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# The Exact Solutions of Stochastic Fractional-Space Kuramoto-Sivashinsky Equation by Using $(\frac{G'}{G})$-Expansion Method

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**Abstract:** In this paper, we consider the stochastic fractional-space Kuramoto–Sivashinsky equation forced by multiplicative noise. To obtain the exact solutions of the stochastic fractional-space Kuramoto–Sivashinsky equation, we apply the $\frac{G'}{G}$-expansion method. Furthermore, we generalize some previous results that did not use this equation with multiplicative noise and fractional space. Additionally, we show the influence of the stochastic term on the exact solutions of the stochastic fractional-space Kuramoto–Sivashinsky equation

**Keywords:** stochastic Kuramoto–Sivashinsky; fractional Kuramoto–Sivashinsky; exact stochastic-fractional solutions; $(\frac{G'}{G})$-expansion method



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## 1. Introduction

In recent decades, fractional derivatives have received a lot of attention because they have been effectively used to problems in finance [1–3], biology [4], physics [5–8], thermodynamic [9,10], hydrology [11,12], biochemistry and chemistry [13]. Since fractional-order integrals and derivatives allow for the representation of the memory and heredity properties of various substances, these new fractional-order models are more suited than the previously used integer-order models [14]. This is the most important benefit of fractional-order models in comparison with integer-order models, where such impacts are ignored.

On the other hand, fluctuations or randomness have now been shown to be important in many phenomena. Therefore, random effects have become significant when modeling different physical phenomena that take place in oceanography, physics, biology, meteorology, environmental sciences, and so on. Equations that consider random fluctuations in time are referred to as stochastic differential equations.

Recently, some studies on the approximation solutions of fractional differential equations with stochastic perturbations have been published, such as those of Taheri et al. [15], Zou [16], Mohammed et al. [17,18], Mohammed [19], Kamrani [20], Li and Yang [21] and Liu and Yan [22], while the exact solutions of stochastic fractional differential equations have not been discussed until now.

In this study, we take into account the following stochastic fractional-space Kuramoto–Sivashinsky (S-FS-KS) equation in one dimension with multiplicative noise in the itô sense:

$$\partial_t u + ruD_x^{\alpha}u + pD_x^{2\alpha}u + qD_x^{4\alpha}u = \rho u\partial_t\beta, \tag{1}$$

where $r$, $p$, and $q$ are nonzero real constants, $\alpha$ is the order of the fractional space derivative, $\rho$ is the noise strength, and $\beta(t)$ is the standard Gaussian process and it depends only on $t$.

The deterministic Kuramoto–Sivashinsky Equation (1) (i.e., $\rho = 0$) with $\alpha = 1$ has been studied by a number of authors to attain its exact solutions by different methods such as the modified tanh-coth method [23], the tanh method and the extended tanh method [24], homotopy analysis method [25], the $(\frac{G'}{G})$-expansion method [26], perturbation method [27], the Weiss–Tabor–Carnevale method [28], Painlevé expansion methods [29], the truncated expansion method [30], the polynomial expansion method [31–37], among many others; see also the references therein.

The motivation of this article is to find the exact solutions of the S-FS-KS (1) derived from multiplicative noise by employing the $(\frac{G'}{G})$-expansion method. The results presented here improve and generalize earlier studies, such as those mentioned in [24]. It is also discussed how multiplicative noise affects these solutions. To the best of our knowledge, this is the first paper to establish the exact solution of the S-FS-KS (1).

In the next section, we define the order $\alpha$ of Jumarie's derivative and we state some significant properties of the modified Riemann–Liouville derivative. In Section 3, we obtain the wave equation for the S-FS-KS Equation (1), while in Section 4 we have the exact stochastic solutions of the S-FS-KS (1) by applying the $(\frac{G'}{G})$-expansion method. In Section 5, we show several graphical representations to demonstrate the effect of stochastic terms on the obtained solutions of the S-FS-KS. Finally, the conclusions of this paper are presented.

## 2. Modified Riemann–Liouville Derivative and Properties

The order $\alpha$ of Jumarie's derivative is defined by [38]:

$$D_x^\alpha g(x) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\zeta)^{-\alpha}(g(\zeta) - g(0))d\zeta, & 0 < \alpha < 1, \\ [g^{(n)}(x)]^{\alpha - n}, & n \le \alpha \le n+1, \;\; n \ge 1, \end{cases}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function but not necessarily first-order differentiable and $\Gamma(.)$ is the Gamma function.

