mathematics

Article

An Analytical Method of Electromagnetic Wave Scattering by a Highly Conductive Sphere in a Lossless Medium with Low-Frequency Dipolar Excitation

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Abstract: The current research involves an analytical method of electromagnetic wave scattering by an impenetrable spherical object, which is immerged in an otherwise lossless environment. The highly conducting body is excited by an arbitrarily orientated time-harmonic magnetic dipole that is located at a reasonable remote distance from the sphere and operates at low frequencies for the physical situation under consideration, wherein the wavelength is much bigger than the size of the object. Upon this assumption, the scattering problem is formulated according to expansions of the implicated magnetic and electric fields in terms of positive integer powers of the wave number of the medium, which is linearly associated to the implied frequency. The static Rayleigh zeroth-order case and the initial three dynamic terms provide an excellent approximation for the obtained solution, while terms of higher orders are of minor significance and are neglected, since we work at the low-frequency regime. To this end, Maxwell's equations reduce to a finite set of interrelated elliptic partial differential equations, each one accompanied by the perfectly electrically conducting boundary conditions on the metal sphere and the necessary limiting behavior as we move towards theoretical infinity, which is in practice very far from the observation domain. The presented analytical technique is based on the introduction of a suitable spherical coordinated system and yields compact fashioned three-dimensional solutions for the scattered components in view of infinite series expansions of spherical harmonic modes. In order to secure the validity and demonstrate the efficiency of this analytical approach, we invoke an example of reducing already known results from the literature to our complete isotropic case.

Keywords: low-frequency scattering; electromagnetic fields; magnetic dipole; spherical scatterer

Citation: Stefanidou, E.; Vafeas, P.; Kariotou, F. An Analytical Method of Electromagnetic Wave Scattering by a Highly Conductive Sphere in a Lossless Medium with Low-Frequency Dipolar Excitation. *Mathematics* **2021**, *9*, 3290. https://doi.org/10.3390/math9243290

Academic Editor: Nikolaos Tsitsas

Received: 8 November 2021
Accepted: 12 December 2021
Published: 17 December 2021

1. Introduction

Real-life physical applications of practical interest in science and engineering are promptly associated with the fundamental principles of advanced electromagnetism [1] and the related information concerning analytical, semi-analytical and numerical techniques towards the solution of important problems. An indivisible part of Maxwell's electromagnetic theory is the low-frequency wave scattering [2] by solid or penetrable bodies of arbitrary geometry, embedded within various kind of media, i.e., conductive or lossless, wherein the source of the produced fields, located nearby, operates at very low frequencies, as is the case for the current assumption. Thus, the issue of retrieving certain anomalies becomes a twofold challenge of solving initially the forward and thereafter the inverse scattering problem. The primary task includes the calculation of the scattered electric and magnetic fields by means of complete knowledge of the physical and geometrical parameters of each situation under consideration. By deciphering such fields, significant information about orientation, size, shape and the magnetic and electric properties of the targets is inferred, which concerns an inverse problem query. However, this is not an easy task, since the inverse problem [3] cannot be tackled in a robust fashion unless integrated models of the field behavior and effective mathematical tools [4] are available. This research is motivated by numerous examples in this direction, such as the detection of inclusions in two-phase composites [5], mineral exploration [6], identification of cavities [7], detection of Unexploded Ordinance [8] and buried objects [9], scattering by chiral bodies [10] and several other cases related to the identification of metallic or non-metallic objects of different shapes and sizes, regardless the nature of the surroundings.

The need to employ effective analytical or even semi-analytical techniques in order to confront such physical situations that capture the essence of the electromagnetic process itself is constantly increasing. Indeed, even though the use of brute computer codes is nowadays inevitable, we should not overlook the fact that the mathematical analysis is the backbone of numerical analysis. Therefore, purely analytical methods, besides the insight they offer to the understanding of the physical background and besides their importance for checking the credibility of numerical methods, keep the mathematical community alive, providing the bases of mathematics. Within this framework, important analytical research has been conducted concerning the identification of metallic impenetrable objects, which are embedded in conductive media, e.g., Earth's subsurface, and they are illuminated by magnetic dipole sources that operate at low frequencies as a fair approximation of such kind of physical applications. For example, the low-frequency electromagnetic fields that are scattered from a perfectly conducting [11] or penetrable [12] sphere in a conductive medium and excited by a magnetic dipole, have been given in closed analytical form, with the analytical results in [11] being followed by a numerical implementation. A demonstration of the utility of such solutions, given in an analytical compact fashion, to the construction of a fast and accurate inverse scheme was accomplished in [13], wherein the low-frequency on-site identification of a highly conductive body buried in Earth from a model ellipsoid was presented. In view of this aspect, cases that involve more complicated geometries for the metallic bodies have been introduced; one can refer, for instance, to [14], in which the authors study the low-frequency interaction with the conductive environment of a ring torus, which scatters off incident waves that are produced by a magnetic dipole source. Different kinds of surroundings have also attracted researchers since the early 1960s [15], where the electromagnetic response of a conductive sphere to a dipole field in a homogeneous dielectric medium has been studied. Though, the interest in occupation with low-frequency scattering by a perfectly electrically conducting target, which is located in an otherwise lossless ambient, e.g., air, has also been on the frontline of research for more than a decade [16–18]. The peculiarities encountered herein have an indivisible character that offers us the challenge of dealing with these problems and becoming familiar with new techniques in solving fundamental physical electromagnetic scattering problems.

The present work is focused on the application of low-frequency diffusive scattering theory in handling the problem of retrieving an impenetrable spherical metallic body in a lossless, i.e., perfect dielectric medium. Such investigation, even concerning highly symmetric geometry, sets the basis of studying the problem for complete spatial isotropy and deepens the understanding of more complicated anisotropic geometries that are introduced in previous models. Moreover, it provides a benchmark, where the more intricate models should meet, by geometrical reduction of the corresponding solution. The most appropriately fitting geometry to this case is introduced by the spherical coordinate system [19], which is adopted for the current status. On the other hand, a time-harmonic magnetic dipole source, acting as the known primary source, operates at the low-frequency regime and produces the three-dimensional incident electromagnetic waves, which propagate towards the spherical scatterer. Then, the spherical-shaped body responds to the excitation and generates the scattered magnetic and electric fields, where their summation comprises the total fields. The nature of the involved physical applications, in which the distance between the source and the object is considered to be significant and the characteristic dimension of the scatterer requires a large enough wavelength to cover the body, justifies the fact that the frequency becomes very small. On that account, we take advantage of this property and adopt the convenient so-called low-frequency technique, by which we expand each one of the incidents, the scattered and, consequently, the total electromagnetic fields, in terms of positive integer powers of the wavenumber at the operation frequency, wherein the coefficients of the expansions denote the corresponding three-dimensional fields for every low-frequency order.

Hence, the initial scattering problem is reduced to a sequence of interconnected boundary value problems, which incorporate elliptic-type Laplace's and Poisson's partial differential equations, being accompanied by the necessary perfectly electrically conducting boundary conditions, which describe the cancellation of the normal component of the magnetic field and the tangential component of the electric field. In addition, bearing in mind that we deal with an exterior problem, whose domain extends practically far away from the neighborhood of the scatterer, the Silver-Müller radiation conditions at infinity are also applied. We restrict our analysis to the important terms of the low-frequency expansions, those being the static Rayleigh term and the first three dynamic terms, since terms of higher orders can be neglected as a fair approximation of the low-frequency hypothesis. The analytical methodology eventually provides solid compact solutions for the surviving electromagnetic fields of interest, i.e., for the first four orders by means of infinite series expansions in terms of spherical harmonic eigenfunctions [20], wherein the leading unknown constant coefficients are calculated either directly via closed-type relationships, or they are embedded into infinite linear algebraic systems and they are determined with the aid of classical cut-off methods. In order to validate the consistency of our results, we propose a reduction methodology of the formulae from the spheroidal case [14] to our circumstance, and we demonstrate this procedure by showing the analytical steps to recover the spherical Rayleigh static magnetic term from the corresponding spheroidal expression. In that sense, all the low-frequency magnetic and electric spheroidal fields can be tackled similarly, so as to obtain the related spherical components.

Having introduced our work briefly in the abstract and more analytically in the current section, the rest of this article is organized as follows. In the next section, the physical background and the main mathematical prerequisites are incorporated via an invariant mode, wherein any formulation is presented independently of the geometry, while the general features of the implied spherical geometry are also included. In the sequel, the corresponding analysis with respect to the hypothesis of operation at low frequencies is presented, and the associated boundary value problems are sketched with respect to the spherical harmonic eigenfunctions. Therein, the section with the main results that follows includes the analytical steps of our solving technique and the developed solutions of each low-frequency problem in terms of compact infinite series expansions. Thereafter, a separate section is devoted to the analytical validation of our method through a reduction sequence, by which the Rayleigh magnetic static component is recovered by the corresponding prolate spheroidal problem. Immediately after, we discuss our results and conclude, while we end this article with an updated reference list.

2. Physical and Mathematical Development

The physical interpretation of this scattering problem is involved with an impenetrable spherical body of approximately infinite conductivity and radius α, which is embedded within an otherwise homogeneous, isotropic and nonmagnetic lossless medium of conductivity $\sigma \cong 0$, dielectric permittivity ε and magnetic permeability μ. On that account, a Cartesian coordinate system (x_1, x_2, x_3) is set conveniently so that its center coincides with the center of the spherical object, and every material point in space is then determined by the position vector $\mathbf{r} = x_1\hat{x}_1 + x_2\hat{x}_2 + x_3\hat{x}_3$, which is expressed via the Cartesian basis $\hat{x}_j$ for $j = 1, 2, 3$. The impenetrable sphere is illuminated by an arbitrarily orientated magnetic dipolar source,

$$\mathbf{m} = \sum_{j=1}^{3} m_j \hat{x}_j \tag{1}$$

which is located at a fixed position $\mathbf{r}_0 = (x_{10}, x_{20}, x_{30})$ and operates at a low circular frequency ω for the current project. Upon this assumption, the wave number [2] of the medium yields

$$k = \omega\sqrt{\varepsilon\mu} \tag{2}$$

which implies low values for k as well, securing the imposed status of low frequencies in this work, provided that the radius of the metal sphere is significantly less than the wavelength $\lambda = 2\pi/k$, which is ensured via the inequality $\alpha \ll \lambda$. In order to comply with the spherical geometry of the solid body, we introduce the spherical coordinate system (r, θ, φ) in view of the radial $r \in [0, +\infty)$, the polar $\theta \in [0, \pi]$ and the azimuthal $\varphi \in [0, 2\pi)$ variables [19], associated with the Cartesian system via

$$x_1 = r\zeta,\ x_2 = r\sqrt{1-\zeta^2}\cos\varphi \text{ and } x_3 = r\sqrt{1-\zeta^2}\sin\varphi \tag{3}$$

where we have set $\zeta \equiv \cos\theta \in [-1, 1]$ for notational clarity in what follows. Obviously, the actual unbounded area $V(\mathbb{R}^3)$ of electromagnetic wave propagation is restricted outside the spherical body, due to its non-penetrable character, which results in no field interaction inside. Thus, the exterior domain of interest is given by

$$\Omega \equiv V\left(\mathbb{R}^3\right) - \{\mathbf{r}_0\} = \{(r, \zeta, \varphi) : r \in (\alpha, +\infty),\ \zeta \in [-1, 1],\ \varphi \in [0, 2\pi)\} - \{(r_0, \zeta_0, \varphi_0)\}, \tag{4}$$

where the singular point $(r_0, \zeta_0, \varphi_0)$, corresponding to (x_{10}, x_{20}, x_{30}) by virtue of (3), is excluded reasonably, as it is considered located far away. The interface that distinguishes the impenetrable body from the scattering region Ω is the smooth spherical surface S for $r = \alpha$ of the perfectly electrically conducting object. On the other hand and without loss of generality, we consider harmonic time-dependence for the implicated electromagnetic fields, multiplying their spatial component by $\exp(-i\omega t)$, where $t \geq 0$ is the time variable and i is the imaginary unit. Therefore, our forthcoming analysis, when the case may be and within each one of the involved relationships, is written solely in terms of the position $\mathbf{r}$ in the three-dimensional regime.

