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The Hankel Determinants from a Singularly Perturbed Jacobi Weight

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Abstract: We study the Hankel determinant generated by a singularly perturbed Jacobi weight $w(x,s) := (1-x)^\alpha(1+x)^\beta e^{-\frac{s}{1-x}}$, $x \in [-1,1]$, $\alpha > 0$, $\beta > 0$ $s \geq 0$. If $s = 0$, it is reduced to the classical Jacobi weight. For $s > 0$, the factor $e^{-\frac{s}{1-x}}$ induces an infinitely strong zero at $x = 1$. For the finite n case, we obtain four auxiliary quantities $R_n(s)$, $r_n(s)$, $\widetilde{R}_n(s)$, and $\widetilde{r}_n(s)$ by using the ladder operator approach. We show that the recurrence coefficients are expressed in terms of the four auxiliary quantities with the aid of the compatibility conditions. Furthermore, we derive a shifted Jimbo–Miwa–Okamoto σ-function of a particular Painlevé V for the logarithmic derivative of the Hankel determinant $D_n(s)$. By variable substitution and some complicated calculations, we show that the quantity $R_n(s)$ satisfies the four Painlevé equations. For the large n case, we show that, under a double scaling, where n tends to ∞ and s tends to 0^+, such that $\tau := n^2 s$ is finite, the scaled Hankel determinant can be expressed by a particular $P_{III'}$.

Keywords: random matrix theory; Hankel determinant; singularly perturbed Jacobi weight; ladder operators; Painlevé V

1. Introduction

Random matrices were introduced in nuclear physics by Wigner in the 1950s to describe the statistics of the energy levels of quantum systems. The theory of random matrices makes the hypothesis that the local statistical behavior of the energy levels is identical to that of the eigenvalues of a random matrix.

In random matrix theory (RMT), it is well known that the joint probability density of the eigenvalues $\{x_j\}_{j=1}^n$ of $n \times n$ Hermitian matrices in the unitary ensemble is [1,2]:

$$p(x_1, x_2, \ldots, x_n)dx_1 dx_2 \cdots dx_n = \frac{1}{D_n[w]}\frac{1}{n!}\prod_{1 \leq j < k \leq n}(x_j - x_k)^2 \prod_{l=1}^n w(x_l)dx_l,$$

where $w(x)$ is a weight or probability density supported on the interval $I \subseteq \mathbb{R}$, and we suppose the moments of all orders,

$$\mu_j := \int_I x^j w(x)dx, \qquad j = 0, 1, 2, \ldots$$

exist. In addition, $D_n[w]$ is the normalization constant or the partition function,

$$D_n[w] = \frac{1}{n!}\int_{I^n}\prod_{1 \leq j < k \leq n}(x_j - x_k)^2 \prod_{l=1}^n w(x_l)dx_l.$$

As a matter of fact, $D_n[w]$ can be expressed as the determinant of the Hankel or moment matrix [3] (2.2.11):

$$D_n[w] = \det\left(\mu_{j+k}\right)_{j,k=0}^{n-1}.$$

Hankel determinants have been studied extensively over the past few years in part due to connections with RMT [4–13]. This is because Hankel determinants compute the most fundamental objects in RMT, such as the partition function of a random matrix ensemble, the probability distribution function of the largest eigenvalue of Hermitian matrices, and the moment-generating function of linear statistics associated with the ensemble.

The motivation of this paper comes from the study of different single or double compression models in recent years. In an early paper, Chen and Its [14] studied the Laguerre weight $x^\alpha e^{-x}$ perturbing a multiplicative factor $e^{-s/x}$, which induces an infinitely strong zero at the origin. Then, Chen and Dai considered the Jacobi weight $x^\alpha(1-x)^\beta$ perturbing a multiplicative factor $e^{-s/x}$ in the paper [15]. Later, Min and Chen studied two kinds of even weights $(1-x^2)^\alpha e^{-s/x^2}$ in [16] and $(1-x^2)^\alpha e^{-s/(1-x^2)}$ in [17]. Zhu and Chen [18] considered a generalized weight function:

$$w(z) = (1-z^2)^\alpha e^{-t/(z^2-k^2)}, z, k \in [-1,1], \alpha, t > 0.$$

The relationshipbetween semiclassical orthogonal polynomials and the (continuous) Painlevé equations was demonstrated by Magnus [19], who showed that the coefficients in the three-term recurrence relation for the Freud weight [20]:

$$\omega(x;t) = \exp\left(-\frac{1}{4}x^4 - tx^2\right), \quad x \in \mathbb{R},$$

with $t \in \mathbb{R}$ a parameter, can be expressed in terms of solutions of P_{IV}. Recently, Wang and Fan [21] studied the large n asymptotics of the monic orthogonal polynomials with respect to another singularly perturbed Pollaczek–Jacobi-type weight:

$$w(x) = x^\alpha(1-x)^\beta e^{-\frac{t}{x(x-1)}}, \quad x \in [0,1], \quad \alpha, \beta > 0, \quad ,t \geq 0.$$

In this paper, we consider the Hankel determinant generated by a singularly perturbed Jacobi weight, namely,

$$D_n(s, \alpha, \beta) := \det\left(\int_{-1}^1 x^{j+k} w(x,s) dx\right)_{j,k=0}^{n-1},$$

where:

$$w(x,s) := (1-x)^\alpha(1+x)^\beta e^{-\frac{s}{1-x}}, \qquad x \in [-1,1], \quad \alpha > 0, \quad \beta > 0, \quad s \geq 0. \tag{1}$$

We call this the single compression model because the weight vanishes infinitely fast at $x = 1$.

The study of this Hankel determinant is motivated in part by the Wigner time-delay distribution in chaotic cavities [22]. In recent years, the asymptotic analysis of orthogonal polynomials and Hankel determinants for the singularly perturbed weights has attracted much interest; see, e.g., [10,12,13,21,23,24].

