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# Novel Fractional Dynamic Hardy–Hilbert-Type Inequalities on Time Scales with Applications

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**Abstract:** The main objective of the present article is to prove some new $\nabla$ dynamic inequalities of Hardy–Hilbert type on time scales. We present and prove very important generalized results with the help of Fenchel–Legendre transform, submultiplicative functions. We prove the $(\gamma, a)$-nabla conformable Hölder's and Jensen's inequality on time scales. We prove several inequalities due to Hardy–Hilbert inequalities on time scales. Furthermore, we introduce the continuous inequalities and discrete inequalities as special case.

**Keywords:** Hardy–Hilbert's inequality; Hölder's and Jensen's inequality; time scale

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## 1. Introduction

In this section, we give several foundational definitions and notations of basic calculus of time scales. Stefan Hilger in his PhD thesis [1] discovered a new calculus named after that time-scale calculus to unify the discrete and continuous analysis (see [2]). Since then, this theory has received a lot of attention. The book by Bohner and Peterson [3], on the subject of time scales, briefs and organizes much of time scale calculus.

We begin with the definition of time scale.

**Definition 1.** *A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the set of all real numbers $\mathbb{R}$.*

Now, we define two operators playing a central role in the analysis on time scales.

**Definition 2.** *If $\mathbb{T}$ is a time scale, then we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

*and*

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

In the above definitions, we put $\inf \emptyset = \sup \mathbb{T}$ (i.e., if $t$ is the maximum of $\mathbb{T}$, then $\sigma(t) = t$) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., if $t$ is the minimum of $\mathbb{T}$, then $\rho(t) = t$), where $\emptyset$ is the empty set.

If $\mathbb{T} \in \{[a, b], [a, \infty), (-\infty, a], \mathbb{R}\}$, then $\sigma(t) = \rho(t) = t$. We note that $\sigma(t)$ and $\rho(t)$ in $\mathbb{T}$ when $t \in \mathbb{T}$ because $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$.

Next, we define the graininess functions as follows:

**Definition 3.** $(i)$ *The forward graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by*

$$\mu(t) = \sigma(t) - t.$$

(*ii*) *The backward graininess function* $\nu : \mathbb{T} \to [0, \infty)$ *is defined by*

$$\nu(t) = t - \rho(t).$$

With the operators defined above, we can begin to classify the points of any time scale depending on the proximity of their neighboring points in the following manner.

**Definition 4.** *Let* $\mathbb{T}$ *be a time scale. A point* $t \in \mathbb{T}$ *is said to be:*

(1) *Right-scattered if* $\sigma(t) > t$*;*
(2) *Left-scattered if* $\rho(t) < t$*;*
(3) *Isolated if* $\rho(t) < t < \sigma(t)$*;*
(4) *Right-dense if* $\sigma(t) = t$*;*
(5) *Left-dense if* $\rho(t) = t$*;*
(6) *Dense if* $\rho(t) = t = \sigma(t)$*.*

The closed interval on time scales is defined by

$$[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T} = \{t \in \mathbb{T} : a \le t \le b\}.$$

Open intervals and half-open intervals are defined similarly.

Two sets we need to consider are $\mathbb{T}^\kappa$ and $\mathbb{T}_\kappa$ which are defined as follows: $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$ if $\mathbb{T}$ has $M$ as a left-scattered maximum and $\mathbb{T}^\kappa = \mathbb{T}$ otherwise. Similarly, $\mathbb{T}_\kappa = \mathbb{T} \setminus \{m\}$ if $\mathbb{T}$ has $m$ as a right-scattered minimum and $\mathbb{T}_\kappa = \mathbb{T}$ otherwise. In fact, we can write

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus \big(\rho(\sup \mathbb{T}), \sup \mathbb{T}\big], & \text{if} \quad \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if} \quad \sup \mathbb{T} = \infty, \end{cases}$$

and

$$\mathbb{T}_\kappa = \begin{cases} \mathbb{T} \setminus \big[\inf \mathbb{T}, \sigma(\inf \mathbb{T})\big), & \text{if} \quad \inf \mathbb{T} > -\infty, \\ \mathbb{T}, & \text{if} \quad \inf \mathbb{T} = -\infty. \end{cases}$$

**Definition 5.** *Let* $f : \mathbb{T} \to \mathbb{R}$ *be a function defined on a time scale* $\mathbb{T}$*. Then we define the function* $f^\sigma : \mathbb{T} \to \mathbb{R}$ *by*

$$f^\sigma(t) = (f \circ \sigma)(t) = f(\sigma(t)), \qquad t \in \mathbb{T},$$

*and the function* $f^\rho : \mathbb{T} \to \mathbb{R}$ *by*

$$f^\rho(t) = (f \circ \rho)(t) = f(\rho(t)), \qquad t \in \mathbb{T}.$$

We introduce the nabla derivative of a function $f : \mathbb{T} \to \mathbb{R}$ at a point $t \in \mathbb{T}_\kappa$ as follows:

**Definition 6.** *Let* $f : \mathbb{T} \to \mathbb{R}$ *be a function and let* $t \in \mathbb{T}_\kappa$*. We define* $f^\nabla(t)$ *as the real number (provided it exists) with the property that for any* $\epsilon > 0$*, there exists a neighborhood* $N$ *of* $t$ *(i.e.,* $N = (t - \delta, t + \delta)_{\mathbb{T}}$ *for some* $\delta > 0$*) such that*

$$\big|[f^\rho(t) - f(s)] - f^\nabla(t)[\rho(t) - s]\big| \le \epsilon|\rho(t) - s| \quad for\ every \quad s \in N.$$

*We say that* $f^\nabla(t)$ *is the nabla derivative of* $f$ *at* $t$*.*

**Theorem 1.** *Let* $f : \mathbb{T} \to \mathbb{R}$ *be a function, and* $t \in \mathbb{T}_\kappa$*. Then:*

(*i*) $f$ *being nabla differentiable at* $t$ *implies* $f$ *is continuous at* $t$*.*

$(ii)$ *$f$ being continuous at left-scattered $t$ implies $f$ is nabla differentiable at $t$ with*

$$f^{\nabla}(t) = \frac{f(t) - f^{\rho}(t)}{\nu(t)}. \tag{1}$$

$(iii)$ *If $t$ is left-dense, then $f$ is nabla differentiable at $t$ if and only if the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

*exists as a finite number. In such a case,*

$$f^{\nabla}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

$(iv)$ *$f^{\rho}(t) = f(t) - \nu(t) f^{\nabla}(t)$ whenever $f$ is nabla differentiable at $t$.*

**Example 1.** $(i)$ *Let $\mathbb{T} = \mathbb{R}$. Then*

$$f^{\nabla}(t) = f'(t).$$

$(ii)$ *Let $\mathbb{T} = \mathbb{Z}$. Then*

$$f^{\nabla}(t) = \nabla f(t) = f(t) - f(t-1),$$

*where $\nabla$ is the backward difference operator.*

**Theorem 2.** *Let $f, g : \mathbb{T} \to \mathbb{R}$ be functions that are nabla differentiable at $t \in \mathbb{T}_{\kappa}$. Then:*

$(i)$ *The sum $f + g : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at $t$ with*

$$(f+g)^{\nabla}(t) = f^{\nabla}(t) + g^{\nabla}(t).$$

$(ii)$ *If $\alpha \in \mathbb{R}$ is a constant, then the function $\alpha f : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at $t$ with*

$$(\alpha f)^{\nabla}(t) = \alpha f^{\nabla}(t).$$

$(iii)$ *The product $fg : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at $t$, and we obtain the product rule*

$$(fg)^{\nabla}(t) = f^{\nabla}(t) g(t) + f^{\rho}(t) g^{\nabla}(t) = f(t) g^{\nabla}(t) + f^{\nabla}(t) g^{\rho}(t).$$

$(iv)$ *The function $\dfrac{1}{f} : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at $t$ with*

$$\left(\frac{1}{f}\right)^{\nabla}(t) = -\frac{f^{\nabla}(t)}{f(t) f^{\rho}(t)}, \qquad f(t) f^{\rho}(t) \neq 0.$$

$(v)$ *The quotient $\dfrac{f}{g} : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at $t$, and we obtain the quotient rule*

$$\left(\frac{f}{g}\right)^{\nabla}(t) = \frac{f^{\nabla}(t) g(t) - f(t) g^{\nabla}(t)}{g(t) g^{\rho}(t)}, \qquad g(t) g^{\rho}(t) \neq 0.$$

**Definition 7.** *We say that a function $F : \mathbb{T} \to \mathbb{R}$ is a nabla antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $F^{\nabla}(t) = f(t)$ for all $t \in \mathbb{T}_{\kappa}$. In this case, the nabla-integral of $f$ is defined by*

$$\int_a^t f(t) \nabla \tau = F(t) - F(a) \qquad for\ all \quad t \in \mathbb{T}_{\kappa}.$$

Now, we introduce the set of all ld-continuous functions to find a class of functions that have nabla antiderivatives.

**Definition 8** (Ld-Continuous Function ($C_{ld}$))**.** *We say that the function $f : \mathbb{T} \to \mathbb{R}$ is ld-continuous if it is continuous at all left-dense points of $\mathbb{T}$ and its right-sided limits exist (finite) at all right-dense points of $\mathbb{T}$.*

**Theorem 3** (Existence of Nabla Antiderivatives)**.** *Every ld-continuous function possess a nabla antiderivative.*

**Theorem 4.** *Let $f : \mathbb{T} \to \mathbb{R}$ be a ld-continuous function, and let $t \in \mathbb{T}_\kappa$. Then*

$$\int_{\rho(t)}^{t} f(\tau)\nabla\tau = \nu(t)f(t).$$

**Theorem 5.** *If $f^\nabla(t) \geq 0$ (respectively, $f^\nabla(t) \leq 0$), then $f$ is nondecreasing (respectively, nonincreasing).*

**Theorem 6.** *If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f, g \in C_{ld}$, then*

$(i)$ $\int_a^b [f(t) + g(t)]\nabla t = \int_a^b f(t)\nabla t + \int_a^b g(t)\nabla t;$

$(ii)$ $\int_a^b \alpha f(t)\nabla t = \alpha \int_a^b f(t)\nabla t;$

$(iii)$ $\int_a^b f(t)\nabla t = -\int_b^a f(t)\nabla t;$

$(iv)$ $\int_a^b f(t)\nabla t = \int_a^c f(t)\nabla t + \int_c^b f(t)\nabla t;$

$(v)$ $\int_a^a f(t)\nabla t = 0;$

$(vi)$ *if* $f(t) \geq g(t)$ *on* $[a, b)_\mathbb{T}$, *then* $\int_a^b f(t)\nabla t \geq \int_a^b g(t)\nabla t;$

$(vii)$ *if* $f(t) \geq 0$ *on* $\in [a, b)_\mathbb{T}$, *then* $\int_a^b f(t)\nabla t \geq 0.$

**Theorem 7.** *Let $f : \mathbb{T} \to \mathbb{R}$ be a ld-continuous function, and $a, b \in \mathbb{T}$.*

$(i)$ *In the case that $\mathbb{T} = \mathbb{R}$, we have*

$$\int_a^b f(t)\nabla t = \int_a^b f(t)dt,$$

*where the integral on the right-hand side is the Riemann integral from calculus.*

$(ii)$ *In the case that $[a, b]_\mathbb{T}$ consists of only isolated points, we have*

$$\int_a^b f(t)\nabla t = \begin{cases} \sum\limits_{t \in (a,b]_\mathbb{T}} \nu(t)f(t), & \text{if} \quad a < b, \\ 0, & \text{if} \quad a = b, \\ -\sum\limits_{t \in (b,a]_\mathbb{T}} \nu(t)f(t), & \text{if} \quad a > b. \end{cases}$$

$(iii)$ *In the case that $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, where $h > 0$, we have*

$$\int_a^b f(t)\nabla t = \begin{cases} \sum\limits_{k=\frac{a}{h}+1}^{\frac{b}{h}} hf(hk), & \text{if} \quad a < b, \\ 0, & \text{if} \quad a = b, \\ -\sum\limits_{k=\frac{b}{h}+1}^{\frac{a}{h}} hf(hk), & \text{if} \quad a > b. \end{cases}$$

$(iv)$ *In the case that* $\mathbb{T} = \mathbb{Z}$, *we have*

$$\int_a^b f(t)\nabla t = \begin{cases} \sum_{t=a+1}^{b} f(t), & \text{if} \quad a < b, \\ 0, & \text{if} \quad a = b, \\ -\sum_{t=b+1}^{a} f(t), & \text{if} \quad a > b. \end{cases}$$

The formula for nabla integration by parts is as follows:

$$\int_a^b f(t)g^{\nabla}(t)\nabla t = (fg)(b) - (fg)(a) - \int_a^b f^{\nabla}(t)g^{\rho}(t)\nabla t.$$

The following theorem gives a relationship between the delta and nabla derivative.

**Theorem 8.** $(i)$ *Let* $f : \mathbb{T} \to \mathbb{R}$ *be delta differentiable on* $\mathbb{T}^{\kappa}$. *Then* $f$ *is nabla differentiable at* $t$ *and* $f^{\nabla}(t) = f^{\Delta}(\rho(t))$ *for any* $t \in \mathbb{T}_{\kappa}$ *that satisfies* $\sigma(\rho(t)) = t$. *If, in addition,* $f^{\Delta}$ *is continuous on* $\mathbb{T}^{\kappa}$, *then* $f$ *is nabla differentiable at* $t$, *and* $f^{\nabla}(t) = f^{\Delta}(\rho(t))$ *for each* $t \in \mathbb{T}_{\kappa}$.

$(ii)$ *Let* $f : \mathbb{T} \to \mathbb{R}$ *be nabla differentiable on* $\mathbb{T}_{\kappa}$. *Then* $f$ *is delta differentiable at* $t$ *and* $f^{\Delta}(t) = f^{\nabla}(\sigma(t))$ *for any* $t \in \mathbb{T}^{\kappa}$ *that satisfies* $\rho(\sigma(t)) = t$. *If, in addition,* $f^{\nabla}$ *is continuous on* $\mathbb{T}_{\kappa}$, *then* $f$ *is delta differentiable at* $t$, *and* $f^{\Delta}(t) = f^{\nabla}(\sigma(t))$ *for each* $t \in \mathbb{T}^{\kappa}$.

We will use the following relations between calculus on time scales $\mathbb{T}$ and either continuous calculus on $\mathbb{R}$ or discrete calculus on $\mathbb{Z}$. Please note that:

$(i)$ If $\mathbb{T} = \mathbb{R}$, then

$$\begin{gathered} \rho(t) = t, \quad \nu(t) = 0, \quad f^{\nabla}(t) = f'(t), \\ \int_a^b f(t)\nabla t = \int_a^b f(t)dt. \end{gathered} \tag{2}$$

$(ii)$ If $\mathbb{T} = \mathbb{Z}$, then

$$\begin{gathered} \rho(t) = t - 1, \quad \nu(t) = 1, \\ f^{\nabla}(t) = \nabla f(t), \\ \sum_{t=a}^{b-1} f(t), \quad \int_a^b f(t)\nabla t = \sum_{t=a+1}^{b} f(t), \end{gathered} \tag{3}$$

where $\nabla$ is the forward difference operators, respectively.