The low-frequency electromagnetic scattering problem is now formulated according to the known [2] primary magnetic field

$$\mathbf{H}^p(\mathbf{r}; \mathbf{r}_0) = \frac{1}{4\pi}\left[\left(k^2 + \frac{ik}{R} - \frac{1}{R^2}\right)\mathbf{m} - \left(k^2 + \frac{3ik}{R} - \frac{3}{R^2}\right)\frac{\mathbf{R}\otimes\mathbf{R}\cdot\mathbf{m}}{R^2}\right]\frac{e^{ikR}}{R} \text{ for } \mathbf{r} \in \Omega \tag{5}$$

and primary electric field

$$\mathbf{E}^p(\mathbf{r}; \mathbf{r}_0) = \left[\frac{\omega\mu k}{4\pi}\left(1 + \frac{i}{kR}\right)\frac{\mathbf{m}\times\mathbf{R}}{R}\right]\frac{e^{ikR}}{R} \text{ for } \mathbf{r} \in \Omega, \tag{6}$$

which are radiated from the foregone dipole source (1), wherein $\mathbf{R} = \mathbf{r} - \mathbf{r}_0$ and $R = |\mathbf{r} - \mathbf{r}_0|$, while the symbol "$\otimes$" denotes a dyadic product. Since no penetration into the sphere is permitted for this physical situation, the incident fields of (5) and (6) are completely perturbed by the metal body, generating the corresponding scattered fields $\mathbf{H}^s$ and $\mathbf{E}^s$. The latter satisfy the reduced spatial Maxwell's equations [2], written for the particular case of the lossless surrounding medium as

$$\nabla \times \mathbf{E}^s(\mathbf{r}) = i\omega\mu\,\mathbf{H}^s(\mathbf{r}) \text{ and } \nabla \times \mathbf{H}^s(\mathbf{r}) = -i\omega\varepsilon\,\mathbf{E}^s(\mathbf{r}) \text{ for } \mathbf{r} \in \Omega, \tag{7}$$

which indicate the solenoidal character of these fields for as much

$$\nabla \cdot \mathbf{H}^s(\mathbf{r}) = \nabla \cdot \mathbf{E}^s(\mathbf{r}) = 0 \text{ for } \mathbf{r} \in \Omega. \tag{8}$$

The gradient operator [20], appearing in (7) and (8), is rendered by

$$\nabla = \sum_{j=1}^{3} \hat{\boldsymbol{x}}_j \frac{\partial}{\partial x_j} = \hat{\boldsymbol{r}} \frac{\partial}{\partial r} - \frac{\sqrt{1-\zeta^2}}{r} \hat{\zeta} \frac{\partial}{\partial \zeta} + \frac{1}{r\sqrt{1-\zeta^2}} \hat{\boldsymbol{\varphi}} \frac{\partial}{\partial \varphi}, \tag{9}$$

given in terms of the orthonormal Cartesian basis and the associated spherical unit normal vectors

$$\hat{\boldsymbol{r}} \equiv \hat{\boldsymbol{r}}(\zeta, \varphi) = \zeta \hat{\boldsymbol{x}}_1 + \sqrt{1-\zeta^2} \cos \varphi \hat{\boldsymbol{x}}_2 + \sqrt{1-\zeta^2} \ \sin \ \varphi \hat{\boldsymbol{x}}_3 \tag{10}$$

which coincides with the normal unit vector on the spherical surface S at $r = \alpha$ and

$$\hat{\zeta} \equiv \hat{\zeta}(\zeta, \varphi) = -\sqrt{1-\zeta^2} \hat{\boldsymbol{x}}_1 + \zeta \cos \varphi \hat{\boldsymbol{x}}_2 + \zeta \sin \varphi \hat{\boldsymbol{x}}_3 \tag{11}$$

$$\hat{\boldsymbol{\varphi}} \equiv \hat{\boldsymbol{\varphi}}(\varphi) = -\sin \varphi \hat{\boldsymbol{x}}_2 + \cos \varphi \hat{\boldsymbol{x}}_3 \tag{12}$$

which constitute the spherical orthonormal basis with position vector $\mathbf{r} = r\,\hat{\boldsymbol{r}}(\zeta, \varphi)$. In order to decouple the magnetic from the electric field, we apply the $\nabla \times$ on both sides of (7), we use repeatedly the operator identity $\nabla \times \nabla \times = \nabla(\nabla \cdot) - \Delta$ and we invoke (8), so as to reach the two independent Helmholtz equations

$$\left(\Delta + k^2\right) \mathbf{H}^s(\mathbf{r}) = 0 \text{ and } \left(\Delta + k^2\right) \mathbf{E}^s(\mathbf{r}) = 0 \text{ for } \mathbf{r} \in \Omega, \tag{13}$$

where k is the wavenumber of the medium, provided in (2). Herein, the Laplacian operator [20] in the relationships in (13) assumes the form

$$\Delta \equiv \nabla \cdot \nabla = \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \zeta} \left[\left(1-\zeta^2\right) \frac{\partial}{\partial \zeta} \right] + \frac{1}{r^2(1-\zeta^2)} \frac{\partial^2}{\partial \varphi^2} \tag{14}$$

in the Cartesian and the spherical coordinate system, respectively. Both differential operators (9) and (14) apply on $\mathbf{r}$ by definition, but they could apply on $\mathbf{r}_0$ as well, where this attribute is designated via the notation $\nabla_{\mathbf{r}_0}$ and $\Delta_{\mathbf{r}_0}$ when appropriate. The summation of the primary and the scattered fields leads to the total fields

$$\mathbf{H}^t(\mathbf{r}) = \mathbf{H}^p(\mathbf{r}; \mathbf{r}_0) + \mathbf{H}^s(\mathbf{r}) \text{ and } \mathbf{E}^t(\mathbf{r}) = \mathbf{E}^p(\mathbf{r}; \mathbf{r}_0) + \mathbf{E}^s(\mathbf{r}) \text{ for } \mathbf{r} \in \Omega, \tag{15}$$

which describe the ensemble of waves that propagate in Ω. To achieve our final goal and obtain $\mathbf{H}^s$ and $\mathbf{E}^s$, the partial differential equations in (13) must be supplemented by the appropriate boundary conditions [2] on the surface S of the spherical body, with those being

$$\hat{\boldsymbol{r}}(\zeta, \varphi) \cdot \mathbf{H}^t(\alpha, \zeta, \varphi) = 0 \text{ and } \hat{\boldsymbol{r}}(\zeta, \varphi) \times \mathbf{E}^t(\alpha, \zeta, \varphi) = 0 \text{ for } \zeta \in [-1, 1] \text{ and } \varphi \in [0, 2\pi), \tag{16}$$

which demand cancellation of the normal component of the total magnetic field and of the tangential component of the total electric field. On the other hand, the proper behavior of the scattered fields far from the object, mathematically speaking at infinity, is ensured via the limiting Silver-Müller radiation conditions [2], yielding

$$\lim_{r \to +\infty} \left[\mathbf{r} \times \nabla \times \begin{pmatrix} \mathbf{H}^s(\mathbf{r}) \\ \mathbf{E}^s(\mathbf{r}) \end{pmatrix} + ikr \begin{pmatrix} \mathbf{H}^s(\mathbf{r}) \\ \mathbf{E}^s(\mathbf{r}) \end{pmatrix} \right] = 0 \text{ for } \mathbf{r} \in \Omega \text{ and } r = |\mathbf{r}|, \tag{17}$$

which are necessary for exterior-type solutions. In conclusion, our aim is to evaluate the electromagnetic scattered fields $\mathbf{H}^s$ and $\mathbf{E}^s$, once the well-posed boundary value problem of (13) with (16) and (17) is evidently solved by virtue of (5), (6) and (15).

3. Electromagnetic Low-Frequency Consideration

Under the aim to facilitate the solution of the aforementioned problem, we take profit from the low frequencies ω, whereupon the magnetic dipole radiates, implying low values for the wavenumber k through (2). In view of this aspect, we handle the produced primary fields (5) and (6) in such a way as to obtain formulae with respect to integral powers of k. Towards this direction, we apply the Maclaurin's series expansion of the exponent e^{ikR}, we reinforce (2) in order to eliminate ω and we perform extended cumbersome manipulations, resulting in the following relations:

$$\mathbf{H}^p(\mathbf{r};\mathbf{r}_0) = \left[\mathbf{H}_0^p(\mathbf{r};\mathbf{r}_0) + \frac{\mathbf{H}_2^p(\mathbf{r};\mathbf{r}_0)}{2}(ik)^2 + \frac{\mathbf{H}_3^p(\mathbf{r};\mathbf{r}_0)}{6}(ik)^3\right] + \left((ik)^4\right) \text{ for } \mathbf{r} \in \Omega \tag{18}$$

and

$$\mathbf{E}^p(\mathbf{r};\mathbf{r}_0) = \left[\mathbf{E}_1^p(\mathbf{r};\mathbf{r}_0)(ik) + \frac{\mathbf{E}_3^p(\mathbf{r};\mathbf{r}_0)}{6}(ik)^3\right] + \left((ik)^4\right) \text{ for } \mathbf{r} \in \Omega, \tag{19}$$

wherein we have collected the related terms as powers of (ik) instead of k for reasons of notational clarity. In the sought relationships, we kept the first four important and, eventually, adequate terms for the magnetic (18) and the electric (19) fields, counting for the orders $n = 0, 1, 2, 3$, while terms of order $n \geq 4$ are of minor significance, since k is very small in the low-frequency regime, addressing [11,13,14] for justification of such an argumentation. The surviving terms within (18) and (19) are proven to have the expressions

$$\mathbf{H}_0^p(\mathbf{r};\mathbf{r}_0) = \frac{\mathbf{m}}{4\pi} \cdot \left(\frac{3\mathbf{R}\otimes\mathbf{R}}{R^2} - \tilde{\mathbf{I}}\right)\frac{1}{R^3} = \frac{\mathbf{m}}{4\pi} \cdot \left(\nabla \otimes \nabla \frac{1}{R}\right) \text{ for } \mathbf{r} \in \Omega, \tag{20}$$

$$\mathbf{H}_2^p(\mathbf{r};\mathbf{r}_0) = -\frac{\mathbf{m}}{4\pi} \cdot \left(\frac{\mathbf{R}\otimes\mathbf{R}}{R^2} + \tilde{\mathbf{I}}\right)\frac{1}{R} = \frac{\mathbf{m}}{4\pi} \cdot \left(\nabla \frac{1}{R} \otimes \mathbf{R} - \frac{\tilde{\mathbf{I}}}{R}\right) \text{ for } \mathbf{r} \in \Omega \tag{21}$$

$$\mathbf{H}_3^p(\mathbf{r};\mathbf{r}_0) = \frac{\mathbf{m}}{4\pi} \cdot \left(-4\tilde{\mathbf{I}}\right) \text{ for } \mathbf{r} \in \Omega, \tag{22}$$

$$\mathbf{E}_1^p(\mathbf{r};\mathbf{r}_0) = \frac{\mathbf{m}}{4\pi}\sqrt{\frac{\mu}{\varepsilon}} \times \frac{\mathbf{R}}{R^3} = -\frac{\mathbf{m}}{4\pi}\sqrt{\frac{\mu}{\varepsilon}} \times \nabla\frac{1}{R} \text{ for } \mathbf{r} \in \Omega, \tag{23}$$

$$\mathbf{E}_3^p(\mathbf{r};\mathbf{r}_0) = -\frac{\mathbf{m}}{4\pi}\sqrt{\frac{\mu}{\varepsilon}} \times \frac{3\mathbf{R}}{R} \text{ for } \mathbf{r} \in \Omega, \tag{24}$$

whereas $\mathbf{H}_1^p = \mathbf{E}_0^p = \mathbf{E}_2^p = 0$, as indicated by the performed analysis. Note that

$$\tilde{\mathbf{I}} = \sum_{j=1}^{3} \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_j = \hat{\mathbf{r}} \otimes \hat{\mathbf{r}} + \hat{\boldsymbol{\zeta}} \otimes \hat{\boldsymbol{\zeta}} + \hat{\boldsymbol{\varphi}} \otimes \hat{\boldsymbol{\varphi}} \tag{25}$$

stands for the unit dyadic, written either in Cartesian or in spherical coordinates, while the easy-to-handle forms on the right-hand side of (20), (21) and (23) emerge from straightforward analytical calculations, based on the fact that

$$\nabla\frac{1}{R} = -\nabla_{\mathbf{r}_0}\frac{1}{R} = -\frac{\mathbf{R}}{R^3}, \text{ where } R = |\mathbf{r} - \mathbf{r}_0| \text{ for } \mathbf{r} \in \Omega, \tag{26}$$

bearing in mind that $\nabla \otimes \mathbf{r} = \tilde{\mathbf{I}}$.

