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Article

New Irregular Solutions in the Spatially Distributed Fermi–Pasta–Ulam Problem

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Abstract: For the spatially-distributed Fermi–Pasta–Ulam (FPU) equation, irregular solutions are studied that contain components rapidly oscillating in the spatial variable, with different asymptotically large modes. The main result of this paper is the construction of families of special nonlinear systems of the Schrödinger type—quasinormal forms—whose nonlocal dynamics determines the local behavior of solutions to the original problem, as $t \to \infty$. On their basis, results are obtained on the asymptotics in the main solution of the FPU equation and on the interaction of waves moving in opposite directions. The problem of "perturbing" the number of N elements of a chain is considered. In this case, instead of the differential operator, with respect to one spatial variable, a special differential operator, with respect to two spatial variables appears. This leads to a complication of the structure of an irregular solution.

Keywords: Fermi–Pasta–Ulam problem; quasinormal forms; asymptotics; special distributed chains

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1. Introduction

The system of equations

$$M\frac{d^2u_j}{dt^2} = F_{j,j+1} - F_{j-1,j}, \tag{1}$$

which is named after Fermi–Pasta–Ulam, was proposed in [1]. Here,

$$F_{j-1,j} = k(\Delta l) + \alpha(\Delta l)^2 + \beta(\Delta l)^3,\ \Delta l = u_j - u_{j-1} \quad (k > 0),$$

M, k, α, β are some coefficients and the coefficients M, k are positive. We can assume that $k = 1$, the index j varies from 1 to N and the 'periodicity' conditions $u_0 \equiv u_N,\ u_{N+1} \equiv u_1$ hold. The values of $u_j(t)$ can be associated with the values of the function of two variables $u(t, x_j)$ at the uniformly distributed on some circle points with the angular coordinate x_j. The basic assumption in this paper is that the value of N is sufficiently large, i. e., the quantity $\varepsilon = 2\pi N^{-1}$ is sufficiently small:

$$\varepsilon = 2\pi N^{-1} \ll 1. \tag{2}$$

It is natural to use the continuous variable $x \in [0, 2\pi]$ instead of the discrete one x_j under the above condition. It is also convenient to preliminary normalize the time $t \to M^{1/2}\varepsilon^{-1}t$. Then, the system (1) takes the following form

$$\varepsilon^2 \frac{\partial^2 y}{\partial t^2} = y(t, x+\varepsilon) - 2y(t, x) + y(t, x-\varepsilon) + \\ + \alpha\Big[y^2(t, x+\varepsilon) - 2y(t, x)y(t, x+\varepsilon) + 2y(t, x)y(t, x-\varepsilon) - y^2(t, x-\varepsilon)\Big] + \\ + \beta\Big[(y(t, x+\varepsilon) - y(t, x))^3 - (y(t, x) - y(t, x-\varepsilon))^3\Big] \quad (3)$$

and the periodicity condition

$$y(t, x+2\pi) \equiv y(t, x) \quad (4)$$

holds.

The Equation (3) has been studied by many authors (see, for example, [1–11]) with the main focus on the so-called regular solutions. We recall that regular solutions were distinguished by the condition of 'good' dependence on the parameter ε. The asymptotic representation

$$u(t, x+\varepsilon) = u(t, x) + \varepsilon \frac{\partial}{\partial x} u(t, x) + \frac{1}{2}\varepsilon^2 \frac{\partial^2}{\partial x^2} u(t, x) + \dots \quad (5)$$

holds for these solutions. The transition from the Equation (3) to a special nonlinear partial differential equation was made to study regular solutions with a certain degree of accuracy, with respect to the parameter ε. The basic results mainly concerned the problems of finding the exact solutions and revealing the integrability properties of the obtained partial differential equations.

We also note that interesting results, in the wave interactions in such equations, were obtained in [11].

We emphasize once again that the value of N determines the parameter ε in (3). We are interested in the study of influence of this value on the asymptotics of the solutions. Let the number of elements in (1) be equal to $N + c$ where c is an arbitrary fixed integer value. Let $\mu = 2\pi(N + c)^{-1}$. Then, the parameter μ appears in the Equation (3) instead of the parameter ε:

$$\varepsilon^2 \frac{\partial^2 y}{\partial t^2} = y(t, x+\mu) - 2y(t, x) + y(t, x-\mu) + \\ + \alpha\Big[y^2(t, x+\mu) - 2y(t, x)y(t, x+\mu) + 2y(t, x)y(t, x-\mu) - y^2(t, x-\mu)\Big] + \\ + \beta\Big[(y(t, x+\mu) - y(t, x))^3 - (y(t, x) - y(t, x-\mu))^3\Big]. \quad (6)$$

We have the asymptotic formula

$$\mu = \varepsilon\Big(1 + \varepsilon c(2\pi)^{-1}\Big)^{-1} = \varepsilon\bigg(1 - \frac{c\varepsilon}{2\pi} + \frac{c^2\varepsilon^2}{4\pi^2} + \dots\bigg) \quad (7)$$

for μ.

