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Supercyclic and Hypercyclic Generalized Weighted Backward Shifts over a Non-Archimedean $c_0(\mathbb{N})$ Space

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Abstract: In the present paper, we propose to study generalized weighted backward shifts $B_{\mathcal{B}}$ over non-Archimedean $c_0(\mathbb{N})$ spaces; here, $\mathcal{B} = (b_{ij})$ is an upper triangular matrix with $\sup_{i,j} |b_{ij}| < \infty$. We investigate the sypercyclic and hypercyclic properties of $B_{\mathcal{B}}$. Furthermore, certain properties of the operator $I + B_{\mathcal{B}}$ are studied as well. To establish the hypercyclic property of $I + B_{\mathcal{B}}$ we have essentially used the non-Archimedeanity of the norm which leads to the difference between the real case.

Keywords: non-Archimedean valuation; hypercylic operator; supercyclic operator; generalized backward shift operator

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1. Introduction

It is well-known [1,2] that linear dynamics began in the 1980s with the thesis of Kitai [3] and the paper of Gethner and Shapiro [4]. The main problem of this theory is to investigate the dynamical properties of a (bounded) linear operator T acting on some (complete) linear space [5]. We notice that linear dynamics was first initiated by the study of the density of orbits, which lead to the notions of hypercyclicity, supercyclicity, and their variants. We point out that the hypercyclicity of linear operators, as one of the most-studied properties in linear dynamics, has become an active area of research [1,2]. One of the interests of linear dynamics is started by examining certain examples of operators which have certain properties [6,7]. Among these examples, the most-studied class is certainly that of weighted shifts [8]. In [9–15], the hypercyclicity and supercyclicity of weighted bilateral (unilateral) shifts were characterized. In [16–19], the existence of hypercyclic subspaces and other properties of hypercyclic operators was explored. Recently, in [20], it was shown that every bilateral weighted shift on (formula-presented) has a factorization $T = AB$, where A and B are hypercyclic bilateral weighted shifts.

In the last few decades, a lot of books have been published that are devoted to the non-Archimedean functional analysis (see, for example, [21,22]). Therefore, recently in [23], a non-Archimedean shift operator was investigated. Later on, in [24], the invariant subspace problem was studied for the class of non-Archimedean compact operators. Furthermore, in [25], we developed the theory of dynamics of linear operators defined on topological vector space over non-Archimedean valued fields. Sufficient and necessary conditions of hypercyclicity (resp. supercyclicity) of linear operators on separable F-spaces have been found. Moreover, we considered shifts on $c_0(\mathbb{Z})$ and $c_0(\mathbb{N})$, respectively, and characterized their hypercyclicity and supercyclicity. Besides, we have considered the operator $\lambda I + \mu B$, where I is the identity and B is the bilateral (unilateral) shift. It turns out that such an operator is not supercyclic on $c_0(\mathbb{Z})$. However, it can be hypercyclic and supercyclic on $c_0(\mathbb{N})$, depending on values of ł and μ. It is natural to replace B with weighted shifts. Therefore, in the present paper, we consider generalized weighted backward shift operators $B_{\mathcal{B}}$ on $c_0(\mathbb{N})$. Such types of operators are generated by upper triangular matrices, and hence, all earlier investigated weighted backward shift operators can be treated as particular cases of the new ones. Furthermore, the supercyclicity and hypercyclicity of $B_{\mathcal{B}}$ are investigated. Moreover, the hypercyclicity and supercyclicity of $I + B_{\mathcal{B}}$ on $c_0(\mathbb{N})$ are studied. In the real setting, the hyperbolicity of such types of operators associated with weighted shifts on $\ell^2(\mathbb{N})$ (and other spaces) has been investigated in [8,26]. It was proved that $I + B_{\mathbf{b}}$ is hypercyclic if the weights are positive. We stress that, in the non-Archimedean setting, all ℓ^p-spaces coincide with c_0. Therefore, in the current paper, we are going to establish the hypercyclicity of $I + B_{\mathcal{B}}$ on $c_0(\mathbb{N})$. Our results are totally different from the real case, since to obtain the main results of this paper, we have essentially used the non-Archimedeanity of the norm of $c_0(\mathbb{N})$ (see Example 1 in the last section). It is stressed that shift operators, in a non-Archimedean setting, have certain applications in p-adic dynamical systems [27,28]. On this point, we mention that p-adic dynamical systems have certain applications in mathematical physics [29–33].

2. Definitions and Preliminary Results

All fields appearing in this paper are commutative. A valuation on a field $\mathbb{K}$ is a map $|\cdot| : \mathbb{K} \to [0, +\infty)$ such that:

(i) $|\lambda| = 0$ if, and only if $\lambda = 0$,
(ii) $|\lambda\mu| = |\lambda| \cdot |\lambda|$ (multiplicativity),
(iii) $|\lambda + \mu| \leq |\lambda| + |\mu|$ (triangle inequality), for all $\lambda, \mu \in \mathbb{K}$. The pair $(\mathbb{K}, |\cdot|)$ is called a valued field. We often write $\mathbb{K}$ instead of $(\mathbb{K}, |\cdot|)$.

Definition 1. *Let $\mathbb{K} = (\mathbb{K}, |\cdot|)$ be a valued field. The valuation $|\cdot|$ is called non-Archimedean, and $\mathbb{K}$ is called a non-Archimedean valued field if $|\cdot|$ satisfies the strong triangle inequality: (iii') $|\lambda + \mu| \leq \max\{|\lambda|, |\mu|\}$, for all $\lambda, \mu \in \mathbb{K}$.*

From the strong triangle inequality, we get the following useful property of non-Archimedean value: If $|\lambda| \neq |\mu|$, then $|\lambda \pm \mu| = \max\{|\lambda|, |\mu|\}$.

We frequently use this property, and call it the non-Archimedean norm's property. A non-Archimedean valued field $\mathbb{K}$ is a metric space, and it is called *ultrametric space*.

Let $a \in \mathbb{K}$ and $r > 0$. The set

$$B(a, r) := \{x \in \mathbb{K} : |x - a| \leq r\}$$

is called the *closed ball with a radius r about a*. (Indeed, $B(a, r)$ is closed in the induced topology). Similarly,

$$B(a, r^-) := \{x \in \mathbb{K} : |x - a| < r\}$$

is called the *open ball with radius r about a*.

We set $|\mathbb{K}| := \{|\lambda| : \lambda \in \mathbb{K}\}$ and $\mathbb{K}^\times := \mathbb{K} \setminus \{0\}$, the *multiplicative group* of $\mathbb{K}$. Additionally, $|\mathbb{K}^\times| := \{|\lambda| : \lambda \in \mathbb{K}^\times\}$ is a multiplicative group of positive real numbers, the *value group* of $\mathbb{K}$. There are two possibilities:

Lemma 1 ([22])**.** *Let $\mathbb{K}$ be a non-Archimedean valued field. Then the value group of $\mathbb{K}$ is either dense or discrete; in the latter case, there is a real number $0 < r < 1$ such that $|\mathbb{K}^\times| = \{r^s : s \in \mathbb{Z}\}$.*

For example, the value group of $\mathbb{Q}_p$ (field of p-adic numbers) is discrete and the value group of $\mathbb{C}_p$ (field of p-adic complex numbers) is dense [34].

