mathematics

MDPI

Article

Framelet Sets and Associated Scaling Sets

Zhihua Zhang

School of Mathematics, Shandong University, Jinan 250100, China; zhangzhihua@sdu.edu.cn

Abstract: In time–frequency analysis, an increasing interest is to develop various tools to split a signal into a set of non-overlapping frequency regions without the influence of their adjacent regions. Although the framelet is an ideal tool for time–frequency analysis, most of the framelets only give an overlapping partition of the frequency domain. In order to obtain a non-overlapping partition of the frequency domain, framelet sets and associated scaling sets are introduced. In this study, we will investigate the relation between framelet (or scaling) sets and the frequency domain of framelets (or frame scaling functions). We find that the frequency domain of any frame scaling function always contains a scaling set and the frequency domain of any FMRA framelet always contains a framelet set. Moreover, we give a simple approach to construct various framelet/scaling sets from band-limited framelets and frame scaling functions.

Keywords: framelets set; frame multiresolution analysis; scaling sets

check for updates

Citation: Zhang, Z. Framelet Sets and Associated Scaling Sets. *Mathematics* **2021**, *9*, 2824. https://doi.org/10.3390/math9212824

Academic Editor: Michael Voskoglou

Received: 7 October 2021
Accepted: 27 October 2021
Published: 8 November 2021

1. Introduction

Frames are an overcomplete version of bases [1–7]. Compared with bases, the redundant representation offered by frames often demonstrates superior performances in time–frequency analysis, feature extraction, data compression and compressed sensing [8,9].

Let $\{h_n\}_1^\infty$ be a sequence in $L^2(\mathbb{R})$. If there exist $A,\ B > 0$ such that

$$A \parallel f \parallel^2 \le \sum_{n=1}^{\infty} |(f,\ h_n)|^2 \le B \parallel f \parallel^2 \qquad \forall\ f \in L^2(\mathbb{R}),$$

then $\{h_n\}_1^\infty$ is called a *frame* for $L^2(\mathbb{R})$ with bounds A and B [1,8]. Let $\psi \in L^2(\mathbb{R})$ and

$$\psi_{j,k}^{(\mu)} := 2^{\frac{jd}{2}}\, \psi^{(\mu)}(2^j \cdot - k), \qquad j \in \mathbb{Z};\ k \in \mathbb{Z}^d;\ \mu = 1, 2, \ldots, r.$$

If the affine system $\{\psi_{j,k}^{(\mu)}\}$ is a frame for $L^2(\mathbb{R})$, then the set $\Psi = \{\psi^{(1)}, \ldots, \psi^{(r)}\}$ is called a *framelet* [9,10]. Framelets are a natural extension of known wavelets. Similar to the construction of wavelets, due to the existence of fast implementation algorithms, a general approach to construct framelets is through frame multiresolution analysis (FMRA) [4–6,10]:

Let $\{V_m\}_{m\in\mathbb{Z}}$ be a sequence of subspaces of $L^2(\mathbb{R})$ such that
(i) $V_m \subset V_{m+1}\ (m \in \mathbb{Z})$, $\quad \overline{\bigcup}_{m\in\mathbb{Z}} V_m = L^2(\mathbb{R}^d)$, $\quad \bigcap_m V_m = \{0\}$;
(ii) $f(\mathbf{t}) \in V_m$ if and only if $f(2t) \in V_{m+1}\ (m \in \mathbb{Z})$;
(iii) there exists a $\varphi(\mathbf{t}) \in V_0$ such that $\{\varphi(\mathbf{t} - \mathbf{n})\}_{\mathbf{n}\in\mathbb{Z}}$ is a frame for V_0.

Then, $\{V_m\}$ is called a *frame multiresolution analysis* (FMRA) and φ is called a *frame scaling function*.

It is well-known [5,10] that the above condition (iii) can be replaced by two conditions: $V_0 = \overline{\text{span}}\{\varphi(t-n),\ n \in \mathbb{Z}\}$ and

$$\sum_k |\widehat{\varphi}(\omega + 2k\pi)|^2 = 0 \text{ or } 1. \tag{1}$$

Furthermore, for any $f \in V_0$, we have

$$f(t) = \sum_{n \in \mathbb{Z}} c_n \varphi(t-n), \text{ where } c_n = \int_{\mathbb{R}} f(t)\overline{\varphi}(t-n)\mathrm{d}t.$$

Since the suitable frequency domain of band-limited FMRAs can mitigate the effects of narrow-band noises well, the perfect reconstruction filter bank associated with a band-limited FMRA can achieve quantization noise reduction simultaneously with reconstruction of a given narrow-band signal [4]. The frequency domain of band-limited frame scaling functions can be characterized as:

Proposition 1 ([6])**.** *Let G be a bounded closed set in $\mathbb{R}$. Then, there is a frame scaling function φ with supp $\widehat{\varphi} = G$ if and only if*

$$\text{(a) } G \subset 2G, \qquad \text{(b) } \bigcup_m 2^m G = \mathbb{R}, \quad \text{and} \quad \text{(c) } \left(G \setminus \tfrac{G}{2}\right) \bigcap \left(\tfrac{G}{2} + 2\pi\mu\right) = \emptyset \ (\mu \in \mathbb{Z})$$