By $E_\pm(t, x, \varepsilon)$ we denote the functions

$$E_\pm(t, x, \varepsilon) = (\delta + \varepsilon\Theta)x \pm \bigg(2\sin\frac{\delta}{2} + \varepsilon\bigg(\Theta - \frac{\delta c}{\pi}\cos\frac{\delta}{2}\bigg)\bigg)t.$$

In this paper, we study the irregular solutions to the boundary value problem (4) and (6). The structure of such solutions consists of the superposition of functions that depend smoothly (regularly) on the parameter ε as well as the functions that depend smoothly on the parameter ε^{-1}. We dwell on this in more detail.

First of all, we note that any identically constant function $u_0(t, x) \equiv const.$ is an equilibrium state of the boundary value problem (4) and (6). We investigate the local behavior of solutions in the vicinity of each equilibrium state mentioned above. Thus, we study the solutions with sufficiently small and ε-independent deviations of their 2π—

periodic and continuous with respect to x initial conditions $u(0,x) = \varphi_1(x)$, $\frac{\partial u}{\partial t}(0,x) = \varphi_2(x)$ and $\max_x(|\varphi_1(x) - \mathrm{const}\,| + |\varphi_2(x)|) \ll 1$. The linearized on equilibrium states boundary value problem

$$\varepsilon^2 \frac{\partial^2 u}{\partial x^2} = u(t, x+\mu) - 2u(t,x) + u(t, x-\mu),\ u(t, x+2\pi) \equiv u(t,x) \tag{8}$$

plays a prominent part in the study of the local behavior of the solutions. In turn, the structure of solutions to the boundary value problem (8) is determined by the location of the roots of the characteristic equation

$$\varepsilon^2\lambda^2 = -4\sin^2\left(\frac{1}{2}\varepsilon\left(1 - \frac{\varepsilon c}{2\pi} + \frac{\varepsilon^2 c^2}{4\pi^2} + \ldots\right)k\right),\ k = 0, \pm 1, \pm 2, \ldots. \tag{9}$$

We consider now the asymptotic behavior of the roots of (9) for the sufficiently large values k of order ε^{-1}.

First of all, we fix arbitrarily the value $\delta > 0$ and assume that

$$\delta \neq 2\pi n \quad (n = 1, 2, \ldots). \tag{10}$$

Below, we denote by $\Theta = \Theta(\delta, \varepsilon) \in [0, 1)$ the value that complements $\delta\varepsilon^{-1}$ to an integer expression.

Let K_ε be a set of integers that are given by the formal relation

$$K_\varepsilon = \left\{\delta\varepsilon^{-1} + \Theta + 2\pi n\varepsilon^{-1} + m;\ m, n = 0, \pm 1, \pm 2, \ldots\right\}.$$

We note that the value $2\pi n\varepsilon^{-1}$ is an integer by virtue of the equality $\varepsilon = 2\pi N^{-1}$.

Below, we consider the question of the solutions to the boundary value problem (4) and (6), the formation of which is based on modes with numbers from K_ε. To find the main parts of such solutions, special partial differential equation systems are going to be obtained, which are systems of two coupled nonlinear Schrödinger equations.

Each element of the set K_ε corresponds to the value of the root $\lambda^{\pm}_{m,n}(\varepsilon)$ of the characteristic Equation (9), and

$$\varepsilon^2\left(\lambda^{\pm}_{m,n}(\varepsilon)\right)^2 = -4\sin^2\left[\frac{1}{2}\varepsilon\left(1 - \frac{\varepsilon c}{2\pi} + \frac{\varepsilon^2 c^2}{4\pi^2} + \ldots\right)\left(\delta\varepsilon^{-1} + \Theta + m + 2\pi n\varepsilon^{-1}\right)\right].$$

From here,

$$\begin{aligned}\lambda^{\pm}_{m,n}(\varepsilon) = \pm 2i\varepsilon^{-1}\Bigg[&\sin\frac{\delta}{2} + \frac{1}{2}\varepsilon\left(\Theta + m - \frac{\delta c}{\pi} - cn\right)\cos\frac{\delta}{2} + \\ &+\varepsilon^2\frac{1}{4}\left(\left(\frac{\delta c^2}{2\pi^2} - \frac{c(\Theta+m)}{\pi} + \frac{nc^2}{\pi}\right)\cos\frac{\delta}{2} + \left(\frac{\delta c}{\pi} - (\Theta+m) + cn\right)^2\sin\frac{\delta}{2}\right)\Bigg] + O\left(\varepsilon^3\right).\end{aligned} \tag{11}$$