In addition, for the unperturbed Jacobi weight, i.e., $s = 0$, we have the explicit formula of $D_n(0)$ [2] (17.6.2):

$$
\begin{aligned}
D_n(0) &= \frac{2^{n(n+\alpha+\beta)}}{n!} \prod_{j=1}^n \frac{\Gamma(j+1)\Gamma(j+\alpha)\Gamma(j+\beta)}{\Gamma(j+n+\alpha+\beta)} \\
&= 2^{n(n+\alpha+\beta)} \frac{G(n+1)G(n+1+\alpha+\beta)G(n+\alpha+1)G(n+\beta+1)}{G(2n+\alpha+\beta+1)G(\alpha+1)G(\beta+1)},
\end{aligned} \tag{2}
$$

where $G(\cdot)$ is the Barnes G-function, which satisfies the relation [25]:

$$G(z+1) = \Gamma(z)G(z), \Re(z) > 0, \qquad G(1) := 1.$$

See also [26] for more properties of this function.

It is well known that $D_n(s)$ can be expressed as the product of the square of the L^2 norms of the monic orthogonal polynomials [27] (2.1.6), namely:

$$D_n(s) = \prod_{j=0}^{n-1} h_j(s), \tag{3}$$

where:

$$h_j(s)\delta_{jk} = \int_{-1}^{1} P_j(x,s)P_k(x,s)w(x,s)dx, \qquad j,k = 0,1,2,\ldots \tag{4}$$

and $P_j(x,s)$, $j = 0,1,2,\ldots$, are the monic polynomials of degree j orthogonal with respect to the weight function $w(x,s)$.

In Section 2, with the aid of supplementary conditions, we obtain a system of difference equations by certain auxiliary quantities $(R_n, r_n, \widetilde{R}_n, \widetilde{r}_n)$ and the recurrence coefficients (α_n, β_n). In Section 3.1, we make use of the results of Section 2 to express α_n and β_n in terms of the auxiliary quantities and show that these reduce to recurrence coefficients of the classical Jacobi weight when $s = 0$. A pair of Toda equations is derived in Section 3.2 by the s dependence of the recurrence coefficients and the auxiliary quantities. In Section 4, we express $\widetilde{r}_n, r_n$ and R_n in terms of:

$$H_n(s) := s\frac{d}{ds}\ln D_n(s)$$

and $H_n'(s)$. We show that:

$$\widetilde{H}_n(t) := H_n(s)|_{s=2t} - n(n + \alpha + \beta)$$

satisfies the Jimbo–Miwa–Okamoto σ-form of a Painlevé V equation. In Section 5, we derive a Painlevé V form by $S_n(t) := 2R_n(s)|_{s=2t}/(2n + \alpha + \beta + 1)$. In Section 6, we use the double-scaling analysis and obtain some new results. Finally, Section 7 ends this paper with a brief conclusion.

2. Preliminaries

The main method of this paper is the ladder operator approach associated with the orthogonal polynomials. This approach has been widely applied to solve the problems on Hankel determinants generated by various weight functions; see [10,14,15,28,29] for reference. Therefore, we introduce some elementary facts about the orthogonal polynomials.

Let $P_n(x)$ be the monic polynomials of degree n in x and orthogonal with respect to the weight function $w(x,s)$ defined in (1), that is:

$$\int_{-1}^{1} P_m(x)P_n(x)w(x,s)dx = h_n\delta_{m,n}. \tag{5}$$

Here, the polynomials $P_n(x)$ and the constant h_n all depend on s, but we suppress the dependence for brevity. An immediate consequence of the orthogonality condition is the recurrence relation [3]:

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x), \quad n = 0,1,\ldots \tag{6}$$

with the initial conditions $P_0(z) := 1$ and $\beta_0 P_{-1}(z) := 0$. The most commonly used orthogonal polynomials are orthogonal for a measure with support in a real interval. The classical orthogonal polynomials include Jacobi polynomials, Laguerre polynomials, Hermite polynomials, and their special cases, Gegenbauer polynomials, Chebyshev polynomials, and Legendre polynomials. Note that $P_n(z)$ has the following form [30]:

$$P_n(z) = z^n + p_1(n)z^{n-1} + \ldots \tag{7}$$

Substituting (7) into (6), we see that:

$$\alpha_n = p_1(n) - p_1(n+1). \tag{8}$$

Taking a telescopic sum of the above equation and noting that $p_1(0) := 0$, we have:

$$-\sum_{j=0}^{n-1} \alpha_j = p_1(n). \tag{9}$$

The Hankel determinant generated by our weight is:

$$\begin{aligned} D_n(s): \quad &= \quad \det(\mu_{j+k}(s))_{j,k=0}^{n-1} \\ &= \quad \prod_{j=0}^{n-1} h_j, \end{aligned} \tag{10}$$

where:

$$\mu_k(s): \quad = \quad \int_{-1}^{1} x^k (1-x)^\alpha (1+x)^\beta e^{-\frac{s}{1-x}} dx.$$

For the case $k = 0$, we have:

$$\mu_0(s) = 2^{\alpha+\beta+1} e^{-\frac{s}{2}} \Gamma(1+\beta) U(1+\beta, -\alpha, \frac{s}{2})$$

and $U(a, b, z)$ is the Kummer function of the second; see [26,31].

The Hankel determinant will turn out to play an important role in our determination of α_n and β_n for the weight given by (1). Furthermore, through several auxiliary variables, we obtain expressions for the α_n and β_n terms of $H_n(s)$ given in (1.2) and its derivatives with respect to s. Equations (5)–(10) can be found in Szegő's treatise [3] on orthogonal polynomials.

In this section, we introduce a pair of ladder operators and the compatibility conditions first. The information below can be found in Chen and Its [14].