By the bi-scale equation of FMRA [4–6], it follows that

$$\widehat{\varphi}(2\omega) = H(\omega)\widehat{\varphi}(\omega) \ (\omega \in \mathbb{R}), \text{ where } H(\omega) \in L^2_{2\pi}.$$

Let $H_1, \ldots, H_r$ be 2π-periodic bounded functions such that

$$\sum_{\mu=0}^{r} |H_\mu(\omega)|^2 = 1, \ \sum_{\mu=0}^{r} H_\mu(\omega)\overline{H}_\mu(\omega+\pi) = 0 \qquad (\omega \in G^{per}), \tag{2}$$

where $G^{per} = G + 2\pi\mathbb{Z}$ and $H_0 = H$. Let $\{\psi_\mu\}_{\mu=1,\ldots,r}$ be such that

$$\widehat{\psi}_\mu(2\omega) = H_\mu(\omega)\widehat{\varphi}(\omega) \qquad (\mu = 1, \ldots, r). \tag{3}$$

By (1) and the matrix extension principle of FMRA [6,11], we know that $\Psi = \{\psi^{(1)}, \ldots, \psi^{(r)}\}$ is a framelet for $L^2(\mathbb{R})$. Since Ψ is generated from an FMRA, Ψ is often called an *FMRA framelet*. By (2) and (3), the FMRA framelet Ψ satisfies [9]

$$\sum_{\mu=1}^{r} |\widehat{\psi}^{(\mu)}(2\omega)|^2 = |\widehat{\varphi}(\omega)|^2 - |\widehat{\varphi}(2\omega)|^2 \ (\omega \in \mathbb{R}), \tag{4}$$

$$|\widehat{\varphi}(\omega)|^2 = \sum_{\mu=1}^{r} \sum_{j=1}^{\infty} |\widehat{\psi}^{(\mu)}(2^j\omega)| \ (\omega \in \mathbb{R}). \tag{5}$$

In time–frequency analysis, there is an increasing interest in developing various tools to split a signal into a set of non-overlapping time/frequency regions without the influence of their adjacent regions: Saito and Remy [11] proposed a new sine transform without overlaps: the polyharmonic local sine transform (PHLST). The core idea of PHLST is to segment any signal into local pieces using the characteristic functions, decompose each block into a polyharmonic component and a residual, and finally expand the residual into a sine series. Yamatani and Saito [12] used a similar approach to improve discrete cosine transform and proposed the polyharmonic local cosine transform (PHLCT). Zhang and Saito [13] improved overlapped discrete wavelet transform and proposed the polyharmonic wavelet transform. Weiss [14] first proposed the concept of the minimally supported frequency (MSF) wavelets which can split a signal into a set of non-overlapping frequency regions. The construction of MSF wavelets has been widely applied [15].

Due to their resilience to background noise, stability of sparse reconstruction, and ability to capture local time–frequency information, the framelet is a better tool for time–frequency analysis than the wavelet. Unfortunately, most of framelets only give an overlapping partition of the frequency domain [1,9,10]. In order to obtain a non-overlapping partition of the frequency domain, we introduce the concepts of framelet sets and associated scaling sets: (a) if the Fourier transform of a framelet $\Psi = \{\psi^{(1)}, \ldots, \psi^{(r)}\}$ is the characteristic function of the point sets $\Omega_1, \ldots, \Omega_r$, then the point set $\Omega = \bigcup_{k=1}^{r} \Omega_k$ is called a *framelet set* of order r; (b) if a band-limited frame scaling function φ whose Fourier transform is a characteristic function of some point set M, we call M a *scaling set*. By using (2), (3) and the splitting trick in [6,11], it is easy to construct framelet sets from scaling sets and these framelet sets can provide an overlapping partition of the frequency domain for any signal.

In this study, we will investigate the relation between framelet (or scaling) sets and the frequency domain of framelets (or frame scaling functions). In Theorems 1 and 2, we find that the frequency domain of any frame scaling function always contains a scaling set and the frequency domain of any FMRA framelet always contains a framelet set. Moreover, we give a simple approach to construct various framelet sets and scaling sets from band-limited framelets and frame scaling functions in the proof of Theorems 1 and 2.

2. Scaling Sets

In this section, we will show that for a band-limited frame scaling function φ, there exists a scaling set M such that $M \subset \mathrm{supp}\widehat{\varphi}$. For this purpose, we introduce the concept of 2π-translation kernels:

Definition 1. *Let E be a set of $\mathbb{R}$. If a set $E^* \subset E$ satisfies the conditions $E^* + 2\pi\mathbb{Z} = E + 2\pi\mathbb{Z}$ and $E^* \bigcap (E^* + 2\pi\nu) = \emptyset\,(\nu \in \mathbb{Z},\, \nu \neq 0)$, then the set E^* is called a 2π-translation kernel of E.*

We give a partition of the frequency domain $G = \mathrm{supp}\widehat{\varphi}$ as follows. Since G is bounded, there is a $k \in \mathbb{Z}_+$ such that

$$G \subset (-2^k\pi, 2^k\pi). \tag{6}$$

Let $G_j = 2^{-j}(G \setminus \frac{G}{2})\,(j = 0, 1, \ldots)$. By $\frac{G}{2} \subset G$, we have

$$G = \left(\bigcup_{j=0}^{k-1} G_j \right) \bigcup (2^{-k}G) \qquad \text{(a disjoint union)}. \tag{7}$$