Definition 2. *Let $\mathbb{K}$ be a non-Archimedean valued field and E be a $\mathbb{K}$-vector space. A norm on E is a map $\|\cdot\|: E \to [0, +\infty)$ such that:*

(i) $\| \boldsymbol{x} \| = \boldsymbol{0}$ *if, and only if* $\boldsymbol{x} = \boldsymbol{0}$,
(ii) $\| \lambda \boldsymbol{x} \| = |\lambda| \, \| \boldsymbol{x} \|$,
(iii) $\| \boldsymbol{x} + \boldsymbol{y} \| \leq \max\{\| \boldsymbol{x} \|, \| \boldsymbol{y} \|\}$,
for all $\boldsymbol{x}, \boldsymbol{y} \in E,\ \lambda \in \mathbb{K}$.

We call $(E, \| \cdot \|)$ a $\mathbb{K}$-normed space or a normed space over $\mathbb{K}$. We frequently write E instead of $(E, \| \cdot \|)$. E is called a $\mathbb{K}$-Banach space or a Banach space over $\mathbb{K}$ if it is complete with respect to the induced ultrametric $(\mathbf{x}, \mathbf{y}) \to \| \mathbf{x} - \mathbf{y} \|$.

Example 1. *Let* $\mathbb{K}$ *be a non-Archimedean valued field. Then*

$$\ell^\infty(\mathbb{N}) := \{\mathbf{x} = (x_n)_{n \in \mathbb{N}} :\ x_n \in \mathbb{K},\ \sup_n |x_n| < \infty\}$$

with pointwise addition and scalar multiplication, and the norm

$$\| \boldsymbol{x} \|_\infty := \sup_n |x_n|$$

is a $\mathbb{K}$*-Banach space.*

Remark 1. *From now on, we often drop the prefix "*$\mathbb{K}$*" and write vector space, normed space, or Banach space instead of* $\mathbb{K}$*-vector space,* $\mathbb{K}$*-normed space, or* $\mathbb{K}$*-Banach space, respectively.*

Let X and Y be topological vector spaces over a non-Archimedean valued field $\mathbb{K}$. By $L(X, Y)$, we denote the set of all continuous linear operators from X to Y. If $X = Y$, then $L(X, Y)$ is denoted by $L(X)$. In what follows, we use the following terminology: T is a linear continuous operator on X, which means that $T \in L(X)$. The T*-orbit* of a vector $\mathbf{x} \in X$, for some operator $T \in L(X)$ is the set

$$O(\mathbf{x}, T) := \{T^n(\mathbf{x}); n \in \mathbb{Z}_+\}.$$

An operator $T \in L(X)$ is called *hypercyclic* if there exists some vector $\mathbf{x} \in X$ such that its T-orbit is dense in X. The corresponding vector $\mathbf{x}$ is called T*-hypercyclic*, and the set of all T-hypercyclic vectors is denoted by $HC(T)$. Similarly, T is called *supercyclic* if there exists a vector $\mathbf{x} \in X$ such that whose projective orbit

$$\mathbb{K} \cdot O(\mathbf{x}, T) := \{\lambda T^n(\mathbf{x}); n \in \mathbb{Z}_+,\ \lambda \in \mathbb{K}\}$$

is dense in X. The set of all T-supercyclic vectors is denoted by $SC(T)$. Finally, we recall that T is called *cyclic* if there exists $\mathbf{x} \in X$ such that

$$\mathbb{K}[T]\mathbf{x} := \mathrm{span} O(\mathbf{x}, T) = \{P(T)\mathbf{x}; P \text{ polynomial}\}$$

is dense in X. The set of all T-cyclic vectors is denoted by $CC(T)$.

Remark 2. *We stress that the notion of hypercyclicity makes sense only if space* X *is separable. Note that one has*

$$HC(T) \subset SC(T) \subset CC(T).$$

Definition 3 ([35])**.** *Let* X *be a topological vector space and let* $T \in L(X)$*. It is said that* T *satisfies the* **Hypercyclicity Criterion** *if there exist an increasing sequence of integers* (n_k)*, two dense sets* $\mathcal{D}_1, \mathcal{D}_2 \subset X$*, and a sequence of maps* $S_{n_k} : \mathcal{D}_2 \to X$ *such that:*

(1) $T^{n_k}(\boldsymbol{x}) \to \boldsymbol{0}$ *for any* $\boldsymbol{x} \in \mathcal{D}_1$;
(2) $S_{n_k}(\boldsymbol{y}) \to \boldsymbol{0}$ *for any* $\boldsymbol{y} \in \mathcal{D}_2$;
(3) $T^{n_k} S_{n_k}(\boldsymbol{y}) \to \boldsymbol{y}$ *for any* $\boldsymbol{y} \in \mathcal{D}_2$.

Note that in the above definition, the maps S_{n_k} are not assumed to be continuous or linear. We will sometimes say that T satisfies the Hypercyclicity Criterion with respect to the sequence (n_k). When it is possible to take $n_k = k$ and $\mathcal{D}_1 = \mathcal{D}_2$, it is usually said that T satisfies *Kitai's Criterion* [3].

Theorem 1 ([25]). *Let $T \in L(X)$, where X is a separable K-Banach space. Assume that T satisfies the Hypercyclicity Criterion. Then the operator T is hypercyclic.*

Definition 4 ([12]). *Let X be a Banach space and let $T \in L(X)$. We say that T satisfies the* **Supercyclic Criterion** *if there exist an increasing sequence of integers (n_k), two dense sets $\mathcal{D}_1, \mathcal{D}_2 \subset X$, and a sequence of maps $S_{n_k} : \mathcal{D}_2 \to X$ such that:*

(1) $\| T^{n_k}(\boldsymbol{x}) \| \cdot \| S_{n_k}(\boldsymbol{y}) \| \to 0$ *for any $\boldsymbol{x} \in \mathcal{D}_1$ and any $\boldsymbol{y} \in \mathcal{D}_2$;*
(2) $T^{n_k} S_{n_k}(\boldsymbol{y}) \to \boldsymbol{y}$ *for any $\boldsymbol{y} \in \mathcal{D}_2$.*

Theorem 2 ([25]). *Let $T \in L(X)$, where X is a separable Banach space. Assume that T satisfies the Supercyclic Criterion. Then T is supercyclic.*

Let us recall some basic definitions of dynamical systems which play a crucial role in our investigations. Assume that $T_0 : X_0 \to X_0$ and $T : X \to X$ are two continuous maps acting on topological spaces X_0 and X. The map T is said to be a *quasi-factor* of T_0 if there exists a continuous map with dense range $J : X_0 \to X$ such that $TJ = JT_0$. When this can be achieved with a homeomorphism $J : X_0 \to X$, we say that T_0 and T are *topological conjugate*. Finally, when $T_0 \in L(X_0)$ and $T \in L(X)$ and the factoring map (resp. the homeomorphism) J can be taken as linear, we say that T is a *linear quasi-factor* of T_0 (resp. that T_0 and T are *linearly conjugate*).