Each root $\lambda^{\pm}_{m,n}(\varepsilon)$ corresponds to the linear boundary value problem (8) solution

$$u^{\pm}_{m,n}(t, x, \varepsilon) = \exp\left[i\varepsilon^{-1}(\delta + \varepsilon(\Theta + m) + 2\pi n)x + \lambda^{\pm}_{m,n}(\varepsilon)t\right].$$

This means that the same boundary value problem has the set of solutions

$$u(t, x, \varepsilon) = \sum_{m,n=-\infty}^{\infty} \xi_{m,n\pm} u^{\pm}_{m,n}(t, x, \varepsilon). \tag{12}$$

We introduce some notation in order to significantly simplify this expression. Let

$$x_{\pm} = x \pm t\cos\frac{\delta}{2},\ y = 2\pi\varepsilon^{-1}x,\ y_{\pm} = y \mp ct\cos\frac{\delta}{2}. \tag{13}$$

Then, (12) transforms to the form $u(t,x,\varepsilon) = u^{+}(t,x,\varepsilon) + u^{-}(t,x,\varepsilon)$, and

$$u^{\pm}(t,x,\varepsilon) = \exp\Big(i\varepsilon^{-1}E_{\pm}(t,x,\varepsilon)\Big) \times$$
$$\times \sum_{m,n=-\infty}^{\infty} \xi_{m,n\pm} \exp\Bigg(i\Bigg(mx_{\pm} + ny_{\pm} \pm \frac{i\varepsilon}{2}\Bigg(\Bigg(\frac{\delta c^2}{2\pi^2} - \frac{(\Theta+m)c}{\pi} + \frac{nc^2}{\pi}\Bigg)\cos\frac{\delta}{2} +$$
$$+\Bigg(\frac{\delta c}{\pi} - (\Theta+m) + cn\Bigg)^2 \sin\frac{\delta}{2}\Bigg) + O(\varepsilon)\Bigg)t\Bigg) =$$
$$= \xi_{+}(\tau, x_{+}, y_{+})\exp\Big(i\varepsilon^{-1}E_{+}(t,x,\varepsilon)\Big) + \xi_{-}(\tau, x_{-}, y_{-})\exp\Big(i\varepsilon^{-1}E_{-}(t,x,\varepsilon)\Big)$$

where $\tau = \varepsilon t$, $\xi_{\pm}(\tau, x_{\pm}, y_{\pm}) = \sum\limits_{m,n=-\infty}^{\infty} \xi_{m,n\pm}(\tau)\exp(imx_{\pm} + iny_{\pm})$,

$$\xi_{m,n\pm}(\tau) = \xi_{m,n\pm}\exp\Bigg(\pm\frac{i}{2}\Bigg(\Bigg(\frac{\delta c^2}{2\pi^2} - \frac{(\Theta+m)c}{\pi} + \frac{nc^2}{\pi}\Bigg)\cos\frac{\delta}{2} +$$
$$+\Bigg(\frac{\delta c}{\pi} - (\Theta+m) + cn\Bigg)^2 \sin\frac{\delta}{2} + O(\varepsilon)\Bigg)\tau\Bigg).$$

In the next section, we formulate the basic result from which follows that the nonlinear boundary value problem (4) and (6) has a set of irregular solutions whose basic terms of the asymptotics are determined by the expression

$$u(t,x,\varepsilon) = \varepsilon\Big(\xi_{+}(\tau, x_{+}, y_{+})\exp\Big(i\varepsilon^{-1}E_{+}(t,x,\varepsilon)\Big) + \overline{cc} +$$
$$+\xi_{-}(\tau, x_{-}, y_{-})\exp\Big(i\varepsilon^{-1}E_{-}(t,x,\varepsilon)\Big) + \overline{cc} + O(\varepsilon^2)\Big).$$

Below, a special system of coupled nonlinear Schrödinger equations is presented to determine the unknown amplitudes $\xi_{\pm}(\tau, x_{\pm}, y_{\pm})$. Here and below we denote by $\overline{cc}$ the expressions that are complex conjugate to the previous term.

The justification of this result is provided in Section 2.2. We separately consider the case when the equality

$$\delta = 2\pi k_0 \tag{14}$$

holds for some integer k_0 in Section 2.3. We note—right away—that this case differs significantly from the case of (10).