The low-frequency attribute of the primary fields obliges us to introduce a similar behavior for the scattered fields, via the expansions

$$\mathbf{H}^s(\mathbf{r};\mathbf{r}_0) = \sum_{n=0}^{+\infty} \mathbf{H}_n^s(\mathbf{r})\frac{(ik)^n}{n!} \text{ and } \mathbf{E}^s(\mathbf{r};\mathbf{r}_0) = \sum_{n=0}^{+\infty} \mathbf{E}_n^s(\mathbf{r})\frac{(ik)^n}{n!} \text{ for } \mathbf{r} \in \Omega, \tag{27}$$

hence, our next task includes the construction of the equivalent to the previous paragraph problem under the low-frequency consideration. We employ the infinite series (27) into relationship of (7) with (8), and we rearrange the sum index appropriately in order to equate same powers of $(ik)^n$, $n \geq 0$ and easily obtain

$$\nabla \times \mathbf{E}_n^s(\mathbf{r}) = n\sqrt{\frac{\mu}{\varepsilon}}\mathbf{H}_{n-1}^s(\mathbf{r}) \text{ and } \nabla \times \mathbf{H}_n^s(\mathbf{r}) = -n\sqrt{\frac{\varepsilon}{\mu}}\,\mathbf{E}_{n-1}^s(\mathbf{r}),\ n \geq 0 \text{ for } \mathbf{r} \in \Omega, \tag{28}$$

followed by

$$\nabla \cdot \mathbf{H}_n^s(\mathbf{r}) = 0 \text{ and } \nabla \cdot \mathbf{E}_n^s(\mathbf{r}) = 0,\ n \geq 0 \text{ for } \mathbf{r} \in \Omega, \tag{29}$$

which are Maxwell's equations, now rewritten in terms of the low-frequency expansion components. Acting in a similar fashion the Helmholtz equations, (13) transforms to

$$\Delta\mathbf{H}_n^s(\mathbf{r}) = n(n-1)\mathbf{H}_{n-2}^s(\mathbf{r}) \text{ and } \Delta\mathbf{E}_n^s(\mathbf{r}) = n(n-1)\mathbf{E}_{n-2}^s(\mathbf{r}),\ n \geq 0 \text{ for } \mathbf{r} \in \Omega, \tag{30}$$

while the set of partial differential equations (30) are accompanied by the respective set of the low-frequency boundary conditions

$$\hat{\boldsymbol{r}}(\zeta,\varphi) \cdot \mathbf{H}_n^t(\alpha,\zeta,\varphi) = 0 \text{ and } \hat{\boldsymbol{r}}(\zeta,\varphi) \times \mathbf{E}_n^t(\alpha,\zeta,\varphi) = 0 \text{ for } \zeta \in [-1,1] \text{ and } \varphi \in [0,2\pi), \tag{31}$$

where the total fields admit $\mathbf{H}_n^t = \mathbf{H}_n^p + \mathbf{H}_n^s$ and $\mathbf{E}_n^t = \mathbf{E}_n^p + \mathbf{E}_n^s$, $n \geq 0$ and the low-frequency limiting conditions

$$\lim_{r\to+\infty}\left[\mathbf{r} \times \nabla \times \begin{pmatrix} \mathbf{H}_n^s(\mathbf{r}) \\ \mathbf{E}_n^s(\mathbf{r}) \end{pmatrix} + n\,r\begin{pmatrix} \mathbf{H}_{n-1}^s(\mathbf{r}) \\ \mathbf{E}_{n-1}^s(\mathbf{r}) \end{pmatrix}\right] = 0,\ n \geq 0 \text{ for } \mathbf{r} \in \Omega \text{ and } r = |\mathbf{r}|, \tag{32}$$

which are readily recovered from (16) and (17), respectively, if we utilize the introduced expansions (27).

The type of the electromagnetic incident fields in (18) and (19), by virtue of (20)–(24), designate the corresponding behavior for the unknown scattered fields, wherein holding the first four contributors of the infinite expansions (27), $\mathbf{H}^s$ and $\mathbf{E}^s$ may assume to be identical to the incident field low-frequency expansions

$$\mathbf{H}^s(\mathbf{r};\mathbf{r}_0) = \left[\mathbf{H}_0^s(\mathbf{r};\mathbf{r}_0) + \frac{\mathbf{H}_2^s(\mathbf{r};\mathbf{r}_0)}{2}(ik)^2 + \frac{\mathbf{H}_3^s(\mathbf{r};\mathbf{r}_0)}{6}(ik)^3\right] + \Big((ik)^4\Big) \text{ for } \mathbf{r} \in \Omega \tag{33}$$

and

$$\mathbf{E}^s(\mathbf{r};\mathbf{r}_0) = \left[\mathbf{E}_1^s(\mathbf{r};\mathbf{r}_0)(ik) + \frac{\mathbf{E}_3^s(\mathbf{r};\mathbf{r}_0)}{6}(ik)^3\right] + \Big((ik)^4\Big) \text{ for } \mathbf{r} \in \Omega, \tag{34}$$

in which we stipulated that $\mathbf{H}_1^s = \mathbf{E}_0^s = \mathbf{E}_2^s = 0$, as an immediate consequence of the absence of related primary fields. It is evident that relationships (33) and (34) restrict our analysis to the evaluation of the non-vanishing low-frequency scattering components $\mathbf{H}_0^s$, $\mathbf{H}_2^s$, $\mathbf{H}_3^s$, $\mathbf{E}_1^s$, $\mathbf{E}_3^s$ up to the third order, instead of the elaborate solution of the initial boundary value problem. This is a satisfactory consideration, since the terms of both the magnetic and electric scattered fields of order $n \geq 4$ are of minor significance as long as the wavenumber k remains low. Thus, we need to construct a sequence of coupled or uncoupled boundary value problems for each one of the low-frequency scattered fields, elaborating on (28)–(32) selectively, from the so-called static term at $n = 0$ (Rayleigh approximation) to all the rest of the dynamic terms of orders up to $n = 3$. In view of the above reasoning, we begin with the Rayleigh term for $n = 0$, leading to the uncoupled Laplace equation for the scattered magnetic field

$$\Delta\mathbf{H}_0^s(\mathbf{r}) = 0 \ \Rightarrow\ \mathbf{H}_0^s(\mathbf{r}) = \nabla\Psi_0^s(\mathbf{r}),\ \text{where } \Delta\Psi_0^s(\mathbf{r}) = 0 \text{ for } \mathbf{r} \in \Omega, \tag{35}$$

since $\nabla \cdot \mathbf{H}_0^s = 0$ and $\nabla \times \mathbf{H}_0^s = 0$, imposing the boundary condition

$$\hat{\boldsymbol{r}}(\zeta,\varphi) \cdot \left[\mathbf{H}_0^p(\alpha,\zeta,\varphi\,;\mathbf{r}_0) + \mathbf{H}_0^s(\alpha,\zeta,\varphi)\right] = 0 \text{ for } \zeta \in [-1,1] \text{ and } \varphi \in [0,2\pi), \tag{36}$$

while there exists no electric field, i.e., $\mathbf{E}_0^s = 0$, since the corresponding term for the incident electric field is absent. On the other hand, the dynamic term for $n = 1$ reveals the existence of the vector harmonic electric field

$$\mathbf{E}_1^s(\mathbf{r}) = -\frac{1}{2}\sqrt{\frac{\mu}{\varepsilon}}\nabla \times \mathbf{H}_2^s(\mathbf{r}) \text{ for } \mathbf{r} \in \Omega, \tag{37}$$

which can be evaluated once the $\mathbf{H}_2^s$ field is calculated in the next step. For as much as there is no primary magnetic term for $n = 1$, there is no scattered field, that is, $\mathbf{H}_1^s = 0$. As for the $n = 2$ case, it involves a Poisson equation for the magnetic field, coupled to the static term, which reads

$$\Delta\mathbf{H}_2^s(\mathbf{r}) = 2\mathbf{H}_0^s(\mathbf{r}) \;\;\Rightarrow\;\; \mathbf{H}_2^s(\mathbf{r}) = X_2^s(\mathbf{r}) + \mathbf{r}\Psi_0^s(\mathbf{r}) \text{ where } \Delta X_2^s(\mathbf{r}) = 0 \text{ for } \mathbf{r} \in \Omega \tag{38}$$

with $\nabla \cdot \mathbf{H}_2^s = 0$, wherein we solve the non-homogeneous equation by introducing the particular solution $\mathbf{r}\Psi_0^s$, since $\Delta\left(\mathbf{r}\Psi_0^s\right) = \Psi_0^s\Delta\mathbf{r} + \mathbf{r}\Delta\Psi_0^s + 2(\nabla \otimes \mathbf{r})^T \cdot \nabla\Psi_0^s = 2\nabla\Psi_0^s = 2\mathbf{H}_0^s$; for this outcome, we used the fact that $\Delta\mathbf{r} = 0$ and $\nabla \otimes \mathbf{r} = \widetilde{\mathbf{I}}$, as well as (35). The vector-type problem (38) is accompanied by three separate boundary conditions, with the first one given by

$$\hat{\boldsymbol{r}}(\zeta,\varphi) \cdot \left[\mathbf{H}_2^p(\alpha,\zeta,\varphi\,;\mathbf{r}_0) + \mathbf{H}_2^s(\alpha,\zeta,\varphi)\right] = 0 \text{ for } \zeta \in [-1,1] \text{ and } \varphi \in [0,2\pi) \tag{39}$$

and the other two borrowed from the $n = 1$ electric conditions (see (31)), i.e.,

$$\hat{\boldsymbol{r}}(\zeta,\varphi) \times \left[\mathbf{E}_1^p(\alpha,\zeta,\varphi\,;\mathbf{r}_0) + \mathbf{E}_1^s(\alpha,\zeta,\varphi)\right] = 0 \text{ for } \zeta \in [-1,1] \text{ and } \varphi \in [0,2\pi) \tag{40}$$

or, due to (37),

$$\hat{\boldsymbol{r}}(\zeta,\varphi) \times \left[\mathbf{E}_1^p(\alpha,\zeta,\varphi\,;\mathbf{r}_0) - \frac{1}{2}\sqrt{\frac{\mu}{\varepsilon}}\nabla \times \mathbf{H}_2^s(\alpha,\zeta,\varphi)\right] = 0 \text{ for } \zeta \in [-1,1] \text{ and } \varphi \in [0,2\pi), \tag{41}$$

while the incident electric field for $n = 2$ does not exist, inheriting the same vanishing character of the scattered one, implying $\mathbf{E}_2^s = 0$. In the sequel, at $n = 3$, both the magnetic and electric fields survive. As far as the magnetic field is concerned, we simply solve an uncoupled Laplace equation

$$\Delta\mathbf{H}_3^s(\mathbf{r}) = 0 \;\;\Rightarrow\;\; \mathbf{H}_3^s(\mathbf{r}) = \nabla\Psi_3^s(\mathbf{r})\,, \text{ where } \Delta\Psi_3^s(\mathbf{r}) = 0 \text{ for } \mathbf{r} \in \Omega, \tag{42}$$

since $\nabla \cdot \mathbf{H}_3^s = 0$ and $\nabla \times \mathbf{H}_3^s = 0$, invoking the boundary condition

$$\hat{\boldsymbol{r}}(\zeta,\varphi) \cdot \left[\mathbf{H}_3^p(\alpha,\zeta,\varphi\,;\mathbf{r}_0) + \mathbf{H}_3^s(\alpha,\zeta,\varphi)\right] = 0 \text{ for } \zeta \in [-1,1] \text{ and } \varphi \in [0,2\pi). \tag{43}$$

However, this is not the case for the problem that refers to the electric field $\mathbf{E}_3^s$, for which the solution is introduced in the form of an integral representation in terms of the fundamental solution of the Laplace equation [20], which is

$$\Delta\mathbf{E}_3^s(\mathbf{r}) = 6\mathbf{E}_1^s(\mathbf{r}) \;\;\Rightarrow\;\; \mathbf{E}_3^s(\mathbf{r}) = X_3^s(\mathbf{r}) + 6\left[-\frac{1}{4\pi}\iiint\limits_{\Omega}\frac{\mathbf{E}_1^s\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}d\Omega'\right], \text{ where } \Delta X_3^s(\mathbf{r}) = 0 \tag{44}$$

for $\mathbf{r} \in \Omega$ and with $\nabla \cdot \mathbf{E}_3^s = 0$. Hence, when the first-order scattered field $\mathbf{E}_1^s$ is obtained via (37), then $\mathbf{E}_3^s$ accepts solution (44), as long as $\Delta\left[-4\pi\left|\mathbf{r}-\mathbf{r}'\right|\right]^{-1} = \delta\left(\mathbf{r}-\mathbf{r}'\right)$, where δ is the delta function. For, once more, since we are dealing with a pure vector field, we are obliged to infer three independent boundary conditions, with two of them arising from the electric field in (31) for $n = 3$, i.e.,

$$\hat{\mathbf{r}}(\zeta,\varphi) \times \left[\mathbf{E}_3^p(\alpha,\zeta,\varphi;\mathbf{r}_0) + \mathbf{E}_3^s(\alpha,\zeta,\varphi)\right] = 0 \text{ for } \zeta \in [-1,1] \text{ and } \varphi \in [0,2\pi), \tag{45}$$

while, due to the lack of a third boundary condition for this situation, we reinforce the divergence-free property of $\mathbf{E}_3^s$ on the surface of the spherical body, giving us

$$\nabla \cdot \mathbf{E}_3^s(\alpha,\zeta,\varphi) = 0 \text{ for } \zeta \in [-1,1] \text{ and } \varphi \in [0,2\pi) \tag{46}$$

without loss of uniqueness and generality of the method. At this point, let us clarify that the limiting conditions (32) for each one of the above low-frequency modes

$$\lim_{r\to+\infty}\left[\mathbf{r} \times \nabla \times \begin{pmatrix} \mathbf{H}_n^s(\mathbf{r}) \\ \mathbf{E}_n^s(\mathbf{r}) \end{pmatrix} + n\,r\begin{pmatrix} \mathbf{H}_{n-1}^s(\mathbf{r}) \\ \mathbf{E}_{n-1}^s(\mathbf{r}) \end{pmatrix}\right] = 0,\ n = 0,1,2,3 \text{ for } \mathbf{r} \in \Omega \text{ and } r = |\mathbf{r}|, \tag{47}$$

are automatically satisfied if we demand the proper descending character of the defined scalar Ψ_0^s, Ψ_3^s and vector X_2^s, X_3^s harmonic potentials, implying exterior-type solutions in order to determine the scattered fields $\mathbf{H}_0^s$, $\mathbf{H}_2^s$, $\mathbf{H}_3^s$, $\mathbf{E}_1^s$, $\mathbf{E}_3^s$ in the confined scattering region Ω of wave propagation.