Lemma 2 ([25]). *Let $T_0 \in L(X_0)$ and $T \in L(X)$. Assume that there exists a continuous map with dense range $J : X_0 \to X$ such that $TJ = JT_0$. Then the following statements are satisfied:*

(1) *hypercyclicity of T_0 implies hypercyclicity of T;*
(2) *Let J be a homeomorphism and T_0 satisfies the Hypercyclicity Criterion; then, T satisfies the Hypercyclicity Criterion;*
(3) *Let J be a linear homeomorphism; then, T is hypercyclic if T_0 is hypercyclic.*

3. Some Basic Properties of Cyclic and Supercyclic Operators

In the present section, we are going to study some basic properties of cyclic/supercyclic operators.

Proposition 1. *Let X be a separable Banach space over a non-Archimedean valued field $\mathbb{K}$ and $T \in L(X)$. Then the following statements hold:*

(i) *T is cyclic if, and only if (for short "iff") λT is cyclic for every $\lambda \in \mathbb{K}^\times$;*
(ii) *T is supercyclic if λT is supercyclic for every $\lambda \in \mathbb{K}^\times$.*

Proof. (i) "If part" is clear. Therefore, we prove the "only if" part.

Let T be cyclic. Take $\mathbf{x} \in CC(T)$. Then for any $\mathbf{y} \in X$ and for every $\varepsilon > 0$, there exists a finite collection of $\mathbb{K}$-numbers $\{a_0, a_1, \dots, a_n\}$ such that

$$\|a_0\mathbf{x} + a_1 T(\mathbf{x}) + \cdots + a_n T^n(\mathbf{x}) - \mathbf{y} \| < \varepsilon.$$

For any $\lambda \in \mathbb{K}^\times$, we define a new finite collection of $\mathbb{K}$-numbers as follows: $b_k = \frac{a_k}{\lambda^k}$, where $k = \overline{0, n}$. Then,

$$\|b_0\mathbf{x} + b_0(\lambda T)(\mathbf{x}) + \cdots + b_n(\lambda T)^n(\mathbf{x}) - \mathbf{y}\| < \varepsilon.$$

This means $\mathbf{x} \in CC(\lambda T)$. The arbitrariness of λ implies the required assertion.

Using the same argument, one can prove (ii). □

Remark 3. *We notice that the hypercyclicity of T does not imply the hypercyclicity of λT, in general. Indeed, in [25] we considered an operator $\alpha I + \beta B$ and proved that it is hypercyclic if* $\max\{|\alpha|, 1\} < |\beta|$. *One can see that if $|\lambda| \leq \frac{1}{|\beta|}$, then $\lambda(\alpha I + \beta B)$ cannot be hypercyclic.*

In the present paper, we consider linear operators on $c_0(\mathbb{N})$, and here,

$$c_0(\mathbb{N}) := \{(x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{K}, |x_n| \to 0 \text{ as } n \to +\infty\}$$

with a norm

$$\| \mathbf{x} \| := \sup_n \{|x_n|\}.$$

It is clear that $c_0(\mathbb{N})$ is a Banach space. In what follows, we always assume that $c_0(\mathbb{N})$ is a separable space. Note that the separability of $c_0(\mathbb{N})$ is equivalent to the separability of $\mathbb{K}$. Let K be a countable dense subset of $\mathbb{K}$. Then the countable set

$$c_{00}(\mathbb{N}) := \{\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \cdots + \lambda_n \mathbf{e}_n : \lambda_k \in K, k \in \mathbb{N}\}$$

is dense in $c_0(\mathbb{N})$, where $\mathbf{e}_n$ is a unit vector such that only the n-th coordinate equals to one and others are zero.

The following lemma plays a crucial role in our further investigations.

Lemma 3. *Let T be a cyclic operator on $c_0(\mathbb{N})$. If $\mathbf{x} \in c_0(\mathbb{N})$ satisfies the following condition*

$$\begin{matrix} (T^n(\mathbf{x}))_i = x_i, \\ (T^n(\mathbf{x}))_j = x_j, \end{matrix} \qquad \text{for some } \ i \neq j, \ \ \forall n \in \mathbb{N}, \tag{1}$$

then $\mathbf{x} \notin CC(T)$.

Proof. The cyclicity of T implies that $CC(T) \neq \varnothing$. Assume that $\mathbf{x} \in c_0(\mathbb{N})$ satisfies (1). For the sake of convenience, we may assume that $|x_i| \geq |x_j|$. If $x_j = 0$, then for any $n \geq 1$ we have $(T^n(\mathbf{x}))_j = 0$. Hence, for any polynomial P, one has $(P(T)(\mathbf{x}))_j = 0$, which yields $P(T)\mathbf{x} \notin B(\mathbf{e}_j, 1^-)$. Consequently, the arbitrariness of P implies $\mathbf{x} \notin CC(T)$.

Now let us consider the case $x_j \neq 0$. Take arbitrary $\alpha, \beta \in \mathbb{K}^\times$ such that $|\alpha| > |x_i|$ and $|\beta| < |x_j|$. Define a set of polynomials as follows:

$$\mathcal{P}_{\mathbf{x},T} = \left\{ P : |(P(T)\mathbf{x})_j - \beta| < \frac{|\beta|}{2} \right\}.$$

One can see that $\mathbf{x} \in CC(T)$ if, and only if $\{P(T)\mathbf{x} : P \in \mathcal{P}_{\mathbf{x},T}\}$ is a dense subset of $c_0(\mathbb{N})$. We notice that $\mathcal{P}_{\mathbf{x},T} = \varnothing$ implies $\mathbf{x} \notin CC(T)$. Thus, we assume that $\mathcal{P}_{\mathbf{x},T} \neq \varnothing$ and pick an arbitrary $P \in \mathcal{P}_{\mathbf{x},T}$. Suppose that P has the following form, $P(T) = a_0 T^0 + a_1 T + \cdots + a_n T^n$. Then

$$\left|a_0 x_j + a_1 (T(\mathbf{x}))_j + \cdots + a_n (T^n(\mathbf{x}))_j - \beta\right| < \frac{|\beta|}{2}. \tag{2}$$

Keeping in mind the second inequality of (1) and using the non-Archimedean norm's property from (2), we obtain

$$|a_0 + a_1 + \cdots + a_n| = \frac{|\beta|}{|x_j|} < 1. \tag{3}$$

Now, we check the norm of $(P(T)\mathbf{x})_i - \alpha$. From the first inequality of (1) together with (3), one finds

$$|a_0 x_i + a_1 (T(\mathbf{x}))_i + \cdots + a_n (T^n(\mathbf{x}))_i| = |(a_0 + a_1 + \cdots + a_n) x_i| < |x_i|. \tag{4}$$

Since $|\alpha| > |x_i|$ and using the strong triangle inequality, we obtain

$$|(P(T)\mathbf{x})_i - \alpha| = |\alpha| > \frac{|\beta|}{2}.$$

This means

$$\| P(T)\mathbf{x} - \alpha\mathbf{e}_i - \beta\mathbf{e}_j \| > \frac{|\beta|}{2}. \tag{5}$$

Now, if P does not belong to $\mathcal{P}_{\mathbf{x},T}$, then (5) holds as well. Hence, by the arbitrariness of P, we arrive at $\mathbf{x} \notin CC(T)$. $\square$

Lemma 4. *Let T be a supercyclic operator on c_0. Assume that for a given $\mathbf{x} \in c_0$, there exists an integer $N \geq 0$ such that*

$$\begin{array}{l} |(T^n(\mathbf{x}))_i| = |(T^m(\mathbf{x}))_i|, \\ \left|(T^n(\mathbf{x}))_j\right| = \left|(T^m(\mathbf{x}))_j\right|, \end{array} \qquad \text{for some } i \neq j, \ \forall n, m \geq N, \tag{6}$$

then $\mathbf{x} \notin SC(T)$.