Suppose that $w(x)$ is a continuous even weight function defined on $[-1, 1]$, and $w(-1) = w(1) = 0$. The monic orthogonal polynomials with respect to $w(x)$ satisfy the lowering operator equation:

$$\left(\frac{d}{dz} + B_n(z)\right) P_n(z) = \beta_n A_n(z) P_{n-1}(z) \tag{11}$$

and the raising operator equation:

$$\left(\frac{d}{dz} - B_n(z) - v'(z)\right) P_{n-1}(z) = -A_{n-1}(z) P_n(z), \tag{12}$$

where $v(z) := -\ln w(z)$, and:

$$A_n(z) := \frac{1}{h_n} \int_{-1}^{1} \frac{v'(z) - v'(y)}{z - y} P_n^2(y) w(y) dy, \tag{13}$$

$$B_n(z) := \frac{1}{h_{n-1}} \int_{-1}^{1} \frac{v'(z) - v'(y)}{z - y} P_n(y) P_{n-1}(y) w(y) dy. \tag{14}$$

The functions $A_n(z)$ and $B_n(z)$ are not independent, but must satisfy the following supplementary conditions valid for $z \in \mathbb{C} \cup \{\infty\}$:

$$B_{n+1}(z) + B_n(z) = (z - \alpha_n) A_n(z) - v'(z), \tag{15}$$

$$1 + (z - \alpha_n)(B_{n+1}(z) - B_n(z)) = \beta_{n+1}A_{n+1}(z) - \beta_n A_{n-1}(z), \tag{16}$$

$$B_n^2(z) + v'(z)B_n(z) + \sum_{j=0}^{n-1} A_j(z) = \beta_n A_n(z)A_{n-1}(z). \tag{17}$$

In our problem, the weight and associated quantities are:

$$w(x, s) = (1 - x)^\alpha (1 + x)^\beta e^{-\frac{s}{1-x}}, \tag{18}$$

$$v(z) = -\ln w(x, s) = \frac{s}{1 - z} - \alpha \ln(1 - z) - \beta \ln(1 + z), \qquad v'(z) = \frac{\alpha}{1 - z} - \frac{\beta}{1 + z} + \frac{s}{(1 - z)^2}, \tag{19}$$

$$\frac{v'(z) - v'(y)}{z - y} = \frac{1}{1 - z}\left(\frac{\alpha}{1 - y} + \frac{s}{(1 - y)^2}\right) + \frac{\beta}{(1 + z)(1 + y)} + \frac{s}{(1 - z)^2(1 - y)}. \tag{20}$$

Substituting the above formula into the definition of $A_n(z)$ and $B_n(z)$ in (13) and (14), we have the following proposition. In the following discussions, we suppress the s dependence for brevity.

Proposition 1. *For our problem, we have:*

$$A_n(z) = \frac{R_n}{1 - z} + \frac{R_n}{1 + z} + \frac{\widetilde{R}_n}{(1 - z)^2}, \tag{21}$$

$$B_n(z) = \frac{r_n + n}{1 - z} + \frac{r_n}{1 + z} + \frac{\widetilde{r}_n}{(1 - z)^2}, \tag{22}$$

where:

$$\widetilde{R}_n(s) := \frac{s}{h_n}\int_{-1}^{1} \frac{P_n^2(y)w(y)}{1 - y}dy, \tag{23}$$

$$R_n(s) := \frac{\beta}{h_n}\int_{-1}^{1} \frac{P_n^2(y)w(y)}{1 + y}dy, \tag{24}$$

$$\widetilde{r}_n(s) := \frac{s}{h_{n-1}}\int_{-1}^{1} \frac{P_n(y)P_{n-1}(y)w(y)}{1 - y}dy, \tag{25}$$

$$r_n(s) := \frac{\beta}{h_{n-1}}\int_{-1}^{1} \frac{P_n(y)P_{n-1}(y)w(y)}{1 + y}dy, \tag{26}$$

Proof. According to the definition of $A_n(z)$ in (13), we obtain:

$$A_n(z) = \frac{1}{1 - z}\frac{1}{h_n}\int_{-1}^{1} P_n^2(y)w(y)\left(\frac{\alpha}{1 - y} + \frac{s}{(1 - y)^2}\right)dy + \frac{1}{1 + z}\frac{\beta}{h_n}\int_{-1}^{1} P_n^2(y)w(y)\frac{dy}{1 + y}$$

$$+ \frac{s}{(1 - z)^2}\frac{1}{h_n}\int_{-1}^{1} P_n^2(y)w(y)\frac{dy}{1 - y}. \tag{27}$$

Applying integration by parts, we have:

$$\int_{-1}^{1} P_n^2(y)w(y)\frac{\alpha}{1 - y}dy = \beta\int_{-1}^{1} P_n^2(y)w(y)\frac{dy}{1 + y} - \int_{-1}^{1} P_n^2(y)w(y)\frac{s}{(1 - y)^2}dy \tag{28}$$

Combining (27) and (28) gives us (21).

Similarly, we obtain (22) from (14). One just needs to take into account the following equality:

$$\int_{-1}^{1} P_n(y) P_{n-1}(y) w(y) \frac{\alpha}{1-y} dy$$

$$= nh_{n-1} + \int_{-1}^{1} P_n(y) P_{n-1}(y) w(y) \frac{\beta}{1+y} dy - \int_{-1}^{1} P_n(y) P_{n-1}(y) w(y) \frac{s}{(1-y)^2} dy \quad (29)$$

This completes the proof. □

Now, we have four more auxiliary quantities R_n, $\widetilde{R}_n$, r_n, $\widetilde{r}_n$, in addition to the two unknowns α_n and β_n. However, from (15)–(17), we obtain relations among these quantities.

Proposition 2. *From (15), we obtain the following equations:*

$$\widetilde{r}_{n+1} + \widetilde{r}_n = (1 - \alpha_n)\widetilde{R}_n - s, \quad (30)$$

$$\widetilde{R}_n = 2R_n - \alpha - \beta - 2n - 1, \quad (31)$$

$$r_{n+1} + r_n = \beta - (1 + \alpha_n)R_n, \quad (32)$$

where the constants R_n, $\widetilde{R}_n$, r_n, and $\widetilde{r}_n$ are defined in (23)–(26), respectively.

