > 0$ and $z_\nu \in J(L^X)$ such that $y_\mu \nleq R_{\frac{s}{2k}}(z_\nu)$ and $R_r(x_\lambda) \leq R_s(z_\nu)$. Note that

$$\begin{aligned} & R_{\frac{s}{2k}}(y_\mu) \geq \bigwedge\{R_{\frac{s}{2k}}(w_l) \mid w_l \nleq R_{\frac{s}{2k}}(z_\nu)\} = \left(R_{\frac{s}{2k}} \odot R_{\frac{s}{2k}}\right)(z_\nu) \\ \geq \quad & R_{k\left(\frac{s}{2k}+\frac{s}{2k}\right)}(z_\nu) = R_s(z_\nu) \geq R_r(x_\lambda). \end{aligned}$$

This shows $R_{\frac{s}{4k^2}}(y_\mu) \in \{R_{\frac{s}{2k}}(z_\nu) \mid R_r(x_\lambda) \leq R_s(z_\nu)\}$. Then, $B \leq R_{\frac{s}{4k^2}}(y_\mu)$. Thus, $B' \in \mathcal{Q}_{y_\mu}$. Combining (i) and (ii), (LQ5) holds.

□

Theorem 9. *If $f : (X, \mathcal{R}^X) \longrightarrow (Y, \mathcal{R}^Y)$ is continuous between pointwise k-remote neighborhood ball spaces, then $f : (X, \mathcal{Q}^{\mathcal{R}^X}) \longrightarrow (Y, \mathcal{Q}^{\mathcal{R}^Y})$ is continuous between L-quasi neighborhood spaces.*

Proof. It needs to check that $\forall x_\lambda \in J(L^X)$, $\forall U \in \mathcal{Q}^{\mathcal{R}^Y}_{f(x)_\lambda}$, $f^{\leftarrow}_L(U) \in \mathcal{Q}^{\mathcal{R}^X}_{x_\lambda}$. For any $U \in \mathcal{Q}^{\mathcal{R}^Y}_{f(x)_\lambda}$, there exists $r > 0$ such that $U' \leq R^Y_r(f(x)_\lambda)$.

By the continuity of pointwise k-remote neighborhood ball spaces and the order-preserving property of $f_L^{\leftarrow}$, we have

$$(f_L^{\leftarrow}(U))' = f_L^{\leftarrow}(U') \leq f_L^{\leftarrow}\left(R_r^Y(f(x)_\lambda)\right) \leq R_r^X(x_\lambda).$$

This shows $f_L^{\leftarrow}(U) \in \mathcal{Q}_{x_\lambda}^{\mathcal{R}^X}$. $\square$

As the category L-**Top** is isomorphic to the category of L-**QNS** [33,36], it is easy to obtain an L-topology induced by $\mathcal{R}$, that is,

$$\mathcal{T}^{\mathcal{R}} = \{A \in L^X \mid \forall x_\lambda \not\leq A',\ \exists r > 0,\ A' \leq R_r^d(x_\lambda)\}.$$

Further, we can get an L-topology induced by a pointwise k-pseudo-quasi metric through Figure 1 as a link,

$$\mathcal{T}^d = \{A \in L^X \mid \forall x_\lambda \not\leq A',\ \exists r > 0,\ \forall y_\mu \leq A',\ d(x_\lambda, y_\mu) \geq r\}.$$

In [33,36], it is also shown that there is a one-to-one correspondence between L-quasi neighborhood systems and L-closure operators. Precisely speaking, if $\mathcal{Q}$ is an L-quasi neighborhood system, then

$$cl^{\mathcal{Q}}(A) = \bigvee\{x_\lambda \in J(L^X) \mid A' \notin \mathcal{Q}_{x_\lambda}\}$$

is an L-closure operator induced by $\mathcal{Q}$. Conversely, if cl is an L-closure operator, then $\mathcal{Q}^{cl} = \{\mathcal{Q}_{x_\lambda}^{cl} \mid x_\lambda \in J(L^X)\}$ is an L-quasi neighborhood system induced by cl, in which $\mathcal{Q}_{x_\lambda}^{cl} = \{A \in L^X \mid x_\lambda \not\leq cl(A')\}$.

As we have already gotten $\mathcal{Q}_{x_\lambda}^{\mathcal{R}} = \{A \in L^X \mid \exists r \in (0, \infty),\ A' \leq R_r(x_\lambda)\}$ in Theorem 8, we have the following conclusions.