Recently, depending just on the basic limit definition of the derivative, Khalil et al. [4] proposed the conformable derivative $T_{\alpha}f(t)$ $(\alpha \in (0,1])$ of a function $f : \mathbb{R}^{+} \to \mathbb{R}$

$$T_{\alpha}f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon},$$

for all $t > 0$, $\alpha \in (0,1]$, this definition found wide resonance in the scientific community interested in fractional calculus, see [5–7]. Iyiola and Nwaeze in [5] proposed an extended mean value theorem and Racetrack type principle for a class of $\alpha$-differentiable functions. Therefore, calculating the derivative by this definition is easy compared to the definitions that are based on integration. The researchers in [4] also suggested a definition for the $\alpha$-conformable integral of a function $\eta$ as follows:

$$\int_a^b \eta(t)d_{\alpha}t = \int_a^b \eta(t)t^{\alpha-1}dt.$$

After that, Abdeljawad [8] studied extensive research of the newly introduced conformable calculus. In his work, he introduced a generalization of the conformable derivative $T^a_\alpha f(t)$ definition. For $t > a \in \mathbb{R}^+$ as $f : \mathbb{R}^+ \to \mathbb{R}$

$$T^a_\alpha f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon(t-a)^{1-\alpha}) - f(t)}{\epsilon}.$$

Benkhettou et al. [9] introduced a conformable calculus on an arbitrary time scale, which is a natural extension of the conformable calculus.

However, in the last few decades, many authors pointed out that derivatives and integrals of non-integer order are very suitable for the description of properties of various real materials, e.g., polymers. Fractional derivatives provides an excellent instrument for the description of memory and hereditary properties of various materials and processes. This is the main advantages of fractional derivatives in comparison with classical integer-order models.

In [10], the authors studied a version of the nabla conformable fractional derivative on arbitrary time scales. Specifically, for a function $f : \mathbb{T} \to \mathbb{R}$, the nabla conformable fractional derivative, $T_{\nabla,\alpha} f(t) \in \mathbb{R}$ of order $\alpha \in (0,1]$ at $t \in \mathbb{T}_\kappa$ and $t > 0$ was defined as: Given any $\epsilon > 0$, there is a $\delta$- neighborhood $U_t \subset \mathbb{T}$ of $t$, $\delta > 0$, such that

$$|[f(\rho(t)) - f(s)]t^{1-\alpha} - T_{\nabla,\alpha}(f)(t)[\rho(t) - s]| \le \varepsilon|\rho(t) - s|$$

for all $s \in U_t$. The nabla conformable fractional integral is defined by

$$\int f(t) \nabla_\alpha t = \int f(t) t^{\alpha-1} \nabla t.$$

Rahmat et al. [11] presented a new type of conformable nabla derivative and integral which involve the time-scale power function $\widehat{G}_n(t,s)$ for $s, t \in \mathbb{T}$.

**Definition 9.** *Suppose $[s,t] \subset \mathbb{T}$ and $s < t$. The generalized time-scale power function $\widehat{G}_n : \mathbb{T} \times \mathbb{T} \longrightarrow \mathbb{R}^+$ for $n \in \mathbb{N}_0$ is defined by*

$$\widehat{G}_n(t,s) = \begin{cases} (t-s)^n, & \text{if } [t,s] \text{ dense;} \\ \prod_{j=0}^{n-1}(t - \rho^j(s)), & \text{if } [t,s] \text{ isolated;} \end{cases} \tag{4}$$

*and its inverse function $\widehat{G}_{-n} : \mathbb{T} \times \mathbb{T} \longrightarrow \mathbb{R}^+$ is then given by*

$$\widehat{G}_{-n}(t,s) = \begin{cases} (t-s)^{-n}, & \text{if } [t,s] \text{ dense;} \\ \frac{1}{\prod_{j=0}^{n-1}(\rho^n(t) - \rho^j(s))}, & \text{if } [t,s] \text{ isolated.} \end{cases} \tag{5}$$

*Notice that:*

$$\widehat{G}_{-n}(\sigma^n(t), s) = \frac{1}{\widehat{G}n(t,s)}. \quad s, t \in \mathbb{T}, s < t. \tag{6}$$

**Corollary 1.** *For $h > 0$, $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, we have $\rho^k(s) = s - kh$. Then*

$$\begin{aligned} \widehat{G}_n(t,s) &= (t-s)_h^{(n)} \\ &= \prod_{j=0}^{n-1}(t - s + jh) \\ &= h^n \left(\frac{t-s}{h}\right)^{(n)}, \qquad n \in \mathbb{N}, \end{aligned} \tag{7}$$

*and*

$$\begin{aligned}\widehat{G}_n(t,s) &= (t-s)_h^{(-n)}\\ &= \frac{1}{\prod_{j=0}^{n-1}(t-s+jh)}\prod_{j=0}^{n-1}(t-n-s+jh)\\ &= h^{-n}\left(\frac{t-n-s}{h}+n\right)^{(n)}, \qquad n\in\mathbb{N},\end{aligned} \tag{8}$$

*where*

$$x^{(n)} = \frac{\Gamma(x+n)}{\Gamma(x)} \;\text{ and }\; x^{(-n)} = \frac{1}{(x-n)^{(n)}} = \frac{\Gamma(x-n)}{\Gamma(x)},\;\; n\in\mathbb{N},\quad x^{(0)}=1.$$

*For* $\mathbb{T} = q^{\mathbb{N}_0}$*, we have* $\rho^k(s) = sq^{-k}$*. Then we write*

$$\begin{aligned}\widehat{G}_n(t,s) &= (t-s)_{\tilde{q}}^{(n)}\\ &= \prod_{j=0}^{n-1}(t-sq^{-j})\\ &= t^n\prod_{j=0}^{n-1}\left(1-\frac{\tilde{q}^j s}{t}\right), \qquad \left(0<\tilde{q}=\frac{1}{q}<1\right).\end{aligned} \tag{9}$$

**Remark 1.** *Regarding the generalization of the power function,* $\widehat{G}_\alpha(t,s)$ *to real values of* $\alpha \geqslant 0$ *(instead of integers), we have*

$$(t-s)_h^{(\alpha)} = h^\alpha\frac{\Gamma(\frac{t-s}{h}+\alpha)}{\Gamma(\frac{t-s}{h})},\;\; (t-s)_{\tilde{q}}^{(\alpha)} = t^\alpha\frac{(s/t,\tilde{q})_\infty}{(\tilde{q}^\alpha s/t,\tilde{q})_\infty}\;\; t\neq 0, \tag{10}$$

*where*

$$(p,\tilde{q})_\infty = \prod_{j=0}^{\infty}(1-p\tilde{q}).$$

**Definition 10** (Conformable nabla derivative)**.** *Given a function* $f:\mathbb{T}\longrightarrow\mathbb{R}$ *and* $a\in\mathbb{T}$*,* $f$ *is* $(\gamma,a)$*-nabla differentiable at* $t>a$*, if it is nabla differentiable at* $t$*, and its* $(\gamma,a)$*-nabla derivative is defined by*

$$\nabla_a^\gamma f(t) = \widehat{G}_{1-\gamma}(t,a)f^{\nabla}(t) \quad t>a, \tag{11}$$

*where the function* $\widehat{G}_{1-\gamma}(t,a)$ *as defined in* (4)*. If* $\nabla_a^\gamma[f(t)]$ *exists in some interval* $(a,a+\epsilon)_{\mathbb{T}},\epsilon>0$*, then we define*

$$\nabla_a^\gamma[f(a)] = \lim_{t\longrightarrow a^+}\nabla_a^\gamma[f(t)]$$

*if the* $\lim_{t\longrightarrow a^+}\nabla_a^\gamma[f(t)]$ *exists. Moreover, we call* $f$ *is* $(\gamma,a)$*-nabla differentiable on* $\mathbb{T}_k$ $(a\in\mathbb{T}_k)$ *provided* $\nabla_a^\gamma[f(t)]$ *exists for all* $t\in\mathbb{T}_k$*. The function* $\nabla_a^\gamma:\mathbb{T}_k\longrightarrow\mathbb{R}$ *is then called* $(\gamma,a)$*-nabla derivative of* $f$ *on* $\mathbb{T}_k$*.*

Next, we present the $(\gamma,a)$-nabla derivatives of products, sums, and quotients as follows.

**Theorem 9.** *Assume* $f,g:\mathbb{T}\longrightarrow\mathbb{R}$ *are* $(\gamma,a)$*-nabla differentiable at* $t\in\mathbb{T}_k$*,* $t>a$*. Then:*

*(i)* *The sum* $f+g:\mathbb{T}\longrightarrow\mathbb{R}$ *is* $(\gamma,a)$*-nabla differentiable at* $t$ *with*

$$\nabla_a^\gamma(rf+sg)(t) = r\nabla_a^\gamma f(t) + s\nabla_a^\gamma g(t).$$

*(ii)* *For all $k \in \mathbb{R}$, then $kf : \mathbb{T} \longrightarrow \mathbb{R}$ is $(\gamma, a)$-nabla differentiable at $t$ with*

$$\nabla_a^\gamma (kf)(t) = k\nabla_a^\gamma f(t).$$

*(iii)* *The product $fg : \mathbb{T} \longrightarrow \mathbb{R}$ is $(\gamma, a)$-nabla differentiable at $t$ with*

$$\nabla_a^\gamma (fg) = [\nabla_a^\gamma f(t)]g(t) + f^\rho(t)[\nabla_a^\gamma g(t)]. \tag{12}$$

*(iv)* *If $g(t)g^\rho(t) \neq 0$, then $f/g$ is $(\gamma, a)$-nabla differentiable at $t$ with*

$$\nabla_a^\gamma \left(\frac{f}{g}\right)(t) = \frac{[\nabla_a^\gamma f(t)]g(t) - f(t)[\nabla_a^\gamma g(t)]}{g(t)g^\rho(t)}.$$

**Lemma 1** (Integration by parts)**.** *Suppose that $d$, $b \in \mathbb{T}$ where $b > d$. If $\eta$, $\xi$ are conformable $(\gamma, a)$- nabla fractional differentiable and $\gamma \in (0, 1]$, then:*

$$\int_d^b \eta(t) \left[\nabla_a^\gamma \xi(t)\right] \nabla_a^\gamma t = \Big[\eta(t)\xi(t)\Big]_d^b - \int_d^b \left[\nabla_a^\gamma \eta(t)\right] \xi^\rho(t) \nabla_a^\gamma t. \tag{13}$$

**Lemma 2** (Chain rule)**.** *Let $g \in C_{ld}^\nabla(\mathbb{T})$ and assume that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable function. Then $(f \circ g) : \mathbb{T} \to \mathbb{R}$ is $(\gamma, a)$-nabla differentiable and satisfies*

$$\nabla_a^\gamma (f \circ g)(t) = \left\{ \int_0^1 f'\big(g(t) - h\nu(t)g^\nabla(t)\big)dh \right\} \nabla_a^\gamma g(t). \tag{14}$$

**Lemma 3.** *Let $\gamma \in (0, 1]$. Assume $\xi : \mathbb{T} \to \mathbb{R}$ is continuous and $(\gamma, a)$-nabla differentiable of order $\gamma$ at $t \in \mathbb{T}^k$, where $t > a$ and $\eta : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there is $c$ in the real interval $[\rho(t), t]$ such that*

$$\nabla_a^\gamma (\eta \circ \xi)(t) = \eta'(\xi(c))\nabla_a^\gamma (\xi(t)). \tag{15}$$

**Definition 11** ($\gamma$-nabla-integral from $a$)**.** *Assume that $0 < \gamma \leqslant 1$, $a$, $t_1$, $t_2 \in \mathbb{T}$, $a \leqslant t_1 \leqslant t_2$ and $f \in C_{ld}(\mathbb{T})$, then we the function $f$ is called $(\gamma, a)$-nabla integrable on $[t_1, t_2]$. if:*

$$\begin{aligned} \nabla_a^{-\gamma} f(t) &= \int_{t_1}^{t_2} f(t)\nabla_a^\gamma t \\ &= \int_{t_1}^{t_2} f(t)\widehat{G}_{\gamma-1}(\sigma^{\gamma-1}(t), a)\nabla t, \end{aligned} \tag{16}$$

*exists and is finite.*

**Theorem 10.** *Suppose $t \in \mathbb{T}$ and $a \in \mathbb{T}_\kappa$ with $\rho(t) > a$ and $f \in C_{ld}(\mathbb{T})$, then*

$$\int_{\rho(t)}^t f(t)\nabla_a^\gamma t = \nu(t)f(t)\widehat{G}_{\gamma-1}(\sigma^{\gamma-1}(t), a). \tag{17}$$

We need the relations between different types of calculus on time scales $\mathbb{T}$ and continuous calculus, discrete calculus and quantum calculus as follows. Please note that: For the case $\mathbb{T} = \mathbb{R}$, we obtain

$$\int_a^t f(t)\nabla_a^\gamma t = \int_a^t f(t)(t-a)^{\gamma-1}dt. \tag{18}$$

If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, we obtain

$$\int_a^t f(t)\nabla_a^\gamma t = \sum_{t \in (a,t]} hf(t)\big(\rho^{\gamma-1}(t) - a\big)_h^{(\gamma-1)}. \tag{19}$$

For $\mathbb{T} = q^{\mathbb{N}_0}$, we have

$$\int_a^t f(t)\nabla_a^\gamma t = \sum_{t\in(a,t]} t(1-\tilde{q})f(t)\left(\rho^{\gamma-1}(t)-a\right)_{\tilde{q}}^{(\gamma-1)}. \tag{20}$$

**Theorem 11.** *Let $\gamma \in (0,1]$ and $a \in \mathbb{T}$. Then, for any ld-continuous function $f : \mathbb{T} \longrightarrow \mathbb{R}$, there exists a function $F : \mathbb{T} \longrightarrow \mathbb{R}$ such that*

$$\nabla_a^\gamma F(t) = f(t) \text{ for all } t \in \mathbb{T}_k.$$

*The function $F$ is called an $(\gamma, a)$- nabla antiderivative of $f$, and we have*

$$\int_{t_1}^{t_2} f(t)\nabla_a^\gamma t = F(t_2) - F(t_1), \quad t_1, t_2 \in \mathbb{T}, \quad t_1 \leq t_2. \tag{21}$$

**Theorem 12.** *Let $\gamma \in (0,1]$. If $a, t_1\, t_2, t_3 \in \mathbb{T}$, $a \leqslant t_1 \leqslant t_2 \leqslant t_3$, $\alpha \in \mathbb{R}$ and $f, g \in C_{ld}(\mathbb{T})$, then*

*(i)* $\int_{t_1}^{t_2}[f(t)+g(t)]\nabla_a^\gamma t = \int_{t_1}^{t_2} f(t)\nabla_a^\gamma t + \int_{t_1}^{t_2} g(t)\nabla_a^\gamma t.$

*(ii)* $\int_{t_1}^{t_2} \alpha f(t)\nabla_a^\gamma t = \alpha \int_{t_1}^{t_2} f(t)\nabla_a^\gamma t.$

*(iii)* $\int_{t_1}^{t_2} f(t)\nabla_a^\gamma t = -\int_{t_2}^{t_1} f(t)\nabla_a^\gamma t.$

*(iv)* $\int_{t_1}^{t_3} f(t)\nabla_a^\gamma s = \int_{t_1}^{t_2} f(t)\nabla_a^\gamma t + \int_{t_2}^{t_3} f(t)\nabla_a^\gamma t.$

*(v)* $\int_{t_1}^{t_1} f(t)\nabla_a^\gamma t = 0.$

*(vi)* *If $|f(t)| \leq g(t)$ on $[t_1, t_2]$, then*

$$\left|\int_{t_2}^{t_3} f(t)\nabla_a^\gamma t\right| \leq \int_{t_2}^{t_3} g(t)\nabla_a^\gamma t.$$

**Lemma 4.** *Let $\gamma \in (0,1]$. Assume $\urcorner : \mathbb{T} \to \mathbb{R}$ is continuous and $(\gamma, a)$-nabla differentiable of order $\gamma$ at $t \in \mathbb{T}^k$, where $t > a$ and $\eta : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there is $c$ in the real interval $[\rho(t), t]$ such that*

$$\nabla_a^\gamma(\eta \circ \urcorner)(t) = \eta'(\urcorner(c))\nabla_a^\gamma(\urcorner(t)). \tag{22}$$

Next, we introduce the Fenchel–Legendre transform [12–14].