Now, let us state some significant properties of modified Riemann–Liouville derivative as follows:

$$D_x^\alpha x^\delta = \frac{\Gamma(1+\delta)}{\Gamma(1+\delta-\alpha)} x^{\delta-\alpha}, \; \delta > 0,$$

$$D_x^\alpha [a g(x)] = a D_x^\alpha g(x),$$

$$D_x^\alpha [a f(x) + b g(x)] = a D_x^\alpha f(x) + b D_x^\alpha g(x),$$

and

$$D_x^\alpha g(u(x)) = \sigma_x \frac{dg}{du} D_x^\alpha u,$$

where $\sigma_x$ is called the sigma indexes [39,40].

## 3. Wave Equation for S-FS-KS Equation

To obtain the wave equation for the SKS Equation (1), we apply the next wave transformation

$$u(x,t) = \varphi(\eta) e^{(\rho\beta(t) - \frac{1}{2}\rho^2 t)}, \;\; \eta = \frac{1}{\Gamma(1+\alpha)} x^\alpha - ct, \tag{2}$$

where $\varphi$ is the deterministic function and $c$ is the wave speed. By differentiating Equation (2) with respect to $x$ and $t$, we obtain

$$\begin{aligned} u_t &= (-c\varphi' + \frac{1}{2}\rho^2\varphi - \frac{1}{2}\rho^2\varphi + \rho\varphi\beta_t) e^{(\rho\beta(t) - \frac{1}{2}\rho^2 t)}, \\ D_x^\alpha u &= \sigma_x \varphi' e^{[\rho\beta(t) - \rho^2 t]}, \; D_x^{2\alpha} u = \sigma_x^2 \varphi'' e^{[\rho\beta(t) - \rho^2 t]}. \\ D_x^{3\alpha} &= \sigma_x^3 e^{(\rho\beta(t) - \frac{1}{2}\rho^2 t)}, \; D_x^{4\alpha} = \sigma_x^4 e^{(\rho\beta(t) - \frac{1}{2}\rho^2 t)}, \end{aligned} \tag{3}$$

where $+\frac{1}{2}\rho^2\varphi$ is the Itô correction term. Now, substituting Equation (3) into Equation (1), we obtain

$$-c\varphi' + \tilde{r}\varphi\varphi' e^{(\rho\beta(t)-\frac{1}{2}\rho^2 t)} + \tilde{p}\varphi'' + \tilde{q}\varphi'''' = 0, \tag{4}$$

where we put $\tilde{r} = \sigma_x r$, $\tilde{p} = \sigma_x^2 p$ and $\tilde{q} = \sigma_x^4 q$. Taking the expectation on both sides and considering that $\varphi$ is deterministic function, we have

$$-c\varphi' + \tilde{r}\varphi\varphi' e^{-\frac{1}{2}\rho^2 t}\mathbb{E}(e^{\rho\beta(t)}) + \tilde{p}\varphi'' + \tilde{q}\varphi'''' = 0. \tag{5}$$

Since $\beta(t)$ is standard Gaussian random variable, then for any real constant $\rho$ we have $\mathbb{E}(e^{\rho\beta(t)}) = e^{\frac{\rho^2}{2}t}$. Now, Equation (5) has the form

$$-c\varphi' + \tilde{r}\varphi\varphi' + \tilde{p}\varphi'' + \tilde{q}\varphi'''' = 0. \tag{6}$$

Integrating Equation (6) once in terms of $\eta$ yields

$$\tilde{q}\varphi''' + \tilde{p}\varphi' + \frac{\tilde{r}}{2}\varphi^2 - c\varphi = 0, \tag{7}$$

where we set the constant of integration as equal to zero.