Proof. Substituting (21) and (22) into (15), we obtain:

$$B_{n+1}(z) + B_n(z) = \frac{r_{n+1} + r_n + 2n + 1}{1 - z} + \frac{r_{n+1} + r_n}{1 + z} + \frac{\widetilde{r}_{n+1} + \widetilde{r}_n}{(1 - z)^2} \quad (33)$$

and:

$$(z - \alpha_n)A_n(z) - v'(z) = (z - \alpha_n)\left[\frac{R_n}{1-z} + \frac{R_n}{1+z} + \frac{\widetilde{R}_n}{(1-z)^2}\right] - \left[\frac{\alpha}{1-z} - \frac{\beta}{1+z} + \frac{s}{(1-z)^2}\right]$$

$$= -R_n + \frac{(1 - \alpha_n)R_n - \alpha - \widetilde{R}_n}{1 - z} + \frac{(z - \alpha_n)R_n + \beta}{1 + z} + \frac{(1 - \alpha_n)\widetilde{R}_n - s}{(1 - z)^2}. \quad (34)$$

Comparing the coefficients in the above two formulas, it follows that:

$$\widetilde{r}_{n+1} + \widetilde{r}_n = (1 - \alpha_n)\widetilde{R}_n - s, \quad (35)$$

$$r_{n+1} + r_n + 2n + 1 = -\alpha + (1 - \alpha_n)R_n - \widetilde{R}_n, \quad (36)$$

$$r_{n+1} + r_n = \beta - (1 + \alpha_n)R_n. \quad (37)$$

Combining the above three formulas immediately proves our proposition. □

Proposition 3. *From (17), we obtain the following equations:*

$$(\widetilde{r}_n)^2 + s\widetilde{r}_n = \beta_n \widetilde{R}_n \widetilde{R}_{n-1}, \quad (38)$$

$$(2\widetilde{r}_n + s)(r_n + n) + \alpha\widetilde{r}_n = \beta_n(R_n\widetilde{R}_{n-1} + R_{n-1}\widetilde{R}_n), \quad (39)$$

$$r_n^2 - \beta r_n = \beta_n R_n R_{n-1} \quad (40)$$

and:

$$2\sum_{j=0}^{n-1} \widetilde{R}_j = (2n + \alpha + \beta)(\widetilde{r}_n - 2r_n) + n(s - 2\alpha - 2n) \quad (41)$$

where the constants R_n, $\widetilde{R}_n$, r_n, and $\widetilde{r}_n$ are defined in (23)–(26), respectively.

Proof. From (21) and (22), we know that:

$$
\begin{aligned}
& B_n^2(z) + \mathrm{v}'(z)B_n(z) + \sum_{j=0}^{n-1} A_j(z) \\
&= \frac{\widetilde{r}_n^2 + s\widetilde{r}_n}{(1-z)^4} + \frac{(2\widetilde{r}_n + s)(r_n + n) + \alpha\widetilde{r}_n}{(1-z)^3} + \frac{(r_n + n)(r_n + n + \alpha)}{(1-z)^2} \\
&+ \frac{r_n^2 - \beta r_n}{(1+z)^2} + \frac{2\widetilde{r}_n r_n - \beta\widetilde{r}_n + sr_n}{(1+z)(1-z)^2} + \frac{(r_n + n)(2r_n - \beta) + \alpha r_n}{(1-z)(1+z)} \\
&+ \sum_{j=0}^{n-1}\left[\frac{R_j}{1-z} + \frac{R_j}{1+z} + \frac{\widetilde{R}_j}{(1-z)^2}\right].
\end{aligned}
\tag{42}
$$

Using (21) again, we obtain:

$$
\begin{aligned}
\beta_n A_n(z)A_{n-1}(z) &= \frac{\beta_n \widetilde{R}_n \widetilde{R}_{n-1}}{(1-z)^4} + \beta_n R_n R_{n-1}\left[\frac{1}{(1-z)^2} + \frac{1}{(1+z)^2} + \frac{2}{(1-z)(1+z)}\right] \\
&+ \beta_n(R_n\widetilde{R}_{n-1} + \widetilde{R}_n R_{n-1})\left[\frac{1}{(1-z)^3} + \frac{1}{(1+z)(1-z)^2}\right].
\end{aligned}
\tag{43}
$$

Note that (42) equals (43) due to (17). Then, let us compare their coefficients. At $\mathcal{O}(1-z)^{-4}$, $\mathcal{O}(1-z)^{-3}$, and $\mathcal{O}(1+z)^{-2}$, equating the coefficients, we have (38)–(40), respectively. At $\mathcal{O}(1-z)^{-2}$, we obtain:

$$
\begin{aligned}
& 2(r_n + n)(r_n + n + \alpha) + 2\widetilde{r}_n r_n - \beta\widetilde{r}_n + sr_n + 2\sum_{j=0}^{n-1}\widetilde{R}_j \\
&= \beta_n(R_n\widetilde{R}_{n-1} + \widetilde{R}_n R_{n-1}) + 2\beta_n R_n R_{n-1}.
\end{aligned}
\tag{44}
$$

Substituting (39) and (40) into (44) gives us (41). □

From (16), using calculations similar to those in the above proposition, we obtain one more equation as follows:

$$
2(r_{n+1} - r_n) - (\widetilde{r}_{n+1} - \widetilde{r}_n) + 1 - \alpha_n = 0.
\tag{45}
$$

To continue, we rewrite (45) as:

$$
\alpha_n = 2(r_{n+1} - r_n) - (\widetilde{r}_{n+1} - \widetilde{r}_n) + 1.
\tag{46}
$$

Performing a telescopic sum and recalling (9), we find the very handy relation:

$$
\mathrm{p}_1(n) = \widetilde{r}_n - 2r_n - n,
\tag{47}
$$

where we used the initial conditions $r_0(t) = \widetilde{r}_0(t) := 0$. As we shall see later, (47) will play a crucial role in the derivation of the Painlevé equation.

3. Ladder Operator Approach

3.1. The Recurrence Coefficients

In this section, we express the recurrence coefficients α_n and β_n in terms of the auxiliary quantities R_n, r_n, and $\widetilde{r}_n$.

Lemma 1. *The diagonal recurrence coefficients α_n are expressed in terms of R_n, r_n, and $\widetilde{r}_n$ as follows:*

$$(2n + \alpha + \beta + 2)\alpha_n = 2(\widetilde{r}_n - 2r_n) - 4R_n + \alpha + 3\beta + 2n + s + 2. \tag{48}$$

Proof. We eliminate $\widetilde{R}_n$ from (30) with the aid of (31) and find:

$$\widetilde{r}_n + \widetilde{r}_{n+1} = (1 - \alpha_n)(2R_n - \alpha - \beta - 2n - 1) - s. \tag{49}$$

Subtracting the above formula from (32), we obtain:

$$\widetilde{r}_n + \widetilde{r}_{n+1} - 2(r_{n+1} + r_n) = (1 - \alpha_n)(-\alpha - \beta - 2n - 1) - s - 2\beta + 4R_n. \tag{50}$$

Recalling (45), we see that the left-hand side of the above formula is $2(\widetilde{r}_n - 2r_n) + 1 - \alpha_n$. Then, (48) immediately follows. $\square$

Remark 1. *For $n = 0$, we find from the definition of $\alpha_0(s)$ and $R_0(s)$ that:*

$$\alpha_0(s) = 1 - \frac{2U(1 + \beta, -\alpha - 1, \frac{s}{2})}{U(1 + \beta, -\alpha, \frac{s}{2})}, \tag{51}$$

$$R_0(s) = \frac{U(\beta, -\alpha, \frac{s}{2})}{2U(1 + \beta, -\alpha, \frac{s}{2})}, \tag{52}$$

where U is the second solution of Kummer's equation; see [26]. We verify the validity of (48) at $n = 0$ by substituting the above two formulas.