2. Results

2.1. Basic Result

Firstly, we introduce some notation. We denote by D, J and J_0 operators, which are defined on the continuously differentiable functions $v(x,y)$ of two variables x and y, according to the rules

$$\begin{aligned} Dv(x,y) &= \frac{\partial v}{\partial x} - c\frac{\partial v}{\partial y}, \\ Jv(x,y) &= \int_0^x v(s, cx+y-cs)ds, \\ J_0v(x,y) &= \frac{1}{2\pi}\int_0^{2\pi} v(s, cx+y-cs)ds. \end{aligned}$$

The above imply the following relations

$$Dv(cx+y)=0,\ D(J-J_0)v(x,y)=DJv(x,y)=v(x,y), \tag{15}$$

which are required below.

We denote by $W_2\big(\tau,t,x_\pm,y_\pm,\varepsilon^{-1}(\delta+\varepsilon\Theta)x\big)$ the function

$$\begin{aligned} w_{21}(\xi_+(\tau,x_+,y_+))^2\exp(2E_+)+\overline{cc}+w_{21}(\xi_-(\tau,x_-,y_-))^2\exp(2E_-)+\overline{cc}+\\ +w_{23}\xi_+(\tau,x_+,y_+)\xi_-(\tau,x_-,y_-)\exp(E_++E_-)+\overline{cc}+\\ +w_{24}\xi_+(\tau,x_+,y_+)\overline{\xi_-}(\tau,x_-,y_-)\exp(E_+-E_-)+\overline{cc}+\\ +\xi_+(\tau,x_+,y_+)f_+(\tau,x,y)\exp(E_+)+\overline{cc}+\xi_-(\tau,x_-,y_-)f_-(\tau,x,y)\exp(E_-)+\overline{cc} \end{aligned}$$

where

$$w_{21}=i\alpha ctg\frac{\delta}{2},\ w_{23}=2itg\frac{\delta}{2},\ w_{24}=i\sin\delta, \tag{16}$$

$$\begin{aligned} f_+(\tau,x,y)&=i\sigma_1(4\sin\delta)^{-1}(J-J_0)|\xi_-(\tau,x,y)|^2,\\ f_-(\tau,x,y)&=-i\sigma_1(4\sin\delta)^{-1}(J-J_0)|\xi_+(\tau,x,y)|^2. \end{aligned} \tag{17}$$

Let us formulate the basic result. We consider the boundary value problem

$$\begin{aligned} 2i\frac{\partial\xi_+}{\partial\tau}=\sin\left(\frac{\delta}{2}\right)D^2\xi_+-2i\Theta\sin\left(\frac{\delta}{2}\right)D\xi_++\Theta^2\sin\left(\frac{\delta}{2}\right)\xi_++\\ +8\sin\left(\frac{\delta}{2}\right)\xi_+\left[\sigma_0|\xi_+|^2+\sigma_1J_0|\xi_-|^2\right], \end{aligned} \tag{18}$$

$$\begin{aligned} -2i\frac{\partial\xi_-}{\partial\tau}=\sin\left(\frac{\delta}{2}\right)D^2\xi_--2i\Theta\sin\left(\frac{\delta}{2}\right)D\xi_-+\Theta^2\sin\left(\frac{\delta}{2}\right)\xi_-+\\ +8\sin\left(\frac{\delta}{2}\right)\xi_-\left[\sigma_0|\xi_-|^2+\sigma_1J_0|\xi_+|^2\right], \end{aligned} \tag{19}$$

$$\xi_\pm(\tau,x+2\pi,y)\equiv\xi_\pm(\tau,x,y+2\pi)\equiv\xi_\pm(\tau,x,y) \tag{20}$$

where

$$\begin{aligned} \sigma_0&=\alpha^2\cos\left(\frac{\delta}{2}\right)+6\beta\sin\left(\frac{\delta}{2}\right),\\ \sigma_1&=-\sin\left(\frac{\delta}{2}\right)2\left[\left(1+2\sin^2\frac{\delta}{2}\right)\alpha^2+6\beta\sin^2\left(\frac{\delta}{2}\right)\right]. \end{aligned}$$

Below, we use $\varepsilon_k(\Theta_0)\quad(k=k_0,k_0+1,\dots)$ for a sequence such that $\varepsilon_k(\Theta_0)\to0$, as $k\to\infty$. The value of Θ does not change on this sequence: $\Theta(\varepsilon_k(\Theta_0))=\Theta_0$.