**Definition 12.** *Suppose $\psi : \mathbb{R}^i \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a function: $\psi \neq +\infty$ i.e., $\mathrm{Dom}(\psi) = \{\tilde{\omega} \in \mathbb{R}^i, |\psi(\tilde{\omega}) < \infty\} \neq \emptyset$. Then the Fenchel–Legendre transform is defined as:*

$$\psi^* : \mathbb{R}^i \longrightarrow \mathbb{R} \cup \{+\infty\}, \quad z \longrightarrow \psi^*(\tilde{z}) = \sup\{< \tilde{z}, \tilde{\omega} > -\psi(\tilde{\omega}), \tilde{\omega} \in \mathrm{Dom}(\psi)\} \tag{23}$$

*The scalar product is denoted by $< .,. >$ on $\mathbb{R}^i$, and $\psi \longrightarrow \psi^*$ is said to be the conjugate operation.*

**Lemma 5** ( [12])**.** *Suppose a function $\psi$ and suppose $\psi^*$ Fenchel–Legendre transform of $\psi$, we obtain*

$$< \tilde{\omega}, \tilde{z} > \leqslant \psi(\tilde{\omega}) + \psi^*(\tilde{z}), \tag{24}$$

*for all $\tilde{\omega} \in \mathrm{Dom}(\psi)$, and $\tilde{z} \in \mathrm{Dom}(\psi^*)$.*

**Definition 13.** *We said $\Omega$ is submultiplicative $[0, \infty)$ if*

$$\Omega(\tilde{\omega}\tilde{z}) \leqslant \Omega(\tilde{\omega})\Omega(\tilde{z}), \ \forall \ \tilde{\omega}, \tilde{z} \geqslant 0. \tag{25}$$

**Lemma 6** ([15])**.** *Let $\vartheta : \mathbb{T} \to \mathbb{R}$ is left-dense continuous function. Then the equality that allows interchanging the order of nabla integration given by*

$$\int_{t_0}^{w}\left(\int_{t_0}^{s}\vartheta(\eta)\nabla\eta\right)\nabla s=\int_{t_0}^{w}\left(\int_{\rho(\eta)}^{w}\vartheta(\eta)\nabla s\right)\nabla\eta=\int_{t_0}^{w}[w-\rho(\eta)]\vartheta(\eta)\nabla\eta \tag{26}$$

*holds for all $s$, $w$, $t_0 \in \mathbb{T}$.*

**Lemma 7** ([16])**.** *Let $w$ and $z \in \mathbb{R}$ be such that $w+z \geqslant 1$ and $0<\gamma$, then*

$$(w+z)^{\frac{1}{\gamma}} \leqslant \left(|w|^{\frac{1}{2\beta}}+|z|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{\gamma}} \quad \text{for all} \quad \frac{1}{2}\leq\beta\leq\alpha. \tag{27}$$

Over several decades Hilbert-type inequalities have been attracted many researchers and several refinements and extensions have been done to the previous results, we refer the reader to the works [15–26].

The celebrated Hardy–Hilbert's integral inequality [27] is

$$\int_0^\infty\int_0^\infty \frac{f(x)g(x)}{x+y}\mathrm{d}x\mathrm{d}y \leq \frac{t}{\sin\frac{t}{p}}\left[\int_0^\infty f^p(x)\mathrm{d}x\right]^{\frac{1}{p}}\left[\int_0^\infty g^q(y)\mathrm{d}y\right]^{\frac{1}{q}}, \tag{28}$$

where $p>1$, $q=\frac{p}{p-1}$ and the constant $\frac{t}{\sin\frac{t}{p}}$ is best possible. As special case, if $p=q=2$, the inequality (29) is reduced to the classical Hilbert integral inequality

$$\int_0^\infty\int_0^\infty \frac{f(x)g(x)}{x+y}\mathrm{d}x\mathrm{d}y \leq t\left[\int_0^\infty f^2(x)\mathrm{d}x\right]^{\frac{1}{2}}\left[\int_0^\infty g^2(y)\mathrm{d}y\right]^{\frac{1}{2}}, \tag{29}$$

where the coefficient $t$ is best possible.

In [28], Pachappte established a discrete Hilbert-type inequality and its integral version as in the following two theorems:

**Theorem 13.** *Let $\{a_m\}$, $\{b_n\}$ be two nonnegative sequences of real numbers defined for $m=1,\dots,k$, and $n=1,\dots,r$ with $a_0=b_0=0$, and let $\{p_m\}$, $\{q_n\}$, be two positive sequences of real numbers defined for $m=1,\dots,k$ and $n=1,\dots,r$ where $k$, $r$ are natural numbers. Define $P_m=\sum_{s=1}^{m}p_s$ and $Q_n=\sum_{t=1}^{n}q_t$. Let $\Phi$ and $\Psi$ be two real-valued nonnegative, convex, and submultiplicative functions defined on $[0,\infty)$. Then*

$$\begin{aligned}\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\Phi(a_m)\Psi(b_n)}{m+n} \leqslant M(k,r)&\left(\sum_{m=1}^{k}(k-m+1)\left(p_m\Phi\left(\frac{\nabla a_m}{p_m}\right)\right)^2\right)^{\frac{1}{2}}\\&\times\left(\sum_{n=1}^{r}(r-n+1)\left(q_n\Psi\left(\frac{\nabla b_n}{q_n}\right)\right)^2\right)^{\frac{1}{2}},\end{aligned} \tag{30}$$

*where*

$$M(k,r)=\frac{1}{2}\left(\sum_{m=1}^{k}\left(\frac{\Phi(P_m)}{p_m}\right)^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{r}\left(\frac{\Psi(Q_n)}{Q_n}\right)^2\right)^{\frac{1}{2}}$$

*and $\nabla a_m=a_m-a_{m-1}$, $\nabla b_n=b_n-b_{n-1}$.*

**Theorem 14.** *Let $f\in C^1[[0,x],\mathbb{R}^+]$, $g\in C^1[[0,y],\mathbb{R}^+]$ with $f(0)=g(0)=0$ and let $p(\xi)$, $q(\tau)$ be two positive functions defined for $\xi\in[0,x)$ and $\tau\in[0,y)$. Let $P(s)=\int_0^s p(\xi)d\xi$ and*

*$Q(t) = \int_0^t p(\tau)d\tau$ for $s \in [0,x)$ and $t \in [0,y)$ where $x, y$ are positive real numbers. Let $\Phi$, and $\Psi$ be as in Theorem 13. Then*

$$\int_0^x \int_0^y \frac{\Phi(f(s))\Psi(g(t))}{s+t} dsdt \leqslant L(x,y)\left(\int_0^x (x-s)\left(p(s)\Phi\left(\frac{f'(s)}{p(s)}\right)^2 ds\right)^{\frac{1}{2}}\right) \times\left(\int_0^y (y-t)\left(q(t)\Psi\left(\frac{g'(t)}{q(t)}\right)^2 dt\right)^{\frac{1}{2}}\right) \quad (31)$$

*where*

$$L(x,y) = \frac{1}{2}\left(\int_0^x \left(\frac{\Phi(P(s))}{P(s)}\right)^2 ds\right)^{\frac{1}{2}}\left(\int_0^y \left(\frac{\Psi(Q(t))}{Q(t)}\right)^2 dt\right)^{\frac{1}{2}}.$$

In [29], Handley et al. gave general versions of inequalities (30) and (31) in the following two theorems:

**Theorem 15.** *Let $\{a_{i,m_i}\}$ $(i = 1,2,\ldots,n)$ be $n$ sequences of nonnegative real numbers defined for $m_i = 1,\ldots,k_i$ with $a_{1,0} = a_{2,0}\ldots a_{n,0} = 0$, and let $\{p_{i,m_i}\}$ be $n$ sequences of positive real numbers defined for $m_i = 1,\ldots,k_i$, where $k_i$ are natural numbers. Set $P_{i,m_i} = \sum_{s_i}^{m_i} p_{i,s_i}$. Let $\Phi_i$ $(i = 1,2,\ldots,n)$ be $n$ real-valued nonnegative convex and submultiplicative functions defined on $(0,\infty)$. Let $\alpha_i \in (0,1)$, and set $\alpha'_i = 1-\alpha_i$, $(i = 1,2,\ldots,n)$, $\alpha = \sum_{i=1}^n \alpha_i$, and $\alpha'_i = \sum_{i=1}^n \alpha'_i, = n-\alpha$. Then*

$$\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n} \frac{\prod_{i=1}^n \Phi_i(a_{i,m_i})}{\left(\sum_{i=1}^n \alpha'_i m_i\right)^{\alpha'}} \leqslant M(k_1,\ldots,k_n)\prod_{i=1}^n\left(\sum_{m_i=1}^{k_i}(k_i-m_i+1)\left(p_{i,m_i}\Phi_i\left(\frac{\nabla a_{i,m_i}}{p_{i,m_i}}\right)^{\frac{1}{\alpha_i}}\right)^{\alpha_i}\right.$$

*where*

$$M(k_1,\ldots,k_n) = \frac{1}{(\alpha')^{\alpha'}}\prod_{i=1}^n\left(\sum_{m_i=1}^{k_i}\left(\frac{\Phi_i(P_{i,m_i})}{P_{i,m_i}}\right)^{\frac{1}{\alpha'_i}}\right)^{\alpha'_i}.$$

**Theorem 16.** *Let $f_i \in C^1([0,k_i],\mathbb{R}_+])$ $i = 1,\ldots,n$, with $f_i(0) = 0$, let $p_i(\xi_i)$ be $n$ positive functions defined for $\xi_i \in [0,x_i]$ $(i = 1,\ldots,n)$. Set $P_i(s_i) = \int_0^{s_i} p_i(\xi_i)d\xi_i$ for $s_i \in [0,x_i]$, where $x_i$ are positive real numbers. Let $\Phi_i$, $\alpha_i$, $\alpha'_i$, $\alpha$, and $\alpha'$ be as in Theorem 15. Then*

$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n \Phi_i(f(s_i))}{\left(\sum_{i=1}^n \alpha'_i s_i\right)^{\alpha'}} ds_1\ldots ds_n \leqslant L(x_1,\ldots,x_n)\prod_{i=1}^n\left(\int_0^{x_i}(x_i-s_i)\left(p_i(s_i)\Phi_i\left(\frac{f'(s_i)}{p(s_i)}\right)^{\frac{1}{\alpha_i}} ds_i\right)^{\alpha_i}\right.,$$

*where*

$$L(x_1,\ldots,x_n) = \frac{1}{(\alpha')^{\alpha'}}\prod_{i=1}^n\left(\int_0^{x_i}\left(\frac{\Phi_i(P_i(s_i))}{P_i(s_i)}\right)^{\frac{1}{\alpha'_i}} ds_i\right)^{\alpha'_i}.$$

Hamiaz et al. [22] discussed the inequalities:

**Theorem 17.** *Let $q, p \geqslant 1$, $\alpha \geqslant \beta \geqslant \frac{1}{2}$ and $(\xi_j)_j \geq 0$ $(\delta_i)_i \geq 0$ be sequences of real numbers. Define $\theta_i = \sum_{s=1}^{i} \delta_s$, $\phi_j = \sum_{t=1}^{j} \xi_t$. Then*

$$\sum_{i=1}^{k}\sum_{j=1}^{r}\frac{\theta_i^{2p}\phi_j^{2q}}{\psi(i)+\psi^*(j)} \leqslant C_1^*(p,q)\left(\sum_{i=1}^{k}(k-i+1)(\delta_i\theta_i^{p-1})^2\right) \times\left(\sum_{j=1}^{r}(r-j+1)(\xi_j\phi_j^{q-1})^2\right)$$

*and*

$$\begin{aligned}\sum_{i=1}^{k}\sum_{j=1}^{r}\frac{\theta_i^{p}\phi_j^{q}}{\left(|\psi(i)|^{\frac{1}{2\beta}}+|\psi^*(j)|^{\frac{1}{2\beta}}\right)^{\alpha}} &\leqslant \sum_{i=1}^{k}\sum_{j=1}^{r}\frac{\theta_i^{p}\phi_j^{q}}{\sqrt{\psi(i)+\psi^*(j)}} \\ &\leqslant C_2^*(p,q,k,r)\left(\sum_{i=1}^{k}(k-i+1)(\delta_i\theta_i^{p-1})^2\right)^{\frac{1}{2}} \\ &\quad\times\left(\sum_{j=1}^{r}(r-j+1)(\xi_j\phi_j^{q-1})^2\right)^{\frac{1}{2}}\end{aligned}$$

*unless $(\delta_i)$ or $(\xi_j)$ is null, where*

$$C_1^*(p,q) = (pq)^2 \ \text{ and } \ C_2^*(p,q,k,r) = pq\sqrt{kr}.$$

In this important article, by implying (24), we study some new dynamic inequalities of Hardy–Hilbert type using nabla-integral on time scales. We further show some relevant inequalities as special cases: discrete inequalities and integral inequalities. These inequalities maybe be used to obtain more generalized results of several obtained inequalities before by replace $\psi$, $\psi^*$ by specific substitution.

## 2. Main Results

In the following, we will let $p > 1$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

We start with a foundational results before introducing the main inequalities.