By Proposition 1(iii), we have $G_j \bigcap (2^{-j-1}G + 2\pi 2^{-j}\mu) = \emptyset\,(j = 0, 1, \ldots,\, \mu \in \mathbb{Z})$. Taking $\mu = 2^j\nu$ $(\nu \in \mathbb{Z})$, we have $G_j \bigcap (2^{-j-1}G + 2\pi\nu) = \emptyset$. By $G \subset 2G$, we further obtain that

$$G_j \bigcap (2^{-j'}G + 2\pi\nu) = \emptyset \qquad (j' > j,\ \nu \in \mathbb{Z}). \tag{8}$$

From this and $G_{j'} \subset 2^{-j'}G$, we have

$$G_j \bigcap (G_{j'} + 2\pi\nu) = \emptyset \qquad (j' > j,\ \nu \in \mathbb{Z}). \tag{9}$$

By the bi-scale equation of FMRA [4–6], it follows that

$$\widehat{\varphi}(2\omega) = H(\omega)\widehat{\varphi}(\omega)\,(\omega \in \mathbb{R}), \quad H(\omega) \in L^2_{2\pi}, \tag{10}$$

where the filter $H(\omega)$ in (10) is not unique. By $G = \mathrm{supp}\widehat{\varphi}$, it follows that $\widehat{\varphi}(\omega) = 0$ for $\omega \notin G + 2\pi\mathbb{Z}$, so in (10), one can take

$$H(\omega) = 0,\ \ \omega \notin G + 2\pi\mathbb{Z}. \tag{11}$$

Lemma 1. *Let $G^{per} = G + 2\pi\mathbb{Z}$. Then, $|H(\omega)|^2 + |H(\omega + \pi)|^2 = \chi_{G^{per}}(2\omega)\,(\omega \in \mathbb{R})$, where $\chi_{G^{per}}$ is the characteristic function of G^{per}.*

Proof. By $G = \text{supp}\widehat{\varphi}$, (1) and (10), we obtain

$$\begin{aligned} \chi_{G^{per}}(\omega) &= \textstyle\sum_k |\widehat{\varphi}(\omega + 2k\pi)|^2 = \sum\limits_k |H\big(\tfrac{\omega}{2} + k\pi\big)\widehat{\varphi}\big(\tfrac{\omega}{2} + k\pi\big)|^2 \\ &= \sum\limits_k |H\big(\tfrac{\omega}{2} + 2k\pi\big)\widehat{\varphi}\big(\tfrac{\omega}{2} + 2k\pi\big)|^2 + \sum\limits_k |H\big(\tfrac{\omega}{2} + \pi + 2k\pi\big)\widehat{\varphi}\big(\tfrac{\omega}{2} + \pi + 2k\pi\big)|^2. \end{aligned}$$

Since $H(\omega)$ is 2π-periodic, we deduce that

$$\begin{aligned} \chi_{G^{per}}(\omega) &= |H\big(\tfrac{\omega}{2}\big)|^2 \sum\limits_k |\widehat{\varphi}\big(\tfrac{\omega}{2} + 2k\pi\big)|^2 + |H\big(\tfrac{\omega}{2} + \pi\big)|^2 \sum\limits_k |\widehat{\varphi}\big(\tfrac{\omega}{2} + \pi + 2k\pi\big)|^2 \\ &= |H\big(\tfrac{\omega}{2}\big)|^2 \chi_{G^{per}}\big(\tfrac{\omega}{2}\big) + |H\big(\tfrac{\omega}{2} + \pi\big)|^2 \chi_{G^{per}}\big(\tfrac{\omega}{2} + \pi\big) \qquad (\omega \in \mathbb{R}). \end{aligned} \tag{12}$$

By (10), we obtain $\text{supp}\widehat{\varphi}(2\cdot) = \text{supp}H \bigcap \text{supp}\widehat{\varphi}$, i.e.,

$$\frac{G}{2} = \text{supp}H \bigcap G.$$

From this and the periodicity of G^{per}, we have $\frac{G}{2} + 2\pi\mathbb{Z} = \text{supp}H \bigcap G^{per}$. Noticing that $\frac{G}{2} + 2\pi\mathbb{Z} \subset G + 2\pi\mathbb{Z} = G^{per}$, by (11), it follows that $\text{supp}H \subset \text{supp}\chi_{G^{per}}$, furthermore, we have $|H(\omega)|^2\chi_{G^{per}}(\omega) = |H(\omega)|^2$ $(\omega \in \mathbb{R})$. Again by (12), we deduce that

$$|H(\omega)|^2 + |H(\omega + \pi)|^2 = \chi_{G^{per}}(2\omega) \qquad (\omega \in \mathbb{R}).$$

□

Theorem 1. *Let φ be a band-limited frame scaling function with $G = supp\widehat{\varphi} \subset (-2^k\pi, 2^k\pi)$. Let*

$$\begin{aligned} &A_k = G_k = 2^{-k}(G \setminus \tfrac{G}{2}), \qquad\qquad D = \{\omega \in \mathbb{R} : |H(\omega)| = 1\}, \\ &B_{k-j} = A_{k-j} \bigcap D, \quad C_{k-j} = A_{k-j} \setminus D, \quad A_{k-j-1} = (2B_{k-j}) \bigcup (2C_{k-j})^* \ (j = 0, 1, \ldots, k-1), \end{aligned} \tag{13}$$

where $(2C_{k-j})^*$ is a 2π-translation kernel of the set $2C_{k-j}$. Denote

$$M = \left(\bigcup_{j=1}^{k} A_{k-j}\right) \bigcup \frac{G}{2^k}. \tag{14}$$

Then, M is a scaling set and $M \subset G$.