## 4. The Exact Solutions of the S-FS-KS Equation

Here, we apply the $\frac{G'}{G}$-expansion method [41] in order to find the solutions of Equation (7). As a result, we have the exact solutions of the S-FS-KS (1). First, we suppose the solution of the S-FS-KS equation, Equation (7), has the form

$$\varphi = \sum_{k=0}^{M} b_k [\frac{G'}{G}]^k, \tag{8}$$

where $b_0$, $b_1$, ..., $b_M$ are uncertain constants that must be calculated later, and $G$ solves

$$G'' + \lambda G' + \mu G = 0, \tag{9}$$

where $\lambda$, $\mu$ are unknown constants. Let us now calculate the parameter $M$ by balancing $\varphi^2$ with $\varphi'''$ in Equation (7) as follows

$$2M = M + 3;$$

hence

$$M = 3. \tag{10}$$

From (10), we can rewrite Equation (8) as

$$\varphi = b_0 + b_1[\frac{G'}{G}] + b_2[\frac{G'}{G}]^2 + b_3[\frac{G'}{G}]^3. \tag{11}$$

Putting Equation (11) into Equation (7) and utilizing Equation (9), we obtain a polynomial with degree 6 of $\frac{G'}{G}$ as follows

$$
\begin{aligned}
&(\frac{1}{2}\tilde{r}b_3^2 - 60\tilde{q}b_3)[\frac{G'}{G}]^6 + (-24\tilde{q}b_2 + \tilde{r}b_2b_3 - 144\tilde{q}\lambda b_3)[\frac{G'}{G}]^5 \\
&+(\frac{1}{2}\tilde{r}b_2^2 - 3\tilde{p}b_3 - 6\tilde{q}b_1 + \tilde{r}b_1b_3 - 111\tilde{q}\lambda^2 b_3 - 114\tilde{q}\mu b_3 - 54\tilde{q}\lambda b_2)[\frac{G'}{G}]^4 \\
&+(-cb_3 + 2\tilde{p}b_2 + \tilde{r}b_0b_3 + \tilde{r}b_1b_2 - 3\tilde{p}\lambda b_3 - 38\tilde{q}\lambda^2 b_2 - 40\tilde{q}\mu b_2 - 27\lambda^3 b_3 \\
&-12\tilde{q}\lambda b_1 - 168\tilde{q}\lambda\mu b_3)[\frac{G'}{G}]^3 + (-cb_2 + \frac{1}{2}\tilde{r}b_1^2 - \tilde{p}b_1 + \tilde{r}b_0b_2 - 2\tilde{p}\lambda b_2 \\
&-3\tilde{p}\mu b_3 - 7\tilde{q}\lambda^2 b_1 - 8\tilde{q}\mu b_1 - 8\tilde{q}\lambda^3 b_2 - 52\tilde{q}\lambda\mu b_2 - 60\tilde{q}\mu^2 b_3 \\
&-57\tilde{q}\lambda^2\mu b_3)[\frac{G'}{G}]^2 + (-cb_1 + \tilde{r}b_0b_1 - \tilde{p}\lambda b_1 - 2\tilde{p}\mu b_2 - \tilde{q}\lambda^3 b_1 \\
&-16\tilde{q}\mu^2 b_2 - 8\tilde{q}\lambda\mu b_1 - 14\tilde{q}\lambda^2\mu b_2 - 36\tilde{q}\mu^2\lambda b_3)[\frac{G'}{G}]+
\end{aligned}
$$

$$
(-cb_0 + \frac{1}{2}\tilde{r}b_0^2 - \tilde{p}\mu b_1 - \tilde{q}\lambda^2\mu b_1 - 6\tilde{q}\mu^2\lambda b_2 - 2\tilde{q}\mu^2 b_1 - 6\tilde{q}\mu^3 b_3) = 0.
$$

By equating each coefficient of $[\frac{G'}{G}]^i$ $(i = 6,5,4,3,2,1,0)$ to zero, we have a system of algebraic equations. By solving this system by using Maple, we obtain two cases:

**First case:**

$$
\begin{aligned}
b_0 &= \pm\frac{30\tilde{p}}{19\tilde{r}}\sqrt{\frac{-\tilde{p}}{19\tilde{q}}},\ b_1 = \frac{90\tilde{p}}{19\tilde{r}},\ b_2 = 0,\ b_3 = \frac{120\tilde{q}}{\tilde{r}}, \\
c &= \pm\frac{30\tilde{p}}{19}\sqrt{\frac{-\tilde{p}}{19\tilde{q}}},\ \lambda = 0,\ \mu = \frac{\tilde{p}}{76\tilde{q}},\ \text{if } \frac{\tilde{p}}{\tilde{q}} < 0. 
\end{aligned} \tag{12}
$$