The next lemma gives an expression for β_n.

Lemma 2. *The off-diagonal recurrence coefficients β_n are expressed in terms of r_n and $\widetilde{r}_n$ as follows:*

$$[(2n + \alpha + \beta)^2 - 1]\beta_n = (\widetilde{r}_n - 2r_n)^2 + (s - 2\alpha - 4n)\widetilde{r}_n - 2(s + 2\beta)r_n - 2ns. \tag{53}$$

Proof. We eliminate $\widetilde{R}_n$ from (38) with the aid of (31) and replace $\beta_n R_n R_{n-1}$ by $r_n^2 - \beta r_n$ with (39) to find:

$$(\widetilde{r}_n)^2 + s\widetilde{r}_n = \beta_n(2R_n - (\alpha + \beta + 2n + 1))(2R_{n-1} - (\alpha + \beta + 2n - 1))$$
$$= 4\beta_n R_n R_{n-1} - 2\beta_n[(\alpha + \beta + 2n + 1)R_{n-1} + (\alpha + \beta + 2n - 1)R_n] + [(2n + \alpha + \beta)^2 - 1]\beta_n.$$

The same substitutions as that made in (40) produce:

$$(r_n + n)(2\widetilde{r}_n + s) + \alpha\widetilde{r}_n = \beta_n[R_n(2R_{n-1} - (\alpha + \beta + 2n - 1)) + R_{n-1}(2R_n - (\alpha + \beta + 2n + 1))]$$
$$= 4\beta_n R_n R_{n-1} - \beta_n[(\alpha + \beta + 2n + 1)R_{n-1} + (\alpha + \beta + 2n - 1)R_n]. \tag{54}$$

Subtracting the above two formulas gives us (53). $\square$

Remark 2. *We consider the case when $s = 0$. In this situation, $\widetilde{R}_n(0) = \widetilde{r}_n(0) = 0$. Therefore, from (31) and (41), we find:*

$$R_n(0) = \frac{\alpha + \beta + 2n + 1}{2} \tag{55}$$

and:

$$r_n(0) = \frac{-n(n + \alpha)}{2n + \alpha + \beta}, \tag{56}$$

respectively. Finally, from (48) and (53), we obtain:

$$\alpha_n(0) = \frac{\beta^2 - \alpha^2}{(2n + \alpha + \beta + 2)(2n + \alpha + \beta)},\tag{57}$$

$$\beta_n(0) = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(\alpha+\beta+2n)^2[(2n+\alpha+\beta)^2 - 1]}.\tag{58}$$

They are in agreement with the classical theory in [32].

3.2. The s Dependence

Note that our weight function depends on s. As a consequence, the coefficients of our polynomials, the recurrence coefficients, and the auxiliary quantities defined in (23)–(26) all depend on s. In this section, we study the evolution of auxiliary quantities in s. First of all, we state a lemma that concerns the derivative of $p_1(n)$ with respect to s.

Lemma 3. *We have:*

$$s\frac{d}{ds}p_1(n) = \widetilde{r}_n.\tag{59}$$

Proof. By the orthogonal property (5), we know that:

$$\int_{-1}^{1} P_n(x)P_{n-1}(x)w(x,s)dx = 0.$$

Differentiating the above formula with respect to s gives:

$$\int_{-1}^{1} \frac{d}{ds}P_n(x)P_{n-1}(x)w(x,s)dx - \int_{-1}^{1} P_n(x)P_{n-1}(x)\frac{d}{ds}w(x,s)dx = 0.$$

Using (1), (5), and (7), we obtain:

$$h_{n-1}\frac{d}{ds}p_1(n) - \int_{-1}^{1} P_n(x)P_{n-1}(x)w(x,s)\frac{1}{1-x}dx = 0.$$

Taking into account (25), we obtain (59) immediately. $\square$

From (47) and the above lemma, it is easily seen that:

$$s\frac{d}{ds}p_1(n) = \widetilde{r}_n = s\frac{d}{ds}\widetilde{r}_n - 2s\frac{d}{ds}r_n\tag{60}$$

or:

$$s\frac{d}{ds}\widetilde{r}_n = \widetilde{r}_n + 2s\frac{d}{ds}r_n.\tag{61}$$

We introduce a quantity related to the logarithmic derivative of the Hankel determinant,

$$H_n(s) := s\frac{d}{ds}\ln D_n(s).\tag{62}$$

Next, we have the following property concerning the Hankel determinant D_n.

Lemma 4. *We have:*

$$H_n(s) = s\frac{d}{ds}\ln D_n(s) = -\sum_{j=0}^{n-1} \widetilde{R}_j,\tag{63}$$

where $\widetilde{R}_j$ is defined in (23).

Proof. Note that the constant h_n defined in (5) depends on the parameter s. Then, we have from (1) and (5):

$$h'_n(s) = -\int_{-1}^{1} P_n^2(x) w(x,s) \frac{dx}{1-x}.$$
(64)

Using (23), we obtain from the above formula:

$$h'_n = -\frac{\widetilde{R}_n h_n}{s},$$
(65)

which gives us:

$$s\frac{d}{ds}\ln h_n = -\widetilde{R}_n.$$
(66)

Then, our lemma follows from the above formula and (10). □

From the above lemmas, we also derive differential relations for the recurrence coefficients α_n and β_n. These are the nonstandard Toda equations.

Lemma 5. *We have:*

$$s\frac{d}{ds}\alpha_n = \widetilde{r}_n - \widetilde{r}_{n+1},$$
(T_1)

$$s\frac{d}{ds}\beta_n = (\widetilde{R}_{n-1} - \widetilde{R}_n)\beta_n,$$
(T_2)

where $\widetilde{R}_n$ and $\widetilde{r}_n$ are defined in (23) and (25), respectively.