Remark 1. *Theorem 1 not only shows that the frequency domain of any band-limited frame scaling function must contain a scaling set, but also indicates how to construct a scaling set from a given band-limited frame scaling function.*

Lemma 2. *Let $\{A_{k-j}\}_{j=1,\ldots,k}$ be stated in (13). Then,*
(i) $A_{k-j} \subset G_{k-j}$ $(j = 1, \ldots, k)$ and (ii) $(A_{k-j} + 2\pi\nu) \bigcap A_{k-j'} = \varnothing$ $(j \neq j',\ \nu \in \mathbb{Z})$.

Proof. By (13), we have

$$A_{k-1} = (2B_k) \bigcup (2C_k)^* \subset 2(B_k \bigcup C_k) = 2G_k = G_{k-1},$$

$$A_{k-2} = (2B_{k-1}) \bigcup (2C_{k-1})^* \subset 2(B_{k-1} \bigcup C_{k-1}) = 2A_{k-1} \subset 2G_{k-1} \subset G_{k-2},$$

$$\vdots$$

Continuing this procedure, we obtain $A_{k-j} \subset G_{k-j}\,(j = 1, \ldots, k)$. Again, by Proposition 1(iii), we deduce that for $j \neq j'$,

$$((A_{k-j} + 2\pi\nu) \bigcap A_{k-j'}) \subset ((G_{k-j} + 2\nu\pi) \bigcap G_{k-j'}) = \emptyset \qquad (\nu \in \mathbb{Z}).$$

□

Lemma 3. *The sets $\{A_{k-j} + 2\nu\pi\}_{\nu\in\mathbb{Z}}$ are pairwise disjoint for $j = 1, \ldots, k$.*

Proof. For $j = 1$, by definition,

$$A_{k-1} + 2\pi\nu = (2B_k + 2\nu\pi) \bigcup ((2C_k)^* + 2\nu\pi). \tag{15}$$

First we prove that

$$(2B_k) \bigcap (2B_k + 2\nu\pi) = \emptyset \qquad (\nu \neq 0). \tag{16}$$

Let $\omega \in B_k$. Since $B_k \subset D$, we have $|H(\omega)| = 1$. From this and Lemma 1, we deduce that $H(\omega + \pi) = 0$. By periodicity, for $\omega \in (B_k + (2\nu + 1)\pi)\,(\nu \in \mathbb{Z})$, $H(\omega) = 0$, and so $\widehat{\varphi}(2\omega) = 0$. Again by $\mathrm{supp}\widehat{\varphi} = G$, we have

$$G \bigcap (2B_k + 4\nu\pi + 2\pi) = \emptyset \qquad (\nu \in \mathbb{Z}). \tag{17}$$

Since $2B_k \subset 2G_k = G_{k-1} \subset G$, we obtain

$$2B_k \bigcap (2B_k + 4\nu\pi + 2\pi) = \emptyset\,(\nu \in \mathbb{Z}). \tag{18}$$

By (6) and (7), it follows that $B_k \subset G_k \subset [-\pi, \pi]$; furthermore, $B_k \bigcap (B_k + 2\nu\pi) = \emptyset\,(\nu \in \mathbb{Z}, \nu \neq 0)$. So, (16) holds.

Next, we prove that

$$(2C_k)^* \bigcap (2B_k + 2\nu\pi) = \emptyset \qquad (\nu \in \mathbb{Z}). \tag{19}$$

By (6) and (7), it follows that $B_k, C_k \subset G_k \subset (-\pi, \pi)$ and $B_k \bigcap C_k = \emptyset$; furthermore, $(B_k + 2\nu\pi) \bigcap C_k = \emptyset\,(\nu \in \mathbb{Z})$. It means that $(2C_k) \bigcap (2B_k + 4\nu\pi) = \emptyset$. By $C_k \subset G_k$, we have $2C_k \subset G_{k-1} \subset G$. By (17), it follows that

$$(2C_k) \bigcap (2B_k + 4\nu\pi + 2\pi) = \emptyset. \tag{20}$$

From these, we obtain (19).

By (15), (16), (19) and Definition 1, we deduce that Lemma 3 holds for $j = 1$.

Now, we use the idea of mathematical induction to prove Lemma 3, i.e., assuming that $\{A_{k-j} + 2\nu\pi\}_{\nu\in\mathbb{Z}}$ are pairwise disjoint, we will prove that$\{A_{k-j-1} + 2\nu\pi\}_{\nu\in\mathbb{Z}}$ are pairwise disjoint.