Theorem 1. *We fix the arbitrarily positive value $\delta\neq\pi k$ $(k=0,1,2,\dots)$, and $\Theta_0\in[0,1)$. Let $(\xi_+(\tau,x,y),\xi_-(\tau,x,y))$ be the solution of the boundary value problem (18)–(20), which is bounded together with the derivative, with respect to τ, and with the first and second derivatives, with respect to x and y for $\tau\to\infty$, $x\in[0,2\pi]$, $y\in[0,2\pi]$ as $\Theta=\Theta_0$. Then, the function*

$$\begin{aligned} u(t,x,\varepsilon_k)=\varepsilon_k(\Theta_0)\Big(\xi_+(\tau,x_+,y_+)\exp(E_+)+\overline{cc}+\xi_-(\tau,x_-,y_-)\exp(E_-)+\overline{cc}\Big)+\\ +\varepsilon_k^2(\Theta_0)W_2\Big(\tau,t,x_\pm,y_\pm,\varepsilon^{-1}(\delta+\varepsilon\Theta)x\Big) \end{aligned} \tag{21}$$

satisfies the boundary value problem (4) and (6) up to $O(\varepsilon_k^3(\Theta_0))$ as $\tau=\varepsilon_k(\Theta_0)t$, $x_\pm=x\pm t\cos\left(\frac{\delta}{2}\right)$, $y_\pm=y\mp ct\cos\left(\frac{\delta}{2}\right)$, $y=2\pi\varepsilon_k^{-1}(\Theta_0)x$.

In terms of this result, we note two circumstances. Firstly, the same values of the arguments τ, x, y in both equations of the boundary value problems (18)–(20) and (17) and the arguments of the functions $\xi_\pm(\tau, x_\pm, y_\pm)$ in the formula (21) are different. We succeeded to bring (18)–(20) and (17) to the same arguments due to the fact that the nonlinear expression $\left(J|\xi_-|^2 - J_0|\xi_-|^2\right)$ in (17) and the nonlinear expression $\left(J|\xi_+|^2 - J_0|\xi_+|^2\right)$ depend on the argument $cx + y$. This fact, in turn, follows from the equalities (13) and (15): $cx + y = cx_+ + y_+ = cx_- + y_-$.

Secondly, using Lyapunov transformations, the boundary value problem can be simplified by 'removing' the terms $\Theta^2 \sin\left(\frac{\delta}{2}\right)\xi_\pm$ and $16\sigma_1 \sin\left(\frac{\delta}{2}\right)\xi_\pm J_0|\xi_\mp|^2$ from it. To do this, we set

$$\xi_\pm(\tau, x, y) \exp\left[\pm i\sigma_1 8 \sin\left(\frac{\delta}{2}\right) \int\limits_0^\tau J_0|\xi_\mp(s, x, y)|^2 ds \mp \frac{i}{2}\Theta^2 \sin\left(\frac{\delta}{2}\right)\tau\right] = \eta_\pm(\tau, x, y) \quad (22)$$

in (19) and (20). As a result, we obtain the 'split' boundary value problem

$$\pm 2i\frac{\partial \eta_\pm}{\partial \tau} = \sin\left(\frac{\delta}{2}\right) D^2\eta_\pm - 2i\Theta \sin\left(\frac{\delta}{2}\right) D\eta_\pm + \\ + 16\sigma_0 \sin\left(\frac{\delta}{2}\right)\eta_\pm|\eta_\pm|^2, \quad \eta(\tau, x + 2\pi, y) \equiv \eta(\tau, x, y + 2\pi) \equiv \eta(\tau, x, y). \quad (23)$$

From (22) it follows that $\xi_\pm(\tau, x_\pm, y_\pm)$ are expressed in terms of $\eta_\pm(\tau, x_\pm, y_\pm)$ using the formula

$$\xi_\pm(\tau, x, y) = \eta_\pm(\tau, x, y) \exp\left[\mp i\sigma_1 8 \sin\left(\frac{\delta}{2}\right) \int\limits_0^\tau J_0|\eta_\mp(s, x, y)|^2 ds \pm \frac{i}{2}\Theta^2 \sin\left(\frac{\delta}{2}\right)\tau\right].$$

We note that the possibility of these substitutions is due to the equalities (15).