**Lemma 8.** *Suppose the time scales $\mathbb{T}$ with $t, t_0 \in \mathbb{T}$ such that $t \geqslant t_0$. Let $\Upsilon : \mathbb{T} \to \mathbb{R}$ be left-dense continuous function with $\Upsilon \geqslant 0$ and $\ell \geqslant 1$. Then*

$$\left(\int_{t_0}^{t}\Upsilon(\imath)\nabla_a^{\gamma}\imath\right)^{\ell} \leqslant \ell\int_{t_0}^{t}\Upsilon(\varsigma)\left(\int_{t_0}^{\varsigma}\Upsilon(\jmath)\nabla_a^{\gamma}\jmath\right)^{\ell-1}\nabla_a^{\gamma}\varsigma. \tag{32}$$

**Proof.** We fix the point $t \in \mathbb{T}$, $t \geq t_0$. Assume $t$ is left-dense, by making a modification of the chain rule, we obtain

$$\left[\left(\int_{t_0}^{t}\Upsilon(\imath)\nabla_a^{\gamma}\imath\right)^{\ell}\right]^{\nabla_a^{\gamma}} = \ell\left(\int_{t_0}^{t}\Upsilon(\imath)\nabla_a^{\gamma}\imath\right)^{\ell-1}\Upsilon(t). \tag{33}$$

Letting $t$ be a left-scattered point. Define $0 \leq \lambda_1, \lambda_2 \in \mathbb{R}$ as

$$\lambda_1 = \int_{t_0}^{\rho(t)}\Upsilon(\imath)\nabla_a^{\gamma}\imath, \tag{34}$$

and

$$\lambda_2 = \int_{\rho(t)}^{t}\Upsilon(\imath)\nabla_a^{\gamma}\imath = \nu(t)\Upsilon(t)\widehat{G}_{\gamma-1}(\sigma^{\gamma-1}(t), a), \tag{35}$$

where we used (17). Using the differentiability of the real-valued function $F : \mathbb{R} \to \mathbb{R}$, where $F = \lambda_1^{\ell}$ and applying the mean value theorem, we obtain

$$(\lambda_1+\lambda_2)^{\ell}-\lambda_1 = \ell\tau^{\ell-1}\lambda_2 \leq \ell(\lambda_1+\lambda_2)^{\ell-1}\lambda_2, \quad \text{for some real} \quad \tau \in [\lambda_1, \lambda_1+\lambda_2]. \quad (36)$$

From (1), (11), (34), (35) and (36), we obtain

$$\begin{aligned}
\left[\left(\int_{t_0}^{t} \Upsilon(\iota)\nabla_a^{\gamma}\iota\right)^{\ell}\right]^{\nabla_a^{\gamma}} &= \left[\left(\int_{t_0}^{t} \Upsilon(\iota)\nabla_a^{\gamma}\iota\right)^{\ell}\right]^{\nabla} \widehat{G}_{1-\gamma}(t,a) \\
&= \frac{1}{\nu(t)}[(\lambda_1+\lambda_2)^{\ell}-\lambda_1^{\ell}]\widehat{G}_{1-\gamma}(t,a) \\
&\leq \frac{1}{\nu(t)}\ell(\lambda_1+\lambda_2)^{\ell-1}\lambda_2\widehat{G}_{1-\gamma}(t,a) \qquad (37)\\
&= \frac{1}{\nu(t)}\ell(\lambda_1+\lambda_2)^{\ell-1}\nu(t)\Upsilon(t)\widehat{G}_{\gamma-1}(\sigma^{\gamma-1}(t),a)\widehat{G}_{1-\gamma}(t,a) \\
&= \ell\left(\int_{t_0}^{t} \Upsilon(\iota)\nabla_a^{\gamma}\iota\right)^{\ell-1}\Upsilon(\iota),
\end{aligned}$$

where we used (6) in the last step. Thus, (37) holds in either case. From (37), taking $\nabla_a^{\gamma}$ integral for both sides, we obtain the required inequality (32). This completes the proof. □

**Lemma 9** (Generalized $(\gamma, a)$-nabla Hölder fractional inequality on timescales)**.** *Let $d, b \in \mathbb{T}$ where $b > d$ . If $\gamma \in (0,1]$ and $\eta, \xi : \mathbb{T} \longrightarrow \mathbb{R}$, then*

$$\int_d^b |\eta(t)\xi(t)|\nabla_a^{\gamma}t \leq \left(\int_d^b |\eta(t)|^p\nabla_a^{\gamma}t\right)^{1/p}\left(\int_d^b |\xi(t)|^q\nabla_a^{\gamma}t\right)^{1/q}, \qquad (38)$$

*where $p, q > 1$ and $1/p + 1/q = 1$. This inequality is reversed if $0 < p < 1$ and if $p < 0$, or $q < 0$.*

**Proof.** Setting

$$A = \frac{|\eta(t)|}{\left(\int_d^b |\eta(s)|^p\nabla_a^{\gamma}s\right)^{\frac{1}{p}}}, \quad and \quad B = \frac{|\xi(t)|}{\left(\int_d^b |\xi(s)|^q\nabla_a^{\gamma}s\right)^{\frac{1}{q}}},$$

and applying the Young inequality $AB \leq A^p/p + B^q/q$, where $A$, $B$ are nonnegative, $p > 1$ and $1/p + 1/q = 1$ we see that

$$\begin{aligned}
\int_d^b A(t)B(t)\nabla_a^{\gamma}t &\leqslant \int_d^b A^p(t)/p + B^q(t)/q\nabla_a^{\gamma}t \\
&= \int_d^b \left[\frac{|\eta(t)|^p}{p\left(\int_d^b |\eta(s)|^p\nabla_a^{\gamma}s\right)} + \frac{|\xi(t)|^q}{q\left(\int_d^b |\xi(s)|^q\nabla_a^{\gamma}s\right)}\right]\nabla_a^{\gamma}t \\
&= 1/p\int_d^b \frac{|\eta(t)|^p}{\left(\int_d^b |\eta(s)|^p\nabla_a^{\gamma}s\right)}\nabla_a^{\gamma}t + 1/q\int_d^b \frac{|\xi(t)|^q}{\left(\int_d^b |\xi(s)|^q\nabla_a^{\gamma}s\right)}\nabla_a^{\gamma}t \\
&= 1/p + 1/q = 1
\end{aligned}$$

which is the desired inequality (38). On the other hand, without loss of generality, we assume that $p < 0$. Set $P = -p/q$ and $Q = 1/q$. Then $1/P + 1/Q = 1$. From (38) we have that

$$\int_d^b |F(t)G(t)|\nabla_a^{\gamma}t \leq \left(\int_d^b |F(t)|^P\nabla_a^{\gamma}t\right)^{1/P}\left(\int_d^b |G(t)|^Q\nabla_a^{\gamma}t\right)^{1/Q}.$$

Letting $F(t) = \eta^{-q}(t)$ and $G(t) = \eta^{q}(t)\xi^{q}(t)$ in the last inequality, we obtain the inverse inequality of (38). The proof is complete. □

**Lemma 10** (Generalized $(\gamma, a)$-nabla Jensen's fractional inequality on timescales)**.** *Let $\delta$, $\xi \in \mathbb{T}$ and $c, d \in \mathbb{R}$. Assume that $\zeta \in C_{ld}([\delta,\xi]_{\mathbb{T}}, [c,d])$ and $r \in C_{ld}([\delta,\xi]_{\mathbb{T}}, \mathbb{R})$ are nonnegative with $\int_\delta^\xi r(t)\nabla_a^\gamma t > 0$. If $\Phi \in C_{ld}((c,d), \mathbb{R})$ is a convex function, then*

$$\Phi\left(\frac{\int_\delta^\xi r(t)\zeta(t)\nabla_a^\gamma t}{\int_\delta^\xi r(t)\nabla_a^\gamma t}\right) \leqslant \frac{\int_\delta^\xi r(t)\Phi(\zeta(t))\nabla_a^\gamma t}{\int_\delta^\xi r(t)\nabla_a^\gamma t}. \tag{39}$$

*It is easy to see that inequality (39) are turned around if $\Phi$ is concave.*

**Proof.** Because the convexity of $\Phi$. For $\imath \in (c,d)$ there exists $\omega_\imath \in \mathbb{R}$ such that

$$\omega_\imath(\chi - \imath) \leq \Phi(\chi) - \Phi(\imath), \quad \text{for all} \quad \chi \in (c,d). \tag{40}$$

Suppose

$$\imath = \frac{\int_\delta^\xi r(t)\zeta(t)\nabla_a^\gamma t}{\int_\delta^\xi r(t)\nabla_a^\gamma t}.$$

From (40) and item (vi) in Theorem 12, we obtain

$$\begin{aligned}
&\int_\delta^\xi r(t)\Phi(\zeta(t))\nabla_a^\gamma t - \left(\int_\delta^\xi r(t)\nabla_a^\gamma t\right)\Phi\left(\frac{\int_\delta^\xi r(t)\zeta(t)\nabla_a^\gamma t}{\int_\delta^\xi r(t)\nabla_a^\gamma t}\right)\\
=\ &\int_\delta^\xi r(t)\Phi(\zeta(t))\nabla_a^\gamma t - \left(\int_\delta^\xi r(t)\nabla_a^\gamma t\right)\Phi(\imath)\\
=\ &\int_\delta^\xi (\Phi(\zeta(t)) - \Phi(\imath)) r(t)\nabla_a^\gamma t\\
\geq\ &\omega_\imath \int_\delta^\xi (\zeta(s) - \imath) r(t)\nabla_a^\gamma t\\
=\ &\omega_\imath\left(\int_\delta^\xi r(t)\zeta(t)\nabla_a^\gamma t - \imath\int_\delta^\xi r(t)\nabla_a^\gamma t\right)\\
=\ &\omega_\imath\left(\int_\delta^\xi r(t)\zeta(t)\nabla_a^\gamma t - \int_\delta^\xi r(t)\zeta(t)\nabla_a^\gamma t\right)\\
=\ &0.
\end{aligned} \tag{41}$$

This gives the required inequality (39). The reversed inequality obtained directly if we put $\Phi = -\Phi$ and apply inequality (39), since $\Phi$ is convex. □

**Theorem 18.** *Suppose the time scales $\mathbb{T}$ with $\ell, \epsilon \geqslant 1$ and $s, t, t_0, w, z \in \mathbb{T}$. Assume $\delta(\tau) \geq 0$ and $\xi(\tau) \geq 0$ are r-d continuous $[t_0, w]_{\mathbb{T}}$ and $[t_0, z]_{\mathbb{T}}$ respectively and define*

$$\theta(s) := \int_{t_0}^{s} \delta(\tau)\nabla_a^\gamma \tau, \ \ and \ \ \phi(t) := \int_{t_0}^{t} \xi(\tau)\nabla_a^\gamma \tau,$$

*then for $s \in [t_0, w]_{\mathbb{T}}$ and $t \in [t_0, z]_{\mathbb{T}}$, we have that*

$$\begin{aligned}
&\int_{t_0}^{w}\int_{t_0}^{z} \frac{\theta^{q(\epsilon-\gamma-1)}(s)\phi^{q(\ell-\gamma-1)}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2q\alpha}{p}}}\nabla_a^{\gamma}s\nabla_a^{\gamma}t \\
&\leqslant C_1(\ell,\epsilon,\gamma,q)\left(\int_{t_0}^{w}(w-\rho(s))\left(\delta(s)\theta^{\epsilon-\gamma}(s)\right)^q\nabla_a^{\gamma}s\right) \\
&\times\left(\int_{t_0}^{z}(z-\rho(t))\left(\xi(t)\phi^{\ell-\gamma}(t)\right)^q\nabla_a^{\gamma}t\right)
\end{aligned} \tag{42}$$

*and*

$$\begin{aligned}
&\int_{t_0}^{w}\int_{t_0}^{z} \frac{\theta^{\epsilon-\gamma+1}(s)\phi^{\ell-\gamma+1}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{p}}}\nabla_a^{\gamma}s\nabla_a^{\gamma}t \\
&\leqslant C_2(\ell,\epsilon,\gamma,p)\left(\int_{t_0}^{w}(w-\rho(s))\left(\theta^{\epsilon-\gamma}(s)\delta(s)\right)^q\nabla_a^{\gamma}s\right)^{\frac{1}{q}} \\
&\times\left(\int_{t_0}^{z}(z-\rho(t))\left(\phi^{\ell-\gamma}(t)\xi(t)\right)^q\nabla_a^{\gamma}t\right)^{\frac{1}{q}}
\end{aligned} \tag{43}$$

*where $C_1(\ell,\epsilon,\gamma,q) = ((\epsilon-\gamma+1)(\ell-\gamma+1))^q$ and $C_2(\ell,\epsilon,\gamma,p) = (\epsilon-\gamma+1)(\ell-\gamma+1)(w-t_0)^{\frac{1}{p}}(z-t_0)^{\frac{1}{p}}$.*

**Proof.** Using the inequality (32), we obtain

$$\theta^{\epsilon-\gamma+1}(s) \leqslant (\epsilon-\gamma+1)\int_{t_0}^{s}\delta(\eta)\theta^{\epsilon-\gamma}(\eta)\nabla_a^{\gamma}\eta, \tag{44}$$

$$\phi^{\ell-\gamma+1}(t) \leqslant (\ell-\gamma+1)\int_{t_0}^{t}\xi(\eta)\phi^{\ell-\gamma}(\eta))\nabla_a^{\gamma}\eta. \tag{45}$$

We use Lemma 9. Then from (44), we obtain

$$\theta^{\epsilon-\gamma+1}(s) \le (\epsilon-\gamma+1)(s-t_0)^{\frac{1}{p}}\left(\int_{t_0}^{s}\left(\delta(\eta)\theta^{\epsilon-\gamma}(\eta)\right)^q\nabla_a^{\gamma}\eta\right)^{\frac{1}{q}}. \tag{46}$$

Applying Lemma 9. Thus, from (45), we obtain

$$\phi^{\ell-\gamma+1}(t) \leqslant (\ell-\gamma+1)(t-t_0)^{\frac{1}{p}}\left(\int_{t_0}^{t}\left(\xi(\eta)\phi^{\ell-\gamma}(\eta)\right)^q\nabla_a^{\gamma}\eta\right)^{\frac{1}{q}}. \tag{47}$$

From (46) and (47), we obtain

$$\begin{aligned}
\theta^{\epsilon-\gamma+1}(s)\phi^{\ell-\gamma+1}(t) &\leqslant (\epsilon-\gamma+1)(\ell-\gamma+1)(s-t_0)^{\frac{1}{p}}(t-t_0)^{\frac{1}{p}} \\
&\times\left(\int_{t_0}^{s}\left(\delta(\eta)\theta^{\epsilon-\gamma}(\eta)\right)^q\nabla_a^{\gamma}\eta\right)^{\frac{1}{q}} \\
&\times\left(\int_{t_0}^{t}\left(\xi(\eta)\phi^{\ell-\gamma}(\eta)\right)^q\nabla_a^{\gamma}\eta\right)^{\frac{1}{q}}.
\end{aligned} \tag{48}$$

From inequality (48), we have

$$\theta^{q(\epsilon-\gamma-1)}(s)\phi^{q(\ell-\gamma-1)}(t) \leqslant ((\epsilon-\gamma+1)(\ell-\gamma+1))^{q}(s-t_0)^{\frac{q}{p}}(t-t_0)^{\frac{q}{p}} \times\left(\int_{t_0}^{s}(\delta(\eta)\theta^{\epsilon-\gamma}(\eta))^{q}\nabla_a^{\gamma}\eta\right) \times\left(\int_{t_0}^{t}(\xi(\eta)\phi^{\ell-\gamma}(\eta))^{q}\nabla_a^{\gamma}\eta\right). \tag{49}$$

Using Lemma 5 in (48) and (49) gives

$$\theta^{\epsilon-\gamma+1}(s)\phi^{\ell-\gamma+1}(t) \leqslant (\epsilon-\gamma+1)(\ell-\gamma+1)\Big(\psi(s-t_0)+\psi^{*}(t-t_0)\Big)^{\frac{1}{p}} \times\left(\int_{t_0}^{s}(\delta(\eta)\theta^{\epsilon-\gamma}(\eta))^{q}\nabla_a^{\gamma}\eta\right)^{\frac{1}{q}} \times\left(\int_{t_0}^{t}(\xi(\eta)\phi^{\ell-\gamma}(\eta))^{q}\nabla_a^{\gamma}\eta\right)^{\frac{1}{q}}, \tag{50}$$

$$\theta^{q(\epsilon-\gamma-1)}(s)\phi^{q(\ell-\gamma-1)}(t) \leqslant ((\epsilon-\gamma+1)(\ell-\gamma+1))^{q}\Big(\psi(s-t_0)+\psi^{*}(t-t_0)\Big)^{\frac{q}{p}} \times\left(\int_{t_0}^{s}(\delta(\eta)\theta^{\epsilon-\gamma}(\eta))^{q}\nabla_a^{\gamma}\eta\right) \times\left(\int_{t_0}^{t}(\xi(\eta)\phi^{\ell-\gamma}(\eta))^{q}\nabla_a^{\gamma}\eta\right). \tag{51}$$