Proof. Assume that for $\mathbf{x} \in c_0$, the equalities (6) hold. Then, there are integers $N_i, N_j \in \{0, 1, \ldots, N\}$ such that $\left|(T^{N_i}(\mathbf{x}))_i\right| \geq |(T^n(\mathbf{x}))_i|$ and $\left|(T^{N_j}(\mathbf{x}))_j\right| \geq \left|(T^n(\mathbf{x}))_j\right|$ for all $n \geq 0$. Without loss of generality, we may assume that $\left|(T^{N_i}(\mathbf{x}))_i\right| \geq \left|(T^{N_j}(\mathbf{x}))_j\right|$. The lemma's claim is obvious if $\left|(T^{N_j}(\mathbf{x}))_j\right| = 0$. Indeed, if $\left|(T^{N_j}(\mathbf{x}))_j\right| = 0$, then $\| T^n(\mathbf{x}) - \mathbf{e}_j \| \geq 1$ for all $n \geq 0$, which means $\mathbf{x} \notin SC(T)$.

Thus, we consider a case $\left|(T^{N_j}(\mathbf{x}))_j\right| \neq 0$. Now, let us take an arbitrary $\alpha \in \mathbb{K}^{\times}$ such that

$$|\alpha| > \frac{|(T^k(\mathbf{x}))_i|}{|(T^k(\mathbf{x}))_j|}, \quad k \in \{0, 1, \ldots, N\}, \ (T^k(\mathbf{x}))_j \neq 0.$$

Then, for any $\lambda \in \mathbb{K}$ and for every $n \geq 0$, we have either $\left|\lambda(T^n(\mathbf{x}))_j - 1\right| \geq 1$ or $|\lambda(T^n(\mathbf{x}))_i - \alpha| \geq |\alpha|$, which yields $\| T^n(\mathbf{x}) - \alpha\mathbf{e}_i - \mathbf{e}_j \| \geq \min\{|\alpha|, 1\}$ for all $n \geq 0$. This shows that $\mathbf{x} \notin SC(T)$. $\square$

4. Generalized Weighted Backward Shift Operators on $c_0(\mathbb{N})$

Let us consider an infinite dimensional upper-triangular matrix $\mathcal{B} = (b_{i,j})_{i,j=1}^{\infty}$ over a non-Archimedean field $\mathbb{K}$, such that

$$\sup_{i,j}\{|b_{i,j}|\} < \infty, \quad b_{k,l} = 0, \quad \forall k \geq l. \tag{7}$$

For a given $\mathcal{B}$ with (7), we define the following linear operator on $c_0(\mathbb{N})$ by

$$B_{\mathcal{B}}(\mathbf{e}_n) = \begin{cases} 0, & \text{if } n = 1; \\ \sum_{j=1}^{n-1} b_{j,n}\mathbf{e}_j, & \text{if } n \geq 2. \end{cases} \tag{8}$$

The linear operator (8) is called *the generalized weighted backward shift* operator. Recall that if matrix $\mathcal{B}$ has the following extra condition $b_{k,l} = 0$ for all $k \neq l + 1$, then the corresponding linear operator $B_{\mathcal{B}}$ is reduced to *weighted backward shift*. In this setting, the operator acts as follows: $B_{\mathbf{a}}(\mathbf{e}_1) = 0$ and $B_{\mathbf{a}}(\mathbf{e}_n) = a_{n-1}\mathbf{e}_{n-1}$ if $n \geq 2$, where $a_{n-1} := b_{n-1,n-1}$ is called a *weighted backward shift*. Here, $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$. The operator $B_{\mathbf{a}}$ is called a *backward shift* if $b_{n-1,n-1} = 1$ for all $n \geq 1$, and such a shift is denoted by B.

We notice that the supercyclicity and hypercyclicity of weighted backward shift operators have been studied in [25].

Theorem 3 ([25])**.** *Let $B_{\mathbf{a}}$ be a weighted backward shift on $c_0(\mathbb{N})$. If $a_k \neq 0$ for all $k \geq 1$, then the following statements hold:*

(i) *$B_{\mathbf{a}}$ is supercyclic;*

(ii) *$B_{\mathbf{a}}$ is hypercyclic if*

$$\limsup_{n\to\infty} \prod_{k=1}^{n} |a_k| = \infty.$$

In this section, we are going to extend the formulated result for generalized weighted backward shift operators on $c_0(\mathbb{N})$.

Theorem 4. *Let $\mathcal{B}$ be a matrix given by (7). Assume that $b_{k,k+1} \neq 0$ for all $k \geq 1$. Then the generalized weighted backward shift operator $B_{\mathcal{B}}$ is supercyclic on $c_0(\mathbb{N})$.*

Proof. Let $B_{\mathcal{B}}$ be a generalized weighted backward shift, and $\mathcal{D}_1 = \mathcal{D}_2 := c_{00}(\mathbb{N})$ be the set of all finitely supported sequences. We define a linear map S on $\mathcal{D}_2$ as follows:

$$S(\mathbf{e}_n) = \begin{cases} \frac{1}{b_{1,2}}\mathbf{e}_2, & \text{if } n = 1, \\ \sum_{j=1}^{n} \alpha_{n,j}\mathbf{e}_{j+1}, & \text{if } n \geq 2, \end{cases} \tag{9}$$

where the coefficients $\alpha_{n,1}, \dots, \alpha_{n,n}$ are given by

$$\alpha_{n,n} = \frac{1}{b_{n,n+1}}, \quad \alpha_{n,n-i} = -\frac{1}{b_{n-i,n-i+1}} \sum_{j=n-i+1}^{n} \alpha_{n,j} b_{n-i,j+1}, \quad i \in \{1, 2, \dots, n-1\}.$$

Then, we put $S_k := S^k$. One can see that $B_{\mathcal{B}}^k S_k = I$ on $\mathcal{D}_2$. Let us pick an arbitrary $\mathbf{x} \in \mathcal{D}_1$. Then there exists an integer $N \geq 1$ such that $x_k = 0$ for all $k > N$. Hence, $B_{\mathcal{B}}^k(\mathbf{x}) = \mathbf{0}$ for every $k > N$. This means that $\|B_{\mathcal{B}}^k(\mathbf{x})\| = 0$ for all $k > N$. Then, for any $\mathbf{y} \in \mathcal{D}_2$, we have $\|B_{\mathcal{B}}^k(\mathbf{x})\| \cdot \|S_k(\mathbf{y})\| = 0$ for every $k > N$. According to Theorem 2, the operator $B_{\mathcal{B}}$ is supercyclic. □

Due to Remark 2 from the last theorem, we can formulate the following fact.