2.2. *Justification of Theorem 1*

We use the technique developed in [11–15]. It is based on the assumption that a certain set of solutions to the boundary value problem (4) and (6) can be represented as an asymptotic series

$$u(t, x, \varepsilon) = \varepsilon\left(\xi_+(\tau, x_+, y_+) \exp\left(i\varepsilon^{-1}E_+\right) + \overline{cc} + \xi_-(\tau, x_-, y_-) \exp\left(i\varepsilon^{-1}E_1\right) + \overline{cc}\right) + \\ + \varepsilon^2 W_2\left(\tau, t, x_\pm, y_\pm, \varepsilon^{-1}(\delta + \varepsilon\Theta)x\right) + \varepsilon^3 W_3\left(\tau, t, x_\pm, y_\pm, \varepsilon^{-1}(\delta + \varepsilon\Theta)x\right) + \dots . \quad (24)$$

We substitute (24) into the boundary value problem (4) and (6) and equate the coefficients of the same powers of ε on the left and right parts of the resulting formal identity. We obtain the correct equality for ε^1 and arrive at the relation

$$\varepsilon^2 \frac{\partial W_2}{\partial t^2} - W_2\left(\tau, t, x_\pm, y_\pm, \varepsilon^{-1}(\delta + \varepsilon\Theta)x + \delta\right) + 2W_2 - W_2\left(\tau, t, x_\pm, y_\pm, \varepsilon^{-1}(\delta + \varepsilon\Theta)x - \delta\right) = \\ = 2i(\sin 2\delta - 2\sin\delta)\Big[\xi_+^2 \exp\left(i2\varepsilon^{-1}E_+\right) - \overline{\xi_+^2} \exp\left(-2i\varepsilon^{-1}E_+\right) + \\ + \xi_-^2 \exp\left(2i\varepsilon^{-1}E_-\right) - \overline{\xi_-^2} \exp\left(-2i\varepsilon^{-1}E_-\right) + 2\xi_+\xi_- \exp\left(i\varepsilon^{-1}(E_+ + E_-)\right) - \\ - 2\overline{\xi_+}\,\overline{\xi_-} \exp\left(-i\varepsilon^{-1}(E_+ + E_-)\right) + 2\xi_+\overline{\xi_-} \exp\left(i\varepsilon^{-1}(E_+ - E_-)\right) - \\ - 2\overline{\xi_+}\xi_- \exp\left(i\varepsilon^{-1}(-E_+ + E_-)\right)\Big] \quad (25)$$

for ε^2. Therefore, the function W_2 is sought in the form

$$W_2 = W_{21}\xi_+^2 \exp\Big(2i\varepsilon^{-1}E_+\Big) + \overline{cc} + W_{21}\xi_-^2 \exp\Big(2i\varepsilon^{-1}E_-\Big) + \overline{cc} + \\ + W_{23}\xi_+\xi_- \exp\Big(i\varepsilon^{-1}(E_+ + E_-)\Big) + \overline{cc} + W_{24}\xi_+\overline{\xi_-} \exp\Big(i\varepsilon^{-1}(E_+ - E_-)\Big) + \overline{cc} + \\ + f_+(\tau,x,y)\exp\Big(i\varepsilon^{-1}E_+\Big) + \overline{cc} + f_-(\tau,x,y)\exp\Big(i\varepsilon^{-1}E_-\Big) + \overline{cc}. \quad (26)$$

From this, we obtain the equalities (16) at once. We do not define the functions $f_\pm(\tau,x,y)$ at this step.

Then, we equate the coefficients at ε^3. As a result, we obtain the equality

$$\varepsilon^2\frac{\partial^2 W_3}{\partial t^2} - W_3\Big(\tau,t,x_\pm,y_\pm,\varepsilon^{-1}(\delta+\varepsilon\Theta)x+\delta\Big) + 2W_3 - \\ - W_3\Big(\tau,t,x_\pm,y_\pm,\varepsilon^{-1}(\delta+\varepsilon\Theta)x-\delta\Big) = B_+(\tau,t,x_+,y_+)\exp\Big(i\varepsilon^{-1}E_+\Big) + \overline{cc} + \\ + B_-(\tau,t,x_-,y_-)\exp\Big(i\varepsilon^{-1}E_-\Big) + \overline{cc} + B_0\Big(\tau,t,x_\pm,y_\pm,\varepsilon^{-1}(\delta+\varepsilon\Theta)x\Big) \quad (27)$$

where the last term is the sum of some coefficients with exponents $\pm 2E_\pm$, $\pm(E_+ + E_-)$, $\pm(E_+ - E_-)$, $\pm 3E_\pm$, $\pm(2E_\pm \pm E_\mp)$. The functions of $t,\tau,x_\pm,y_\pm$ are 2π-periodic with respect to $x_\pm$, $y_\pm$ and $\pi\Big(\sin\frac{\delta}{2}\Big)^{-1}$-periodic with respect to t coefficients at these exponents. Let $W_3 = W_{30} + W_{31}$ in (27). The function W_{30} is the solution to the equation

$$\varepsilon^2\frac{\partial^2 W_{30}}{\partial t^2} - W_{30}\Big(\tau,t,x_\pm,y_\pm,\varepsilon^{-1}(\delta-\varepsilon\Theta)x+\delta\Big) + 2W_{30} - \\ - W_{30}\Big(\tau,t,x_\pm,y_\pm,\varepsilon^{-1}(\delta+\varepsilon\Theta)x-\delta\Big) = B_0\Big(\tau,t,x_\pm,y_\pm,\varepsilon^{-1}(\delta+\varepsilon\Theta)x\Big). \quad (28)$$

It has the same structure as the B_0 function and is explicitly defined by (28). We do not present its explicit form here as unnecessary.