Noticing that

$$A_{k-j-1} + 2\nu\pi = (2B_{k-j} + 2\nu\pi) \bigcup ((2C_{k-j})^* + 2\nu\pi),$$

$$(2C_k)^* \bigcap ((2C_k)^* + 2\nu\pi) = \emptyset \qquad (\nu \neq 0),$$

we only need prove that

(i) $\{2B_{k-j} + 2\nu\pi\}_{\nu\in\mathbb{Z}}$ are pairwise disjoint;

(ii) $(2B_{k-j} + 2\nu\pi) \bigcap (2C_{k-j})^* = \emptyset\ (\nu \in \mathbb{Z})$.

Since $B_{k-j} \subset D$ and $2B_{k-j} \subset G$, an argument similar to (18) shows that

$$(2B_{k-j}) \bigcap (2B_{k-j} + 4\nu\pi + 2\pi) = \emptyset \qquad (\nu \in \mathbb{Z}). \tag{21}$$

Since $B_{k-j} \subset A_{k-j}$ and $\{A_{k-j} + 2\nu\pi\}_{\nu\in\mathbb{Z}}$ are pairwise disjoint, we deduce that $\{B_{k-j} + 2\nu\pi\}_{\nu\in\mathbb{Z}}$ are pairwise disjoint. So $(2B_{k-j}) \bigcap (2B_{k-j} + 4\nu\pi) = \emptyset\,(\nu \in \mathbb{Z})$. Again by (21), we have $(2B_{k-j}) \bigcap (2B_{k-j} + 2\nu\pi) = \emptyset\,(\nu \in \mathbb{Z})$ and (i) follows.

Since $D + 2\pi\mathbb{Z} = D$, we have

$$B_{k-j} + 2\nu\pi = (A_{k-j} + 2\nu\pi) \bigcap (D + 2\nu\pi) = (A_{k-j} + 2\nu\pi) \bigcap D\,(\nu \in \mathbb{Z}).$$

By $C_{k-j} = A_{k-j} \setminus D$, we have

$$(B_{k-j} + 2\nu\pi) \bigcap C_{k-j} = \emptyset \qquad (\nu \in \mathbb{Z}). \tag{22}$$

Since $B_{k-j} \subset D$ and $2C_{k-j} \subset G$, an argument similar to (20) shows that

$$(2B_{k-j} + 4\nu\pi + 2\pi) \bigcap (2C_{k-j}) = \emptyset\,(\nu \in \mathbb{Z}).$$

From this and (22), we obtain $(2B_{k-j} + 2\nu\pi) \bigcap (2C_{k-j}) = \emptyset\,(\nu \in \mathbb{Z})$. Again, by $(2C_{k-j})^* \subset 2C_{k-j}$, we obtain (ii). Finally, by mathematical induction, we obtain Lemma 3. $\square$

Proof of Theorem 1. By the construction of M, we have

$$M + 2\nu\pi = \left(\bigcup_{j=1}^{k} (A_{k-j} + 2\nu\pi)\right) \bigcup \left(\frac{G}{2^k} + 2\nu\pi\right) \qquad (\nu \in \mathbb{Z}).$$

By Lemmas 1 and 2, the sets $\{A_{k-j} + 2\nu\pi\}_{\nu\in\mathbb{Z}}$ are pairwise disjoint. By $A_{k-\nu} \subset G_{k-\nu}$ and (8),

$$\left((A_{k-j} + 2\nu\pi) \bigcap \left(\frac{G}{2^k} + 2\nu'\pi\right)\right) \subset \left((G_{k-j} + 2\nu\pi) \bigcap \left(\frac{G}{2^k} + 2\nu'\pi\right)\right) = \emptyset \qquad (j = 1, \ldots, k;\ \nu \neq \nu').$$

Again, by (6), we have $(\frac{G}{2^k} + 2\nu\pi) \bigcap (\frac{G}{2^k} + 2\nu'\pi) = \emptyset\,(\nu \neq \nu')$. From this, we deduce that $\{M + 2\nu\pi\}_{\nu\in\mathbb{Z}}$ are pairwise disjoint.

By Proposition 1.1(ii), we have $\bigcup_m 2^m G = \mathbb{R}$. Again, by $M \supset \frac{G}{2^k}$, we deduce that

$$\mathbb{R} \supset \bigcup_m 2^m M \supset \bigcup_m 2^{m-k} G = \bigcup_m 2^m G = \mathbb{R},$$

i.e., $\bigcup_m 2^m M = \mathbb{R}$.

Finally, by (13), we have

$$A_{k-j} = (2B_{k-j+1}) \bigcup (2C_{k-j+1})^* \subset 2(B_{k-j+1} \bigcup C_{k-j+1}) = 2A_{k-j+1} \qquad (j = 2, \ldots, k).$$

Again, by (14) and $G_j = 2^{-j}(G \setminus \frac{G}{2})$, we deduce that

$$M \subset \left(\bigcup_{j=1}^{k-1} (2A_{k-j})\right) \bigcup G_{k-1} \bigcup \frac{G}{2^k} = 2\left(\bigcup_{j=1}^{k-1} A_{k-j}\right) \bigcup \left(\frac{G}{2^{k-1}}\right) \subset 2\left(\left(\bigcup_{j=1}^{k-1} A_{k-j}\right) \bigcup \frac{G}{2^k}\right) \subset 2M.$$

Define a function φ_M such that its Fourier transform is $\widehat{\varphi}_M = \chi_M$. By using all the above properties on M, it is easy to check that φ_M is a frame scaling function and M is a scaling set.