Corollary 1. *Let $\mathcal{B}$ be a matrix given by (7). Assume that $b_{k,k+1} \neq 0$ for all $k \geq 1$. Then the generalized weighted backward shift $B_{\mathcal{B}}$ is cyclic on $c_0(\mathbb{N})$.*

Now we study the hypercyclic phenomena of the generalized weighted backward shift operator on $c_0(\mathbb{N})$.

Theorem 5. *Let $\mathcal{B}$ be a matrix given by (7) with an extra condition,*

$$\sup_j \{|b_{k,j}|\} = |b_{k,k+1}| \neq 0, \ \textit{for any} \ k \geq 1. \tag{10}$$

Then, $B_{\mathcal{B}}$ is hypercyclic on $c_0(\mathbb{N})$ if

$$\limsup_{n\to\infty} \prod_{i=1}^{n} |b_{i,i+1}| = \infty. \tag{11}$$

Proof. For a given matrix $\mathcal{B} = (b_{i,j})_{i,j=1}^{\infty}$, we define a sequence $\mathbf{b} = \{b_{n,n+1}\}_{n=1}^{\infty}$. Now, we consider the weighted backward shift operator $B_{\mathbf{b}}$ and show the existence of a linear homeomorphism P on $c_0(\mathbb{N})$ such that $B_{\mathbf{b}}P = PB_{\mathcal{B}}$.

Let us consider the linear operator

$$P(\mathbf{e}_n) = \sum_{j=1}^{n} p_{j,n}\mathbf{e}_j, \quad \forall n \geq 1, \tag{12}$$

where

$$p_{i+1,i+1} = p_{i,i}, \quad p_{i+1,j} = \frac{1}{b_{i,i+1}} \sum_{k=i}^{j-1} b_{k,j} p_{i,k}, \quad \text{for every } 1 < i+1 < j. \tag{13}$$

For convenience, we assume that $p_{1,j} = 1$ for all $j \geq 1$. Then by (10), and applying the strong triangle inequality into the second equality of (13), we obtain $|p_{i,j}| \leq 1$ for all $1 \leq i \leq j$, which guarantees the boundedness of P. Thus, $P \in L(c_0(\mathbb{N}))$.

Now we are going to establish the invertibility of P and $P^{-1} \in L(c_0(\mathbb{N}))$. Let us fist consider the following linear operator

$$\hat{P}(\mathbf{e}_n) = \sum_{j=1}^{n} \hat{p}_{j,n} \mathbf{e}_j, \quad \forall n \geq 1,$$

where $\hat{p}_{ij}$ coefficients satisfy the following recurrence formula:

$$\hat{p}_{i,i} = p_{i,i}, \quad \hat{p}_{i,j} = -\sum_{k=i+1}^{j} p_{i,k} \hat{p}_{k,j}, \quad \text{for every } 1 \leq i < j. \tag{14}$$

We notice that (14) is equivalent to the following

$$: \hat{p}_{i,i} = p_{i,i}, \quad \hat{p}_{i,j} = -\sum_{k=i}^{j-1} \hat{p}_{i,k} p_{k,j}, \quad \text{for every } 1 \leq i < j. \tag{15}$$

From (14) and (15), we obtain $\hat{P}P = P\hat{P} = I$, which shows that $P^{-1} = \hat{P}$. Moreover, due to $|p_{i,j}| \leq 1$ for all $i, j \in \mathbb{N}$, from (14) it follows that $|\hat{p}_{i,j}| \leq 1$ for any $i, j \in \mathbb{N}$. This means that the operator $\hat{P}$ is bounded, that is, $P^{-1} \in L(c_0(\mathbb{N}))$.

Thus, we have shown that P is a linear homeomorphism on $c_0(\mathbb{N})$. One can see that (13) implies $B_{\mathbf{b}} P = P B_{\mathcal{B}}$. Then, thanks to Lemma 2, we infer that $HC(B_{\mathcal{B}}) \neq \varnothing$ if, and only if $HC(B_{\mathbf{b}}) \neq \varnothing$. Then, by Theorem 3, the hypercyclicity of the generalized weighted backward shift operator is equivalent to (11). This completes the proof. □

Remark 4. *We stress that a similar kind of result does not exist in the real/complex setting. In the proof of Theorem 5, we have essentially used the non-Archimedean property.*

5. $I + B_{\mathcal{B}}$ Operator on $c_0(\mathbb{N})$

In many areas of mathematics, an operator $I + T$ appears, where I is an identity and T is a given operator. In this paper, we consider the generalized weighted backward shift operator $B_{\mathcal{B}}$ instead of T. Our aim is going to study the supercyclicity of such types of operators on $c_0(\mathbb{N})$.

For a given $\mathcal{B}$ with (7), we denote

$$T_{\mathcal{B}} := I + B_{\mathcal{B}}, \tag{16}$$

that is, for any $\mathbf{x} \in c_0(\mathbb{N})$

$$(T_{\mathcal{B}}(\mathbf{x}))_k = x_k + \sum_{j=k+1}^{\infty} b_{k,j} x_j, \quad \forall k \in \mathbb{N}. \tag{17}$$

Proposition 2. *Let $T_{\mathcal{B}}$ be an operator given by (16). Then the following assertions hold:*

(i) *If $CC(T_{\mathcal{B}}) \neq \varnothing$, then*

$$card(\{n \in \mathbb{N} : (b_{n1}, b_{n2}, \dots) = \mathbf{0}\}) \leq 1.$$

Here, $card(A)$ stands for the cardinality of a set A.

(ii) *If $SC(T_{\mathcal{B}}) \neq \varnothing$, then*
$$(b_{k1}, b_{k2}, b_{k3}, \dots) \neq \mathbf{0}, \quad \forall k \geq 2.$$
(iii) *If $HC(T_{\mathcal{B}}) \neq \varnothing$, then*
$$(b_{k1}, b_{k2}, b_{k3}, \dots) \neq \mathbf{0}, \quad \forall k \geq 1.$$

Proof. Let $T_{\mathcal{B}}$ be a cyclic operator on $c_0(\mathbb{N})$. Suppose that there exist two distinct positive integers k and m, such that $b_{kj} = b_{mj} = 0$ for all $j \geq 1$. Then for any $n \geq 1$, one has

$$\begin{matrix} (T^n_{\mathcal{B}}(\mathbf{x}))_k = x_k, \\ (T^n_{\mathcal{B}}(\mathbf{x}))_m = x_m, \end{matrix} \quad \forall \mathbf{x} \in c_0(\mathbb{N}).$$

This means that a vector $\mathbf{x}$ satisfies (1), then Lemma 3 implies $\mathbf{x} \notin CC(T_{\mathcal{B}})$. Hence, the arbitrariness of $\mathbf{x}$ yields $c_0(\mathbb{N}) \cap CC(T_{\mathcal{B}}) = \varnothing$, which contradicts the cyclicity of $T_{\mathcal{B}}$.