Proof. (T_1) follows from (8) and (59). (T_2) follows from (65) and the fact that $\beta_n = h_n/h_{n-1}$. □

4. The Jimbo–Miwa–Okamoto σ-form of the Painlevé V

In this section, we study the logarithmic derivative of the Hankel determinant. Usually, this quantity satisfies a second-order differential equation, which is related to the Jimbo–Miwa–Okamoto σ-form of a particular Painlevé equation. For this purpose, we express $\widetilde{r}_n$, r_n, and R_n in terms of H_n and its derivative with respect to s and obtain a functional equation involving H_n, H'_n, and H''_n. Firstly, we express r_n and $\widetilde{r}_n$ in terms of H_n and H'_n in the next lemma.

Lemma 6.

$$\widetilde{r}_n = \frac{-s(2H'_n + n)}{2n + \alpha + \beta},$$
(67)

$$r_n = \frac{H_n - sH'_n - n(n+\alpha)}{2n + \alpha + \beta}.$$
(68)

Proof. From (41) and (63), we obtain:

$$-2H_n = ns - 2(\alpha+n)n + (2n+\alpha+\beta)(\widetilde{r}_n - 2r_n)$$
$$= ns - 2(\alpha+n)n + (2n+\alpha+\beta)(n + \mathrm{p}_1(n)).$$
(69)

Taking the derivative of the above formula with respect to s and using (59), we have:

$$-2H'_n = n + (2n+\alpha+\beta)\frac{\widetilde{r}_n}{s}.$$
(70)

Equation (67) then follows from the above one. Equation (68) follows from eliminating $\widetilde{r}_n$ from (69) and (70). □

Remark 3. *The logarithmic derivative of the Hankel determinant $H_n(s)$ plays an important role. The auxiliary quantities $\widetilde{r}_n$ and r_n can be expressed in terms of H_n and its derivative.*

Then, we try to obtain a similar lemma for R_n. To achieve this, we first derive the relation among R_n, r_n, and $\widetilde{r}_n$.

Proposition 4.

$$\frac{\alpha + \beta + 2n + 1}{R_n}(r_n^2 - \beta r_n) + (\alpha + \beta + 2n - 1)\beta_n R_n = 4(r_n^2 - \beta r_n) - \alpha \widetilde{r}_n - (r_n + n)(2\widetilde{r}_n + s) \tag{71}$$

and:

$$\frac{\Delta(r_n^2 - \beta r_n)}{R_n} - \Delta \beta_n R_n = -4(r_n^2 - \beta r_n) + \alpha \widetilde{r}_n + (r_n + n)(2\widetilde{r}_n + s) - 2s(\alpha + \beta + 2n)\frac{dr_n}{ds}, \tag{72}$$

where $\Delta := (2n + \alpha + \beta)^2 - 1$.

Proof. Substituting (39) into (54) gives us (71). Then, we look back to (T_2) by using (31) and (39):

$$\begin{aligned} s\frac{d}{ds}\beta_n &= (\widetilde{R}_{n-1} - \widetilde{R}_n)\beta_n \\ &= (2R_{n-1} - 2R_n + 2)\beta_n \\ &= 2\beta_n - 2R_n\beta_n + 2\frac{r_n^2 - \beta r_n}{R_n}. \end{aligned} \tag{73}$$

Applying $s\frac{d}{ds}$ to (53) gives us:

$$(2n + \alpha + \beta + 1)(2n + \alpha + \beta - 1)s\frac{d}{ds}\beta_n$$
$$= (2\widetilde{r}_n + s)(\widetilde{r}_n - 2r_n) - 2sn + (s - 2\alpha - 4n)\widetilde{r}_n - 4s(\alpha + \beta + 2n)\frac{dr_n}{ds}, \tag{74}$$

where we made use of (60) to arrive at the last step. Substituting (73) into the above formula and eliminating β_n with (53) gives (72). $\quad\square$

From the above proposition, we express R_n and $1/R_n$ in terms of r_n, $\widetilde{r}_n$, and $r'_n(s)$.

Proposition 5.

$$R_n = \frac{(2n + \alpha + \beta + 1)[4(r_n^2 - \beta r_n) - \alpha \widetilde{r}_n - (r_n + n)(2\widetilde{r}_n + s) + 2sr'_n(s)]}{2[(\widetilde{r}_n - 2r_n)^2 + (s - 2\alpha - 4n)\widetilde{r}_n - 2(s + 2\beta)r_n - 2sn]}, \tag{75}$$

$$\frac{1}{R_n} = \frac{4(r_n^2 - \beta r_n) - \alpha \widetilde{r}_n - (r_n + n)(2\widetilde{r}_n + s) - 2sr'_n(s)}{2(\alpha + \beta + 2n + 1)(r_n^2 - \beta r_n)}. \tag{76}$$

Proof. These are found by solving for R_n and $1/R_n$ from (71) and (72). $\quad\square$

Finally, we arrive at the following theorem.

Theorem 1. *The logarithmic derivative of the Hankel determinant with respect to s,*

$$H_n(s) := s\frac{d}{ds}\ln D_n(s),$$

satisfies a shifted Jimbo–Miwa–Okamoto σ-form of the Painlevé equation:

$$4(sH_n'')^2 = -16s(H_n')^3 + (H_n')^2[16H_n + (s + 2\alpha)^2 + 8\beta s]$$

$$+ 2H_n'[(-s - 2\alpha - 4\beta)H_n + (s + 2\alpha)n(n + \alpha + \beta)] + [H_n - n(n + \alpha + \beta)]^2, \quad (77)$$

with the boundary conditions:

$$H_n(0) = 0, \quad H_n'(0) = \frac{-n(n + \alpha + \beta)}{2\alpha}.$$

Proof. Multiplying (75) and (76) gives us:

$$4s^2(r_n')^2 = s^2 r_n^2 + r_n[2ns^2 + 2s(\alpha + 2\beta + 4n)\widetilde{r}_n$$

$$+ 4(\alpha + \beta + 2n)\widetilde{r}_n^2] + [(\alpha + 2n)\widetilde{r}_n + ns]^2. \quad (78)$$