It remains to consider the equation for W_{31}:

$$\varepsilon^2\frac{\partial^2 W_{31}}{\partial t^2} - W_{31}\Big(\tau,t,x_\pm,y_\pm,\varepsilon^{-1}(\delta+\varepsilon\Theta)x+\delta\Big) + 2W_{31} - \\ - W_{31}\Big(\tau,t,x_\pm,y_\pm,\varepsilon^{-1}(\delta+\varepsilon\Theta)x-\delta\Big) = B_+(\tau,t,x_+,y_+)\exp\Big(i\varepsilon^{-1}E_+\Big) + \overline{cc} + \\ + B_-(\tau,t,x_-,y_-)\exp\Big(i\varepsilon^{-1}E_-\Big) + \overline{cc} \quad (29)$$

where

$$B_\pm(\tau,x,t) = \pm 2i\sin\Big(\frac{\delta}{2}\Big)\frac{\partial\xi_\pm}{\partial\tau} + \Big(\cos^2\Big(\frac{\delta}{2}\Big) - \cos\delta\Big)D^2\xi_\pm + \\ + 2i\Theta\Big(\cos^2\Big(\frac{\delta}{2}\Big) - \cos\delta\Big)D\xi_\pm + \Theta^2\Big(\cos\delta - \cos^2\Big(\frac{\delta}{2}\Big)\Big)\xi_\pm + \\ + \xi_\pm\Big[\sigma_0|\xi_\pm|^2 + \sigma_1|\xi_\mp|^2\Big] + 2i\sin\Big(\frac{\delta}{2}\Big)\xi_\pm D f_\pm. \quad (30)$$

The Equation (29) has a solution in the indicated class of function under the condition

$$B_+(\tau,x,t) \equiv B_-(\tau,x,t) \equiv 0 \quad (31)$$

only. Each of these equalities contains the unknown functions $f_\pm(\tau,x,y)$. We choose these functions in such a way as to simplify the corresponding expressions $B_\pm$ as much as possible. The dependence of B_+ only on τ, x_+, y_+ and of B_- only on τ, x_-, y_- define this simplification. From the above and from (30) and (31) the equalities

$$\mp 2iDf_\pm = \sigma_1\Big(|\xi_\mp|^2 - J_0|\xi_\mp|^2\Big)$$

arise. We obtain (17) from them. Considering these formulas in (26), we obtain the resulting expressions (18) and (19).

The theorem is proved.

2.3. Case of $\delta = 2\pi n_0$ Results

The set of integers K_ε has the form

$$K_\varepsilon = \left\{ m + 2\pi n \varepsilon^{-1};\ m, n = 0, \pm 1, \pm 2, \dots \right\}$$

in this case. The asymptotic equalities

$$\lambda^{\pm}_{m,n}(\varepsilon) = \pm i \left[\varepsilon \left(1 - \frac{\varepsilon c}{2\pi} \right)(m - nc) - \frac{1}{6}\varepsilon^3 (m - nc)^3 \right] + \dots$$

hold for the roots $\lambda^{\pm}_{m,n}(\varepsilon)$ of the characteristic Equation (9).

Based on the structure of the solutions to the linearized boundary value problem (8) with modes from K_ε, we seek solutions to the nonlinear boundary value problem (4) and (6) in the form

$$u(t, x, \varepsilon) = \varepsilon(\xi_+(\tau, x_+, y_+) + \xi_-(\tau, x_-, y_-)) + \\ + \varepsilon^2 (W_{2+}(\tau, x_+, y_+) + W_{2-}(\tau, x_-, y_-) + W_{20}(\tau, x_\pm, y_\pm)) + \dots \quad (32)$$

where $\tau = \varepsilon^2 t$, $x_\pm = x \pm t$, $y_\pm = y \pm ct$.