Before we proceed to the mathematical treatment of the individual cases for $n = 0,1,2,3$ and under the aim to satisfy (47), we have to provide the adequate exterior expansions of the appeared harmonic potentials, since there are no electromagnetic fields in the spherical object for the reasons described earlier. Thus, we introduce the exterior $u_{\ell,ex}^{m/q}$ (regular as $r \to +\infty$), either even ($q = e$) or odd ($q = o$) spherical harmonic eigenfunctions of degree $\ell \geq 0$ and of order $m = 0,1,2,\ldots,\ell$, in terms of the surface spherical harmonics $Y_\ell^{m/q}$ [20], which are given by the expression

$$u_{\ell,ex}^{m/q}(\mathbf{r}) = \frac{1}{r^{\ell+1}} Y_\ell^{m/q}(\zeta,\varphi) \text{ with } Y_\ell^{m/q}(\zeta,\varphi) = P_\ell^m(\zeta) f_m^q(\varphi) \text{ for } \mathbf{r} \in \Omega, \tag{48}$$

wherein the associated Legendre functions of the first kind P_ℓ^m [20] are employed, being regular for $\zeta = \pm 1$, while the azimuthal angular dependence is incorporated into the trigonometric function

$$f_m^q(\varphi) = \begin{cases} \cos m\varphi, & q = e \\ \sin m\varphi, & q = o \end{cases} \quad\Rightarrow\quad f_m^{q\,\prime}(\varphi) = \begin{cases} -m\sin m\varphi, & q = e \\ m\cos m\varphi, & q = o \end{cases} \text{ for any } \varphi \in [0,2\pi), \tag{49}$$

the prime denoting derivation with respect to the argument. Otherwise, the surface spherical harmonics are orthogonal with respect to the integral

$$\int_0^{2\pi}\int_{-1}^{+1} Y_\ell^{m/q}(\zeta,\varphi) Y_{\ell'}^{m'/q'}(\zeta,\varphi)\,d\zeta\,d\varphi = \frac{1}{\varepsilon_m}\frac{4\pi}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell\ell'}\delta_{mm'}\delta_{qq'} \tag{50}$$

on the surface $r = \alpha$ and for $\ell,\ell' \geq 0$, $(m,m') = 0,1,2,\ldots,(\ell,\ell')$ and $(q,q') = (e,o)$, where $\delta_{\ell\ell'}$, $\delta_{mm'}$, $\delta_{qq'}$ are the deltas of Kronecker and $\varepsilon_0 = 1$, while $\varepsilon_m = 2$ when $m \geq 1$. Therefore, on account that the functions Ψ_0^s, X_2^s, Ψ_3^s, X_3^s belong to the kernel space of the Laplace operator and admit exterior expansions, they are written as

$$\Psi_0^s(\mathbf{r}) = \sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o} a_{\ell,ex}^{m/q} u_{\ell,ex}^{m/q}(\mathbf{r}) \text{ for } \mathbf{r} \in \Omega, \tag{51}$$

$$X_2^s(\mathbf{r}) = \sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o} \mathbf{b}_{\ell,ex}^{m/q} u_{\ell,ex}^{m/q}(\mathbf{r}) \text{ for } \mathbf{r} \in \Omega, \tag{52}$$

$$\Psi_3^s(\mathbf{r}) = \sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o} c_{\ell,ex}^{m/q} u_{\ell,ex}^{m/q}(\mathbf{r}) \text{ for } \mathbf{r} \in \Omega, \tag{53}$$

$$X_3^s(\mathbf{r}) = \sum_{\ell=0}^{+\infty} \sum_{m=0}^{\ell} \sum_{q=e,o} \mathbf{d}_{\ell,ex}^{m/q} u_{\ell,ex}^{m/q}(\mathbf{r}) \text{ for } \mathbf{r} \in \Omega, \tag{54}$$

respectively, in which $a_{\ell,ex}^{m/q}$, $c_{\ell,ex}^{m/q}$ are the scalar and $\mathbf{b}_{\ell,ex}^{m/q}$, $\mathbf{d}_{\ell,ex}^{m/q}$ are the vector unknown constant coefficients that must be evaluated by solving the set of boundary value problems (35)–(47). To this end, herein we invoke the crucial aspect for the forthcoming analysis, Green's function expansion for spherical geometry [20], in which, for our purpose, the observation region is constrained at $\mathbf{r} < \mathbf{r}_0$ (or equivalently $r < r_0$), yielding the formula

$$\frac{1}{R} \equiv \frac{1}{|\mathbf{r} - \mathbf{r}_0|} = \sum_{\ell=0}^{+\infty} \sum_{m=0}^{\ell} \sum_{q=e,o} \rho_{\ell,ex}^{m/q}(\mathbf{r}_0) r^{\ell} Y_{\ell}^{m/q}(\zeta, \varphi) \text{ for } \mathbf{r} \in \Omega, \tag{55}$$

where at the singular point $\mathbf{r}_0 = (r_0, \zeta_0, \varphi_0)$ we define

$$\rho_{\ell,ex}^{m/q}(\mathbf{r}_0) = \frac{(\ell - m)!}{(\ell + m)!} \varepsilon_m u_{\ell,ex}^{m/q}(r_0) \text{ with } \ell \geq 0,\ m = 0, 1, 2, \ldots, \ell \text{ and } q = e, o, \tag{56}$$

ending our prerequisite manipulation of the particular wave scattering problem. In the sequel, the next section is devoted to the main and most important steps of the long and tedious calculations to obtain compact solutions for the electromagnetic scattered fields. Therein, we start from the magnetic modes $\mathbf{H}_0^s$, $\mathbf{H}_2^s$, $\mathbf{H}_3^s$ and proceed to the electric modes $\mathbf{E}_1^s$, $\mathbf{E}_3^s$, keeping in mind the vanishing fields $\mathbf{H}_1^s = \mathbf{E}_0^s = \mathbf{E}_2^s = 0$ that do not contribute to the reflected field.

4. Non-Trivial Magnetic and Electric Scattered Components

We initiate the procedure of obtaining the scattered low-frequency modes with the analytical calculation of the surviving Rayleigh magnetic term that corresponds to $n = 0$ wherein combining the result (35) and the associated potential (51) with (48), we immediately reach at

$$\mathbf{H}_0^s(\mathbf{r}) = \sum_{\ell=0}^{+\infty} \sum_{m=0}^{\ell} \sum_{q=e,o} a_{\ell,ex}^{m/q} \nabla \left[r^{-(\ell+1)} Y_{\ell}^{m/q}(\zeta, \varphi) \right] \text{ for } \mathbf{r} \in \Omega, \tag{57}$$

where $a_{\ell,ex}^{m/q}$ for $\ell \geq 0$, $m = 0, 1, 2, \ldots, \ell$ and $q = e, o$ denote the constant coefficients to be evaluated from boundary condition (36). To this direction and by virtue of (57) and (9), we first calculate for $r = \alpha$ the simple inner product

$$\begin{aligned} \hat{r}(\zeta, \varphi) \cdot \mathbf{H}_0^s(\alpha, \zeta, \varphi) &= - \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \sum_{q=e,o} a_{\ell,ex}^{m/q} \left(\frac{\partial}{\partial r} r^{-(\ell+1)} \right)_{r=\alpha} Y_{\ell}^{m/q}(\zeta, \varphi) \\ &= - \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \sum_{q=e,o} a_{\ell,ex}^{m/q} (\ell + 1) \alpha^{-(\ell+2)} Y_{\ell}^{m/q}(\zeta, \varphi) \end{aligned} \tag{58}$$

for $\zeta \in [-1, 1]$ and $\varphi \in [0, 2\pi)$. On the other hand, the corresponding inner product with the primary field (20) is handled conveniently by using the fact that the dyadic $\nabla \otimes \nabla(1/R)$ is symmetric and by implying the useful formula (26) so as to obtain

$$\begin{aligned} \hat{\mathbf{r}}(\zeta, \varphi) \cdot \mathbf{H}_0^p(\alpha, \zeta, \varphi; \mathbf{r}_0) &= \left\{ \frac{\partial}{\partial r} \left[\nabla \frac{1}{|\mathbf{r} - \mathbf{r}_0|} \right] \cdot \frac{\mathbf{m}}{4\pi} \right\}_{r=\alpha} \\ &= - \left\{ \frac{\partial}{\partial r} \left[\nabla_{\mathbf{r}_0} \frac{1}{|\mathbf{r} - \mathbf{r}_0|} \right] \cdot \frac{\mathbf{m}}{4\pi} \right\}_{r=\alpha} \\ &= - \frac{\mathbf{m}}{4\pi} \cdot \sum_{\ell=0}^{+\infty} \sum_{m=0}^{\ell} \sum_{q=e,o} \nabla_{\mathbf{r}_0} \rho_{\ell,ex}^{m/q}(\mathbf{r}_0) \left(\frac{\partial}{\partial r} r^{\ell} \right)_{r=\alpha} Y_{\ell}^{m/q}(\zeta, \varphi) \\ &= - \sum_{\ell=0}^{+\infty} \sum_{m=0}^{\ell} \sum_{q=e,o} \left(\frac{\mathbf{m}}{4\pi} \cdot \nabla_{\mathbf{r}_0} \rho_{\ell,ex}^{m/q}(r_0) \right) \ell \alpha^{\ell-1} Y_{\ell}^{m/q}(\zeta, \varphi) \end{aligned} \tag{59}$$

for $\zeta \in [-1,1]$ and $\varphi \in [0,2\pi)$, wherein eigen expansion (55), with the accompanied definition (56), was employed. Substituting (58) and (59) into the vanishing normal component of the zero-order magnetic field (36), we bring out the common factor $Y_\ell^{m/q}$, and we have

$$\sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o}\left[\left(\frac{\mathbf{m}}{4\pi}\cdot\nabla_{\mathbf{r}_0}\rho_{\ell,ex}^{m/q}(r_0)\right)\ell\alpha^{\ell-1}+a_{\ell,ex}^{m/q}(\ell+1)\alpha^{-(\ell+2)}\right]Y_\ell^{m/q}(\zeta,\varphi)=0 \quad (60)$$

for every $\zeta \in [-1,1]$ and $\varphi \in [0,2\pi)$. Using trivial orthogonality arguments with respect to (50), we reach

$$a_{\ell,ex}^{m/q}=-\frac{\ell\alpha^{2\ell+1}}{\ell+1}\left(\frac{\mathbf{m}}{4\pi}\cdot\nabla_{\mathbf{r}_0}\rho_{\ell,ex}^{m/q}(\mathbf{r}_0)\right) \text{ for } \ell\geq 0,\ m=0,1,2,\ldots,\ell \text{ and } q=e,o, \quad (61)$$

which is inserted into (57) and provides us with the solution, that is

$$\mathbf{H}_0^s(\mathbf{r})=-\sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o}\frac{\ell\alpha^{2\ell+1}}{\ell+1}\left(\frac{\mathbf{m}}{4\pi}\cdot\nabla_{\mathbf{r}_0}\rho_{\ell,ex}^{m/q}(\mathbf{r}_0)\right)\nabla\left[r^{-(\ell+1)}Y_\ell^{m/q}(\zeta,\varphi)\right] \text{ for } \mathbf{r}\in\Omega, \quad (62)$$

wherein the gradient operator is given by (9) with (10)–(12), and the magnetic moment $\mathbf{m}$ satisfies (1), while, obviously, the quantity $\nabla_{\mathbf{r}_0}\rho_{\ell,ex}^{m/q}(r_0)$ is known by virtue of (56).

In the sequel, we elaborate on the non-vanishing magnetic field for $n=2$, which exhibits a particular difficulty, mainly due to coupling with the static term (57), as it is indicated by the general solution (38), which admits

$$\mathbf{H}_2^s(\mathbf{r})=X_2^s(\mathbf{r})+\mathbf{r}\Psi_0^s(\mathbf{r}) \text{ for } \mathbf{r}\in\Omega, \quad (63)$$

in which, from the previous case for $n=0$, we recover

$$\Psi_0^s(\mathbf{r})=-\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\sum_{q=e,o}\frac{\ell\alpha^{2\ell+1}}{\ell+1}\left(\frac{\mathbf{m}}{4\pi}\cdot\nabla_{\mathbf{r}_0}\rho_{\ell,ex}^{m/q}(\mathbf{r}_0)\right)r^{-(\ell+1)}Y_\ell^{m/q}(\zeta,\varphi) \text{ for } \mathbf{r}\in\Omega, \quad (64)$$

as the zero-order potential. In addition, the embedded harmonic function (52) with the aid of (48) is rewritten as

$$X_2^s(\mathbf{r})=\sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o}\mathbf{b}_{\ell,ex}^{m/q}\left[r^{-(\ell+1)}Y_\ell^{m/q}(\zeta,\varphi)\right] \text{ for } \mathbf{r}\in\Omega \quad (65)$$

when the vector unknown constant coefficients

$$\begin{aligned}\mathbf{b}_{\ell,ex}^{m/q}=\sum_{j=1}^{3}b_{\ell,j}^{m/q}\hat{x}_j &= b_{\ell,1}^{m/q}\hat{x}_1+b_{\ell,2}^{m/q}\hat{x}_2+b_{\ell,3}^{m/q}\hat{x}_3\\ &=\left(b_{\ell,1}^{m/q}\zeta+b_{\ell,2}^{m/q}\sqrt{1-\zeta^2}\cos\varphi+b_{\ell,3}^{m/q}\sqrt{1-\zeta^2}sin\varphi\right)\hat{r}\\ &+\left(-b_{\ell,1}^{m/q}\sqrt{1-\zeta^2}+b_{\ell,2}^{m/q}\zeta\cos\varphi+b_{\ell,3}^{m/q}\zeta sin\varphi\right)\hat{\zeta}\\ &+\left(-b_{\ell,2}^{m/q}\sin\varphi+b_{\ell,3}^{m/q}\cos\varphi\right)\hat{\varphi}\end{aligned} \quad (66)$$