In this situation, the solution of Equation (7) is

$$
\varphi(\eta) = b_0 + b_1[\frac{G'}{G}] + b_3[\frac{G'}{G}]^3. \tag{13}
$$

By solving Equation (9) with $\lambda = 0,\ \mu = \frac{\tilde{p}}{76\tilde{q}}$ if $\frac{\tilde{p}}{\tilde{q}} < 0$, we obtain

$$
G(\eta) = c_1\exp(\sqrt{\frac{-\tilde{p}}{76\tilde{q}}}\eta) + c_2\exp(-\sqrt{\frac{-\tilde{p}}{76\tilde{q}}}\eta), \tag{14}
$$

where $c_1$ and $c_2$ are constants. Putting Equation (14) into Equation (13), we have

$$
\begin{aligned}
\varphi(\eta) &= \pm\frac{30\tilde{p}}{19\tilde{r}}\sqrt{\frac{-\tilde{p}}{19\tilde{q}}} + \frac{90\tilde{p}}{19\tilde{r}}\sqrt{\frac{-\tilde{p}}{76\tilde{q}}}[\frac{c_1\exp(\sqrt{\frac{-\tilde{p}}{76\tilde{q}}}\eta) - c_2\exp(-\sqrt{\frac{-\tilde{p}}{76\tilde{q}}}\eta)}{c_1\exp(\sqrt{\frac{-\tilde{p}}{76\tilde{q}}}\eta) + c_2\exp(-\sqrt{\frac{-\tilde{p}}{76\tilde{q}}}\eta)}] \\
&+\frac{120\tilde{q}}{\tilde{r}}(\sqrt{\frac{-\tilde{p}}{76\tilde{q}}})^3[\frac{c_1\exp(\sqrt{\frac{-\tilde{p}}{76\tilde{q}}}\eta) - c_2\exp(-\sqrt{\frac{-\tilde{p}}{76\tilde{q}}}\eta)}{c_1\exp(\sqrt{\frac{-\tilde{p}}{76\tilde{q}}}\eta) + c_2\exp(-\sqrt{\frac{-\tilde{p}}{76\tilde{q}}}\eta)}]^3.
\end{aligned}
$$

Hence, the exact solution in this case of the S-FS-KS (1), by using (2), has the form

$$
\begin{aligned}
u_1(x,t) &= e^{(\rho\beta(t) - \frac{1}{2}\rho^2 t)}\{\pm\frac{30\tilde{p}\hbar}{19\tilde{r}} \\
&+\frac{90\tilde{p}\hbar}{19\tilde{r}}[\frac{c_1\exp(\hbar(\frac{1}{\Gamma(1+\alpha)}x^\alpha - ct)) - c_2\exp(-\hbar(\frac{1}{\Gamma(1+\alpha)}x^\alpha - ct))}{c_1\exp(\hbar(\frac{1}{\Gamma(1+\alpha)}x^\alpha - ct)) + c_2\exp(-\hbar(\frac{1}{\Gamma(1+\alpha)}x^\alpha - ct))}] \\
&+\frac{120\tilde{q}\hbar^3}{\tilde{r}}[\frac{c_1\exp(\frac{\hbar}{\Gamma(1+\alpha)}x^\alpha - c\hbar t) - c_2\exp(-\hbar(\frac{1}{\Gamma(1+\alpha)}x^\alpha - ct))}{c_1\exp(\frac{\hbar}{\Gamma(1+\alpha)}x^\alpha - c\hbar t) + c_2\exp(-\hbar(\frac{1}{\Gamma(1+\alpha)}x^\alpha - ct))}]^3\},
\end{aligned} \tag{15}
$$

where $c = \pm\frac{30\tilde{p}}{19}\sqrt{\frac{-\tilde{p}}{19\tilde{q}}}$, $\hbar = \sqrt{\frac{-\tilde{p}}{76\tilde{q}}}$ and $\frac{\tilde{p}}{\tilde{q}} < 0$.