Substituting (67) and (68) into (78) gives us (77). $\square$

Remark 4. *It turns out that the quantity $H_n(s)$ satisfies a shifted Jimbo–Miwa–Okamoto σ-form of the Painlevé V for a special choice for parameters. If:*

$$\tilde{H}_n(t) := H_n(2t) - n(n + \alpha + \beta), \quad (79)$$

then (77) becomes:

$$(t\tilde{H}_n'')^2 = -4t(\tilde{H}_n')^3 + (\tilde{H}_n')^2\left[4\tilde{H}_n + (\alpha + 2\beta + t)^2 + 4n(n + \alpha + \beta) - 4\beta(\alpha + \beta)\right]$$

$$- 2\tilde{H}_n'\left[(\alpha + 2\beta + t)\tilde{H}_n + 2n\beta(n + \alpha + \beta)\right] + \tilde{H}_n^2,$$

which is the Jimbo–Miwa–Okamoto σ-form of the Painlevé V, with parameters $\nu_0 = 0$, $\nu_1 = -(n + a + b)$, $\nu_2 = n$, $\nu_3 = -b$, following the convention in [33] (C.45).

5. Painlevé V

In this section, we obtain a second-order ordinary differential equation for $R_n(s)$, which is expected for P_V since we saw that $H_n(s)$ satisfies the Jimbo–Miwa–Okamoto σ-form of the Painlevé V. For this purpose, we state the next lemma, a Riccati equation satisfied by $R_n(s)$.

Lemma 7.

$$-4sR_n'(s) = 4R_n(\widetilde{r}_n - 2r_n) + (2n + 1 + \alpha + \beta)(4r_n + R_n - 2\beta) + 2R_n(-4R_n + 2\beta + s). \quad (80)$$

Proof. First, we apply $s\frac{d}{ds}$ to (48) and make use of (T_1) to replace $s\frac{d}{ds}\alpha_n$ by $\widetilde{r}_n - \widetilde{r}_{n+1}$. In the next step, we replace $\widetilde{r}_{n+1}$ by $(1 - \alpha_n)\widetilde{R}_n - s - \widetilde{r}_n$ using (30). Finally, noting (48) and (31), we arrive at (80). $\square$

From (80), we see that:

$$\widetilde{r}_n = \frac{-4sR_n'(s) - (2n + 1 + \alpha + \beta)(4r_n + R_n - 2\beta)}{4R_n} + \frac{4r_n + 4R_n - 2\beta - s}{2}. \quad (81)$$

Substituting the above formula into (75) and (76), we find a pair of linear equations in $r_n(s)$ and $r_n'(s)$. Solving this system, we have:

$$r_n(s) = F(R_n, \widetilde{R}_n), \quad (82)$$

$$r_n'(s) = G(R_n, \widetilde{R}_n). \quad (83)$$

Because the expressions are unwieldy, we decided to take a different approach to obtain the Painlevé equation. In the paper of Chen and Dai [15], they studied the Pollaczek–Jacobi-type weight, i.e.,

$$\hat{w}(x,t) = x^{\alpha}(1-x)^{\beta}e^{-\frac{t}{x}}, \ x \in [0,1], \ \alpha, \beta \geq 0, \ t \geq 0.$$

The orthogonality with the Pollaczek–Jacobi-type weight is:

$$\int_0^1 \hat{P}_j(x,\alpha,\beta)\hat{P}_k(x,\alpha,\beta)x^{\alpha}(1-x)^{\beta}e^{-\frac{t}{x}}dx = \hat{h}_j(t,\alpha,\beta)\delta_{jk},$$

where $\hat{P}_j(x,\alpha,\beta)$, $j = 0, 1, 2, \ldots$, are the monic polynomials of degree j orthogonal with respect to the weight $\hat{w}(x,t)$. Obviously, $\hat{P}_j(x,\alpha,\beta)$ should also depend on t, while we do not write it down for simplicity.

Based on the variable substitution method for the integral, we obtained the following result.

Lemma 8. *We have:*
$$h_n(s)|_{s=2t} = 2^{2n+\alpha+\beta+1}\hat{h}_n(t), \tag{84}$$

and:
$$R_n(s)|_{s=2t} = \frac{1}{2}\hat{R}_n(t), \tag{85}$$

where $\hat{R}_n(t)$ is the quantity $R_n(t)$ in the paper of Chen and Dai [15].

Proof. Replacing x by $\frac{1-y}{2}$, we obtain:

$$
\begin{aligned}
\hat{h}_n(t) &= \int_0^1 \hat{P}_n^2(x)\hat{w}(x,t)dx \\
&= \int_1^{-1} \hat{P}_n^2(\frac{1-y}{2})(\frac{1-y}{2})^{\alpha}(\frac{1+y}{2})^{\beta}e^{-\frac{2t}{1-y}}\frac{-dy}{2} \\
&= \frac{1}{2^{2n+\alpha+\beta+1}}\int_{-1}^1 P_n^2(y)w(y,2t)dy \\
&= \frac{h_n(2t)}{2^{2n+\alpha+\beta+1}}.
\end{aligned}
\tag{86}
$$

Similarly, we obtain the relation for $R_n(s)$ and $\hat{R}_n(t)$. □

Remark 5. *By the variable substitution method for the integral, we obtain the relations of the auxiliary quantities between this paper and the paper of Chen and Dai [15] as follows:*

$$\widetilde{R}_n(s)|_{s=2t} = \hat{R}_n^*(t), \ r_n(s)|_{s=2t} = -\hat{r}_n(t), \ \widetilde{r}_n(s)|_{s=2t} = -2\hat{r}_n^*(t),$$

$$\alpha_n(s)|_{s=2t} = 1 - 2\hat{\alpha}_n(t), \ \beta_n(s)|_{s=2t} = 4\hat{\beta}_n(t). \tag{87}$$

By Theorem (7.2) in the paper of Chen and Dai [15], we have the following result :