(ii) Let $SC(T_{\mathcal{B}}) \neq \varnothing$. Suppose that there exists an integer $k \geq 2$ such that

$$(b_{k1}, b_{k2}, b_{k3}, \dots) = \mathbf{0}.$$

Take an arbitrary $\mathbf{x} \in SC(T_{\mathcal{B}})$. Then,

$$(T^n_{\mathcal{B}}(\mathbf{x}))_k = x_k, \quad \forall n \geq 1. \tag{18}$$

We denote $b = \max\left\{1, \sup_{i,j}\{|b_{ij}|\}\right\}$, and due to $\mathbf{x} \in SC(T_{\mathcal{B}})$, one can find an integer $N \geq 1$ such that $\left|(T^N_{\mathcal{B}}(\mathbf{x}))_{k-1}\right| > \left|b(T^N_{\mathcal{B}}(\mathbf{x}))_i\right|$ for all $i \geq k$. Then, keeping in mind

$$(T_{\mathcal{B}}(\mathbf{x}))_{k-1} = x_{k-1} + \sum_{j=k}^{\infty} b_{k-1,j} x_j,$$

and by the strong triangle inequality, we obtain

$$|(T^n_{\mathcal{B}}(\mathbf{x}))_{k-1}| > |b(T^n_{\mathcal{B}}(\mathbf{x}))_i|$$

for all $n \geq N$, and for every $i \geq k$. Hence,

$$|(T^n_{\mathcal{B}}(\mathbf{x}))_{k-1}| = \left|(T^N_{\mathcal{B}}(\mathbf{x}))_{k-1}\right|, \quad \forall n \geq N.$$

The last equality together with (18), thanks to Lemma 4, yields that $\mathbf{x} \notin SC(T_{\mathcal{B}})$, which is a contradiction.

(iii) Let $HC(T_{\mathcal{B}}) \neq \varnothing$. Then, $SC(T_{\mathcal{B}}) \neq \varnothing$ which by (ii) yields $(b_{k1}, b_{k2}, b_{k3}, \dots) \neq \mathbf{0}$ for all $k \geq 2$. Thus, it is enough to show that $(b_{11}, b_{12}, b_{13}, \dots) \neq \mathbf{0}$.

Take $\mathbf{x} \in HC(T_{\mathcal{B}})$. Assume that $(b_{11}, b_{12}, b_{13}, \dots) = \mathbf{0}$. Then for any $n \geq 1$, one gets $(T^n_{\mathcal{B}}(\mathbf{x}))_1 = x_1$. Hence, for any $\alpha \in \mathbb{K}$ with $|\alpha| > |x_1|$, we get $\| T^n_{\mathcal{B}}(\mathbf{x}) - \alpha \mathbf{e}_1 \| \geq |\alpha|$. This contradicts $\mathbf{x} \in HC(T_{\mathcal{B}})$. Thus, we conclude that $(b_{11}, b_{12}, b_{13}, \dots) \neq \mathbf{0}$. □

The proved proposition clearly shows (see (i) and (ii)) the difference between the supercyclicity and cyclicity of $T_{\mathcal{B}}$.

Theorem 6. *Let $\mathcal{B}$ be a matrix given by (7). If*

$$\sup_{j \geq 1}\{|b_{1,j}|\} = 0, \tag{19}$$

then the following statements are equivalent:

(i) *$T_{\mathcal{B}}$ is supercyclic on $c_0(\mathbb{N})$;*
(ii) *$T_{\mathcal{B}'}$ is hypercyclic on $c_0(\mathbb{N})$, where $\mathcal{B}' = (b'_{i,j})^{\infty}_{i,j}$ and $b'_{i,j} = b_{i+1,j+1}$ for all $i, j \in \mathbb{N}$.*

Proof. Assume that (19) holds. Let us establish the implication $(i) \Rightarrow (ii)$. Pick any $\mathbf{x} \in SC(T_{\mathcal{B}})$. Then $(T_{\mathcal{B}}(\mathbf{x}))_1 = x_1$. We notice that $x_1 \neq 0$, otherwise it is not a supercyclic vector for $T_{\mathcal{B}}$. Indeed, for $x_1 = 0$, we can easily check that $\mathbf{e}_1 \notin \overline{\mathbb{K} \cdot O(\mathbf{x}, T_{\mathcal{B}})}$.

Due to $\mathbf{x} \in SC(T_{\mathcal{B}})$, for any $\mathbf{y} \in c_0(\mathbb{N})$ with $y_1 = x_1$ and for any $0 < \varepsilon < |x_1|$, there exist $\lambda \in \mathbb{K}^\times$ and $n \in \mathbb{N}$ such that

$$\begin{aligned} &|(\lambda - 1)x_1| < \varepsilon, \\ &|\lambda (T^n_{\mathcal{B}}(\mathbf{x}))_j - y_j| < \varepsilon, \quad \forall j \geq 2. \end{aligned} \tag{20}$$

From the first inequality of (20), we get $|\lambda - 1| < \frac{\varepsilon}{|x_1|}$ and $|\lambda| = 1$. Then, from the second inequality of (20) together with $|(\lambda - 1)y_j| < \frac{\varepsilon |y_j|}{|x_1|}$, one finds

$$|\lambda\big((T^n_{\mathcal{B}}(\mathbf{x}))_j - y_j\big)| < \max\left\{\varepsilon, \frac{\varepsilon |y_j|}{|x_1|}\right\}, \quad \forall j \geq 2.$$

Hence, using $|\lambda| = 1$,

$$|(T^n_{\mathcal{B}}(\mathbf{x}))_j - y_j| < \varepsilon \cdot \max\left\{1, \frac{\| \mathbf{y} \|}{|x_1|}\right\}, \quad \forall j \geq 2. \tag{21}$$

We denote $\mathbf{x}' = (x_2, x_3, x_4, \dots)$. Then, by rewriting (21)

$$|(T^n_{\mathcal{B}'}(\mathbf{x}'))_j - y_{j+1}| < \varepsilon \cdot \max\left\{1, \frac{\| \mathbf{y} \|}{|x_1|}\right\}, \quad \forall j \geq 1.$$

The last one together with the arbitrariness of $\mathbf{y}$ implies that $\mathbf{x}' = (x_2, x_3, x_4, \dots)$ is a hypercyclic vector for $T_{\mathcal{B}'}$.

$(ii) \Rightarrow (i)$. Now, we assume that $HC(T_{\mathcal{B}'}) \neq \varnothing$. Pick $\mathbf{x} \in HC(T_{\mathcal{B}'})$ and show that for any $\alpha \neq 0$ the following vector

$$\bar{\mathbf{x}}_j = \begin{cases} \alpha, & j = 1, \\ x_{j-1}, & j > 1. \end{cases}$$

is supercyclic for $T_{\mathcal{B}}$.