**Second case:**

$$\begin{aligned} b_0 &= \pm\frac{30\tilde{p}}{19\tilde{r}}\sqrt{\frac{11}{19\tilde{q}}},\ b_1 = \frac{-270\tilde{p}}{19\tilde{r}},\ b_2 = 0,\ b_3 = \frac{120\tilde{q}}{\tilde{r}}, \\ c &= \pm\frac{30\tilde{p}}{19}\sqrt{\frac{11\tilde{p}}{19\tilde{q}}},\ \lambda = 0,\ \mu = \frac{-11\tilde{p}}{76\tilde{q}},\ \text{if } \frac{\tilde{p}}{\tilde{q}} > 0. \end{aligned} \tag{16}$$

In this situation, the solution of Equation (7) is

$$\varphi(\eta) = b_0 + b_1[\frac{G'}{G}] + b_3[\frac{G'}{G}]^3. \tag{17}$$

Solving Equation (9) with $\lambda = 0,\ \mu = \frac{-11p}{76\tilde{q}}$, if $\frac{\tilde{p}}{\tilde{q}} > 0$, we obtain

$$G(\eta) = c_1 \exp(\sqrt{\frac{11\tilde{p}}{76\tilde{q}}}\eta) + c_2 \exp(-\sqrt{\frac{11\tilde{p}}{76\tilde{q}}}\eta). \tag{18}$$

Substituting Equation (14) into Equation (13), we have

$$\begin{aligned} \varphi(\eta) &= \pm\frac{30\tilde{p}}{19\tilde{r}}\sqrt{\frac{11\tilde{p}}{19\tilde{q}}} - \frac{270\tilde{p}}{19\tilde{r}}\sqrt{\frac{11\tilde{p}}{76\tilde{q}}}[\frac{c_1\exp(\sqrt{\frac{11\tilde{p}}{76\tilde{q}}}\eta) - c_2\exp(-\sqrt{\frac{11\tilde{p}}{76\tilde{q}}}\eta)}{c_1\exp(\sqrt{\frac{11\tilde{p}}{76\tilde{q}}}\eta) + c_2\exp(-\sqrt{\frac{11\tilde{p}}{76\tilde{q}}}\eta)}] \\ &+ \frac{120\tilde{q}}{\tilde{r}}(\sqrt{\frac{11\tilde{p}}{76\tilde{q}}})^3[\frac{c_1\exp(\sqrt{\frac{11\tilde{p}}{76\tilde{q}}}\eta) - c_2\exp(-\sqrt{\frac{11\tilde{p}}{76\tilde{q}}}\eta)}{c_1\exp(\sqrt{\frac{11\tilde{p}}{76\tilde{q}}}\eta) + c_2\exp(-\sqrt{\frac{11\tilde{p}}{76\tilde{q}}}\eta)}]^3. \end{aligned}$$

Therefore, by using (2), the exact solution in this case of the S-FS-KS (1) has the form

$$\begin{aligned} u_2(x,t) &= e^{(\rho\beta(t) - \frac{1}{2}\rho^2 t)}\{\pm\frac{30\tilde{p}}{19\tilde{r}}\sqrt{\frac{11\tilde{p}}{19\tilde{q}}} \\ &- \frac{270\tilde{p}\hbar}{19\tilde{r}}[\frac{c_1\exp(\hbar(\frac{x^\alpha}{\Gamma(1+\alpha)} - ct)) - c_2\exp(-\hbar(\frac{x^\alpha}{\Gamma(1+\alpha)} - ct))}{c_1\exp(\sqrt{\frac{11\tilde{p}}{76\tilde{q}}}(\frac{1}{\Gamma(1+\alpha)}x^\alpha - ct)) + c_2\exp(-\hbar(\frac{x^\alpha}{\Gamma(1+\alpha)} - ct))}] \\ &+ \frac{120\tilde{q}\hbar^3}{\tilde{r}}[\frac{c_1\exp(\hbar(\frac{x^\alpha}{\Gamma(1+\alpha)} - ct)) - c_2\exp(-\hbar(\frac{x^\alpha}{\Gamma(1+\alpha)} - ct))}{c_1\exp(\hbar(\frac{x^\alpha}{\Gamma(1+\alpha)} - ct)) + c_2\exp(-\hbar(\frac{x^\alpha}{\Gamma(1+\alpha)} - ct))}]^3\}, \end{aligned} \tag{19}$$

where $c = \pm\frac{30\tilde{p}}{19}\sqrt{\frac{11\tilde{p}}{19\tilde{q}}}$, $\hbar = \sqrt{\frac{11\tilde{p}}{76\tilde{q}}}$ and $\frac{\tilde{p}}{\tilde{q}} > 0$.