We substitute (32) into (6) and equate the coefficients of the same powers of ε. We obtain the correct equality for ε^1. At the next step, we arrive at the equation for W_{20}, $W_{2\pm}$. We find out from it that

$$W_{20} = -\frac{1}{2}\alpha D(\xi_+ \xi_-).$$

From the condition of solvability of the equations, with respect to $W_{2\pm}$, we obtain the relations for $\xi_\pm(\tau, x, y)$:

$$\pm 2 \frac{\partial}{\partial \tau} D\xi_\pm = D^4 \xi_\pm - 2\alpha D\xi_\pm D^2 \xi_\pm, \quad (33)$$

$$\xi_\pm(\tau, x + 2\pi, y) \equiv \xi_\pm(\tau, x, y + 2\pi) \equiv \xi_\pm(\tau, x, y). \quad (34)$$

Hence, the resulting statement follows:

Theorem 2. *Let the boundary value problem (33) and (34) has a bounded as $\tau \to \infty$, $x \in [0, 2\pi]$, $y \in [0, 2\pi]$ solution $\xi_\pm(\tau, x, y)$ that is continuously differentiable, with respect to τ, and four times continuously differentiable, with respect to x and y. Then, the function*

$$u(t, x, \varepsilon) = \varepsilon(\xi_+(\tau, x_+, y_+) + \xi_-(\tau, x_-, y_-)) - \varepsilon^2 \frac{1}{2}\alpha D(\xi_+(\tau, x_+, y_+)\xi_-(\tau, x_-, y_-))$$

satisfies the boundary value problem (4) and (6) up to $O(\varepsilon^3)$.

We note that the Equations (33) can be simplified. Let

$$\eta_\pm(\tau, x_\pm, y_\pm) = D\xi_\pm(\tau, x_\pm, y_\pm).$$

As a result, we obtain the boundary value problem

$$\pm 2 \frac{\partial \eta_\pm}{\partial \tau} = D^3 \eta_\pm - 2\alpha \eta_\pm D\eta_\pm,\ \eta(\tau, x + 2\pi, y) = \eta(\tau, x, y + 2\pi) \equiv \eta(\tau, x, y). \quad (35)$$

It is natural to call this boundary value problem the Korteweg–de Vries equation in a two-dimensional spatial domain.

The functions $\xi_\pm$ can be expressed in terms of $\eta_\pm$:

$$\xi_\pm(\tau, x, y) = (J - J_0)\eta_\pm(\tau, x, y).$$

3. Discussion

The proposed approach allows one to take into account higher in order ε terms for constructing the asymptotics of solutions. In addition, problems with several parameters $\delta_1, \ldots, \delta_j$ can be considered even, in particular, if there are resonance relations among them.

We note that there is an 'internal' parameter Θ in the boundary value problems (18) and (19), which varies infinitely many times from 0 to 1 as $\varepsilon \to 0$. This points to the fact that an unlimited process of straight and reverse reconstructions of phase portraits can occur.

It is shown that various partial differential equations arise while describing the leading approximations of solutions in different domains of the phase space of the boundary value problem (3).

For $\delta \neq 2\pi n$ and $\delta = 2\pi n_0$, special nonlinear boundary value problems are constructed to find the slowly varying amplitudes $\xi_\pm(\tau, x_\pm, y_\pm)$. These boundary value problems are different for each of these two cases. In the first of them, systems of two Schrödinger equations were obtained, in contrast to the second case where a system of two Korteweg–de Vries equations was obtained. Asymptotic representations of the irregular solutions studied above contain superposition of functions depending on the following: the 'slow' time $\tau = \varepsilon t$ (or $\tau = \varepsilon^2 t$), the 'medium' time t, and 'fast' time $\tilde{\varepsilon}^{-1} t$. In addition, they contain 2π-periodic with respect to the spatial variables x, $y = 2\pi\varepsilon^{-1} x$ components, which rapidly oscillate with respect to the variable $\tilde{\varepsilon}^{-1} x$.

It follows from the above formulas that the mutual influence of the functions ξ_+ and ξ_- leads only to the phase components change. If $\delta = 2\pi n_0$, then this influence is much weaker [11] when the higher infinitesimal order terms in the corresponding boundary value problems are taken into account. Therein, one can trace some analogies with the conclusions from [16–19].

The coefficients σ_0 and σ_1 in (18) and (19) depend both on the parameter α and on the parameter β as $\delta \neq 2\pi n$. Only the parameter α appears in (33) and (34) when $\delta = 2\pi n_0$. The 'orders' of the 'slow' time τ also turned out to be different. In the first case, $\tau = \varepsilon t$, and in the second, $\tau = \varepsilon^2 t$, i.e., the processes are sufficiently slower as $\delta = 2\pi n_0$. We also note that the amplitude of the principal terms of the solutions is of the order ε in both cases.

A wave moving in one direction mainly affects only the phase coordinate of a wave moving in the opposite direction. In this regard, a number of conclusions from the theory of solitons [16–19] remain valid for irregular waves. We also note that the mutual effect of waves on each other differs significantly from the regular case of $\delta = 0$.

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