for $\ell\geq 0$, $m=0,1,2,\ldots,\ell$ and $q=e,o$, as well as the position vector

$$\mathbf{r}=\sum_{j=1}^{3}x_j\hat{x}_j=r\hat{r} \text{ for } \mathbf{r}\in\Omega \quad (67)$$

imply Cartesian and spherical representations. Therefore, we substitute (64)–(67) into (63), we use one condition from the r-component of (39) and two conditions from the ζ,φ-components of (41) by assigning the second-order magnetic (21) and first-order electric (23) incident fields, and we apply the expansion (55) with (56) when necessary. The cumbersome calculation effort is based on straightforward manipulations of classical vector analysis in terms of the gradient operator (9) with (10)–(12). However, we avoid presenting the complete analysis; instead, we choose to show the outcome of this action, which leads us to three independent relationships for the unknown constant coefficients (66), i.e.,

$$\sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o}\left[\sum_{j=1}^{3} f_{\ell,j}^{m/q,\kappa}(\zeta,\varphi)b_{\ell,j}^{m/q} - g_{\ell}^{m/q,\kappa}(\zeta,\varphi\,;\mathbf{r}_0)\right] = 0 \text{ with } \kappa = 1,2,3 \tag{68}$$

and for any $\zeta \in [-1,1]$ and $\varphi \in [0,2\pi)$. As for the implicated functions in (68) and for $\kappa = 1$, we have

$$f_{\ell,1}^{m/q,1}(\zeta,\varphi) = \alpha^{-(\ell+1)}\zeta Y_{\ell}^{m/q}(\zeta,\varphi), \tag{69}$$

$$f_{\ell,2}^{m/q,1}(\zeta,\varphi) = \alpha^{-(\ell+1)}\sqrt{1-\zeta^2}\cos\varphi Y_{\ell}^{m/q}(\zeta,\varphi), \tag{70}$$

$$f_{\ell,3}^{m/q,1}(\zeta,\varphi) = \alpha^{-(\ell+1)}\sqrt{1-\zeta^2}\sin\varphi Y_{\ell}^{m/q}(\zeta,\varphi) \tag{71}$$

and

$$g_{\ell}^{m/q,1}(\zeta,\varphi\,;\mathbf{r}_0) = \left\{\rho_{\ell,ex}^{m/q}(\mathbf{r}_0)\frac{\mathbf{m}}{4\pi}\cdot\left[-\frac{\ell}{\alpha}(\mathbf{r}(\alpha,\zeta,\varphi)-\mathbf{r}_0)+\hat{\mathbf{r}}\right] + \frac{\ell\alpha}{\ell+1}M_{\ell,ex}^{m/q}(\mathbf{r}_0)\right\}\alpha^{\ell}Y_{\ell}^{m/q}(\zeta,\varphi), \tag{72}$$

for $\kappa = 2$, it holds that

$$f_{\ell,1}^{m/q,2}(\zeta,\varphi) = \alpha^{-(\ell+2)}\left[-(\ell+1)\sqrt{1-\zeta^2}P_{\ell}^{m}(\zeta) - \zeta\sqrt{1-\zeta^2}P_{\ell}^{m\prime}(\zeta)\right]f_m^q(\varphi), \tag{73}$$

$$f_{\ell,2}^{m/q,2}(\zeta,\varphi) = \alpha^{-(\ell+2)}\left[(\ell+1)\zeta P_{\ell}^{m}(\zeta) - \left(1-\zeta^2\right)P_{\ell}^{m\prime}(\zeta)\right]\cos\varphi f_m^q(\varphi), \tag{74}$$

$$f_{\ell,3}^{m/q,2}(\zeta,\varphi) = \alpha^{-(\ell+2)}\left[(\ell+1)\zeta P_{\ell}^{m}(\zeta) - \left(1-\zeta^2\right)P_{\ell}^{m\prime}(\zeta)\right]\sin\varphi f_m^q(\varphi) \tag{75}$$

and

$$g_{\ell}^{m/q,2}(\zeta,\varphi\,;\mathbf{r}_0) = -\frac{\ell\alpha^{\ell}}{\ell+1}M_{\ell,ex}^{m/q}(\mathbf{r}_0)\sqrt{1-\zeta^2}P_{\ell}^{m\prime}(\zeta)f_m^q(\varphi) + 2\left(\hat{\mathbf{r}}\times\mathbf{M}_{\ell,ex}^{m/q}(\mathbf{r}_0)\cdot\hat{\zeta}\right)\alpha^{\ell}Y_{\ell}^{m/q}(\zeta,\varphi), \tag{76}$$

while for $\kappa = 3$, it is

$$f_{\ell,1}^{m/q,3}(\zeta,\varphi) = \alpha^{-(\ell+2)}\frac{\zeta}{\sqrt{1-\zeta^2}}f_m^{q\prime}(\varphi)P_{\ell}^{m}(\zeta), \tag{77}$$

$$f_{\ell,2}^{m/q,3}(\zeta,\varphi) = \alpha^{-(\ell+2)}\left[-(\ell+1)sin\varphi f_m^q(\varphi) + \cos\varphi f_m^{q\prime}(\varphi)\right]P_{\ell}^{m}(\zeta), \tag{78}$$

$$f_{\ell,3}^{m/q,3}(\zeta,\varphi) = \alpha^{-(\ell+2)}\left[(\ell+1)\cos\varphi f_m^q(\varphi) + sin\varphi f_m^{q\prime}(\varphi)\right]P_{\ell}^{m}(\zeta) \tag{79}$$

and

$$g_{\ell}^{m/q,3}(\zeta,\varphi\,;\mathbf{r}_0) = M_{\ell,ex}^{m/q}(\mathbf{r}_0)\frac{\ell\alpha^{\ell}}{\ell+1}\frac{P_{\ell}^{m}(\zeta)}{\sqrt{1-\zeta^2}}f_m^{q\prime}(\varphi) + 2\left(\hat{\mathbf{r}}\times\mathbf{M}_{\ell,ex}^{m/q}(\mathbf{r}_0)\cdot\hat{\varphi}\right)\alpha^{\ell}Y_{\ell}^{m/q}(\zeta,\varphi). \tag{80}$$

The definitions in (69)–(80) are given for $\zeta \in [-1,1]$ and $\varphi \in [0,2\pi)$, in which we defined the convenient functions of the dipole's position as

$$M_{\ell,ex}^{m/q}(\mathbf{r}_0) = \frac{\mathbf{m}}{4\pi}\cdot\nabla_{\mathbf{r}_0}\rho_{\ell,ex}^{m/q}(\mathbf{r}_0) \text{ and } \mathbf{M}_{\ell,ex}^{m/q}(\mathbf{r}_0) = \frac{\mathbf{m}}{4\pi}\times\nabla_{\mathbf{r}_0}\rho_{\ell,ex}^{m/q}(\mathbf{r}_0) \tag{81}$$

for $\ell \geq 0$, $m = 0,1,2,\ldots,\ell$ and $q = e,o$, while all derivatives denoted by the prime are with respect to the argument. Notwithstanding relationships, (68), accompanied by definitions in (69)–(80), contains easy-to-handle functions, which are very complicated in terms of accepting elaboration with respect to recurrence relations, since, in the end, a numerical implementation for the resulting systems is inevitable. Nevertheless, in order to obtain a compact solution from these systems of equations, we work as follows. We write the functions (69)–(80) as series expansions via the orthonormal basis $Y_{\ell'}^{m'/q'}$ for $\ell' \geq 0$, $m' = 0,1,2,\ldots,\ell'$ and $q' = e,o$, according to

$$f_{\ell,j}^{m/q,\kappa}(\zeta,\varphi) = \sum_{\ell'=0}^{+\infty}\sum_{m'=0}^{\ell'}\sum_{q'=e,o}\lambda_{(\ell,\ell'),j}^{(m,m')/(q,q'),\kappa}Y_{\ell'}^{m'/q'}(\zeta,\varphi) \text{ with } \kappa, j = 1,2,3 \tag{82}$$

and

$$g_{\ell}^{m/q,\kappa}(\zeta,\varphi;\mathbf{r}_0) = \sum_{\ell'=0}^{+\infty}\sum_{m'=0}^{\ell'}\sum_{q'=e,o}\mu_{(\ell,\ell')}^{(m,m')/(q,q'),\kappa}(\mathbf{r}_0)Y_{\ell'}^{m'/q'}(\zeta,\varphi) \text{ with } \kappa = 1,2,3, \tag{83}$$

wherein $\ell \geq 0$, $m = 0,1,2,\ldots,\ell$ and $q = e,o$, while $\zeta \in [-1,1]$ and $\varphi \in [0,2\pi)$. Next, taking advantage of the orthogonality property (50), we obtain the leading coefficients within (82) and (83) in the integral form

$$\lambda_{(\ell,\ell'),j}^{(m,m')/(q,q'),\kappa} = \frac{2\ell'+1}{4\pi}\frac{(\ell'-m')!}{(\ell'+m')!}\varepsilon_{m'}\int_{-1}^{+1}\int_{0}^{2\pi} f_{\ell,j}^{m/q,\kappa}(\zeta,\varphi)Y_{\ell'}^{m'/q'}(\zeta,\varphi)d\varphi\, d\zeta \tag{84}$$

and

$$\mu_{(\ell,\ell')}^{(m,m')/(q,q'),\kappa}(\mathbf{r}_0) = \frac{2\ell'+1}{4\pi}\frac{(\ell'-m')!}{(\ell'+m')!}\varepsilon_{m'}\int_{-1}^{+1}\int_{0}^{2\pi} g_{\ell}^{m/q,\kappa}(\zeta,\varphi;\mathbf{r}_0)Y_{\ell'}^{m'/q'}(\zeta,\varphi)d\varphi\, d\zeta \tag{85}$$

for every $(\ell,\ell') \geq 0$, $(m,m') = 0,1,2,\ldots,(\ell,\ell')$ and $(q,q') = e,o$ with $\kappa, j = 1,2,3$. In the final step of this procedure, we invoke formulae (82) (with (84)) and (83) (with (85)) into (68), and once more we make use of the orthogonality relation (50) in order to reach the set of algebraic equations

$$\sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o}\left[\sum_{j=1}^{3}\lambda_{(\ell,\ell'),j}^{(m,m')/(q,q'),\kappa}b_{\ell,j}^{m/q} - \mu_{(\ell,\ell')}^{(m,m')/(q,q'),\kappa}(\mathbf{r}_0)\right] = 0 \text{ with } \kappa = 1,2,3, \tag{86}$$

wherein $\ell' \geq 0$, $m' = 0,1,2,\ldots,\ell'$ and $q' = e,o$, which includes the three components ($j = 1,2,3$) of the unknown constants $b_{\ell,j}^{m/q}$ for $\ell \geq 0$, $m = 0,1,2,\ldots,\ell$, $q = e,o$ and the known expressions (84) and (85). Doing so, we manage to transfer the apparent difficulty from the boundary conditions to the simple calculation of integrals (84) and (85), which just incorporate the trivial trigonometric and associated Legendre functions. The set of relationships (86) represent systems of linear algebraic equations, which can be solved via usual cut-off techniques by the imposition of an indispensable common upper limit L for both the degree indexes, that is $\ell = \ell' = 0,1,2,\ldots,L$. Thereafter, (86) can be transformed to quadrature systems of the form

$$\mathbb{A}_L x_L = z_L(\mathbf{r}_0) \;\Rightarrow\; x_L = \mathbb{A}_L^{-1} z_L(\mathbf{r}_0), \tag{87}$$

where for $m = m' = 0,1,2,\ldots,\ell$ and $q = q' = e,o$, as well as $\kappa, j = 1,2,3$, we have

$$\mathbb{A}_L = \begin{bmatrix} \ddots & \vdots & \\ \cdots & \lambda_{(\ell,\ell),j}^{(m,m)/(q,q),\kappa} & \cdots \\ & \vdots & \ddots \end{bmatrix},\; x_L = \begin{bmatrix} \vdots \\ b_{\ell,j}^{m/q} \\ \vdots \end{bmatrix} \text{ and } z_L(\mathbf{r}_0) = \begin{bmatrix} \vdots \\ \mu_{(\ell,\ell),j}^{(m,m)/(q,q),\kappa}(\mathbf{r}_0) \\ \vdots \end{bmatrix}, \tag{88}$$

which denote the $3(L+1) \times m \times q$ squared-type invertible matrix of the coefficients of the unknowns, the vector of the unknown coefficients and the vector of the known constants, respectively. The solution $\mathbf{x}_L$, provided in (87), corresponds to the constant coefficients $\mathbf{b}_{\ell,ex}^{m/q} = b_{\ell,1}^{m/q}\hat{\mathbf{x}}_1 + b_{\ell,2}^{m/q}\hat{\mathbf{x}}_2 + b_{\ell,3}^{m/q}\hat{\mathbf{x}}_3$ for $\ell = 0,1,2,\ldots,L$, $m = 0,1,2,\ldots,\ell$ and $q = e,o$, where the value of L is determined in accordance to the desired accuracy of the final series expansions. Thus, by virtue of (64) and (65), the magnetic term of second order (63) implies

$$\mathbf{H}_2^s(\mathbf{r}) = \sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o}\left\{\left[b_{\ell,ex}^{m/q} - \frac{\ell\alpha^{2\ell+1}}{\ell+1}\left(\frac{\mathbf{m}}{4\pi}\cdot\nabla_{\mathbf{r}_0}\rho_{\ell,ex}^{m/q}(r_0)\right)\mathbf{r}\right]r^{-(\ell+1)}Y_\ell^{m/q}(\zeta,\varphi)\right\} \text{ for } \mathbf{r}\in\Omega, \tag{89}$$

upon solution of (87) with (88) and expression of (56), which embody both the general harmonic part and the particular solution of the Poisson equation (38).