Indeed, take an arbitrary $\mathbf{z} \in c_0(\mathbb{N})$. Then for any $\varepsilon > 0$, one finds $\lambda \in \mathbb{K}^\times$ such that $|\lambda\alpha - z_1| < \varepsilon$. On the other hand, by $\mathbf{x} \in HC(T_{\mathcal{B}'})$ there is an integer $n \geq 1$ with

$$\left|(T^n_{\mathcal{B}'}(\mathbf{x}))_j - \lambda^{-1} z_{j+1}\right| < \frac{\varepsilon}{|\lambda|}, \quad \forall j \geq 1,$$

which is equivalent to

$$|\lambda(T^n_{\mathcal{B}}(\bar{\mathbf{x}}))_j - z_j| < \varepsilon, \quad \forall j \geq 2.$$

The last inequality together with $|\lambda\alpha - z_1| < \varepsilon$ yields that $\lambda T^n_{\mathcal{B}}(\bar{\mathbf{x}}) \in B(\mathbf{z}, \varepsilon^-)$. The arbitrariness of $\mathbf{z}$ and ε imply $\bar{\mathbf{x}} \in SC(T_{\mathcal{B}})$. □

Proposition 3. *Assume that $\mathbf{a} \in \ell^\infty(\mathbb{N})$. Let $B_{\mathbf{a}}$ be a weighted backward shift operator and $T_{B_{\mathbf{a}}}$ be an operator given by* (16)*. If*

$$\lim_{n \to \infty} \prod_{j=1}^{n} |a_j| = \infty, \tag{22}$$

then $HC(T_{B_{\mathbf{a}}}) \neq \varnothing$;

Proof. Let (32) be satisfied. First, we show that

$$\bigcup_{n=1}^{\infty} T^{-n}_{B_{\mathbf{a}}}(\mathbf{0})$$

is dense in $c_0(\mathbb{N})$.

We notice that $T^n_{B_{\mathbf{a}}}(\mathbf{x}) = 0$ has non-zero solution for any $n \geq 1$. Indeed, $T^n_{B_{\mathbf{a}}}(\mathbf{x}) = 0$ is equivalent to

$$x_j + \sum_{k=1}^{n} \binom{n}{k} x_{j+k} \prod_{i=1}^{k} a_{j+i-1} = 0, \quad \forall j \geq 1, \tag{23}$$

where $x_m \to 0$ as $m \to \infty$.

It is clear that for any $(c_1, c_2, \ldots, c_n) \in \mathbb{K}^n$ the sequence $(x_m)_{m\geq 1}$ defined by

$$x_m = \begin{cases} c_m, & \text{if } m \leq n, \\ -\frac{x_{m-n} + \sum_{k=1}^{n-1} \binom{n}{k} x_{m-n+k} \prod_{i=1}^{k} a_{m-n+i-1}}{\prod_{i=0}^{n-1} a_{m-n+k}}, & \text{if } m > n, \end{cases} \tag{24}$$

satisfies (23). We need to check the sequence (24) belongs to $c_0(\mathbb{N})$. Let $n \geq 1$, then by (24) one finds

$$|x_{n+k}| \leq \max_{k \leq j \leq n+k-1} \frac{|x_j|}{\prod_{i=j}^{n+k-1} |a_i|}, \quad \forall k \geq 1. \tag{25}$$

Using (25) by induction, one can prove the following

$$|x_{n+k}| \leq \max_{1 \leq j \leq n} \frac{|x_j|}{\prod_{i=j}^{n+k-1} |a_i|}, \quad \forall k \geq 1. \tag{26}$$

Then, keeping in mind (22) from (26), one finds $|x_{n+k}| \to 0$ as $k \to \infty$, hence, $(x_m)_{m\geq 1} \in c_0(\mathbb{N})$. Thus, we have shown that $T^n_{B_{\mathbf{a}}}(\mathbf{x}) = \mathbf{0}$ has a non-zero solution. Moreover, we can find all solutions of that equation.

Now, let us establish that $\overline{\bigcup_{n=1}^{\infty} T^{-n}_{B_{\mathbf{a}}}(\mathbf{0})} = c_0(\mathbb{N})$. Pick any $\mathbf{y} \in c_0(\mathbb{N})$. Then, for any $\varepsilon > 0$, there exists a positive integer n_0 such that $|y_m| < \varepsilon$ for all $m > n_0$. Due to (22), one can find an integer $N > n_0$ such that

$$\min_{1 \leq j \leq N} \prod_{i=j}^{N+k-1} |a_i| > \frac{\| \mathbf{y} \|}{\varepsilon}, \quad \forall k \geq 1. \tag{27}$$

Take $\mathbf{x} = (x_m)_{m\geq 1}$ as follows:

$$x_m = \begin{cases} y_m, & \text{if } m \leq N, \\ -\frac{x_{m-N} + \sum_{k=1}^{N-1} \binom{N}{k} x_{m-N+k} \prod_{i=1}^{k} a_{m-N+i-1}}{\prod_{i=0}^{N-1} a_{m-N+k}}, & \text{if } m > N, \end{cases} \tag{28}$$

Then $T^N_{B_{\mathbf{a}}}(\mathbf{x}) = \mathbf{0}$, that is, $\mathbf{x} \in \bigcup_{n=1}^{\infty} T^{-n}_{B_{\mathbf{a}}}(\mathbf{0})$. By (27), (28) together with (26), one gets $|x_m| < \varepsilon$ for any $m > N$. Hence, by (28) we infer $\| \mathbf{x} - \mathbf{y} \| < \varepsilon$. By the arbitrariness of $\mathbf{y}$ we arrive at the required assertion.

Now, we are going to show that $T_{B_{\mathbf{a}}}$ satisfies the Hypercyclic Criterion. Indeed, let us denote

$$\mathcal{D}_1 := \bigcup_{n=1}^{\infty} T^{-n}_{B_{\mathbf{a}}}(\mathbf{0}), \quad \mathcal{D}_2 := c_{00}(\mathbb{N}).$$

It isobvious that these sets are dense in $c_0(\mathbb{N})$. We recall that, by the construction of $\mathcal{D}_1$, for any $\mathbf{x} \in \mathcal{D}_1$ there exists $k \geq 1$ such that $T^n_{B_{\mathbf{a}}}(\mathbf{x}) = 0$ for all $n \geq k$.