Using Lemma 7 in (50) and (51) gives

$$\theta^{\epsilon-\gamma+1}(s)\phi^{\ell-\gamma+1}(t) \leqslant (\epsilon-\gamma+1)(\ell-\gamma+1)\Big(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^{*}(t-t_0)|^{\frac{1}{2\beta}}\Big)^{\frac{2\alpha}{p}} \times\left(\int_{t_0}^{s}(\delta(\eta)\theta^{\epsilon-\gamma}(\eta))^{q}\nabla_a^{\gamma}\eta\right)^{\frac{1}{q}} \times\left(\int_{t_0}^{t}(\xi(\eta)\phi^{\ell-\gamma}(\eta))^{q}\nabla_a^{\gamma}\eta\right)^{\frac{1}{q}}, \tag{52}$$

$$\theta^{q(\epsilon-\gamma-1)}(s)\phi^{q(\ell-\gamma-1)}(t) \leqslant ((\epsilon-\gamma+1)(\ell-\gamma+1))^{q}\Big(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^{*}(t-t_0)|^{\frac{1}{2\beta}}\Big)^{\frac{2q\alpha}{p}} \times\left(\int_{t_0}^{s}(\delta(\eta)\theta^{\epsilon-\gamma}(\eta))^{q}\nabla_a^{\gamma}\eta\right) \times\left(\int_{t_0}^{t}(\xi(\eta)\phi^{\ell-\gamma}(\eta))^{q}\nabla_a^{\gamma}\eta\right). \tag{53}$$

Dividing both sides of (52) and (53) by $\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{p}}$ and $\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2q\alpha}{p}}$ respectively, we obtain that

$$\begin{aligned}
&\frac{\theta^{\epsilon-\gamma+1}(s)\phi^{\ell-\gamma+1}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{p}}}\\
&\leqslant (\epsilon-\gamma+1)(\ell-\gamma+1)\left(\int_{t_0}^{s}\left(\delta(\eta)\theta^{\epsilon-\gamma}(\eta)\right)^q\nabla_a^\gamma\eta\right)^{\frac{1}{q}}\\
&\times\left(\int_{t_0}^{t}\left(\xi(\eta)\phi^{\ell-\gamma}(\eta)\right)^q\nabla_a^\gamma\eta\right)^{\frac{1}{q}},
\end{aligned}\tag{54}$$

$$\begin{aligned}
&\frac{\theta^{q(\epsilon-\gamma-1)}(s)\phi^{q(\ell-\gamma-1)}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2q\alpha}{p}}}\\
&\leqslant ((\epsilon-\gamma+1)(\ell-\gamma+1))^q\left(\int_{t_0}^{s}\left(\delta(\eta)\theta^{\epsilon-\gamma}(\eta)\right)^q\nabla_a^\gamma\eta\right)\\
&\times\left(\int_{t_0}^{t}\left(\xi(\eta)\phi^{\ell-\gamma}(\eta)\right)^q\nabla_a^\gamma\eta\right).
\end{aligned}\tag{55}$$

From (54) using Lemma 9 we obtain

$$\begin{aligned}
&\int_{t_0}^{w}\int_{t_0}^{z}\frac{\theta^{\epsilon-\gamma+1}(s)\phi^{\ell-\gamma+1}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{p}}}\nabla_a^\gamma s\nabla_a^\gamma t\\
&\leqslant (\epsilon-\gamma+1)(\ell-\gamma+1)(w-t_0)^{\frac{1}{p}}(z-t_0)^{\frac{1}{p}}\left(\int_{t_0}^{w}\left(\int_{t_0}^{s}\left(\delta(\eta)\theta^{\epsilon-\gamma}(\eta)\right)^q\nabla_a^\gamma\eta\right)\nabla_a^\gamma s\right)^{\frac{1}{q}}\\
&\times\left(\int_{t_0}^{z}\left(\int_{t_0}^{t}\left(\xi(\eta)\phi^{\ell-\gamma}(\eta)\right)^q\nabla_a^\gamma\eta\right)\nabla_a^\gamma t\right)^{\frac{1}{q}}\\
&\leqslant (\epsilon-\gamma+1)(\ell-\gamma+1)(w-t_0)^{\frac{1}{p}}(z-t_0)^{\frac{1}{p}}\left(\int_{t_0}^{w}(w-\rho(s))\left(\delta(s)\theta^{\epsilon-\gamma}(s)\right)^q\nabla_a^\gamma s\right)^{\frac{1}{q}}\\
&\times\left(\int_{t_0}^{z}(z-\rho(t))\left(\xi(t)\phi^{\ell-\gamma}(t)\right)^q\nabla_a^\gamma t\right)^{\frac{1}{q}},
\end{aligned}$$

From (55), we obtain

$$\begin{aligned}
&\int_{t_0}^{w}\int_{t_0}^{z}\frac{\theta^{q(\epsilon-\gamma-1)}(s)\phi^{q(\ell-\gamma-1)}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2q\alpha}{p}}}\nabla_a^\gamma s\nabla_a^\gamma t\\
&\leqslant ((\epsilon-\gamma+1)(\ell-\gamma+1))^q\int_{t_0}^{w}\left(\int_{t_0}^{s}\left(\delta(\eta)\theta^{\epsilon-\gamma}(\eta)\right)^q\nabla_a^\gamma\eta\right)\nabla_a^\gamma s\\
&\times\int_{t_0}^{z}\left(\int_{t_0}^{t}\left(\xi(\eta)\phi^{\ell-\gamma}(\eta)\right)^q\nabla_a^\gamma\eta\right)\nabla_a^\gamma t\\
&\leqslant ((\epsilon-\gamma+1)(\ell-\gamma+1))^q\left(\int_{t_0}^{w}(w-\rho(s))\left(\delta(s)\theta^{\epsilon-\gamma}(s)\right)^q\nabla_a^\gamma t\right)\\
&\times\left(\int_{t_0}^{z}(z-\rho(s))\left(\xi(t)\phi^{\ell-\gamma}(t)\right)^q\nabla_a^\gamma t\right).
\end{aligned}$$

This completes the proof. □

**Remark 2.** *In (43), as special case, if we take* $\psi(w) = \dfrac{w^2}{2}$, *we have* $\psi^*(w) = \dfrac{w^2}{2}$ *see [13], so we get*

$$\begin{aligned}
&\int_{t_0}^{w}\int_{t_0}^{z} \frac{\theta^{\epsilon-\gamma+1}(s)\phi^{\ell-\gamma+1}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{p}}}\nabla_a^\gamma s \nabla_a^\gamma t \\
&= \int_{t_0}^{w}\int_{t_0}^{z} \frac{\theta^{\epsilon-\gamma+1}(s)\phi^{\ell-\gamma+1}(t)}{\left((s-t_0))^{\frac{1}{\beta}}+(t-t_0)^{\frac{1}{\beta}}\right)^{\frac{2\alpha}{p}}}\nabla_a^\gamma s \nabla_a^\gamma t \qquad (56)\\
&\leqslant \left(\frac{1}{2}\right)^{\frac{2\alpha}{p\beta}} C_2(\ell,\epsilon,\gamma,p)\left(\int_{t_0}^{w}(w-\rho(s))\left(\theta^{\epsilon-\gamma}(s)\delta(s)\right)^q \nabla_a^\gamma s\right)^{\frac{1}{q}} \\
&\times\left(\int_{t_0}^{z}(z-\rho(t))\left(\phi^{\ell-\gamma}(t)\xi(t)\right)^q \nabla_a^\gamma t\right)^{\frac{1}{q}}.
\end{aligned}$$

*Consequently, for* $\alpha = \beta = 1$, *inequality (56) produces*

$$\begin{aligned}
&\int_{t_0}^{w}\int_{t_0}^{z} \frac{\theta^{\epsilon-\gamma+1}(s)\phi^{\ell-\gamma+1}(t)}{\left(s+t-2t_0\right)^{\frac{2}{p}}}\nabla_a^\gamma s \nabla_a^\gamma t \\
&\leqslant \left(\frac{1}{2}\right)^{\frac{2}{p}} C_2(\ell,\epsilon,\gamma,p)\left(\int_{t_0}^{w}(w-\rho(s))\left(\theta^{\epsilon-\gamma}(s)\delta(s)\right)^q \nabla_a^\gamma s\right)^{\frac{1}{q}} \\
&\quad\times\left(\int_{t_0}^{z}(z-\rho(t))\left(\phi^{\ell-\gamma}(t)\xi(t)\right)^q \nabla_a^\gamma t\right)^{\frac{1}{q}}.
\end{aligned}$$

*Putting* $p = q = 2$, *we obtain*

$$\begin{aligned}
&\int_{t_0}^{w}\int_{t_0}^{z} \frac{\theta^{\epsilon-\gamma+1}(s)\phi^{\ell-\gamma+1}(t)}{t+s-2t_0}\nabla_a^\gamma s \nabla_a^\gamma t \\
&\leqslant \frac{1}{2}(\epsilon-\gamma+1)(\ell-\gamma+1)\left((w-t_0)\int_{t_0}^{w}(w-\rho(s))\left(\theta^{\epsilon-\gamma}(s)\delta(s)\right)^q \nabla_a^\gamma s\right)^{\frac{1}{2}} \\
&\quad\times\left((z-t_0)\int_{t_0}^{z}(z-\rho(t))\left(\phi^{\ell-\gamma}(t)\xi(t)\right)^q \nabla_a^\gamma t\right)^{\frac{1}{2}}.
\end{aligned}$$

**Remark 3.** *In Remark 2, if we take* $\gamma = 1$, *we obtain [15] [Theorem 3.3].*

**Theorem 19.** *Suppose* $\xi(\eta)$, $\theta(s)$, $\phi(t)$ *and* $\delta(\tau)$, *are defined as in Theorem 18, thus*

$$\begin{aligned}
&\int_{t_0}^{w}\int_{t_0}^{z} \frac{\theta^{q}(s)\phi^{q}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2q\alpha}{p}}}\nabla_a^\gamma s \nabla_a^\gamma t \\
&\leqslant \left(\int_{t_0}^{w}(w-\rho(s))\delta^q(s)\nabla_a^\gamma s\right)\left(\int_{t_0}^{z}(z-\rho(t))\xi^q(t)\nabla_a^\gamma t\right)
\end{aligned}$$

*and*

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\theta(s)\phi(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{p}}}\nabla_a^{\gamma}s\nabla_a^{\gamma}t$$

$$\leqslant (w-t_0)^{\frac{1}{p}}(z-t_0)^{\frac{1}{p}}\left(\int_{t_0}^{w}(w-\rho(s))\delta^q(s)\nabla_a^{\gamma}s\right)^{\frac{1}{q}}\left(\int_{t_0}^{z}(z-\rho(t))\xi^q(t)\nabla_a^{\gamma}t\right)^{\frac{1}{q}}.$$

**Proof.** In (42) and (43) taking $\epsilon=\ell=1$. This gives our claim. □

In Theorem 18, if we chose $\mathbb{T}=\mathbb{R}$, then the next results:

**Corollary 2.** *If $\delta(s)\geq 0$, $\xi(t)\geq 0$. Define $\theta(s):=\int_0^s\delta(\eta)(\eta-a)^{\gamma-1}d\eta$ and $\phi(t):=\int_0^t\xi(\eta)(\eta-a)^{\gamma-1}d\eta$, then*

$$\int_0^w\int_0^z\frac{\theta^{q(\epsilon-\gamma-1)}(s)\phi^{q(\ell-\gamma-1)}(t)}{\left(|\psi(s)|^{\frac{1}{2\beta}}+|\psi^*(t)|^{\frac{1}{2\beta}}\right)^{\frac{2q\alpha}{p}}}[(s-a)(t-a)]^{\gamma-1}dsdt$$

$$\leqslant C_1(\ell,\epsilon,\gamma,q)\left(\int_0^w(w-s)\big(\delta(s)\theta^{\epsilon-\gamma}(s)\big)^q(s-a)^{\gamma-1}ds\right)$$

$$\times\left(\int_0^z(z-t)\big(\xi(t)\phi^{\ell-\gamma}(t)\big)^q(t-a)^{\gamma-1}dt\right).$$

$$\int_0^w\int_0^z\frac{\theta^{\epsilon-\gamma+1}(s)\phi^{\ell-\gamma+1}(t)}{\left(|\psi(s)|^{\frac{1}{2\beta}}+|\psi^*(t)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{p}}}[(s-a)(t-a)]^{\gamma-1}dsdt$$

$$\leqslant C_3(\ell,\epsilon,\gamma,p)\left(\int_0^w(w-s)\left(\theta^{\epsilon-\gamma}(s)\delta(s)\right)^q(s-a)^{\gamma-1}ds\right)^{\frac{1}{q}}$$

$$\times\left(\int_0^z(z-t)\left(\phi^{\ell-\gamma}(t)\xi(t)\right)^q(t-a)^{\gamma-1}dt\right)^{\frac{1}{q}}$$

*where*

$$C_3(\ell,\epsilon,\gamma,p)=(\epsilon-\gamma+1)(\ell-\gamma+1)(wz)^{\frac{1}{p}}.$$

In Theorem 18, if we chose $\mathbb{T}=\mathbb{Z}$, and the next result:

**Corollary 3.** *Let $(\xi_j)_{0\leq j\leq N}$, $(\delta_i)_{0\leq i\leq M}$ be sequences of nonnegative real numbers where $N,M\in\mathbb{N}$. Define*

$$\theta(i)=\sum_{s=0}^{i}\delta(s)(\rho^{\gamma-1}(s)-a)^{\gamma-1},\quad \phi(j)=\sum_{a_1=0}^{j}\xi(a)(\rho^{\gamma-1}(a)-a)^{\gamma-1}.$$

*Then*

$$\sum_{i=1}^{N}\sum_{j=1}^{M}\frac{\theta^{q(\ell-\gamma-1)}(i)\phi^{q(\epsilon-\gamma-1)}(j)(\rho^{\gamma-1}(i)-a)^{\gamma-1}(\rho^{\gamma-1}(j)-a)^{\gamma-1}}{\left(|\psi(i+1)|^{\frac{1}{2\beta}}+|\psi^{*}(j+1)|^{\frac{1}{2\beta}}\right)^{\frac{2q\alpha}{p}}}$$
$$\leqslant C_1(\epsilon,\ell,q)\left(\sum_{i=1}^{N}(N-(i+1))(\delta(i)\theta^{\ell-\gamma}(i))^{q}(\rho^{\gamma-1}(i)-a)^{\gamma-1}\right)$$
$$\times\left(\sum_{j=1}^{M}(M-(j+1))(\xi(j)\phi^{\ell-\gamma}(j))^{q}(\rho^{\gamma-1}(j)-a)^{\gamma-1}\right)$$

$$\sum_{i=1}^{N}\sum_{j=1}^{M}\frac{\theta^{\ell-\gamma+1}(i)\phi^{\epsilon-\gamma+1}(j)(\rho^{\gamma-1}(i)-a)^{\gamma-1}(\rho^{\gamma-1}(j)-a)^{\gamma-1}}{\left(|\psi(i+1)|^{\frac{1}{2\beta}}+|\psi^{*}(j+1)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{p}}}$$
$$\leqslant C_4(\epsilon,\ell,p)\left(\sum_{i=1}^{N}(N-(i+1))(\delta(i)\theta^{\ell-\gamma}(i))^{q}(\rho^{\gamma-1}(i)-a)^{\gamma-1}\right)^{\frac{1}{q}}$$
$$\times\left(\sum_{j=1}^{M}(M-(j+1))(\xi(j)\phi^{\ell-\gamma}(j))^{q}(\rho^{\gamma-1}(j)-a)^{\gamma-1}\right)^{\frac{1}{q}}$$