Noticing that $A_{k-j} \subset G_{k-j}$, by (13) and (14), we have

$$M = \left(\left(\bigcup_{j=1}^{k} A_{k-j}\right) \bigcup \frac{G}{2^k}\right) \subset \left(\left(\bigcup_{j=1}^{k} G_{k-j}\right) \bigcup \frac{G}{2^k}\right) = G.$$

Theorem 1 is proved. $\square$

Example 1. *For a region*

$$G = [-\frac{8}{6}\pi, -\frac{7}{6}\pi] \bigcup [-\frac{5}{6}\pi, \frac{5}{6}\pi] \bigcup [\frac{7}{6}\pi, \frac{8}{6}\pi],$$

we construct a frame scaling function φ whose Fourier transform is

$$\widehat{\varphi}(\boldsymbol{\omega}) = \begin{cases} 1, & \boldsymbol{\omega} \in [-\frac{4}{6}\pi, \frac{4}{6}\pi], \\ \frac{1}{\sqrt{2}}, & \boldsymbol{\omega} \in [-\frac{8}{6}\pi, -\frac{7}{6}\pi] \bigcup [-\frac{5}{6}\pi, -\frac{4}{6}\pi] \bigcup [\frac{4}{6}\pi, \frac{5}{6}\pi] \bigcup [\frac{7}{6}\pi, \frac{8}{6}\pi], \\ 0, & otherwise. \end{cases}$$

Taking $M = [-\frac{5}{6}\pi, \frac{5}{6}\pi]$, it is clear that M is a scaling set and $M \subset G$

3. Framelet Sets

In this study, we will show that the frequency domain of any FMRA framelets always contains a framelet set. At first, we need some lemmas.

Lemma 4. *Assume that the framelet $\Phi = \{\psi^{(1)}, \ldots, \psi^{(r)}\}$ is generated from the frame scaling function φ. Denote $G = supp\widehat{\varphi}$ and $D = \{\omega : |H(\omega)| = 1\}$. Then, $|\widehat{\varphi}(\omega)| = |\widehat{\varphi}(2\omega)|$ if and only if $\omega \in (\mathbb{R} \setminus G) \bigcup D$.*

Proof. Denote

$$\begin{aligned} S &= \{\omega : \quad |\widehat{\varphi}(\omega)| = |\widehat{\varphi}(2\omega)|\}, \\ P_1 &= \{\omega : \quad |\widehat{\varphi}(2\omega)| = |\widehat{\varphi}(\omega)| = 0\}, \\ P_2 &= \{\omega : \quad |\widehat{\varphi}(2\omega)| = |\widehat{\varphi}(\omega)| \neq 0\}. \end{aligned}$$

Then,

$$S = P_1 \bigcup P_2. \tag{23}$$

If $\omega \in P_2$, by $\widehat{\varphi}(2\omega) = H(\omega)\widehat{\varphi}(\omega)$, we have $|H(\omega)| = 1$, and so $\omega \in D$. This implies that $P_2 \subset D$. On the other hand, for $\omega \in D$, we have $|\widehat{\varphi}(2\omega)| = |H(\omega)||\widehat{\varphi}(\omega)| = |\widehat{\varphi}(\omega)|$ and so $\omega \in S$, i.e., $D \subset S$. Hence, $P_2 \subset D \subset S$. Again, by (3.1), we obtain

$$S = (P_1 \bigcup P_2) \subset (P_1 \bigcup D) \subset S.$$

This means that $S = P_1 \bigcup D$.

From $G = \mathrm{supp}\widehat{\varphi}$, it follows that $\widehat{\varphi}(2\omega) = 0\,(\omega \in \mathbb{R} \setminus \frac{G}{2})$. From this and $\frac{G}{2} \subset G$, we obtain

$$P_1 = (\mathbb{R} \setminus G) \bigcap \left(\mathbb{R} \setminus \frac{G}{2}\right) = \mathbb{R} \setminus G,$$

and so $S = (\mathbb{R} \setminus G) \bigcup D$. Lemma 4 holds. □

Lemma 5. *Under the conditions of Theorem 1, we have $\chi_M(2\omega) = \chi_M(\omega)$ for $\omega \in D$, where χ_M is the characteristic function of M.*

Proof. First, we compute $M \bigcap D$.

By $B_{k-\nu} = A_{k-\nu} \bigcap D\,(\nu = 1, \ldots, k-1)$ and (14), we deduce that

$$M \bigcap D = \left(A_0 \bigcup \left(\bigcup_{j=1}^{k-1} A_{k-j}\right) \bigcup \frac{G}{2^k}\right) \bigcap D = (A_0 \bigcap D) \bigcup \left(\bigcup_{j=1}^{k-1} B_{k-j}\right) \bigcup \left(\frac{G}{2^k} \bigcap D\right). \tag{24}$$

We compute the first term on the right-hand side of (24): for $\omega \in G_0 = G \setminus \frac{G}{2}$, we have $\widehat{\varphi}(\omega) \neq 0$ and $\widehat{\varphi}(2\omega) = 0$. By the bi-scale equation, we have $H(\omega) = 0$, and so $\omega \notin D$. This implies that $G_0 \bigcap D = \emptyset$. Again, by $A_0 \subset G_0$, we deduce that

$$A_0 \bigcap D = \emptyset. \tag{25}$$

We compute the last term $\frac{G}{2^k} \bigcap D$ on the right-hand side of (24):

$$\frac{G}{2^k} \bigcap D = \left(G_k \bigcup \frac{G}{2^{k+1}} \right) \bigcap D = (G_k \bigcap D) \bigcup \left(\frac{G}{2^{k+1}} \bigcap D \right) = B_k \bigcup \left(\frac{G}{2^{k+1}} \bigcap D \right). \tag{26}$$