**Special Cases:**

**Case 1:** If we choose $c_1 = c_2 = 1$, then Equations (15) and (19) become

$$\begin{aligned} u_1(x,t) &= e^{(\rho\beta(t) - \frac{1}{2}\rho^2 t)}[\pm\frac{30\tilde{p}}{19\tilde{r}}\sqrt{\frac{-\tilde{p}}{19\tilde{q}}} + \frac{90\tilde{p}\hbar}{19\tilde{r}}\tanh(\hbar(\frac{x^\alpha}{\Gamma(1+\alpha)} - ct)) \\ &+ \frac{120\tilde{q}\hbar^3}{\tilde{r}}\tanh^3(\hbar(\frac{x^\alpha}{\Gamma(1+\alpha)} - ct))], \end{aligned} \tag{20}$$

where $c = \pm\frac{30\tilde{p}}{19}\sqrt{\frac{-\tilde{p}}{19\tilde{q}}}$, $\hbar = \sqrt{\frac{-\tilde{p}}{76\tilde{q}}}$ and $\frac{\tilde{p}}{\tilde{q}} < 0$, and

$$\begin{aligned} u_2(x,t) &= e^{(\rho\beta(t)-\frac{1}{2}\rho^2 t)}[\pm\frac{30\tilde{p}}{19\tilde{r}}\sqrt{\frac{11\tilde{p}}{19\tilde{q}}} - \frac{270\tilde{p}\hbar}{19\tilde{r}}\tanh(\frac{\hbar x^\alpha}{\Gamma(1+\alpha)} - c\hbar t) \\ &+ \frac{120\tilde{q}}{\tilde{r}}\hbar^3\tanh^3(\frac{\hbar x^\alpha}{\Gamma(1+\alpha)} - c\hbar t)], \end{aligned} \tag{21}$$

where $c = \pm\frac{30\tilde{p}}{19}\sqrt{\frac{11\tilde{p}}{19\tilde{q}}}$, $\hbar = \sqrt{\frac{11\tilde{p}}{76\tilde{q}}}$ and $\frac{\tilde{p}}{\tilde{q}} > 0$.

**Case 2:** If we choose $c_1 = 1$ and $c_2 = -1$, then Equations (15) and (19) become

$$\begin{aligned} u_1(x,t) &= e^{(\rho\beta(t)-\frac{1}{2}\rho^2 t)}\{\pm\frac{30p}{19\tilde{r}}\sqrt{\frac{-\tilde{p}}{19\tilde{q}}} + \frac{90\tilde{p}\hbar}{19\tilde{r}}\coth(\frac{\hbar x^\alpha}{\Gamma(1+\alpha)} - c\hbar t) \\ &+ \frac{120\tilde{q}\hbar^3}{\tilde{r}}\coth^3(\frac{\hbar x^\alpha}{\Gamma(1+\alpha)} - c\hbar t))\}, \end{aligned} \tag{22}$$

where $c = \pm\frac{30\tilde{p}}{19}\sqrt{\frac{-\tilde{p}}{19\tilde{q}}}$, $\hbar = \sqrt{\frac{-\tilde{p}}{76\tilde{q}}}$ and $\frac{\tilde{p}}{\tilde{q}} < 0$, and

$$\begin{aligned} u_2(x,t) &= e^{(\rho\beta(t)-\frac{1}{2}\rho^2 t)}\{\pm\frac{30\tilde{p}}{19\tilde{r}}\hbar - \frac{270\tilde{p}\hbar}{19\tilde{r}}\coth(\frac{\hbar x^\alpha}{\Gamma(1+\alpha)} - c\hbar t) \\ &+ \frac{120\tilde{q}\hbar^3}{\tilde{r}}\coth^3(\frac{\hbar x^\alpha}{\Gamma(1+\alpha)} - c\hbar t)\}, \end{aligned} \tag{23}$$

where $c = \pm\frac{30\tilde{p}}{19}\sqrt{\frac{11\tilde{p}}{19\tilde{q}}}$, $\hbar = \sqrt{\frac{11\tilde{p}}{76\tilde{q}}}$ and $\frac{\tilde{p}}{\tilde{q}} > 0$.