*where*

$$C_4(\epsilon,\ell,p)=(\epsilon-\gamma+1)(\ell-\gamma+1)(NM)^{\frac{1}{p}}.$$

**Corollary 4.** *With the hypotheses of Theorem 18, we have:*

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\theta^{q(\epsilon-\gamma-1)}(s)\phi^{q(\ell-\gamma-1)}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^{*}(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2q\alpha}{p}}}\nabla_a^{\gamma}s\nabla_a^{\gamma}t$$
$$\leqslant C_1(\ell,\epsilon,\gamma,q)\left\{\psi\left(\int_{t_0}^{w}(w-\rho(s))\big(\delta(s)\theta^{\epsilon-\gamma}(s)^{q}\nabla_a^{\gamma}s\right)\right.$$
$$\left.+\psi^{*}\left(\int_{t_0}^{z}(z-\rho(t))\big(\xi(t)\phi^{\ell-\gamma}(t)^{q}\nabla_a^{\gamma}t\right)\right\}$$

*and*

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\theta^{\epsilon-\gamma+1}(s)\phi^{\ell-\gamma+1}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^{*}(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{p}}}\nabla_a^{\gamma}s\nabla_a^{\gamma}t$$
$$\leqslant C_2(\ell,\epsilon,\gamma,p)\left\{\psi\left(\int_{t_0}^{w}(w-\rho(s))\left(\theta^{\epsilon-\gamma}(s)\delta(s)\right)^{q}\nabla_a^{\gamma}s\right)\right.$$
$$\left.+\psi^{*}\left(\int_{t_0}^{z}(z-\rho(t))\left(\phi^{\ell-\gamma}(t)\xi(t)\right)^{q}\nabla_a^{\gamma}t\right)\right\}^{\frac{1}{q}}.$$

**Proof.** Using the Fenchel-Young inequality (24) in (42) and (43). This proves the claim. □

**Theorem 20.** *Assuming the time scale $\mathbb{T}$ with $s, t, t_0, w, z \in \mathbb{T}$, $\theta(s)$ and $\phi(t)$ are defined as in Theorem 18. Suppose $\vartheta(\tau)\geq 0$ and $\zeta(\eta)\geq 0$ are right-dense continuous functions on $[t_0,w]_{\mathbb{T}}$ and*

$[t_0, z]_{\mathbb{T}}$ *respectively. Suppose that* $\Phi \geq 0$ *and* $\Psi \geq 0$ *are convex, and submultiplicative on* $[0, \infty)$. *Furthermore, assume that*

$$F(s) := \int_{t_0}^{s} \vartheta(\tau) \nabla_a^{\gamma} \tau, \ \ \textit{and} \ \ G(t) := \int_{t_0}^{t} \zeta(\eta) \nabla_a^{\gamma} \eta, \tag{57}$$

*then for* $s \in [t_0, w]_{\mathbb{T}}$ *and* $t \in [t_0, z]_{\mathbb{T}}$, *we have that*

$$\begin{aligned}
&\int_{t_0}^{w} \int_{t_0}^{z} \frac{\Phi(\theta(s))\Psi(\phi(t))}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}} + |\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{p}}} \nabla_a^{\gamma} s \nabla_a^{\gamma} t \\
&\leqslant M_1(p) \left( \int_{t_0}^{w} (w - \rho(s)) \left( \vartheta(s) \Phi\left[ \frac{\delta(s)}{\vartheta(s)} \right] \right)^{q} \nabla_a^{\gamma} s \right)^{\frac{1}{q}} \\
&\quad \times \left( \int_{t_0}^{z} (z - \rho(t)) \left( \zeta(t) \Psi\left[ \frac{\xi(t)}{\zeta(t)} \right] \right)^{q} \nabla_a^{\gamma} t \right)^{\frac{1}{q}}
\end{aligned} \tag{58}$$

*where*

$$M_1(p) = \left\{ \int_{t_0}^{w} \left[ \frac{\Phi(F(s))}{F(s)} \right]^{p} \nabla_a^{\gamma} s \right\}^{\frac{1}{p}} \left\{ \int_{t_0}^{z} \left[ \frac{\Psi(G(t))}{G(t)} \right]^{p} \nabla_a^{\gamma} t \right\}^{\frac{1}{p}}.$$

**Proof.** From the properties of $\Phi$ and using (10), we obtain

$$\begin{aligned}
\Phi(\theta(s)) &= \Phi\left( \frac{F(s) \int_{t_0}^{s} \vartheta(\tau) \frac{\delta(\tau)}{\vartheta(\tau)} \nabla_a^{\gamma} \tau}{\int_{t_0}^{s} \vartheta(\tau) \nabla_a^{\gamma} \tau} \right) \\
&\leqslant \Phi(F(s) \Phi\left( \frac{\int_{t_0}^{s} \vartheta(\tau) \frac{\delta(\tau)}{\vartheta(\tau)} \nabla_a^{\gamma} \tau}{\int_{t_0}^{s} \vartheta(\tau) \nabla_a^{\gamma} \tau} \right) \\
&\leqslant \frac{\Phi(F(s))}{F(s)} \int_{t_0}^{s} \vartheta(\tau) \Phi\left( \frac{\delta(\tau)}{\vartheta(\tau)} \right) \nabla_a^{\gamma} \tau.
\end{aligned} \tag{59}$$

Using (9) in (59), we see that

$$\Phi(\theta(s)) \leqslant \frac{\Phi(F(s))}{F(s)} (s - t_0)^{\frac{1}{p}} \left( \int_{t_0}^{s} \left( \vartheta(\tau) \Phi\left[ \frac{\delta(\tau)}{\vartheta(\tau)} \right] \right)^{q} \nabla_a^{\gamma} \tau \right)^{\frac{1}{q}}. \tag{60}$$

Additionally, from the convexity and submultiplicative property of $\Psi$, we obtain using (10) and (9):

$$\Psi(\phi(t)) \leqslant \frac{\Psi(G(t))}{G(t)} (t - t_0)^{\frac{1}{p}} \left( \int_{t_0}^{t} \left( \zeta(\eta) \Psi\left[ \frac{\xi(\eta)}{\zeta(\eta)} \right] \right)^{q} \nabla_a^{\gamma} \eta \right)^{\frac{1}{q}}. \tag{61}$$

From (60) and (61), we have

$$\begin{aligned}
\Phi(\theta(s))\Psi(\phi(t)) \ &\leqslant \ (s - t_0)^{\frac{1}{p}} (t - t_0)^{\frac{1}{p}} \left( \frac{\Phi(F(s))}{F(s)} \left( \int_{t_0}^{s} \left( \vartheta(\tau) \Phi\left[ \frac{\delta(\tau)}{\vartheta(\tau)} \right] \right)^{q} \nabla_a^{\gamma} \tau \right)^{\frac{1}{q}} \right) \\
&\quad \times \left( \frac{\Psi(G(t))}{G(t)} \left( \int_{t_0}^{t} \left( \zeta(\eta) \Psi\left[ \frac{\xi(\eta)}{\zeta(\eta)} \right] \right)^{q} \nabla_a^{\gamma} \eta \right)^{\frac{1}{q}} \right)
\end{aligned} \tag{62}$$

Using (24) on $(s-t_0)^{\frac{1}{p}}(t-t_0)^{\frac{1}{p}}$ gives:

$$\begin{aligned}\Phi(\theta(s))\Psi(\phi(t)) \quad &\leqslant \quad \left(\psi(s-t_0)+\psi^*(t-t_0)\right)^{\frac{1}{p}}\left(\frac{\Phi(F(s))}{F(s)}\left(\int_{t_0}^{s}\left(\vartheta(\tau)\Phi\left[\frac{\delta(\tau)}{\vartheta(\tau)}\right]\right)^{q}\nabla_a^{\gamma}\tau\right)^{\frac{1}{q}}\right) \\ &\times\left(\frac{\Psi(G(t))}{G(t)}\left(\int_{t_0}^{t}\left(\zeta(\eta)\Psi\left[\frac{\xi(\eta)}{\zeta(\eta)}\right]\right)^{q}\nabla_a^{\gamma}\eta\right)^{\frac{1}{q}}\right). \end{aligned} \tag{63}$$

Applying Lemma 7 on the right-hand side of (63), we see that

$$\begin{aligned}\Phi(\theta(s))\Psi(\phi(t)) &\leqslant \left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{p}} \\ &\times\left(\frac{\Phi(F(s))}{F(s)}\left(\int_{t_0}^{s}\left(\vartheta(\tau)\Phi\left[\frac{\delta(\tau)}{\vartheta(\tau)}\right]\right)^{q}\nabla_a^{\gamma}\tau\right)^{\frac{1}{q}}\right) \\ &\times\left(\frac{\Psi(G(t))}{G(t)}\left(\int_{t_0}^{t}\left(\zeta(\eta)\Psi\left[\frac{\xi(\eta)}{\zeta(\eta)}\right]\right)^{q}\nabla_a^{\gamma}\eta\right)^{\frac{1}{q}}\right). \end{aligned} \tag{64}$$

From (64), we have

$$\begin{aligned}\frac{\Phi(\theta(s))\Psi(\phi(t))}{\left(|\psi(s-t_0|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{p}}} &\leqslant \left(\frac{\Phi(F(s))}{F(s)}\left(\int_{t_0}^{s}\left(\vartheta(\tau)\Phi\left[\frac{\delta(\tau)}{\vartheta(\tau)}\right]\right)^{q}\nabla_a^{\gamma}\tau\right)^{\frac{1}{q}}\right) \\ &\times\left(\frac{\Psi(G(t))}{G(t)}\left(\int_{t_0}^{t}\left(\zeta(\eta)\Psi\left[\frac{\xi(\eta)}{\zeta(\eta)}\right]\right)^{q}\nabla_a^{\gamma}\eta\right)^{\frac{1}{q}}\right). \end{aligned} \tag{65}$$

From (65), we obtain

$$\begin{aligned}&\int_{t_0}^{w}\int_{t_0}^{z}\frac{\Phi(\theta(s))\Psi(\phi(t))}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{p}}}\nabla_a^{\gamma}s\nabla_a^{\gamma}t \\ &\leqslant \int_{t_0}^{w}\frac{\Phi(F(s))}{F(s)}\left(\int_{t_0}^{s}\left(\vartheta(\tau)\Phi\left[\frac{\delta(\tau)}{\vartheta(\tau)}\right]\right)^{q}\nabla_a^{\gamma}\tau\right)^{\frac{1}{q}}\nabla_a^{\gamma}s \\ &\quad\times\int_{t_0}^{z}\frac{\Psi(G(t))}{G(t)}\left(\int_{t_0}^{t}\left(\zeta(\eta)\Psi\left[\frac{\xi(\eta)}{\zeta(\eta)}\right]\right)^{q}\nabla_a^{\gamma}\eta\right)^{\frac{1}{q}}\nabla_a^{\gamma}t. \end{aligned} \tag{66}$$

From (66), using (9), we have

$$\begin{aligned}&\int_{t_0}^{w}\int_{t_0}^{z}\frac{\Phi(\theta(s))\Psi(\phi(t))}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{p}}}\nabla_a^{\gamma}s\nabla_a^{\gamma}t \\ &\leqslant \left\{\int_{t_0}^{w}\left(\frac{\Phi(F(s))}{F(s)}\right)^{p}\nabla_a^{\gamma}s\right\}^{\frac{1}{p}}\left(\int_{t_0}^{w}\int_{t_0}^{s}\left(\vartheta(\tau)\Phi\left[\frac{\delta(\tau)}{\vartheta(\tau)}\right]\right)^{q}\nabla_a^{\gamma}\tau\nabla_a^{\gamma}s\right)^{\frac{1}{q}} \\ &\times\left\{\int_{t_0}^{z}\left(\frac{\Psi(G(t))}{G(t)}\right)^{p}\nabla_a^{\gamma}t\right\}^{\frac{1}{p}}\left(\int_{t_0}^{z}\int_{t_0}^{t}\left(\zeta(\eta)\Psi\left[\frac{\xi(\eta)}{\zeta(\eta)}\right]\right)^{q}\nabla_a^{\gamma}\eta\nabla_a^{\gamma}t\right)^{\frac{1}{q}}. \end{aligned} \tag{67}$$

From (67), using Lemma 6, we obtain

$$\begin{aligned}&\int_{t_0}^{w}\int_{t_0}^{z}\frac{\Phi(\theta(s))\Psi(\phi(t))}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{p}}}\nabla_a^{\gamma}s\nabla_a^{\gamma}t \\ &\leqslant M_1(p)\left(\int_{t_0}^{w}(w-\rho(s))\left(\vartheta(s)\Phi\left[\frac{\delta(s)}{\vartheta(s)}\right]\right)^{q}\nabla_a^{\gamma}s\right)^{\frac{1}{q}} \\ &\quad\times\left(\int_{t_0}^{z}(z-\rho(t))\left(\zeta(t)\Psi\left[\frac{\xi(t)}{\zeta(t)}\right]\right)^{q}\nabla_a^{\gamma}t\right)^{\frac{1}{q}}. \end{aligned}$$

where

$$M_1(p) = \left\{ \int_{t_0}^{w} \left( \frac{\Phi(F(s))}{F(s)} \right)^p \nabla_a^\gamma s \right\}^{\frac{1}{p}} \left\{ \int_{t_0}^{z} \left( \frac{\Psi(G(t))}{G(t)} \right)^p \nabla_a^\gamma t \right\}^{\frac{1}{p}}.$$

This completes the proof. □

**Remark 4.** *In Theorem 20, as special case, if we take* $\psi(w) = \frac{w^2}{2}$, $\psi^*(w) = \frac{w^2}{2}$, *and by following the same procedure employed in Remark 2, then we obtain [15] [Theorem 3.5].*

In Theorem 20, taking $\mathbb{T} = \mathbb{R}$ and $\gamma = 1$ we have the result:

**Corollary 5.** *Assume that* $\delta(s) \geq 0$, $\xi(t) \geq 0$, $\vartheta(\tau) \geq 0$ *and* $\zeta(\eta) \geq 0$, *we define*

$$\theta(s) := \int_0^s \delta(\eta) d\eta, \quad \phi(t) := \int_0^t \xi(\eta) d\eta, \quad F(s) := \int_0^s \vartheta(\tau) d\tau, \quad and \quad G(t) := \int_0^t \zeta(\eta) d\eta.$$

*Then*

$$\int_0^w \int_0^z \frac{\Phi(\theta(s)\Psi(\phi(t))}{\left( |\psi(s)|^{\frac{1}{2\beta}} + |\psi^*(t)|^{\frac{1}{2\beta}} \right)^{\frac{2\alpha}{p}}} ds dt \leqslant M_2(p) \left( \int_0^w (w-s) \left( \vartheta(s) \Phi\left( \frac{\delta(s)}{\vartheta(s)} \right) \right)^q ds \right)^{\frac{1}{q}} \times \left( \int_0^z (z-t) \left( \zeta(t) \Psi\left( \frac{\xi(t)}{\zeta(t)} \right) \right)^q dt \right)^{\frac{1}{q}}$$

*where*

$$M_2(p) = \left\{ \int_0^w \left( \frac{\Phi(F(s))}{F(s)} \right)^p ds \right\}^{\frac{1}{p}} \left\{ \int_0^z \left( \frac{\Psi(G(t))}{G(t)} \right)^p dt \right\}^{\frac{1}{p}}.$$

In Theorem 20, taking $\mathbb{T} = \mathbb{Z}$, and $\gamma = 1$ gives the result:

**Corollary 6.** *Let* $(\xi_j)_{0 \leq j \leq N}$, $(\delta_i)_{0 \leq i \leq M}$, $(\vartheta_j)_{0 \leq j \leq N}$, $(\zeta_i)_{0 \leq i \leq M}$ *be sequences of nonnegative real numbers where* $N, M \in \mathbb{N}$. *Define*

$$\theta(i) = \sum_{s=0}^{i} \delta(s), \quad \phi(j) = \sum_{a=0}^{j} \xi(a), \quad F(i) = \sum_{s=0}^{i} \vartheta(s) \quad and \quad G(j) = \sum_{a=0}^{j} \zeta(a).$$