By (8), we have $(\frac{G}{2^k} + 2\nu\pi) \bigcap G_j = \emptyset \, (j = 0, 1, \ldots, k-1;\ \nu \in \mathbb{Z})$. By (6), we have

$$\left(\frac{G}{2^k} + 2\nu\pi \right) \bigcap \frac{G}{2^k} = \emptyset \qquad (\nu \neq 0).$$

Again, by (7), we deduce that for $\nu \neq 0$,

$$\left(\frac{G}{2^k} + 2\nu\pi \right) \bigcap G = \left(\bigcup_{j=0}^{k-1} \left(\frac{G}{2^k} + 2\nu\pi \right) \bigcap G_j \right) \bigcup \left(\left(\frac{G}{2^k} + 2\nu\pi \right) \bigcap \frac{G}{2^k} \right) = \emptyset. \tag{27}$$

By $\operatorname{supp}\widehat{\varphi} = G$ and (1.1), we have

$$\sum_{\nu} |\widehat{\varphi}(\omega + 2\nu\pi)|^2 = \chi_{G^{per}}(\omega) \; (\omega \in \mathbb{R}),$$

where $G^{per} = \operatorname{supp}\widehat{\varphi} + 2\pi\mathbb{Z}$. By (27), we deduce that $\widehat{\varphi}(\omega + 2\nu\pi) = 0$ for $\omega \in \frac{G}{2^k}$ and $\nu \neq 0$. Furthermore,

$$\chi_{G^{per}}(\omega) = \sum_{\nu} |\widehat{\varphi}(\omega + 2\nu\pi)|^2 = |\widehat{\varphi}(\omega)|^2 \qquad (\omega \in \frac{G}{2^k}). \tag{28}$$

Again, by $G \subset 2G$, we obtain $|\widehat{\varphi}(2\omega)| = |\widehat{\varphi}(\omega)| = 1 \, (\omega \in \frac{G}{2^{k+1}})$. By the bi-scale equation $\widehat{\varphi}(2\omega) = H(\omega)\widehat{\varphi}(\omega)$, we obtain $H(\omega) = 1 \, (\omega \in \frac{G}{2^{k+1}})$, and so $\frac{G}{2^{k+1}} \subset D$. From this and (26), the last term $\frac{G}{2^k} \bigcap D$ on the right-hand side of (24) becomes

$$\frac{G}{2^k} \bigcap D = B_k \bigcup \frac{G}{2^{k+1}}. \tag{29}$$

By (24), (25) and (29), we know that

$$M \bigcap D = \left(\bigcup_{\nu=0}^{k-1} B_{k-\nu} \right) \bigcup \frac{G}{2^{k+1}}. \tag{30}$$

By (13), it follows that

$$B_k \subset G_k \subset \tfrac{G}{2^k},$$

$$B_{k-\nu} \subset A_{k-\nu} \qquad (\nu = 1, \ldots, k-1),$$

$$2B_{k-\nu} \subset A_{k-\nu-1} \qquad (\nu = 0, \ldots, k-1).$$

From these and $M = \left(\bigcup_{j=1}^{k} A_{k-j} \right) \bigcup \frac{G}{2^k}$, we see that

$$\chi_M(\omega) = \chi_M(2\omega) = 1 \, (\omega \in B_{k-\nu},\ \nu = 0, \ldots, k-1).$$

By $\frac{G}{2^{k+1}} \subset \frac{G}{2^k}$, we obtain $\chi_M(\omega) = \chi_M(2\omega) = 1$ $(\omega \in \frac{G}{2^{k+1}})$. Finally, by (30),

$$\chi_M(\omega) = \chi_M(2\omega) \qquad (\omega \in M \bigcap D). \tag{31}$$

By $\frac{M}{2} \subset M$, we know that for $\omega \notin M$, we have $\omega \notin \frac{M}{2}$, i.e.,

$$\chi_M(\omega) = \chi_M(2\omega) = 0 \qquad (\omega \notin M).$$

From this and (31), we obtain $\chi_M(\omega) = \chi_M(2\omega)$ $(\omega \in D)$. Lemma 5 is proved. □

Theorem 2. *Let $\Psi = \{\psi^{(1)}, \dots, \psi^{(r)}\}$ be a band-limited FMRA framelet corresponding to a frame scaling function φ. Then, there exists a framelet set W such that $\Omega_M \subset \Omega$, where Ω is the whole frequency domain of Ψ: $\Omega = \bigcup_{\mu=1}^r \mathrm{supp}\widehat{\psi}^{(\mu)}$.*

Proof. Since Ψ is band-limited, by (5), we know that φ is band-limited. Let $G = \mathrm{supp}\widehat{\varphi}$ and the point sets D and M be stated in Theorem 1. It is clear that $M \subset G$. Let φ_M be such that its Fourier transform satisfies $|\widehat{\varphi}_M| = \chi_M$. Since M is a scaling set, φ_M is a frame scaling function. By Lemma 5, it follows that

$$|\widehat{\varphi}_M(2\omega)| = |\widehat{\varphi}_M(\omega)| \qquad (\omega \in D). \tag{32}$$