**Remark 1.** *If we put $\rho = 0$ (i.e., Equation (1) without noise) and $\alpha = 1$ in Equations (20)–(23), then we obtain the same results stated in [24].*

## 5. The Influence of Noise on the S-FS-KS Solutions

Here, we discuss the influence of stochastic term on the exact solutions of the S-FS-KS Equation (1) and fix the parameters $\tilde{r} = \tilde{p} = \tilde{q} = 1$. We present a number of simulations for different values of $\rho$ (noise intensity). We utilize the MATLAB program to plot the solution $u_2(t,x)$ defined in Equation (21) for $t \in [0,5]$ and $x \in [0,6]$ as follows:

In Figures 1–3, as seen in the first graph in each figure, the surface becomes less flat when the noise intensity is equal to zero. However, when noise appears and the strength of the noise grows ($\rho = 1,2,3$), we notice that the surface becomes more planar after minor transit behaviors. This indicates that the solutions are stable due to the multiplicative noise effects.

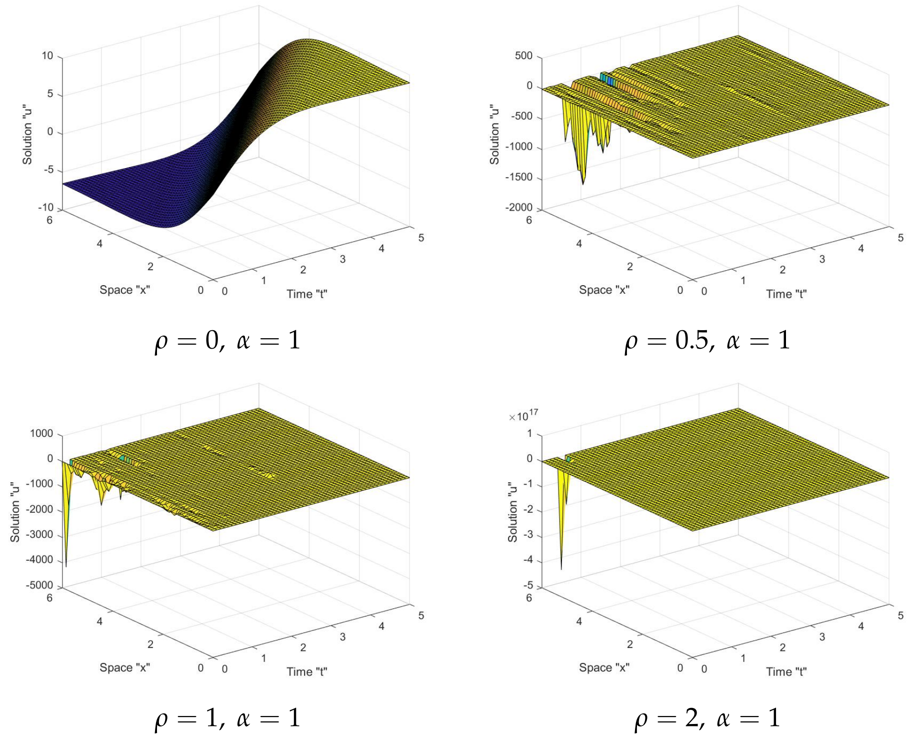