Ending the calculation sequence of the surviving magnetic scattered modes, we move to a quite simple case for $n = 3$, whose simplicity is due to the fact that the corresponding primary field (22) is a constant vector. Herein, we work similarly to the Rayleigh case, since we deal with a harmonic field, given via the gradient of a scalar harmonic potential, as (42) indicates with the Neumann-type boundary condition (43). Hence, using the expression (53) with the eigenfunctions (48), the scattered field under evaluation for this case reads

$$\mathbf{H}_3^s(\mathbf{r}) = \sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o} c_{\ell,ex}^{m/q}\nabla\left[r^{-(\ell+1)}Y_\ell^{m/q}(\zeta,\varphi)\right] \text{ for } \mathbf{r}\in\Omega, \tag{90}$$

where $c_{\ell,ex}^{m/q}$ for $\ell \geq 0$, $m = 0,1,2,\ldots,\ell$ and $q = e,o$ stand for the constant coefficients to be determined from boundary condition (43). The latter combines (90) and (22) in order to reveal that

$$-\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\sum_{q=e,o} c_{\ell,ex}^{m/q}(\ell+1)\alpha^{-(\ell+2)}Y_\ell^{m/q}(\zeta,\varphi) - \frac{m_1}{\pi}\zeta - \frac{m_2}{\pi}\sqrt{1-\zeta^2}\cos\varphi - \frac{m_3}{\pi}\sqrt{1-\zeta^2}\sin\varphi = 0 \tag{91}$$

for every $\zeta \in [-1,1]$ and $\varphi \in [0,2\pi)$. Though, if we identify $\zeta = Y_1^{0/e}$, $\sqrt{1-\zeta^2}\cos\varphi = Y_1^{1/e}$ and $\sqrt{1-\zeta^2}\sin\varphi = Y_1^{1/o}$, then by application of the orthogonality (50), we are led to the vanishing of the constant coefficients

$$c_{0,ex}^{0/e} = 0 \text{ and } c_{\ell,ex}^{m/q} = 0 \text{ for every } \ell \geq 2,\ m = 0,1,2,\ldots,\ell \text{ and } q = e,o, \tag{92}$$

while the three surviving constants for $\ell = 1$ are derived as

$$c_{1,ex}^{0/e} = -\frac{m_1\alpha^3}{2\pi},\ c_{1,ex}^{1/e} = -\frac{m_2\alpha^3}{2\pi} \text{ and } c_{1,ex}^{1/o} = -\frac{m_3\alpha^3}{2\pi}, \tag{93}$$

which determine the three terms of the third-order scattered field. Invoking (92) and (93) into the initial field (90), reading the spherical coordinates (3), taking into account (1) and applying the gradient operator either in Cartesian or in spherical form, we get

$$\begin{aligned}
\mathbf{H}_3^s(\mathbf{r}) &= c_{1,ex}^{0/e}\nabla\left[r^{-2}Y_1^{0/e}(\zeta,\varphi)\right] + c_{1,ex}^{1/e}\nabla\left[r^{-2}Y_1^{1/e}(\zeta,\varphi)\right] + c_{1,ex}^{1/o}\nabla\left[r^{-2}Y_1^{1/o}(\zeta,\varphi)\right] \\
&= -\tfrac{m_1\alpha^3}{2\pi}\nabla\left[r^{-2}\zeta\right] - \tfrac{m_2\alpha^3}{2\pi}\nabla\left[r^{-2}\sqrt{1-\zeta^2}\cos\varphi\right] - \tfrac{m_3\alpha^3}{2\pi}\nabla\left[r^{-2}\sqrt{1-\zeta^2}\sin\varphi\right] \\
&= -\tfrac{m_1\alpha^3}{2\pi}\nabla\left[r^{-3}x_1\right] - \tfrac{m_2\alpha^3}{2\pi}\nabla\left[r^{-3}x_2\right] - \tfrac{m_3\alpha^3}{2\pi}\nabla\left[r^{-3}x_3\right] \\
&= -\tfrac{\alpha^3}{2\pi}\sum_{j=1}^{3} m_j\nabla\left[r^{-3}x_j\right] = \tfrac{\alpha^3}{2\pi}\sum_{j=1}^{3} m_j\left(3x_jr^{-4}\hat{\boldsymbol{r}} - r^{-3}\hat{\boldsymbol{x}}_j\right) \\
&= \tfrac{1}{2\pi}\left(\tfrac{\alpha}{r}\right)^3\sum_{j=1}^{3} m_j\left(3x_jr^{-1}\hat{\boldsymbol{r}} - \hat{\boldsymbol{x}}_j\right) = \tfrac{1}{2\pi}\left(\tfrac{\alpha}{r}\right)^3\left[3r^{-1}\hat{\boldsymbol{r}}\sum_{j=1}^{3} m_jx_j - \sum_{j=1}^{3} m_j\hat{\boldsymbol{x}}_j\right] \\
&= \tfrac{1}{2\pi}\left(\tfrac{\alpha}{r}\right)^3\left[3r^{-1}(\mathbf{m}\cdot\mathbf{r})\hat{\boldsymbol{r}} - \mathbf{m}\right] = \tfrac{1}{2\pi}\left(\tfrac{\alpha}{r}\right)^3\left[3(\mathbf{m}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \mathbf{m}\right]
\end{aligned} \tag{94}$$

or

$$\mathbf{H}_3^s(\mathbf{r}) = \left(\frac{\alpha}{r}\right)^3\frac{m}{2\pi}\cdot\left(3\hat{\boldsymbol{r}}\otimes\hat{\boldsymbol{r}} - \widetilde{I}\right) \text{ for } \mathbf{r}\in\Omega, \tag{95}$$

which provides the relative scattered field in a closed-form dyadic expression, bearing in mind that $\mathbf{r} = r\hat{\boldsymbol{r}}$ in spherical coordinates.

Proceeding to the low-frequency scattered electric modes that do not vanish, we start with the easy case for $n = 1$, being immediately related with the calculated $\mathbf{H}_2^s$ field, as indicated by (37). The latter, by virtue of (89) and implying a trivial vector differential identity, renders

$$\mathbf{E}_1^s(\mathbf{r}) = -\frac{1}{2}\sqrt{\frac{\mu}{\varepsilon}}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\sum_{q=e,o}\left\{\nabla\left[r^{-(\ell+1)}Y_\ell^{m/q}(\zeta,\varphi)\right]\times\left[\mathbf{b}_{\ell,ex}^{m/q} - \frac{\ell\alpha^{2\ell+1}}{\ell+1}\left(\frac{\mathbf{m}}{4\pi}\cdot\nabla_{\mathbf{r}_0}\rho_{\ell,ex}^{m/q}(\mathbf{r}_0)\right)\mathbf{r}\right]\right\} \tag{96}$$

for $\mathbf{r} \in \Omega$, since $\nabla \times \mathbf{r} = 0$ and $\nabla \times \mathbf{b}_{\ell,ex}^{m/q} = 0$ for $\ell \geq 0$, $m = 0,1,2,\ldots,\ell$ and $q = e,o$, wherein the constant coefficients of the second-order scattered magnetic field were computed in previous step.

The task that completes our analysis is associated with the evaluation of the electric scattered field for $n = 3$, whereas the main difficulty comes from the manipulation of the inserted integral in the solution (44), which incorporates the $\mathbf{E}_1^s$ field from (96), yielding

$$\mathbf{E}_3^s(\mathbf{r}) = X_3^s(\mathbf{r}) - \frac{3}{2\pi}\iiint_\Omega \frac{\mathbf{E}_1^s\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}d\Omega',\ \sin\text{ce}\ \Delta\left[-\frac{3}{2\pi}\iiint_\Omega \frac{\mathbf{E}_1^s\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}d\Omega'\right] = 6\mathbf{E}_1^s(\mathbf{r}) \tag{97}$$

for $\mathbf{r} \in \Omega$, where (54) with the aim of (48) results in

$$X_3^s(\mathbf{r}) = \sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o}\mathbf{d}_{\ell,ex}^{m/q}r^{-(\ell+1)}Y_\ell^{m/q}(\zeta,\varphi),\ \sin\text{ce}\ \Delta X_3^s(\mathbf{r}) = 0\text{ for }\mathbf{r}\in\Omega. \tag{98}$$

The unknown constant coefficients in (98), written in both the Cartesian and the spherical fashion, similar to (66), as

$$\begin{aligned}\mathbf{d}_{\ell,ex}^{m/q} &= \sum_{j=1}^{3} d_{\ell,j}^{m/q}\hat{\mathbf{x}}_j = d_{\ell,1}^{m/q}\hat{\mathbf{x}}_1 + d_{\ell,2}^{m/q}\hat{\mathbf{x}}_2 + d_{\ell,3}^{m/q}\hat{\mathbf{x}}_3 \\ &= \left(d_{\ell,1}^{m/q}\zeta + d_{\ell,2}^{m/q}\sqrt{1-\zeta^2}\cos\varphi + d_{\ell,3}^{m/q}\sqrt{1-\zeta^2}sin\varphi\right)\hat{\mathbf{r}} \\ &+\left(-d_{\ell,1}^{m/q}\sqrt{1-\zeta^2} + d_{\ell,2}^{m/q}\zeta\cos\varphi + d_{\ell,3}^{m/q}\zeta sin\varphi\right)\hat{\boldsymbol{\zeta}} \\ &+\left(-d_{\ell,2}^{m/q}\sin\varphi + d_{\ell,3}^{m/q}\cos\varphi\right)\hat{\boldsymbol{\varphi}}\end{aligned} \tag{99}$$

for $\ell \geq 0$, $m = 0,1,2,\ldots,\ell$ and $q = e,o$, must be calculated when we reinforce the three boundary conditions (45) and (46). First, let us elaborate a bit further on the integral in (97), which in view of the elementary volume $d\Omega' = r'^2dr'd\zeta'd\varphi'$ for $r' \in [\alpha,+\infty)$, $\zeta' \in [-1,1]$ and $\varphi' \in [0,2\pi)$, becomes

$$\begin{aligned}-\frac{3}{2\pi}\iiint_\Omega \frac{\mathbf{E}_1^s\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}d\Omega' &= -\frac{3}{2\pi}\int_0^{2\pi}\int_{-1}^{1}\int_\alpha^{+\infty}\frac{\mathbf{E}_1^s\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}r'^2dr'd\zeta'd\varphi' \\ &= -\frac{3}{2\pi}\left\{\lim_{e\to0}\int_0^{2\pi}\int_{-1}^{1}\int_\alpha^{r_0-e}\frac{\mathbf{E}_1^s\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}r'^2dr'd\zeta'd\varphi'\right. \\ &+\lim_{e\to0}\int_0^{2\pi}\int_{-1}^{1}\int_{r_0-e}^{r_0+e}\frac{\mathbf{E}_1^s\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}r'^2dr'd\zeta'd\varphi' \\ &\left.+\lim_{e\to0}\int_0^{2\pi}\int_{-1}^{1}\int_{r_0+e}^{+\infty}\frac{\mathbf{E}_1^s\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}r'^2dr'd\zeta'd\varphi'\right\}\text{ for }\mathbf{r}\in\Omega,\end{aligned} \tag{100}$$

in which, for the family of singular points that correspond to a sphere of radius $r' = r_0$ we introduce a classical limiting technique, based on the exclusion of this area by bounding it with a spherical shell of thickness $2e$, when $0 < e \ll 1$ is a very small positive number. Otherwise and without loss of generality, we presume that $(\zeta',\varphi') \neq (\zeta_0,\varphi_0)$, hence we are not obliged to apply the aforementioned limiting procedure for the two angular variables.