Over the set $\mathcal{D}_2$, we define a linear operator $S_{\mathbf{b}}$ by

$$(S_{\mathbf{a}}(\mathbf{y}))_j = \begin{cases} 0, & \text{if } j = 1, \\ \sum_{k=1}^{j-1} (-1)^{j-1-k} y_k \prod_{i=k}^{j-1} a_i^{-1}, & \text{if } j > 1 \end{cases} \tag{29}$$

We are going to estimate the norm of $S_{\mathbf{a}}^n(\mathbf{y})$ for any $n \neq 1$ and $\mathbf{y} \in \mathcal{D}_2$. Given any $\mathbf{y} \in \mathcal{D}_2$, one finds an integer $m \geq 1$ such that $y_m \neq 0$ and $y_k = 0$ for all $k > m$. For any $\varepsilon > 0$, we get $n_0 \in \mathbb{N}$ such that

$$\min\left\{\prod_{k=j}^{n}|a_k| : 1 \leq j \leq m\right\} > \frac{1}{\varepsilon}, \quad \forall n > n_0.$$

By (29) together with The last inequalit, for any $n \geq n_0$, we obtain

$$\| S_{\mathbf{a}}^n(\mathbf{y}) \| \leq \varepsilon \cdot \| \mathbf{y} \| . \tag{30}$$

Hence, for every $\mathbf{y} \in \mathcal{D}_2$,

$$\| S_{\mathbf{a}}^n(\mathbf{y}) \| \to 0 \quad \text{as } n \to \infty. \tag{31}$$

On the other hand, it is easy to check that $T_{B_{\mathbf{a}}}^n S_{\mathbf{b}}^n(\mathbf{y}) = \mathbf{y}$ for every $\mathbf{y} \in c_{00}(\mathbb{N})$ and $n \geq 1$. Consequently, we have shown that $T_{B_{\mathbf{a}}}$ satisfies the Hypercyclic Criterion. Thus, due to Theorem 1, we infer that $T_{B_{\mathbf{a}}}$ is hypercyclic. This completes the proof. □

Theorem 7. *Let $\mathcal{B}$ be an infinite dimensional matrix (7) with the extra condition (10). Then for the operator $T_{\mathcal{B}}$ given by (16) the following statements hold:*

(i) *if*

$$\lim_{n\to\infty}\prod_{j=1}^{n}|b_{j,j+1}| = \infty, \tag{32}$$

then $HC(T_{\mathcal{B}}) \neq \varnothing$;

(ii) *if*

$$\lim_{n\to\infty}\prod_{j=2}^{n}|b_{j,j+1}| = \infty, \tag{33}$$

then $SC(T_{\mathcal{B}}) \neq \varnothing$.

Proof. Let P be a linear homeomorphism given by (12), (13). Then, by Theorem 5, we know that $B_{\mathbf{b}}P = PB_{\mathcal{B}}$, where $\mathbf{b} = \{b_{k,k+1}\}_{k=1}^{\infty}$. Hence, $T_{B_{\mathbf{b}}}P = PT_{\mathcal{B}}$. Therefore, the hypercyclicity (supercyclicity) of $T_{\mathcal{B}}$ is equivalent to the hypercyclicity (resp. supercyclicity) of $T_{B_{\mathbf{b}}}$. Due to Proposition 3, we obtain the statement (i). The assertion (ii) immediately follows from Theorem 6 and the statement (i). This completes the proof. □

Example 2. *Let us consider the following example, which has not any classical analogue. Assume that $K = \mathbb{Q}_p$, and define the matrix $\mathcal{B}$ as follows: $b_{k,k+1} = 1/p$, $b_{ij} = 1$, $j - i \geq 2$ and $b_{ij} = 0$, if $i - j \geq 0$. Clearly, the defined matrix is upper triangular, and (10) is satisfied. Then, the operator $T_{\mathcal{B}}$ has the following form:*

$$(T_{\mathcal{B}}(\boldsymbol{x}))_k = x_k + \frac{x_{k+1}}{p} + \sum_{j=k+2}^{\infty} x_j, \quad \forall k \in \mathbb{N}. \tag{34}$$

Moreover, by Theorem 7 (i) the operator $T_{\mathcal{B}}$ is hypercyclic. However, if one looks at this operator on $\ell^2(\mathbb{N})$ (over $\mathbb{R}$ or $\mathbb{C}$), then it is unbounded, so its hypercyclicity is another story.

We notice that in [23] various functional models of the unilateral shift operator B have been given. Let us provide an application of the result.

Example 3. *Let $\mathbb{Z}_p$ be the unit ball in $\mathbb{Q}_p$. By $C(\mathbb{Z}_p, \mathbb{C}_p)$ we denote the space of all continuous functions on $\mathbb{Z}_p$ with values in $\mathbb{C}_p$ endowed with "sup"-norm. Consider a linear operator $K : C(\mathbb{Z}_p, \mathbb{C}_p) \to C(\mathbb{Z}_p, \mathbb{C}_p)$ defined by*

$$(Kf)(x) = f(x+1) - f(x), \ \ (x \in \mathbb{Z}_p), \ \ f \in C(\mathbb{Z}_p, \mathbb{C}_p).$$

This operator K can be interpreted as the annihilation operators in a p-adic representation of the canonical commutation relations of quantum mechanics [36].

It is well-known [34] that the Mahler polynomials

$$P_n(x) = \frac{x(x-1)\cdots(x-n+1)}{n!}, \quad n \in \mathbb{N}; \ P_0(x) = 1,$$

form an orthonormal basis in $C(\mathbb{Z}_p, \mathbb{C}_p)$. Then, K acts on the Mahler polynomials as follows:

$$KP_n = P_{n-1}, \ n \in \mathbb{N}; \ KP_0 = 0.$$

It is known that the spaces $C(\mathbb{Z}_p, \mathbb{C}_p)$ and $c_0(\mathbb{N})$ are isomorphic via the isomorphism

$$\sum_{n=0}^{\infty} x_n P_n \to (x_0, x_1, \dots, x_n, \dots)$$

therefore, the operator K is transformed to the shift operator B.

Now, let us consider an operator $I + \lambda K$ on $C(\mathbb{Z}_p, \mathbb{C}_p)$. By Theorem 7, one can establish that the operator $I + \lambda K$ is hypercyclic if, and only if $|\lambda| > 1$. Assume that $g \in C(\mathbb{Z}_p, \mathbb{C}_p)$ be a hypercyclic vector for $I + \lambda K$ provided $|\lambda| > 1$. Then the hyperbolicity of $I + \lambda K$ implies that the set $\{(I + \lambda K)^n g\}$ is dense in $C(\mathbb{Z}_p, \mathbb{C}_p)$. On the other hand, we have

$$(K^m g)(x) = \sum_{j=0}^{m} \binom{m}{j} g(x + m - j). \tag{35}$$

So,

$$\begin{aligned} ((I + \lambda K)^n g)(x) &= \sum_{k=0}^{n} \binom{n}{k} \lambda^k (K^k g)(x) \\ &= \sum_{k,j=0}^{n} \binom{n}{k} \binom{k}{j} \lambda^k g(x + k - j). \end{aligned} \tag{36}$$

Due to the hypercyclicity of $I + \lambda K$ we infer that for arbitrary $\varepsilon > 0$ and any function $f \in C(\mathbb{Z}_p, \mathbb{C}_p)$ there is $N \in \mathbb{N}$ such that

$$\left\| f - \sum_{k,j=0}^{n} \binom{N}{k} \binom{k}{j} \lambda^k g(\cdot + k - j) \right\| < \varepsilon.$$

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