**Figure 1.** Graph of solution $u_2$ in Equation (21) with $\alpha = 1$.

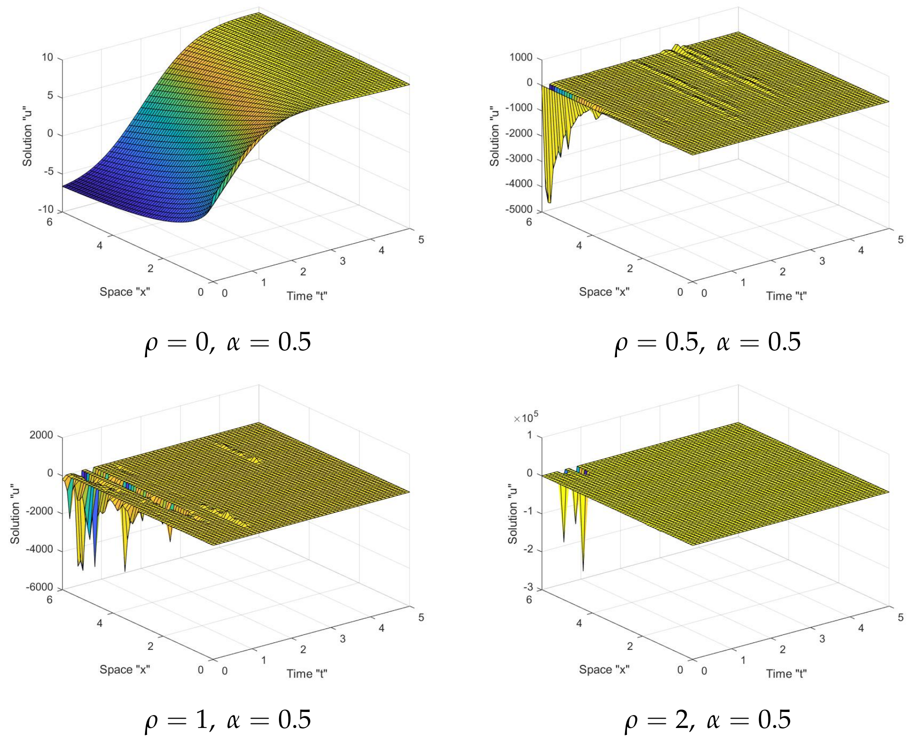


**Figure 2.** Graph of solution $u_2$ in Equation (21) with $\alpha = 0.5$.

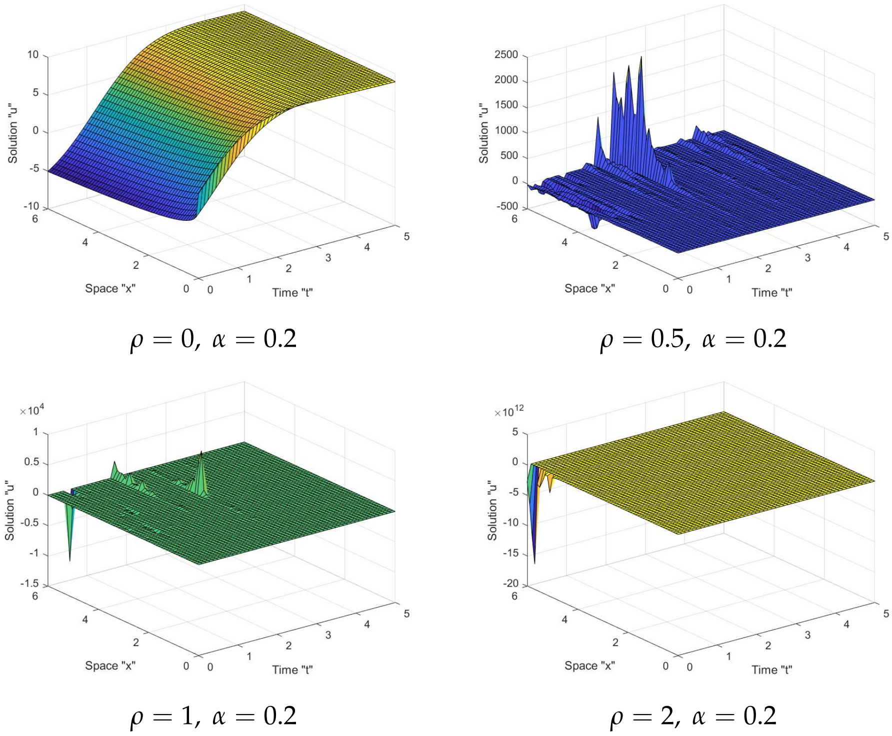


**Figure 3.** Graph of solution $u_2$ in Equation (21) with $\alpha = 0.2$.

## 6. Conclusions

In this paper, we presented different exact solutions of the stochastic fractional-space Kuramoto–Sivashinsky equation, Equation (1), forced by multiplicative noise. Moreover, several results were extended and improved such as those described in [24]. These types of solutions can be utilized to explain a variety of fascinating and complex physical phenomena. Finally, we used the MATLAB program to generate some graphical representations to show the effect of the stochastic term on the solutions of the S-FS-KS (1). In this paper, we considered the multiplicative noise and fractional space. In future work, we can consider the additive noise and fractional time.

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