The first and the third integrals of (100) are analytic, since they exclude the singular point, consequently using twofold the Green's expansion [20] in spherical coordinates

$$\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}=\begin{cases}\sum\limits_{\ell=0}^{+\infty}\sum\limits_{m=0}^{\ell}\sum\limits_{q=e,o}\frac{(\ell-m)!}{(\ell+m)!}\varepsilon_m\frac{r^{\ell}}{r'^{\ell+1}}Y_{\ell}^{m/q}(\zeta,\varphi)Y_{\ell}^{m/q}(\zeta',\varphi'), & r'<r_0-e\\ \sum\limits_{\ell=0}^{+\infty}\sum\limits_{m=0}^{\ell}\sum\limits_{q=e,o}\frac{(\ell-m)!}{(\ell+m)!}\varepsilon_m\frac{r'^{\ell}}{r^{\ell+1}}Y_{\ell}^{m/q}(\zeta,\varphi)Y_{\ell}^{m/q}(\zeta',\varphi'), & r'>r_0+e\end{cases} \tag{101}$$

for $\mathbf{r}\in\Omega$ and applying (101) appropriately in each domain, these integrals are rewritten in the form

$$-\frac{3}{2\pi}\lim_{e\to 0}\int_0^{2\pi}\int_{-1}^{1}\int_{\alpha}^{r_0-e}\frac{\mathbf{E}_1^s\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}r'^2dr'd\zeta'd\varphi'=\sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o}s_{\ell,-}^{m/q}r^{\ell}Y_{\ell}^{m/q}(\zeta,\varphi)\text{ for }\mathbf{r}\in\Omega \tag{102}$$

and

$$-\frac{3}{2\pi}\lim_{e\to 0}\int_0^{2\pi}\int_{-1}^{1}\int_{r_0+e}^{+\infty}\frac{\mathbf{E}_1^s\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}r'^2dr'd\zeta'd\varphi'=\sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o}s_{\ell,+}^{m/q}r^{-(\ell+1)}Y_{\ell}^{m/q}(\zeta,\varphi)\text{ for }\mathbf{r}\in\Omega, \tag{103}$$

which admit harmonic expressions, where the leading constants in (102) and (103) for $\ell\geq 0$, $m=0,1,2,\ldots,\ell$ and q=e,o are

$$s_{\ell,-}^{m/q}=-\frac{3}{2\pi}\frac{(\ell-m)!}{(\ell+m)!}\varepsilon_m\lim_{e\to 0}\int_0^{2\pi}\int_{-1}^{1}\int_{\alpha}^{r_0-e}r'^{-(\ell+1)}Y_{\ell}^{m/q}(\zeta',\varphi')\mathbf{E}_1^s\left(\mathbf{r}'\right)r'^2dr'd\zeta'd\varphi' \tag{104}$$

and

$$s_{\ell,+}^{m/q}=-\frac{3}{2\pi}\frac{(\ell-m)!}{(\ell+m)!}\varepsilon_m\lim_{e\to 0}\int_0^{2\pi}\int_{-1}^{1}\int_{r_0+e}^{+\infty}r'^{\ell}Y_{\ell}^{m/q}(\zeta',\varphi')\mathbf{E}_1^s\left(\mathbf{r}'\right)r'^2dr'd\zeta'd\varphi', \tag{105}$$

respectively. Then again, the second singular integral of (100) is responsible for the generation of the non-homogeneous part of (44), since

$$\Delta\left\{-\frac{3}{2\pi}\lim_{e\to 0}\int_0^{2\pi}\int_{-1}^{+1}\int_{r_0-e}^{r_0+e}\frac{\mathbf{E}_1^s\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}d\Omega'\right\}=6\lim_{e\to 0}\int_0^{2\pi}\int_{-1}^{+1}\int_{r_0-e}^{r_0+e}\delta\left(\mathbf{r}-\mathbf{r}'\right)\mathbf{E}_1^s\left(\mathbf{r}'\right)d\Omega'=6\mathbf{E}_1^s(\mathbf{r})\text{ for }\mathbf{r}\in\Omega, \tag{106}$$

due to the property of the Dirac function $\delta\left(\mathbf{r}-\mathbf{r}'\right)$. Herein, the procedure that is followed is completely different than the one presented among (101)–(105), wherein we choose to expand the involved integral in terms of the orthonormal basis $Y_{\ell}^{m/q}$ via

$$-\frac{3}{2\pi}\lim_{e\to 0}\int_0^{2\pi}\int_{-1}^{1}\int_{r_0-e}^{r_0+e}\frac{\mathbf{E}_1^s\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}r'^2dr'd\zeta'd\varphi'=\sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o}\mathbf{S}_{\ell}^{m/q}(r)Y_{\ell}^{m/q}(\zeta,\varphi)\text{ for }\mathbf{r}\in\Omega, \tag{107}$$

where, due to the orthogonality relation (50), the radial-dependent functions imply

$$\mathbf{S}_{\ell}^{m/q}(r)=-\frac{3(2\ell+1)}{8\pi^2}\varepsilon_m\frac{(\ell-m)!}{(\ell+m)!}\int_0^{2\pi}\int_{-1}^{1}\left[\lim_{e\to 0}\int_0^{2\pi}\int_{-1}^{1}\int_{r_0-e}^{r_0+e}\frac{\mathbf{E}_1^s\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}r'^2dr'd\zeta'd\varphi'\right]Y_{\ell}^{m/q}(\zeta,\varphi)d\zeta d\varphi \tag{108}$$

for every $r \geq \alpha$. Thus, we substitute (102) with (104), (103) with (105) and (107) with (108) into the principal equation (100) in order to obtain the handy expression

$$-\frac{3}{2\pi}\iiint_{\Omega}\frac{\mathbf{E}_1^s\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}d\Omega' = \sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o}\left[s_{\ell,-}^{m/q}r^{\ell} + S_{\ell}^{m/q}(r) + s_{\ell,+}^{m/q}r^{-(\ell+1)}\right]Y_{\ell}^{m/q}(\zeta,\varphi) \quad (109)$$

for $\mathbf{r} \in \Omega$, which can accept orthogonality arguments. On the other hand and for reasons of mathematical convenience, in our forthcoming calculations, we wish to expand the corresponding primary field (24) in the same manner; hence, we expand (55) with (56) if $\mathbf{r} < \mathbf{r}_0$ and we recall that $\mathbf{R} = \mathbf{r} - \mathbf{r}_0$, such that

$$\mathbf{E}_3^p(\mathbf{r};\mathbf{r}_0) = \sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o}\left[\mathbf{I}_{\ell,ex}^{m/q}(r,\zeta,\varphi\,;\mathbf{r}_0)r^{\ell}Y_{\ell}^{m/q}(\zeta,\varphi)\right] \text{ for } \mathbf{r} \in \Omega, \quad (110)$$

where

$$\mathbf{I}_{\ell}^{m/q}(r,\zeta,\varphi\,;\mathbf{r}_0) = -\frac{3}{4\pi}\sqrt{\frac{\mu}{\varepsilon}}\mathbf{m} \times [\mathbf{r}(r,\zeta,\varphi) - \mathbf{r}_0]\rho_{\ell,ex}^{m/q}(\mathbf{r}_0) \text{ for } \mathbf{r} \in \Omega, \quad (111)$$

in which the position vector $\mathbf{r}(r,\zeta,\varphi)$ holds from relation (67). Putting all the analytical tools together, we import the information developed in (97)–(113) into the three boundary conditions, two that correspond to (45) and one from (46), and we follow the very same process, which was analyzed in (68)–(88) for the second-order scattered magnetic field, so as to calculate the unknown constant coefficients $d_{\ell,ex}^{m/q}$ for $\ell \geq 0$, $m = 0,1,2,\ldots,\ell$ and $q = e,o$. Using the above reasoning, we are led to the three independent relations

$$\sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o}\left[\sum_{j=1}^{3}\overline{f}_{\ell,j}^{m/q,\kappa}(\zeta,\varphi)d_{\ell,j}^{m/q} - \overline{g}_{\ell}^{m/q,\kappa}(\zeta,\varphi\,;\mathbf{r}_0)\right] = 0 \text{ with } \kappa = 1,2,3 \quad (112)$$

for every $\zeta \in [-1,1]$ and $\varphi \in [0,2\pi)$. The involved functions in (112) have amenable expressions; hence, for $\kappa = 1$, we have

$$\overline{f}_{\ell,1}^{m/q,1}(\zeta,\varphi) = \alpha^{-(\ell+2)}\left[-(\ell+1)\zeta P_{\ell}^{m}(\zeta) + \left(1-\zeta^2\right)P_{\ell}^{m\prime}(\zeta)\right]f_m^q(\varphi), \quad (113)$$

$$\begin{aligned}\overline{f}_{\ell,2}^{m/q,1}(\zeta,\varphi) = \alpha^{-(\ell+2)}\Big[&\left(-(\ell+1)P_{\ell}^{m}(\zeta) - \zeta P_{\ell}^{m\prime}(\zeta)\right)\sqrt{1-\zeta^2}\cos\varphi f_m^q(\varphi) \\ &-\tfrac{P_{\ell}^{m}(\zeta)}{\sqrt{1-\zeta^2}}sin\varphi f_m^{q}{}'(\varphi)\Big],\end{aligned} \quad (114)$$

$$\overline{f}_{\ell,3}^{m/q,1}(\zeta,\varphi) = \alpha^{-(\ell+2)}\left[\left(-(\ell+1)P_{\ell}^{m}(\zeta) - \zeta P_{\ell}^{m\prime}(\zeta)\right)\sqrt{1-\zeta^2}\sin\varphi f_m^q(\varphi) + \frac{P_{\ell}^{m}(\zeta)}{\sqrt{1-\zeta^2}}\cos\varphi f_m^{q}{}'(\varphi)\right], \quad (115)$$

and

$$\begin{aligned}\overline{g}_{\ell}^{m/q,1}(\zeta,\varphi\,;\mathbf{r}_0) = -&\left[s_{\ell,-}^{m/q} + \tfrac{1}{\alpha^{\ell}}S_{\ell}^{m/q}(\alpha)\right]\cdot\nabla_{\alpha}\left[\alpha^{\ell}Y_{\ell}^{m/q}(\zeta,\varphi)\right] - s_{\ell,+}^{m/q}\cdot\nabla_{\alpha}\left[\tfrac{1}{\alpha^{\ell+1}}Y_{\ell}^{m/q}(\zeta,\varphi)\right] - \\ &-\hat{r}\cdot\left[S_{\ell}^{m/q}{}'(\alpha) - \tfrac{\ell}{\alpha}S_{\ell}^{m/q}(\alpha)\right]Y_{\ell}^{m/q}(\zeta,\varphi),\end{aligned} \quad (116)$$

in which ∇_{α} stands for the gradient operator (9), where instead of the classic differentiation over the r variable, we urge differentiation with respect to α. On the other hand, for $\kappa = 2$, we have

$$\overline{f}_{\ell,1}^{m/q,2}(\zeta,\varphi) = 0, \quad (117)$$

$$\overline{f}_{\ell,2}^{m/q,2}(\zeta,\varphi) = \alpha^{-(\ell+1)}sin\varphi Y_{\ell}^{m/q}(\zeta,\varphi), \quad (118)$$

$$\overline{f}_{\ell,3}^{m/q,2}(\zeta,\varphi) = -\alpha^{-(\ell+1)}\cos\varphi Y_{\ell}^{m/q}(\zeta,\varphi) \quad (119)$$

and

$$\overline{g}_{\ell}^{m/q,2}(\zeta,\varphi\,;\mathbf{r}_0) = -\hat{\mathbf{r}} \times \left[\left(\mathbf{s}_{\ell,-}^{m/q} + \mathbf{I}_{\ell}^{m/q}(\alpha,\zeta,\varphi\,;\mathbf{r}_0)\right)\alpha^{\ell} + \mathbf{s}_{\ell,+}^{m/q}\alpha^{-(\ell+1)} + \mathbf{S}_{\ell}^{m/q}(\alpha)\right]\cdot\hat{\boldsymbol{\zeta}}\,Y_{\ell}^{m/q}(\zeta,\varphi), \quad (120)$$

while the $\kappa = 3$ case uses the formulae

$$\overline{f}_{\ell,1}^{m/q,3}(\zeta,\varphi) = -\alpha^{-(\ell+1)}\sqrt{1-\zeta^2}Y_\ell^{m/q}(\zeta,\varphi), \tag{121}$$

$$\overline{f}_{\ell,2}^{m/q,3}(\zeta,\varphi) = \alpha^{-(\ell+1)}\zeta\cos\varphi Y_\ell^{m/q}(\zeta,\varphi), \tag{122}$$

$$\overline{f}_{\ell,3}^{m/q,3}(\zeta,\varphi) = \alpha^{-(\ell+1)}\zeta sin\varphi Y_\ell^{m/q}(\zeta,\varphi) \tag{123}$$

and

$$\overline{g}_\ell^{m/q,3}(\zeta,\varphi;\mathbf{r}_0) = -\hat{\boldsymbol{r}}\times\left[\left(\mathbf{s}_{\ell,-}^{m/q}+\mathbf{I}_\ell^{m/q}(\alpha,\zeta,\varphi;\mathbf{r}_0)\right)\alpha^\ell+\mathbf{s}_{\ell,+}^{m/q}\alpha^{-(\ell+1)}+\mathbf{S}_\ell^{m/q}(\alpha)\right]\cdot\hat{\boldsymbol{\varphi}}\;Y_\ell^{m/q}(\zeta,\varphi), \tag{124}$$

all of the above are provided in terms of (104), (105), (108) and (111), while all derivatives denoted by the prime are with respect to the argument as usual. Bearing in mind the same argumentation that was developed for similar kinds of functions (69)–(80), with standard orthogonality rules almost impossible to apply on (112), with the aid of recurrence relationships, we choose to expand (113)–(124) in terms of the orthonormal eigenfunctions $Y_{\ell'}^{m'/q'}$ for $\ell' \geq 0$, $m' = 0,1,2,\ldots,\ell'$ and $q' = e,o$ via

$$\overline{f}_{\ell,j}^{m/q,\kappa}(\zeta,\varphi) = \sum_{\ell'=0}^{+\infty}\sum_{m'=0}^{\ell'}\sum_{q'=e,o}\overline{\lambda}_{(\ell,\ell'),j}^{(m,m')/(q,q'),\kappa}Y_{\ell'}^{m'/q'}(\zeta,\varphi) \text{ with } \kappa,j=1,2,3 \tag{125}$$

