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On Rings of Weak Global Dimension at Most One

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Abstract: A ring R is of weak global dimension at most one if all submodules of flat R-modules are flat. A ring R is said to be arithmetical (resp., right distributive or left distributive) if the lattice of two-sided ideals (resp., right ideals or left ideals) of R is distributive. Jensen has proved earlier that a commutative ring R is a ring of weak global dimension at most one if and only if R is an arithmetical semiprime ring. A ring R is said to be centrally essential if either R is commutative or, for every noncentral element $x \in R$, there exist two nonzero central elements $y, z \in R$ with $xy = z$. In Theorem 2 of our paper, we prove that a centrally essential ring R is of weak global dimension at most one if and only is R is a right or left distributive semiprime ring. We give examples that Theorem 2 is not true for arbitrary rings.

Keywords: ring of weak global dimension at most one; centrally essential ring; arithmetical ring; right distributive ring; left distributive ring

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1. Introduction

We consider only nonzero associative unital rings. For a ring R, we write w.gl.dim. $R \leq 1$ if R is a *ring of weak global dimension at most one*, i.e., R satisfies the following equivalent (The equivalence of the conditions is well known; e.g., see the conditions in [1] (Theorem 6.12)).

- For every finitely generated right ideal X of R and each finitely generated left ideal Y of R, the natural group homomorphism $X \otimes_R Y \to XY$ is an isomorphism.
- Every finitely generated right (resp., left) ideal of R is a flat right (resp., left) R-module.
- Every right (resp., left) ideal of R is a flat right (resp., left) R-module.
- Every submodule of any flat right (resp., left) R-module is flat.
- $\mathrm{Tor}_2^R(A, B) = 0$ for all right (resp., left) R-modules A and B.

Since every projective module is flat, any right or left (semi)hereditary ring is of weak global dimension at most one. (a module M is said to be hereditary (resp., semihereditary) if all submodules (resp., finitely generated submodules) of M are projective.) We also recall that a ring R is of weak global dimension zero if and only if R is a Von Neumann regular ring, i.e., $r \in rRr$ for every element r of R. Von Neumann regular rings are widely used in mathematics; see [2,3].

A ring R is said to be *arithmetical* if the lattice of two-sided ideals of R is distributive, i.e., $X \cap (Y + Z) = X \cap Y + X \cap Z$ for any three ideals X, Y, Z of R. A ring R is said to be *semiprime* (resp., *prime*) if R does not have nilpotent nonzero ideals (resp., the product of any two nonzero ideals of R are nonzero).

Theorem 1 (C.U.Jensen ([4], Theorem))**.** *A commutative ring R is a ring of weak global dimension at most one if and only if R is an arithmetical semiprime ring.*

A ring R with center C is said to be *centrally essential* if R_C is an essential extension of the module C_C, i.e., for every nonzero element $r \in R$, there exist two nonzero central elements $x, y \in R$ with $rx = y$. Centrally essential rings are studied in many papers; e.g., see [5].

There are many noncommutative centrally essential rings. For example, if F is the field $\mathbb{Z}/2\mathbb{Z}$ and Q_8 is the quaternion group of order 8, then the group algebra FQ_8 is a finite noncommutative centrally essential ring; see [5].

Let F be the field $\mathbb{Z}/3\mathbb{Z}$, and let V be a vector F-space with basis e_1, e_2, e_3. It is known that the exterior algebra of the space V is a finite centrally essential noncommutative ring. It is known that there exists a centrally essential ring R such that the factor ring $R/J(R)$ with respect to the Jacobson radical is not a PI ring (in particular, the ring $R/J(R)$ is not commutative).

A module M is said to be *distributive* (resp., *uniserial*) if the submodule lattice of M is distributive (resp., is a chain). It is clear that a commutative ring is right (resp., left) distributive if and only if the ring is arithmetical.

The main result of this work is Theorem 2.

Theorem 2. *For a centrally essential ring R, the following conditions are equivalent.*

1. *R is a ring of weak global dimension at most one.*
2. *R is a right (resp., left) distributive semiprime ring.*
3. *R is an arithmetical semiprime ring*