*Then*

$$\sum_{i=1}^{N} \sum_{j=1}^{M} \frac{\Phi(\theta(i))\Psi(\phi(j))}{\left( |\psi(i+1)|^{\frac{1}{2\beta}} + |\psi^*(j+1)|^{\frac{1}{2\beta}} \right)^{\frac{2\alpha}{p}}} \leqslant M_3(p) \left\{ \sum_{i=1}^{N} (N-(i+1)) \left( \vartheta(i) \Phi\left[ \frac{\delta(i)}{\vartheta(i)} \right] \right)^q \right\}^{\frac{1}{q}} \times \left\{ \sum_{j=1}^{M} (M-(j+1)) \left( \zeta(j) \Psi\left[ \frac{\xi(j)}{\zeta(j)} \right] \right)^q \right\}^{\frac{1}{q}}$$

*where*

$$M_3(p) = \left\{ \sum_{i=1}^{N} \left( \frac{\Phi(F(i))}{F(i)} \right)^p \right\}^{\frac{1}{p}} \left\{ \sum_{j=1}^{M} \left( \frac{\Psi(G(j))}{G(j)} \right)^p \right\}^{\frac{1}{p}}$$

**Remark 5.** *In Corollary 6, if* $p = q = 2$ *we obtain the result due to Hamiaz and Abuelela [22] [Theorem 5].*

**Corollary 7.** *Under the hypotheses of Theorem 20 the following inequality hold:*

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\Phi(\theta(s))\Psi(\phi(t))}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{p}}}\nabla_a^{\gamma}s\nabla_a^{\gamma}t$$
$$\leqslant M_1(p)\left[\psi\left(\int_{t_0}^{w}(w-\rho(s))\left(\vartheta(s)\Phi\left(\frac{\delta(s)}{\vartheta(s)}\right)\right)^{q}\nabla_a^{\gamma}s\right)\right.$$
$$\left.+\psi^*\left(\int_{t_0}^{z}(z-\rho(t))\left(\zeta(t)\Psi\left(\frac{\xi(t)}{\zeta(t)}\right)\right)^{q}\nabla_a^{\gamma}t\right)\right]^{\frac{1}{q}}.$$

**Proof.** Using (24) in (58). This proves our claim. □

**Lemma 11.** *With hypotheses of Theorem 20, we obtain:*

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\Phi(\theta(s))^2\Psi(\phi(t))^2}{\left(\psi(s-t_0)+\psi^*(t-t_0)\right)}\nabla_a^{\gamma}s\nabla_a^{\gamma}t$$
$$\leqslant M_4\left\{\int_{t_0}^{w}(w-\rho(s))\left(\vartheta(s)\Phi\left[\frac{\delta(s)}{\vartheta(s)}\right]\right)^{4}\nabla_a^{\gamma}s\right\}^{\frac{1}{2}}\left\{\int_{t_0}^{z}(z-\rho(t))\left(\zeta(t)\Psi\left[\frac{\xi(t)}{\zeta(t)}\right]\right)^{4}\nabla_a^{\gamma}t\right\}^{\frac{1}{2}}\tag{68}$$

*where*

$$M_4=\left\{\int_{t_0}^{w}\left(\frac{\Phi(F(s))^4}{(F(s))^4}\right)(s-t_0)\nabla_a^{\gamma}s\right\}^{\frac{1}{2}}\left\{\int_{t_0}^{z}\left(\frac{\Psi(G(t))^4}{(G(t))^4}\right)(t-t_0)\nabla_a^{\gamma}t\right\}^{\frac{1}{2}}.$$

**Proof.** From (60) and (61) and using Fenchel-Young inequality with $p=q=2$ we have

$$\begin{aligned}&\Phi(\theta(s))^2\Psi(\phi(t))^2\\&\leqslant\left(\psi(s-t_0)+\psi^*(t-t_0)\right)\left(\frac{\Phi(F(s))^2}{(F(s))^2}\left(\int_{t_0}^{s}\left(\vartheta(\tau)\Phi\left[\frac{\delta(\tau)}{\vartheta(\tau)}\right]\right)^{2}\nabla_a^{\gamma}\tau\right)\right)\\&\quad\times\left(\frac{\Psi(G(t))^2}{(G(t))^2}\left(\int_{t_0}^{t}\left(\zeta(\eta)\Psi\left[\frac{\xi(\eta)}{\zeta(\eta)}\right]\right)^{2}\nabla_a^{\gamma}\eta\right)\right).\end{aligned}\tag{69}$$

From (69), using (9) with $p=q=2$, we obtain

$$\begin{aligned}&\int_{t_0}^{w}\int_{t_0}^{z}\frac{\Phi(\theta(s))^2\Psi(\phi(t))^2}{\left(\psi(s-t_0)+\psi^*(t-t_0)\right)}\nabla_a^{\gamma}s\nabla_a^{\gamma}t\\
&\leqslant\int_{t_0}^{w}\left(\frac{\Phi(F(s))^2}{(F(s))^2}\left(\int_{t_0}^{s}\left(\vartheta(\tau)\Phi\left[\frac{\delta(\tau)}{\vartheta(\tau)}\right]\right)^{2}\nabla_a^{\gamma}\tau\right)\nabla_a^{\gamma}s\right.\\
&\times\int_{t_0}^{z}\frac{\Psi(G(t))^2}{(G(t))^2}\left(\int_{t_0}^{t}\left(\zeta(\eta)\Psi\left[\frac{\xi(\eta)}{\zeta(\eta)}\right]\right)^{2}\nabla_a^{\gamma}\eta\right)\nabla_a^{\gamma}t\\
&\leqslant\int_{t_0}^{w}\left(\frac{\Phi(F(s))^2}{(F(s))^2}\right)(s-t_0)^{\frac{1}{2}}\left(\int_{t_0}^{s}\left(\vartheta(\tau)\Phi\left[\frac{\delta(\tau)}{\vartheta(\tau)}\right]\right)^{4}\nabla_a^{\gamma}\tau\right)^{\frac{1}{2}}\Bigg)\nabla_a^{\gamma}s\\
&\times\int_{t_0}^{z}\frac{\Psi(G(t))^2}{(G(t))^2}(t-t_0)^{\frac{1}{2}}\left(\int_{t_0}^{t}\left(\zeta(\eta)\Psi\left[\frac{\xi(\eta)}{\zeta(\eta)}\right]\right)^{4}\nabla_a^{\gamma}\eta\right)^{\frac{1}{2}}\nabla_a^{\gamma}t\\
&\leqslant\left\{\int_{t_0}^{w}\left(\frac{\Phi(F(s))^4}{(F(s))^4}\right)(s-t_0)\nabla_a^{\gamma}s\right\}^{\frac{1}{2}}\left\{\int_{t_0}^{w}\left(\int_{t_0}^{s}\left(\vartheta(\tau)\Phi\left[\frac{\delta(\tau)}{\vartheta(\tau)}\right]\right)^{4}\nabla_a^{\gamma}\tau\right)\nabla_a^{\gamma}s\right\}^{\frac{1}{2}}\\
&\times\left\{\int_{t_0}^{z}\left(\frac{\Psi(G(t))^4}{(G(t))^4}\right)(t-t_0)\nabla_a^{\gamma}t\right\}^{\frac{1}{2}}\left\{\int_{t_0}^{z}\left(\int_{t_0}^{t}\left(\zeta(\eta)\Psi\left[\frac{\xi(\eta)}{\zeta(\eta)}\right]\right)^{4}\nabla_a^{\gamma}\eta\right)\nabla_a^{\gamma}t\right\}^{\frac{1}{2}}.\end{aligned}\tag{70}$$

Applying Lemma 6 on (70), we obtain

$$\int_{t_0}^{w}\int_{t_0}^{z}\frac{\Phi(\theta(s))^2\Psi(\phi(t))^2}{\Big(\psi(s-t_0)+\psi^*(t-t_0)\Big)}\nabla_a^\gamma s\nabla_a^\gamma t$$

$$\leqslant\left\{\int_{t_0}^{w}\left(\frac{\Phi(F(s))^4}{(F(s))^4}\right)(s-t_0)\nabla_a^\gamma s\right\}^{\frac{1}{2}}\left\{\int_{t_0}^{w}(w-\rho(s))\left(\vartheta(s)\Phi\left[\frac{\delta(s)}{\vartheta(s)}\right]\right)^4\nabla_a^\gamma s\right\}^{\frac{1}{2}}$$

$$\times\left\{\int_{t_0}^{z}\left(\frac{\Psi(G(t))^4}{(G(t))^4}\right)(t-t_0)\nabla_a^\gamma t\right\}^{\frac{1}{2}}\left\{\int_{t_0}^{z}(z-\rho(t))\left(\zeta(t)\Psi\left[\frac{\xi(t)}{\zeta(t)}\right]\right)^4\nabla_a^\gamma t\right\}^{\frac{1}{2}}$$

$$=M_4\left\{\int_{t_0}^{w}(w-\rho(s))\left(\vartheta(s)\Phi\left[\frac{\delta(s)}{\vartheta(s)}\right]\right)^4\nabla_a^\gamma s\right\}^{\frac{1}{2}}\left\{\int_{t_0}^{z}(z-\rho(t))\left(\zeta(t)\Psi\left[\frac{\xi(t)}{\zeta(t)}\right]\right)^4\nabla_a^\gamma t\right\}^{\frac{1}{2}}.$$

where

$$M_4=\left\{\int_{t_0}^{w}\left(\frac{\Phi(F(s))^4}{(F(s))^4}\right)(s-t_0)\nabla_a^\gamma s\right\}^{\frac{1}{2}}\left\{\int_{t_0}^{z}\left(\frac{\Psi(G(t))^4}{(G(t))^4}\right)(t-t_0)\nabla_a^\gamma t\right\}^{\frac{1}{2}}.$$

This proves our claim.

□

**Theorem 21.** *Let $\delta$, $\xi$, $G$, $F$, $\zeta$, $\vartheta$, $\Psi$ and $\Phi$ be as in Theorem 20. Furthermore, assume that for $t$, $s$, $t_0$, $w$, $z\in\mathbb{T}$*

$$\theta(s):=\frac{1}{F(s)}\int_{t_0}^{s}\delta(\tau)\vartheta(\tau)\nabla_a^\gamma\tau,\ \ and\ \ \phi(t):=\frac{1}{G(t)}\int_{t_0}^{t}\xi(\eta)\zeta(\eta)\nabla_a^\gamma\eta,\tag{71}$$

*then for $s\in[t_0,w]_{\mathbb{T}}$ and $t\in[t_0,z]_{\mathbb{T}}$, we have that*

$$\begin{aligned}\int_{t_0}^{w}\int_{t_0}^{z}&\frac{\Phi(\theta(s))\Psi(\phi(t))F(s)G(t)}{\Big(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\Big)^{\frac{2\alpha}{p}}}\nabla_a^\gamma s\nabla_a^\gamma t\\&\leqslant M_5(p)\left(\int_{t_0}^{w}(w-\rho(s))\Big(\vartheta(s)\Phi\Big(\delta(s)\Big)\Big)^q\nabla_a^\gamma s\right)^{\frac{1}{q}}\\&\quad\times\left(\int_{t_0}^{z}(z-\rho(t))\Big(\zeta(t)\Psi\Big(\xi(t)\Big)\Big)^q\nabla_a^\gamma t\right)^{\frac{1}{q}}\end{aligned}\tag{72}$$

*where*

$$M_5(p)=(w-t_0)^{\frac{1}{p}}(z-t_0)^{\frac{1}{p}}.$$

**Proof.** From (71), we see that

$$\Phi(\theta(s))=\Phi\left(\frac{1}{F(s)}\int_{t_0}^{s}\vartheta(\tau)\delta(\tau)\nabla_a^\gamma\tau\right).\tag{73}$$

Applying Lemma 9 on (73), we obtain

$$\Phi(\theta(s))\leqslant\frac{(s-t_0)^{\frac{1}{p}}}{F(s)}\left(\int_{t_0}^{s}\Big(\vartheta(\tau)\Phi[\delta(\tau)]\Big)^q\nabla_a^\gamma\tau\right)^{\frac{1}{q}}.\tag{74}$$

From (74), we obtain

$$\Phi(\theta(s))F(s)\leqslant(s-t_0)^{\frac{1}{p}}\left(\int_{t_0}^{s}\Big(\vartheta(\tau)\Phi[\delta(\tau)]\Big)^q\nabla_a^\gamma\tau\right)^{\frac{1}{q}}.\tag{75}$$

Similarly, we obtain

$$\Psi(\phi(t))G(t) \leqslant (t-t_0)^{\frac{1}{p}} \left( \int_{t_0}^{t} \left( \zeta(\eta)\Psi[\xi(\eta)] \right)^{q} \nabla_a^{\gamma}\eta \right)^{\frac{1}{q}}. \tag{76}$$

From (75) and (76), we observe that

$$\begin{aligned} \Phi(\theta(s))\Psi(\phi(t))G(t)F(s) &\leqslant (s-t_0)^{\frac{1}{p}}(t-t_0)^{\frac{1}{p}} \\ &\times \left( \int_{t_0}^{s} \left( \vartheta(\tau)\Phi[\delta(\tau)] \right)^{q} \nabla_a^{\gamma}\tau \right)^{\frac{1}{q}} \left( \int_{t_0}^{t} \left( \zeta(\eta)\Psi[\xi(\eta)] \right)^{q} \nabla_a^{\gamma}\eta \right)^{\frac{1}{q}}. \end{aligned} \tag{77}$$

Applying Lemma 5 on the term $(s-t_0)^{\frac{1}{p}}(t-t_0)^{\frac{1}{p}}$, gives:

$$\begin{aligned} \Phi(\theta(s))\Psi(\phi(t))G(t)F(s) \quad &\leqslant \left( \psi(s-t_0) + \psi^*(t-t_0) \right)^{\frac{1}{p}} \left( \int_{t_0}^{s} \left( \vartheta(\tau)\Phi[\delta(\tau)] \right)^{q} \nabla_a^{\gamma}\tau \right)^{\frac{1}{q}} \\ &\times \left( \int_{t_0}^{t} \left( \zeta(\eta)\Psi[\xi(\eta)] \right)^{q} \nabla_a^{\gamma}\eta \right)^{\frac{1}{q}}. \end{aligned} \tag{78}$$

From 7 and (78), we obtain

$$\begin{aligned} \Phi(\theta(s))\Psi(\phi(t))G(t)F(s) &\leqslant \left( |\psi(s-t_0)|^{\frac{1}{2\beta}} + |\psi^*(t-t_0)|^{\frac{1}{2\beta}} \right)^{\frac{2\alpha}{p}} \\ &\times \left( \int_{t_0}^{s} \left( \vartheta(\tau)\Phi[\delta(\tau)] \right)^{q} \nabla_a^{\gamma}\tau \right)^{\frac{1}{q}} \left( \int_{t_0}^{t} \left( \zeta(\eta)\Psi[\xi(\eta)] \right)^{q} \nabla_a^{\gamma}\eta \right)^{\frac{1}{q}}. \end{aligned} \tag{79}$$