Theorem 10. *Let $(X, \mathcal{R})$ be a pointwise k-remote neighborhood ball space. Define $cl^{\mathcal{R}} : L^X \longrightarrow L^X$ by*

$$cl^{\mathcal{R}}(A) = \bigvee\{x_\lambda \in J(L^X) \mid \forall r > 0, A \not\leq R_r(x_\lambda)\}.$$

Then, $cl^{\mathcal{R}}$ is an L-closure operator.

By Figure 1, we know that an L-closure operator induced by a pointwise k-pseudo metric d can be expressed by

$$cl^d(A) = \bigvee\{x_\lambda \in J(L^X) \mid \forall r > 0, \exists y_\mu \leq A, d(x_\lambda, y_\mu) < r\}.$$

In the following, we shall give a formula of $int^{\mathcal{R}}$.

Theorem 11. *Let $(X, \mathcal{R})$ be a pointwise k-remote neighborhood ball space. Define $int^{\mathcal{R}} : L^X \longrightarrow L^X$ by*

$$int^{\mathcal{R}}(A) = \bigvee\{x_\lambda \in J(L^X) \mid \exists r > 0, \forall y_\mu \not\leq A, x_\lambda \leq R_r(y_\mu)\}.$$

Then $int^{\mathcal{R}}$ is an L-interior operator.

Proof. We need to check (LI1)-(LI4) in Definition 5.

(LI1), (LI2) are obvious.

(LI3) It is clear that $int^{\mathcal{R}}(A \wedge B) \leq int^{\mathcal{R}}(A) \wedge int^{\mathcal{R}}(B)$, since $int^{\mathcal{R}} : L^X \longrightarrow L^X$ is order-preserving. What remains is to prove $int^{\mathcal{R}}(A \wedge B) \geq int^{\mathcal{R}}(A) \wedge int^{\mathcal{R}}(B)$.

Take any $x_\lambda \in J(L^X)$ with $x_\lambda \prec int^{\mathcal{R}}(A) \wedge int^{\mathcal{R}}(B)$, we have $x_\lambda \prec int^{\mathcal{R}}(A)$ and $x_\lambda \prec int^{\mathcal{R}}(B)$. Then there exist $r > 0, s > 0$ such that $x_\lambda \leq R_r(y_\mu)$ for any $y_\mu \not\leq A$ and $x_\lambda \leq R_s(z_\nu)$ for any $z_\nu \not\leq B$.

Let $t = r \wedge s$. Suppose that $w_l \not\leq A \wedge B$ (i.e., $w_l \not\leq A$ or $w_l \not\leq B$). If $w_l \not\leq A$, then $x_\lambda \leq R_r(w_l) \leq R_t(w_l)$. If $w_l \not\leq B$, then $x_\lambda \leq R_s(w_l) \leq R_t(w_l)$. Hence $x_\lambda \leq R_t(w_l)$ for any $w_l \not\leq A \wedge B$. This shows $x_\lambda \leq int^{\mathcal{R}}(A \wedge B)$. From the arbitrariness of x_λ, we obtain $int^{\mathcal{R}}(A) \wedge int^{\mathcal{R}}(B) \leq int^{\mathcal{R}}(A \wedge B)$.
(LI4) It suffices to prove that $int^{\mathcal{R}}(A) \leq int^{\mathcal{R}}(int^{\mathcal{R}}(A))$.

Take any $x_\lambda \in J(L^X)$ with $x_\lambda \prec int^{\mathcal{R}}(A)$, there exist $r > 0$ such that $x_\lambda \leq R_r(y_\mu)$ for any $y_\mu \not\leq A$. In order to show $x_\lambda \prec int^{\mathcal{R}}(int^{\mathcal{R}}(A))$, we need to prove whether there exists $\widetilde{r} > 0$ such that $x_\lambda \leq R_{\widetilde{r}}(z_\nu)$ for any $z_\nu \not\leq int^{\mathcal{R}}(A)$.

Let $\widetilde{r} = \frac{r}{2k}$. For any $z_\nu \not\leq int^{\mathcal{R}}(A)$, there exists $\widetilde{y_\mu} \not\leq A$ such that $z_\nu \not\leq R_s(\widetilde{y_\mu})$ for all $s > 0$. Fix $s = \frac{r}{2k} > 0$. Then, $z_\nu \not\leq R_{\frac{r}{2k}}(\widetilde{y_\mu})$ and $x_\lambda \leq R_r(\widetilde{y_\mu})$. As

$$(R_{\frac{r}{2k}} \odot R_{\frac{r}{2k}})(\widetilde{y_\mu}) = \bigwedge\{R_{\frac{r}{2k}}(z_\nu) \mid z_\nu \not\leq R_{\frac{r}{2k}}(\widetilde{y_\mu})\}$$

and (LKR3), it follows that

$$x_\lambda \leq R_r(\widetilde{y_\mu}) \leq (R_{\frac{r}{2k}} \odot R_{\frac{r}{2k}})(\widetilde{y_\mu}) \leq R_{\frac{r}{2k}}(z_\nu).$$