Theorem 2. *Let:*
$$S_n(t) := \frac{2R_n(2t)}{2n+\alpha+\beta+1}. \tag{88}$$

Then, $S_n(t)$ satisfies the following differential equation:

$$
\begin{aligned}
S_n'' &= \frac{3S_n - 1}{2S_n(S_n - 1)}(S_n')^2 - \frac{S_n'}{t} + \frac{(S_n - 1)^2}{t^2}\left[\frac{(2n + 1 + \alpha + \beta)^2}{2}S_n - \frac{\beta^2}{2S_n}\right] \\
&\quad + \frac{\alpha S_n}{t} - \frac{S_n(S_n + 1)}{2(S_n - 1)},
\end{aligned}
\tag{89}
$$

where is $P_V((2n + 1 + \alpha + \beta)^2/2, -\beta^2/2, \alpha, -1/2)$ with the initial conditions:

$$
S_n(0) = 1, \quad S_n'(0) = \frac{1}{\alpha}.
$$

Proof. Equation (89) follows from the relation for $R_n(s)$ and $\hat{R}_n(t)$. □

Corollary 6. *Let:*

$$
y_n(s) := \frac{2R_n(s)}{2n + \alpha + \beta + 1}.
\tag{90}
$$

Then, $y_n(s)$ satisfies the following differential equation:

$$
\begin{aligned}
y_n'' &= \frac{3y_n - 1}{2y_n(y_n - 1)}(y_n')^2 - \frac{y_n'}{s} + \frac{(y_n - 1)^2}{s^2}\left[\frac{(2n + 1 + \alpha + \beta)^2}{2}y_n - \frac{\beta^2}{2y_n}\right] \\
&\quad + \frac{\alpha y_n}{2s} - \frac{y_n(y_n + 1)}{8(y_n - 1)},
\end{aligned}
\tag{91}
$$

with the initial conditions:

$$
y_n(0) = 1, \quad y_n'(0) = \frac{1}{2\alpha}.
$$

Remark 6. *We made use of the ladder operator approach with the associated compatibility conditions (15), (16), and (17), since it is straightforward to express the recurrence coefficients in terms of the auxiliary variables. This allows us to investigate the s evolution of such quantities and discover the Painvelé V of our problem. The Riemann–Hilbert approach of Deift et al. [34] is particularly suited for the asymptotic analysis. However, the appearance of an irregular singularity at the right endpoint requires further study.*

6. Double-Scaling Analysis

In this section, our interest lies in a double-scaling analysis, where $s \to 0^+$ and $n \to \infty$, such that $\tau = n^2 s$ is finite. Recalling (41) and (63), we have:

$$
H_n(s) = s\frac{d}{ds}\ln\frac{D_n(s)}{D_n(0)} = (2n + \alpha + \beta)(r_n - \tilde{r}_n/2) + n(\alpha + n - s/2).
\tag{92}
$$

We are concerned with the behavior of the Hankel determinant, as n, the dimension of the Hankel matrix, tends to infinity. For this purpose, a double-scaling scheme is introduced, namely sending $n \to \infty$, $s \to 0^+$ and such that $\tau := n^2 s$ remains fixed. Here, the main idea comes from the paper [35].

Carrying out the double scaling above and combining with Theorem 1, we find that the infinite-dimensional Hankel determinant satisfies another σ-form of the corresponding Painlevé equation.

Theorem 3. *Let:*

$$
\tau := n^2 s,
$$

with $s \to 0^+$ and $n \to \infty$, such that $\tau \in (0, +\infty)$. If:

$$\mathcal{H}(\tau) := \lim_{n \to \infty} H_n\left(\frac{\tau}{n^2}\right), \tag{93}$$

then $\mathcal{H}(\tau)$ satisfies,

$$(\tau \mathcal{H}'')^2 + 4(\mathcal{H}')^2(\tau \mathcal{H}' - \mathcal{H}) - \left(\alpha \mathcal{H}' + \frac{1}{2}\right)^2 = 0, \tag{94}$$

with the initial conditions $\mathcal{H}(0) = 0$, $\mathcal{H}'(0) = -\frac{1}{2\alpha}$.

Proof. Plugging $\tau = n^2 s$ into (77), we see that the limit of $H_n(s)$ satisfies a σ-form Painlevé Equation (94). □

Similarly, carrying out the double scaling above and combining with Corollary 6, the scaled quantity satisfies a Painlevé III' in the next theorem.

Theorem 4. *Let:*

$$y_n(s) := 1 + \frac{g_n(s)}{n^2}, \quad \tau := n^2 s, \tag{95}$$

with $s \to 0^+$ and $n \to \infty$, such that $\tau \in (0, +\infty)$. If:

$$g(\tau) := \lim_{n \to \infty} g_n\left(\frac{\tau}{n^2}\right), \tag{96}$$

then $g(\tau)$ satisfies,

$$g''(\tau) = \frac{g'^2(\tau)}{g(\tau)} - \frac{g'(\tau)}{\tau} + \frac{2g^2(\tau)}{\tau^2} + \frac{\alpha}{2\tau} - \frac{1}{4g(\tau)}, \tag{97}$$

with the initial data $g(0) = 0$, $g'(0) = \frac{1}{2\alpha}$.

Proof. Substituting the definition (95) into (91), we see that $g(\tau)$ satisfies $P_{III'}(8, 2\alpha, 0, -1)$. □

7. Conclusions

In this work, we studied the weight, $w(x, s) := (1 - x)^\alpha (1 + x)^\beta e^{-\frac{s}{1-x}}$, $\alpha, \beta > 0$, $x \in [-1, 1]$, $s \geq 0$. We call this the single compression model because the weight vanishes infinitely fast at $x = 1$. For the finite n case, we derived a second-order nonlinear differential equation for the logarithmic derivative of the Hankel determinant. This quantity can be expressed in terms of the Jimbo–Miwa–Okamoto σ-function of a particular Painlevé V. Furthermore, we showed that the quantity $R_n(s)$ satisfies the four Painlevé equations. For the large n case, we used double-scaling analysis and obtained similar results to the paper [35].

The main idea of this paper was to study the relationship between the Painlevé equations and the correlation coefficients, and then, we analyzed the asymptotic property of the auxiliary quantity by the double-scaling method, and finally, we obtained the asymptotic expression of the Hankel determinant.

In the future, we want to use the Riemann–Hilbert method to obtain the asymptotic expression of the orthogonal polynomials directly; thus, we can obtain the asymptotic property of the recursion coefficients of the associated orthogonal polynomials and the asymptotic expression of the Hankel determinant.

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