Dividing both sides of (79) by $\left( |\psi(s-t_0)|^{\frac{1}{2\beta}} + |\psi^*(t-t_0)|^{\frac{1}{2\beta}} \right)^{\frac{2\alpha}{p}}$, we obtain

$$\begin{aligned} \frac{\Phi(\theta(s))\Psi(\phi(t))G(t)F(s)}{\left( |\psi(s-t_0)|^{\frac{1}{2\beta}} + |\psi^*(t-t_0)|^{\frac{1}{2\beta}} \right)^{\frac{2\alpha}{p}}} \quad &\leqslant \left( \int_{t_0}^{s} \left( \vartheta(\tau)\Phi[\delta(\tau)] \right)^{q} \nabla_a^{\gamma}\tau \right)^{\frac{1}{q}} \\ &\times \left( \int_{t_0}^{t} \left( \zeta(\eta)\Psi[\xi(\eta)] \right)^{q} \nabla_a^{\gamma}\eta \right)^{\frac{1}{q}}. \end{aligned} \tag{80}$$

Taking double nabla-integral for (80), yields:

$$\begin{aligned} &\int_{t_0}^{w}\int_{t_0}^{z} \frac{\Phi(\theta(s))\Psi(\phi(t))G(t)F(s)}{\left( |\psi(s-t_0)|^{\frac{1}{2\beta}} + |\psi^*(t-t_0)|^{\frac{1}{2\beta}} \right)^{\frac{2\alpha}{p}}} \nabla_a^{\gamma}s\nabla_a^{\gamma}t \\ &\leqslant \left( \int_{t_0}^{w} \left( \int_{t_0}^{s} \left( \vartheta(\tau)\Phi[\delta(\tau)] \right)^{q} \nabla_a^{\gamma}\tau \right)^{\frac{1}{q}} \nabla_a^{\gamma}s \right) \left( \int_{t_0}^{z} \left( \int_{t_0}^{t} \left( \zeta(\eta)\Psi[\xi(\eta)] \right)^{q} \nabla_a^{\gamma}\eta \right)^{\frac{1}{q}} \nabla_a^{\gamma}t \right). \end{aligned} \tag{81}$$

Using Lemma 9 in (81), yield:

$$\begin{aligned}
&\int_{t_0}^{w}\int_{t_0}^{z}\frac{\Phi(\theta(s))\Psi(\phi(t))G(t)F(s)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{p}}}\nabla_a^\gamma s\nabla_a^\gamma t \\
&\leqslant (w-t_0)^{\frac{1}{p}}(z-t_0)^{\frac{1}{p}}\left(\int_{t_0}^{w}\left(\int_{t_0}^{s}\left(\vartheta(\tau)\Phi[\delta(\tau)]\right)^q\nabla_a^\gamma\tau\right)\nabla_a^\gamma s\right)^{\frac{1}{q}} \\
&\times\left(\int_{t_0}^{z}\left(\int_{t_0}^{t}\left(\zeta(\eta)\Psi[\xi(\eta)]\right)^q\nabla_a^\gamma\eta\right)\nabla_a^\gamma t\right)^{\frac{1}{q}} \\
&= M_5(p)\left(\int_{t_0}^{w}\left(\int_{t_0}^{s}\left(\vartheta(\tau)\Phi[\delta(\tau)]\right)^q\nabla_a^\gamma\tau\right)\nabla_a^\gamma s\right)^{\frac{1}{q}} \\
&\times\left(\int_{t_0}^{z}\left(\int_{t_0}^{t}\left(\zeta(\eta)\Psi[\xi(\eta)]\right)^q\nabla_a^\gamma\eta\right)\nabla_a^\gamma t\right)^{\frac{1}{q}}.
\end{aligned} \tag{82}$$

From Lemma 6 and (82), we obtain:

$$\begin{aligned}
&\int_{t_0}^{w}\int_{t_0}^{z}\frac{\Phi(\theta(s))\Psi(\phi(t))G(t)F(s)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}}+|\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{p}}}\nabla_a^\gamma s\nabla_a^\gamma t \\
&= M_5(p)\left(\int_{t_0}^{w}(w-\rho(s))\left(\vartheta(s)\Phi[\delta(s)]\right)^q\nabla_a^\gamma s\right)^{\frac{1}{q}} \\
&\times\left(\int_{t_0}^{z}(z-\rho(t))\left(\zeta(t)\Psi[\xi(t)]\right)^q\nabla_a^\gamma t\right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof.

□

**Remark 6.** *In Theorem 21, as special case, if we take $\psi(w)=\dfrac{w^2}{2}$, $\psi^*(w)=\dfrac{w^2}{2}$, and by following the same procedure employed in Remark 2, then we obtain [15] [Theorem 3.7].*

Taking $\mathbb{T}=\mathbb{R}$ and $\gamma=1$ in Theorem 21, we have:

**Corollary 8.** *Assume $\zeta(t)\geq 0$, $\xi(t)\geq 0$, $\vartheta(s)\geq 0$, $\delta(s)\geq 0$. Define*

$$\theta(s):=\frac{1}{F(s)}\int_0^s\vartheta(\tau)\delta(\tau)d\tau \ \ \text{and}\ \ \phi(t):=\frac{1}{G(t)}\int_0^t\zeta(\tau)\xi(\tau)d\tau,$$

$$F(s):=\int_0^s\vartheta(\tau)d\tau \ \ \text{and}\ \ G(t):=\int_0^t\zeta(\tau)d\tau.$$

*Then*

$$\begin{aligned}
\int_0^w\int_0^z\frac{\Phi(\theta(s))\Psi(\phi(t))F(s)G(t)}{\left(|\psi(s)|^{\frac{1}{2\beta}}+|\psi^*(t)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{p}}}dsdt \quad &\leqslant M_6(p)\left(\int_0^w(w-s)\left(\vartheta(s)\Phi\left(\delta(s)\right)\right)^q ds\right)^{\frac{1}{q}}. \\
&\times\left(\int_0^z(z-t)\left(\zeta(t)\Psi\left(\xi(t)\right)\right)^q dt\right)^{\frac{1}{q}}
\end{aligned}$$

*where*

$$M_6(p)=(w)^{\frac{1}{p}}(z)^{\frac{1}{p}}$$

Taking $\mathbb{T} = \mathbb{Z}$ and $\gamma = 1$ in Theorem 21, gives:

**Corollary 9.** *Let $(\xi_j)_{0\leq j\leq N}$, $(\delta_i)_{0\leq i\leq M}$, $(\vartheta_j)_{0\leq j\leq N}$, $(\zeta_i)_{0\leq i\leq M}$ be sequences of nonnegative real numbers where $N, M \in \mathbb{N}$. Define*

$$\theta(i) := \frac{1}{F(i)} \sum_{s=0}^{i} \vartheta(s)\delta(s) \;\; and \;\; \phi(j) := \frac{1}{G(j)} \sum_{a=0}^{j} \zeta(a)\xi(a).$$

$$F(i) := \sum_{s=0}^{i} \vartheta(s) \;\; and \;\; G(j) := \sum_{a=0}^{j} \zeta(a).$$

*Then*

$$\sum_{i=1}^{N}\sum_{j=1}^{M} \frac{\Phi(\theta(i))\Psi(\phi(j))F(i)G(j)}{\left(|\psi(i+1)|^{\frac{1}{2\beta}} + |\psi^*(j+1)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{p}}} \leqslant M_7(p)\left(\sum_{i=1}^{N}(N-(i+1))\left(\vartheta(i)\Phi\left(\delta(i)\right)\right)^q\right)^{\frac{1}{q}}.$$
$$\times\left(\sum_{j=1}^{M}(M-(j+1))\left(\zeta(j)\Psi\left(\xi(j)\right)\right)^q\right)^{\frac{1}{q}}$$

*where*

$$M_7(p) = (NM)^{\frac{1}{p}}.$$

**Remark 7.** *In Corollary 9, if $p = q = 2$ we obtain the result due to Hamiaz and Abuelela [22] [Theorem 7].*

**Corollary 10.** *With the hypotheses of Theorem 21, we obtain:*

$$\int_{t_0}^{w}\int_{t_0}^{z} \frac{\Phi(\theta(s))\Psi(\phi(t))F(s)G(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}} + |\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{p}}} \nabla_a^\gamma s \nabla_a^\gamma t$$
$$\leqslant M_5(p)\left\{\psi\left(\int_{t_0}^{w}(w-\rho(s))\left(\vartheta(s)\Phi\left(\delta(s)\right)\right)^q \nabla_a^\gamma s\right)\right.$$
$$\left. +\psi^*\left(\int_{t_0}^{z}(z-\rho(t))\left(\zeta(t)\Psi\left(\xi(t)\right)\right)^q \nabla_a^\gamma t\right)\right\}^{\frac{1}{q}}$$

**Proof.** We apply the Fenchel-Young inequality (24) in (72). This completes the proof. □

## 3. Some Applications

We can apply our inequalities to obtain different formulas of inequalities by suggesting $\psi^*(z)$ and $\psi(w)$ by some functions:

In (43), as special case, if we take $\psi(w) = \dfrac{w^2}{2}$, we have $\psi^*(w) = \dfrac{w^2}{2}$ see [13], so we get

$$\begin{aligned}
&\int_{t_0}^{w}\int_{t_0}^{z} \frac{\theta^{\epsilon-\gamma+1}(s)\phi^{\ell-\gamma+1}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}} + |\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{p}}} \nabla_a^{\gamma} s \nabla_a^{\gamma} t \\
&= \int_{t_0}^{w}\int_{t_0}^{z} \frac{\theta^{\epsilon-\gamma+1}(s)\phi^{\ell-\gamma+1}(t)}{\left((s-t_0))^{\frac{1}{\beta}} + (t-t_0)^{\frac{1}{\beta}}\right)^{\frac{2\alpha}{p}}} \nabla_a^{\gamma} s \nabla_a^{\gamma} t \\
&\leqslant \left(\frac{1}{2}\right)^{\frac{\alpha}{p\beta}} C_2(\ell,\epsilon,\gamma,p)\left(\int_{t_0}^{w}(w-\rho(s))\left(\theta^{\epsilon-\gamma}(s)\delta(s)\right)^{q}\nabla_a^{\gamma} s\right)^{\frac{1}{q}} \\
&\times\left(\int_{t_0}^{z}(z-\rho(t))\left(\phi^{\ell-\gamma}(t)\xi(t)\right)^{q}\nabla_a^{\gamma} t\right)^{\frac{1}{q}}.
\end{aligned} \tag{83}$$

Consequently, for $\alpha = \beta = 1$, inequality (83) produces

$$\begin{aligned}
&\int_{t_0}^{w}\int_{t_0}^{z} \frac{\theta^{\epsilon-\gamma+1}(s)\phi^{\ell-\gamma+1}(t)}{\left((s-t_0)) + (t-t_0)\right)^{\frac{2}{p}}} \nabla_a^{\gamma} s \nabla_a^{\gamma} t \\
&\leqslant \left(\frac{1}{2}\right)^{\frac{1}{p}} C_2(\ell,\epsilon,\gamma,p)\left(\int_{t_0}^{w}(w-\rho(s))\left(\theta^{\epsilon-\gamma}(s)\delta(s)\right)^{q}\nabla_a^{\gamma} s\right)^{\frac{1}{q}} \\
&\quad\times\left(\int_{t_0}^{z}(z-\rho(t))\left(\phi^{\ell-\gamma}(t)\xi(t)\right)^{q}\nabla_a^{\gamma} t\right)^{\frac{1}{q}}.
\end{aligned} \tag{84}$$

On the other hand, if we take $\psi(i) = \frac{i^r}{r}, r > 1$, then $\psi^*(j) = \frac{j^a}{a}$ where $\frac{1}{r} + \frac{1}{a} = 1$ and $i, j\ \mathbb{R}_+$, then (43) gives

$$\begin{aligned}
&\int_{t_0}^{w}\int_{t_0}^{z} \frac{\theta^{\epsilon-\gamma+1}(s)\phi^{\ell-\gamma+1}(t)}{\left(|\psi(s-t_0)|^{\frac{1}{2\beta}} + |\psi^*(t-t_0)|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{p}}} \nabla_a^{\gamma} s \nabla_a^{\gamma} t \\
&= \int_{t_0}^{w}\int_{t_0}^{z} \frac{\theta^{\epsilon-\gamma+1}(s)\phi^{\ell-\gamma+1}(t)}{\left((a(s-t_0)^r)^{\frac{1}{2\beta}} + (r(t-t_0)^a)^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{p}}} \nabla_a^{\gamma} s \nabla_a^{\gamma} t \\
&\leqslant \left(\frac{1}{rk}\right)^{\frac{\alpha}{p\beta}} C_2(\ell,\epsilon,\gamma,p)\left(\int_{t_0}^{w}(w-\rho(s))\left(\theta^{\epsilon-\gamma}(s)\delta(s)\right)^{q}\nabla_a^{\gamma} s\right)^{\frac{1}{q}} \\
&\quad\times\left(\int_{t_0}^{z}(z-\rho(t))\left(\phi^{\ell-\gamma}(t)\xi(t)\right)^{q}\nabla_a^{\gamma} t\right)^{\frac{1}{q}}.
\end{aligned} \tag{85}$$

Clearly, when $\beta = \frac{1}{2\alpha}$, the inequality (85) becomes

$$\int_{t_0}^{w}\int_{t_0}^{z} \frac{\theta^{\epsilon-\gamma+1}(s)\phi^{\ell-\gamma+1}(t)}{\left((a(s-t_0)^r)^{\alpha}+(r(t-t_0)^a)^{\alpha}\right)^{\frac{2\alpha}{p}}}\nabla_a^{\gamma}s\nabla_a^{\gamma}t$$
$$\leqslant \left(\frac{1}{rk}\right)^{\frac{2\alpha^2}{p}} C_2(\ell,\epsilon,\gamma,p)\left(\int_{t_0}^{w}(w-\rho(s))\left(\theta^{\epsilon-\gamma}(s)\delta(s)\right)^{q}\nabla_a^{\gamma}s\right)^{\frac{1}{q}} \tag{86}$$
$$\times\left(\int_{t_0}^{z}(z-\rho(t))\left(\phi^{\ell-\gamma}(t)\xi(t)\right)^{q}\nabla_a^{\gamma}t\right)^{\frac{1}{q}}.$$

If $\beta = \alpha = 1$. From (85), we obtain

$$\int_{t_0}^{w}\int_{t_0}^{z} \frac{\theta^{\epsilon-\gamma+1}(s)\phi^{\ell-\gamma+1}(t)}{\left((a(s-t_0)^r)^{\frac{1}{2}}+(r(t-t_0)^a)^{\frac{1}{2}}\right)^{\frac{2}{p}}}\nabla_a^{\gamma}s\nabla_a^{\gamma}t$$
$$\leqslant \left(\frac{1}{rk}\right)^{\frac{1}{p}} C_2(\ell,\epsilon,\gamma,p)\left(\int_{t_0}^{w}(w-\rho(s))\left(\theta^{\epsilon-\gamma}(s)\delta(s)\right)^{q}\nabla_a^{\gamma}s\right)^{\frac{1}{q}}$$
$$\times\left(\int_{t_0}^{z}(z-\rho(t))\left(\phi^{\ell-\gamma}(t)\xi(t)\right)^{q}\nabla_a^{\gamma}t\right)^{\frac{1}{q}}.$$

## 4. Conclusions

In this manuscript, by employing the $\nabla$-conformable calculus, several $\nabla$-conformable Hardy–Hilbert-type inequalities on time scales are introduced. The results proved here, extend several dynamic inequalities known in the literature, and it also can yield to some original continuous, discrete and quantum inequalities. For the sake of completeness, we applied the main results to some nonuniform time scales. To illustrate the benefits of our results we introduced many special cases of time scales such as $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \mathbb{R}$.

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