Thus, $x_\lambda \leq R_{\frac{r}{2k}}(z_\nu)$. Therefore, $x_\lambda \leq int(int(A))$. From the arbitrariness of x_λ, we obtain $int^{\mathcal{R}}(A) \leq int^{\mathcal{R}}(int^{\mathcal{R}}(A))$. $\square$

By Figure 1, we know that an L-interior operator induced by a pointwise k-pseudo metric d can be expressed by

$$int^d(A) = \bigvee\{x_\lambda \mid \exists r > 0, \forall y_\mu \not\leq A,\ d(y_\mu, x_\lambda) \geq r\}.$$

Finally, we shall discuss whether a pointwise k-remote neighborhood ball system can induce a pointwise quasi-uniformity or not. Before answering this question, some concepts related to a pointwise quasi-uniformity introduced in [37] are recalled.

Let $\mathcal{F} = \{f : J(L^X) \longrightarrow L^X | f \text{ is order-preserving}\}$ such that $x_\lambda \not\leq f(x_\lambda)$. For any $f, g \in \mathcal{F}$, define

(1) $f \leq g \Leftrightarrow \forall x_\lambda \in J(L^X), f(x_\lambda) \leq g(x_\lambda)$;
(2) $(f \vee g)(x_\lambda) = f(x_\lambda) \vee g(x_\lambda)$;
(3) $(f \odot g)(x_\lambda) = \wedge\{f(y_\mu) \mid y_\mu \not\leq g(x_\lambda)\}$.

It is not difficult to prove that $f \vee g \in \mathcal{F}$, $f \odot g \in \mathcal{F}$ and the operators $\vee$ and $\odot$ satisfy the associativity law.

Definition 10 ([37])**.** *A mapping $f \in \mathcal{F}$ is said to be symmetric if it satisfies the following condition:*

$$y_\mu \not\leq \bigwedge_{\gamma \not\leq \lambda'} f(x_\gamma) \Leftrightarrow x_\lambda \not\leq \bigwedge_{\nu \not\leq \mu'} f(y_\nu).$$

Definition 11 ([37])**.** *A non-empty subset $\mathcal{U} \subseteq \mathcal{F}$ is called a pointwise quasi-uniformity on L^X if it satisfies*

(LU1) $\forall f \in \mathcal{F}, \forall g \in \mathcal{U}, f \leq g$ implies $f \in \mathcal{U}$;
(LU2) $\forall f, g \in \mathcal{U}$ implies $f \vee g \in \mathcal{U}$;
(LU3) $\forall f \in \mathcal{U}$ implies $\exists g \in \mathcal{U}$ such that $g \odot g \geq f$.

A subset $\mathcal{A} \subseteq \mathcal{U}$ is called a basis of $\mathcal{U}$ if $\forall f \in \mathcal{U}$, $\exists g \in \mathcal{A}$ such that $f \leq g$, namely, $\mathcal{U} = \{f \in \mathcal{F} \mid \exists g \in \mathcal{A}, s.t. f \leq g\}$. A pointwise quasi-uniformity is called a pointwise uniformity if it has a symmetric basis.

Definition 12 ([37]). *An order homomorphism $F : X \longrightarrow Y$ is said to be pointwise quasi-uniformly continuous with respect to pointwise quasi-uniformities $\mathcal{U}_X$ and $\mathcal{U}_Y$ if for each $g \in \mathcal{U}_Y$, there exists $f \in \mathcal{U}_X$ such that*

$$\forall x_\lambda, y_\mu \in J(L^X), y_\mu \not\leq f(x_\lambda) \Rightarrow F_L^{\rightarrow}(y_\mu) \not\leq g(F_L^{\rightarrow}(x_\lambda)).$$

Theorem 12 ([37]). *Let $F : X \longrightarrow Y$ be an order homomorphism. Then, $F : (X, \mathcal{U}_X) \longrightarrow (Y, \mathcal{U}_Y)$ is quasi-uniformly continuous if and only if $\forall g \in \mathcal{U}_Y$, $\exists f \in \mathcal{U}_X$ such that $F_L^{\leftarrow} \circ g \circ F_L^{\rightarrow} \leq f$.*

By the conditions in Definition 8 and Proposition 2, it is easy to know that a (symmetric) pointwise k-remote neighborhood ball system $\mathcal{R} = \{R_r : J(L^X) \longrightarrow L^X \mid r > 0\}$ is a (symmetric) basis of a pointwise uniformity. Then, we have the following theorems.