2. Remarks and Proof of Theorem 2

Example 1. *The implication (1) $\Rightarrow$ (2) of Theorem 2 is not true for arbitrary rings. There exists a right hereditary ring R of weak global dimension at most one that is neither right distributive nor semiprime; in particular, the right hereditary ring R is of weak global dimension at most one. Let F be a field, and let R be the* 5*-dimensional F-algebra consisting of all* 3×3 *matrices of the following form:* $\begin{pmatrix} f_{11} & f_{12} & f_{13} \\ 0 & f_{22} & 0 \\ 0 & 0 & f_{33} \end{pmatrix}$*, where* $f_{ij} \in F$*. The ring R is not semiprime, since the following set is a nonzero nilpotent ideal of R:* $\left\{ \begin{pmatrix} 0 & f_{12} & f_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$*. Let* e_{11}*,* e_{22}*, and* e_{33} *be ordinary matrix units. The ring R is not right or left distributive, since every idempotent of a right or left distributive ring is central (see [6]), but the matrix unit* e_{11} *of R is not central. To prove that the ring R is right hereditary, it is sufficient to prove that* R_R *is a direct sum of hereditary right ideals. We have that* $R_R = e_{11}R \oplus e_{22}R \oplus e_{33}R$*, where* $e_{22}R$ *and* $e_{33}R$ *are projective simple R-modules; in particular,* $e_{22}R$ *and* $e_{33}R$ *are hereditary R-modules. Any direct sum of hereditary modules is hereditary; see ([7], 39.7, p. 332). Therefore, it remains to show that the R-module* $e_{11}R = e_{11}F + e_{12}F + e_{13}F$ *is hereditary, which is directly verified.*

The following lemma is well known; e.g., see ([1], Assertion 6.13).

Lemma 1. *Let R be a ring in which the principal right ideals are flat. If r and s are two elements of R with* $rs = 0$*, then there exist two elements* $a, b \in R$ *such that* $a + b = 1$*,* $ra = 0$*, and* $bs = 0$*.*

Lemma 2. *Let R be a centrally essential ring in which the principal right ideals are flat. Then, the ring R does not have nonzero nilpotent elements.*

Proof. Indeed, let us assume that there exists a nonzero element $r \in R$ with $r^2 = 0$. Since the ring R is centrally essential, there exist two nonzero central elements $x, y \in R$ with $rx = y$. Since $r^2 = 0$, we have that $y^2 = (rx)^2 = r^2x^2 = 0$. Since $y^2 = 0$, it follows from Lemma 1 that there exist two elements $a, b \in R$ such that $a + b = 1$, $ry = 0$, and $by = yb = 0$. Then, $y = y(a + b) = ya + yb = 0$. This is a contradiction. □

Lemma 3. *There exists right and left uniserial prime rings R that habe a non-flat principal right ideal.*

Proof. There exists right and left uniserial prime rings R with two nonzero elements $r, s \in R$ such that $rs = 0$; see ([8], p. 234, Corollary). The uniserial ring R is local; therefore, the invertible elements of R form the Jacobson radical $J(R)$ of R. The ring R is not a ring in which the principal right ideals are flat. Indeed, let us assume the contrary. By Lemma 1, there exist two elements $a, b \in R$ such that $a + b = 1$, $ra = 0$, and $bs = 0$. We have that either $aR \subseteq bR$ or $bR \subseteq aR$; in addition, $aR + bR = R = Ra + Rb$. Therefore, at least one of the elements a, b of the local ring R is invertible; in particular, this invertible element is not a right or left zero-divisor. This contradicts to the relations $ra = 0$ and $bs = 0$. □

Remark 1. *It follows from Lemma 3 that the implication (2) ⇒ (1) of Theorem 2 is not true for arbitrary rings.*

Lemma 4. *Every centrally essential semiprime ring R is commutative.*

Proof. Assume the contrary. Then, the ring R does not coincide with its center C and $xy - yx \neq 0$ for some $x, y \in R$. We note that $A = \{c \in C \colon xc \in C\}$ is an ideal of the ring C. The set $d \in C \mid dA = 0$ is not empty, since we can take $d = 0$. We take any element $d \in C$ with $dA = 0$. If $xd \neq 0$, then $xdz \in C \setminus \{0\}$ for some $z \in C$. Hence $dz \in A$, and therefore, $d(dz) = 0$ and $(dz)^2 = 0$. Thus, $dz = 0$ and $xdz = 0$; this is a contradiction. Therefore, $xd = 0$, and thus, $d \in A$. Therefore, $d^2 = 0$ and $d = 0$. This implies that $\mathrm{Ann}_C(A) = 0$. For any $a \in A$, we have that $xa = ax \in C$. Thus,

$$(xy - yx)a = x(ya) - y(xa) = xay - xay = 0$$

and $(xy - yx)A = 0$. However, $c_1(xy - yx) = c_2$ for some nonzero elements $c_1, c_2 \in C$, so $c_2A = 0$ and, hence, $\mathrm{Ann}_C(A) \neq 0$; this is a contradiction. Thus, R is commutative. □

The Completion of the Proof of Theorem 2

Proof. (1) ⇒ (2). Since R is a centrally essential ring of weak global dimension at most one, it follows from Lemma 2 that the ring R does not have nonzero nilpotent elements. By Lemma 4, the centrally essential semiprime ring R is commutative. By Theorem 1, R is an arithmetical semiprime ring. Any commutative arithmetical ring is right and left distributive.

The implication (2) ⇒ (3) follows from the property that every right or left distributive ring is arithmetical.

(3) ⇒ (1). Since R is a centrally essential semiprime ring, it follows from Lemma 4 that the ring R is commutative; in particular, R is centrally essential. In addition, R is arithmetical. By Theorem 1, the ring R is of weak global dimension at most one. □

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