By (4) and $\widehat{\varphi}_M = \chi_M$,

$$\sum_{\mu=1}^r |\widehat{\psi}_M^{(\mu)}(2\omega)|^2 = |\widehat{\varphi}_M(\omega)|^2 - |\widehat{\varphi}_M(2\omega)|^2 = \chi_M(\omega) - \chi_{\frac{M}{2}}(\omega) = \chi_{M \setminus \frac{M}{2}}(\omega),$$

and so $\sum_{\mu=1}^r |\widehat{\psi}_M^{(\mu)}(\omega)|^2 = \chi_{2M\setminus M}(\omega)$, i.e., $\Omega_M = 2M \setminus M$ is a framelet set. So, we have

$$Q_M(\omega) := \sum_{\mu=1}^r |\widehat{\psi}_M^{(\mu)}(2\omega)|^2 = 0 \qquad (\omega \notin M).$$

Since $M \subset G$, it follows that

$$Q_M(\omega) = 0 \qquad (\omega \in \mathbb{R} \setminus G). \tag{33}$$

By (32), $Q_M(\omega) = 0$ $(\omega \in D)$, and so

$$Q_M(\omega) = 0 \qquad (\omega \in (\mathbb{R} \setminus G) \bigcup D). \tag{34}$$

By Lemma 4 and (4), it follows that $\sum_{\mu=1}^r |\widehat{\psi}^{(\mu)}(2\omega)|^2 = 0$ if and only if $\omega \in (\mathbb{R} \setminus G) \bigcup D$. Again, by (34), we obtain that if $\sum_{\mu=1}^r |\widehat{\psi}^{(\mu)}(2\omega)|^2 = 0$, then

$$Q_M(\omega) = \sum_{\mu=1}^r |\widehat{\psi}_M^{(\mu)}(2\omega)|^2 = 0,$$

i.e.,

$$W = \bigcup_{\mu=1}^r (\mathrm{supp}\psi_M^{(\mu)}) \subset \bigcup_{\mu=1}^r (\mathrm{supp}\widehat{\psi}^{(\mu)}) = \Omega.$$

Theorem 2 is proved. □

Funding: This research was partially supported by the European Commission Horizon2020 Framework (Program No 861584) and Taishan Distinguished Professor Fund.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Mallat, S. *A Wavelet Tour of Signal Processing*; Academic Press: San Diego, CA, USA, 2008.
2. Antolin, A.S.; Zalik, R.A. Compactly supported Parseval framelets with symmetry associated to E(2)d(Z) matrices. *Appl. Math. Comput.* **2018**, *325*, 179–190.
3. Atreas, N.; Karantzas, N.; Papadakis, M.; Stavropoulos, T. On the design of multi-dimensional compactly supported Parseval framelets with directional characteristics. *Linear Algebra Appl.* **2019**, *582*, 1–36. [CrossRef]
4. Benedetto, J.J.; Li, S. The theory of multiresolution analysis frames and applications to filter banks. *Appl. Comp. Harmon. Anal.* **1998**, *5*, 389–427. [CrossRef]
5. Mu, L.; Zhang, Z.; Zhang, P. On the Higher-dimensional Wavelet Frames. *Appl. Comput. Harmon. Anal.* **2004**, *16*, 1. [CrossRef]
6. Zhang, Z. Characterization of Frequency Domains of Bandlimited Frame Multiresolution Analysis. *Mathematics* **2021**, *9*, 1050. [CrossRef]
7. Chui, C.K.; He, W. Compactly supported tight frames associated with refinable function. *Appl. Comput. Harmon. Anal.* **2000**, *8*, 293–319. [CrossRef]
8. Li, T. *Time Series with Mixed Spectra*; Chapman and Hall/CRC: Boca Raton, FL, USA, 2019. [CrossRef]
9. Zhang, Z.; Jorgenson, P.E.T. *Frame Theory in Data Science*; Springer: Heidelberg, Germany, 2022; in press.
10. Walter, G.G. *Wavelet and Other Orthogonal Systems with Applications*; CRC Press: Boca Raton, FL, USA, 2018.
11. Saito, N.; Remy, J.F. The polyharmonic local sine transform: A new tool for local image analysis and synthesis without edge effect. *Appl. Comput. Harmon. Anal.* **2006**, *20*, 41–73.
12. Yamatani, K.; Saito, N. Improvement of DCT-based compression algorithms using Poisson's equation. *IEEE Trans. Image Process.* **2006**, *15*, 672–3689. [CrossRef]
13. Zhang, Z.; Saito, N. Harmonic wavelet transform and image approximation. *J. Math. Imaging Vis.* **2010**, *38*, 14–34. [CrossRef] [PubMed]
14. Hernandez, E.; Weiss, G. *A First Course on Wavelets*; CRC Press: Boca Raton, FL, USA, 1996. [CrossRef]
15. Vyas, A.; Kim, G. Minimally Supported Frequency (MSF) d-Dilation Wavelets. *Mathematics* **2021**, *9*, 1284. [CrossRef]