Theorem 13. *Let $(X, \mathcal{R})$ be a (symmetric) pointwise k-remote neighborhood ball space. Define $\mathcal{U}^{\mathcal{R}} \subseteq \mathcal{F}$ by*

$$\mathcal{U}^{\mathcal{R}} = \{f \in \mathcal{F} \mid \exists r > 0,\ f \leq R_r\}$$

Then, $\mathcal{U}^{\mathcal{R}}$ is a pointwise quasi-uniformity (pointwise uniformity).

By Figure 1, we know that a pointwise (quasi)-uniformity induced by a pointwise k-(quasi) pseudo metric d can be expressed by

$$\mathcal{U}^d = \{f \in \mathcal{F} \mid \exists r > 0, \forall y_\mu \leq f(x_\lambda),\ d(x_\lambda, y_\mu) \geq r\}.$$

Theorem 14. *If $F : (X, \mathcal{R}^X) \longrightarrow (Y, \mathcal{R}^Y)$ is continuous between pointwise k-remote neighborhood ball spaces, then $F : (X, \mathcal{U}^{\mathcal{R}^X}) \longrightarrow (Y, \mathcal{U}^{\mathcal{R}^Y})$ is quasi-uniformly continuous between pointwise quasi-uniform spaces.*

Proof. For any $g \in \mathcal{U}^{\mathcal{R}^Y}$, there exists $r > 0$ such that $g \leq R_r^Y$. By the continuity of pointwise k-remote neighborhood ball spaces and the order-preserving property of $F_L^{\leftarrow}$, we have

$$\begin{aligned} (F_L^{\leftarrow} \circ g \circ F_L^{\rightarrow})(x_\lambda) &= F_L^{\leftarrow} \circ g(F(x)_\lambda) \leq F_L^{\leftarrow}(R_r^Y(F(x)_\lambda)) \\ &\leq R_r^X(F_L^{\leftarrow}(F(x)_\lambda)) = R_r^X(x_\lambda) \end{aligned}$$

As $R_r^X \in \mathcal{U}^{\mathcal{R}^X}$, it follows that F is quasi-uniformly continuous between pointwise uniform spaces $(X, \mathcal{U}^{\mathcal{R}^X})$ and $(Y, \mathcal{U}^{\mathcal{R}^Y})$. □

At the end of the paper, we present a diagram illustrating the obtained here results about L-structures induced by k-quasi-pseudo metrics (see Figure 2).

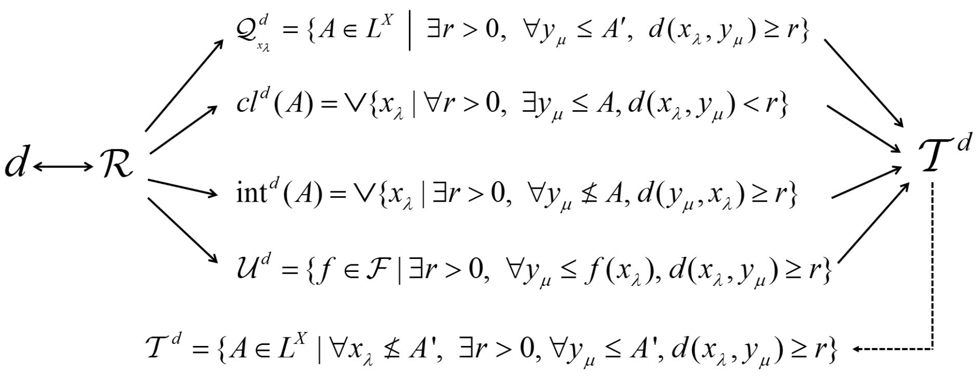

Figure 2. L-structures induced by d.

5. Conclusions

In this paper, the definition of a pointwise k-(quasi) pseudo metric and a pointwise k-remote neighborhood ball system were introduced. We showed that the category of pointwise k-pseudo metric spaces is isomorphic to the category of symmetric pointwise k-remote neighborhood ball spaces. Besides, we discussed some L-topological structures induced by a pointwise k-quasi-pseudo metric and investigated their properties.

Some research works about the concept of an (L, M)-fuzzy k-metric and its induced (L, M)-fuzzy structures would be our interest in the future. Furthermore, we plan to generalize an (L, M)-fuzzy k-metric to an (L, M)-fuzzy partial k-metric and study its properties.

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