and

$$\overline{g}_\ell^{m/q,\kappa}(\zeta,\varphi;\mathbf{r}_0) = \sum_{\ell'=0}^{+\infty}\sum_{m'=0}^{\ell'}\sum_{q'=e,o}\overline{\mu}_{(\ell,\ell')}^{(m,m')/(q,q'),\kappa}(\mathbf{r}_0)Y_{\ell'}^{m'/q'}(\zeta,\varphi) \text{ with } \kappa=1,2,3, \tag{126}$$

wherein $\ell \geq 0$, $m = 0,1,2,\ldots,\ell$ and $q = e,o$, while $\zeta \in [-1,1]$ and $\varphi \in [0,2\pi)$. Subsequently, we use again the orthogonal property of the surface spherical harmonics so as to calculate the leading coefficients in (125) and (126) as

$$\overline{\lambda}_{(\ell,\ell'),j}^{(m,m')/(q,q'),\kappa} = \frac{2\ell'+1}{4\pi}\frac{(\ell'-m')!}{(\ell'+m')!}\varepsilon_{m'}\int_{-1}^{+1}\int_0^{2\pi}\overline{f}_{\ell,j}^{m/q,\kappa}(\zeta,\varphi)Y_{\ell'}^{m'/q'}(\zeta,\varphi)d\varphi\,d\zeta \tag{127}$$

and

$$\overline{\mu}_{(\ell,\ell')}^{(m,m')/(q,q'),\kappa}(\mathbf{r}_0) = \frac{2\ell'+1}{4\pi}\frac{(\ell'-m')!}{(\ell'+m')!}\varepsilon_{m'}\int_{-1}^{+1}\int_0^{2\pi}\overline{g}_\ell^{m/q,\kappa}(\zeta,\varphi;\mathbf{r}_0)Y_{\ell'}^{m'/q'}(\zeta,\varphi)d\varphi\,d\zeta \tag{128}$$

for every $(\ell,\ell') \geq 0$, $(m,m') = 0,1,2,\ldots,(\ell,\ell')$ and $(q,q') = e,o$ with $\kappa,j = 1,2,3$. Apparently, the evaluation of these coefficients is feasible in an easy manner, since they involve integrals of simple functions. Thus, our next action merges (125) (with (127)) and (126) (with (128)) into the principal relation (112) in order to conclude that

$$\sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o}\left[\sum_{j=1}^{3}\overline{\lambda}_{(\ell,\ell'),j}^{(m,m')/(q,q'),\kappa}d_{\ell,j}^{m/q}-\overline{\mu}_{(\ell,\ell')}^{(m,m')/(q,q'),\kappa}(\mathbf{r}_0)\right] = 0 \text{ with } \kappa=1,2,3, \tag{129}$$

which comprises a complete set of algebraic equations for any $\ell' \geq 0$, $m' = 0,1,2,\ldots,\ell'$ and $q' = e,o$, while it embodies the three components ($j = 1,2,3$) of the unknown constants $d_{\ell,j}^{m/q}$ for $\ell \geq 0$, $m = 0,1,2,\ldots,\ell$, $q = e,o$ and the calculable expressions (127) and (128). Thus, once more, the insuperable difficulty in dealing with these boundary conditions is reduced to the simple elaboration of the easy-to-handle integrals (127) and (128), which are involved with trigonometric and associated Legendre functions. Our task is then limited to the manipulation of the corresponding systems of linear algebraic equations, arising from

(129), whose solution is achieved when classical cut-off techniques are applied in view of the determination of an appropriate common upper limit $\overline{L}$ for both the degree indexes, which means $\ell = \ell' = 0, 1, 2, \ldots, \overline{L}$. Under this aim, the set of relations (129) yields the matrix form

$$\overline{\mathbb{A}}_L \overline{\mathbf{x}}_L = \overline{\mathbf{z}}_L(\mathbf{r}_0) \;\; \Rightarrow \;\; \overline{\mathbf{x}}_L = \overline{\mathbb{A}}_L^{-1} \overline{\mathbf{z}}_L(\mathbf{r}_0), \tag{130}$$

where for $m = m' = 0, 1, 2, \ldots, \ell$, $q = q' = e, o$ and $\kappa, j = 1, 2, 3$, we get

$$\overline{\mathbb{A}}_L = \begin{bmatrix} \ddots & \vdots & \\ \cdots & \overline{\lambda}_{(\ell,\ell),j}^{(m,m)/(q,q),\kappa} & \cdots \\ & \vdots & \ddots \end{bmatrix}, \; \overline{\mathbf{x}}_L = \begin{bmatrix} \vdots \\ d_{\ell,j}^{m/q} \\ \vdots \end{bmatrix} \text{ and } \overline{\mathbf{z}}_L(\mathbf{r}_0) = \begin{bmatrix} \vdots \\ \overline{\mu}_{(\ell,\ell),j}^{(m,m)/(q,q),\kappa}(\mathbf{r}_0) \\ \vdots \end{bmatrix}, \tag{131}$$

which are the $3(\overline{L}+1) \times m \times q$ quadratic invertible matrix of the coefficients of the unknowns, the vector of the unknown coefficients and the vector of the known constants, respectively. The unique solution $\mathbf{x}_L$, given in (130), corresponds to the constant coefficients $\mathbf{d}_{\ell,ex}^{m/q} = d_{\ell,1}^{m/q}\hat{\mathbf{x}}_1 + d_{\ell,2}^{m/q}\hat{\mathbf{x}}_2 + d_{\ell,3}^{m/q}\hat{\mathbf{x}}_3$ for $\ell = 0, 1, 2, \ldots, \overline{L}$, $m = 0, 1, 2, \ldots, \ell$ and $q = e, o$, wherein $\overline{L}$ is defined such that the expected accuracy of the series expansions is achieved. Consequently, combining (98) and (109), we substitute into (97) to obtain the third-order electric field as

$$\mathbf{E}_3^s(\mathbf{r}) = \sum_{\ell=0}^{+\infty} \sum_{m=0}^{\ell} \sum_{q=e,o} \left[\mathbf{S}_\ell^{m/q}(r) + \mathbf{s}_{\ell,-}^{m/q} r^\ell + \left(\mathbf{s}_{\ell,+}^{m/q} + \mathbf{d}_{\ell,ex}^{m/q} \right) r^{-(\ell+1)} \right] Y_\ell^{m/q}(\zeta, \varphi) \text{ for } \mathbf{r} \in \Omega \tag{132}$$

by virtue of the solution of (130) with (131) and in terms of the known expressions (104), (105) and (108), once $\mathbf{E}_1^s$ is recovered from (96).

5. Analytical Validation of the Method

The current research belongs to the applied mathematics area but also covers the corresponding engineering science field, hence the validation of the produced results for the sake of completeness. Towards this direction and aiming to demonstrate the correctness and the applicability of the presented methodology, we invoke an application through which the already published results of reference [16] for the case of the electromagnetic wave scattering by a highly conductive prolate spheroidal body [19] in a lossless medium with low-frequency dipolar excitation are reduced to our formulae. However, due to the elaborate work of this task and without loss of generality, we choose to show the detailed analytical steps in order to recover the Rayleigh static magnetic term for the low-frequency order $n = 0$, which is given by the compact relationship (62), and we claim that the corresponding recovery stands also for the terms $n = 1, 2, 3$.

Doing so, we need to present briefly the basic tools and prerequisites in order to infer information about the Rayleigh magnetic field, which is scattered by a solid prolate spheroidal body and is embedded in a lossless medium and excited by a low-frequency magnetic dipole $\mathbf{m}$, identical to that given in (1). Bearing in mind that the best fitted geometry for this situation is the prolate spheroidal one and given the semifocal distance of this system $c > 0$, we introduce the transformed prolate spheroidal coordinates (τ, ζ, φ) for every $1 \leq \tau < +\infty$, $-1 \leq \zeta \leq 1$ and $0 \leq \varphi < 2\pi$ via the relations

$$x_1 = c\tau\zeta, \; x_2 = c\sqrt{\tau^2 - 1}\sqrt{1 - \zeta^2}\cos\varphi \text{ and } x_3 = c\sqrt{\tau^2 - 1}\sqrt{1 - \zeta^2}\sin\varphi \tag{133}$$

with position vector $\mathbf{r} = (x_1, x_2, x_3)$, where $\mathbf{r}_0 = (x_{10}, x_{20}, x_{30})$ stands for the fixed location of the source. Then, the impenetrable surface S_{ps} of the prolate spheroidal target is designated by $\tau = \tau_s \equiv a_1/c$, and the exterior domain of wave propagation is

$$\Omega_{ps} \equiv V\left(\mathbb{R}^3\right) - \{\mathbf{r}_0\} = \{(\tau, \zeta, \varphi) : \tau \in [\tau_s, +\infty), \; \zeta \in [-1, 1], \; \varphi \in [0, 2\pi)\} - \{(\tau_0, \zeta_0, \varphi_0)\}, \tag{134}$$

where $(\tau_0, \zeta_0, \varphi_0)$ corresponds to (x_{10}, x_{20}, x_{30}). For this circumstance, the prolate spheroidal zero-order low-frequency magnetic field that is scattered within the region (134) and corresponds to the Rayleigh term is evaluated in [16] as

$$\begin{aligned}\mathbf{H}^s_{0,sp}(\mathbf{r}) = & \sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o}(2\ell+1)\left[\tfrac{(\ell-m)!}{(\ell+m)!}\right]^2(-1)^m\varepsilon_m\tfrac{P_\ell^{m\prime}(\tau_s)}{Q_\ell^{m\prime}(\tau_s)} \\ & \left[\tfrac{\mathbf{m}}{4\pi c}\cdot\nabla_{sp,\mathbf{r}_0}\left(Q_\ell^m(\tau_0)P_\ell^m(\zeta_0)f_m^q(\varphi_0)\right)\right]\nabla_{sp}\left(Q_\ell^m(\tau)P_\ell^m(\zeta)f_m^q(\varphi)\right),\end{aligned} \tag{135}$$

in which the gradient ∇_{sp} in prolate spheroidal geometry yields

$$\nabla_{sp} = \frac{1}{c\sqrt{\tau^2-\zeta^2}}\left[\sqrt{\tau^2-1}\,\hat{\boldsymbol{\tau}}\frac{\partial}{\partial\tau} - \sqrt{1-\zeta^2}\,\hat{\boldsymbol{\zeta}}\frac{\partial}{\partial\zeta}\right] + \frac{1}{c\sqrt{\tau^2-1}\sqrt{1-\zeta^2}}\hat{\boldsymbol{\varphi}}\frac{\partial}{\partial\varphi}, \tag{136}$$

in view of the unit normal vectors

$$\hat{\boldsymbol{\tau}} \equiv \hat{\boldsymbol{\tau}}(\tau,\zeta,\varphi) = \frac{1}{\sqrt{\tau^2-\zeta^2}}\left(\zeta\sqrt{\tau^2-1}\hat{\mathbf{x}}_1 + \tau\sqrt{1-\zeta^2}\cos\varphi\hat{\mathbf{x}}_2 + \tau\sqrt{1-\zeta^2}\sin\varphi\hat{\mathbf{x}}_3\right), \tag{137}$$

$$\hat{\boldsymbol{\zeta}} \equiv \hat{\boldsymbol{\zeta}}(\tau,\zeta,\varphi) = \frac{1}{\sqrt{\tau^2-\zeta^2}}\left(-\tau\sqrt{1-\zeta^2}\hat{\mathbf{x}}_1 + \zeta\sqrt{\tau^2-1}\cos\varphi\hat{\mathbf{x}}_2 + \zeta\sqrt{\tau^2-1}\sin\varphi\hat{\mathbf{x}}_3\right), \tag{138}$$

$$\hat{\boldsymbol{\varphi}} \equiv \hat{\boldsymbol{\varphi}}(\varphi) = -\sin\varphi\hat{\mathbf{x}}_2 + \cos\varphi\hat{\mathbf{x}}_3, \tag{139}$$

where from (136), the gradient could operate on $\mathbf{r}_0$, providing us with $\nabla_{sp,\mathbf{r}_0}$ as well. On the other hand, expression (135) is written in terms of the exterior prolate spheroidal harmonics [20], wherein the associated Legendre functions of the first P_ℓ^m and second Q_ℓ^m kind of degree $\ell \geq 0$ and order $m = 0, 1, 2, \ldots, \ell$, as well the functions of the azimuthal angle (49) (being the same in both the spherical and the prolate spheroidal eigenfunctions) are involved, while $\varepsilon_0 = 1$, $\varepsilon_m = 2$ when $m \geq 1$ and the prime denotes argument differentiation.

Under the aim of the above, we can proceed to the analytical validation via the reduction of formula (135) of the prolate spheroidal case to formula (62) of the spherical case. Therein, the prolate spheroidal geometry degenerates to the spherical one as $c \to 0^+$. For the corresponding analytical reduction, the limiting process involves an appropriate combination of the semifocal distance with the coordinate variable $\tau \geq \tau_s$, such as

$$\lim_{c\to 0^+} c\tau = r \text{ and } \lim_{c\to 0^+}\frac{1}{2c}\ln\frac{\tau+1}{\tau-1} = \frac{1}{r}, \tag{140}$$

where $r \geq \alpha$ is the radial component of the spherical coordinate system (3), which is readily recovered from (133) by applying (140), given that the other two radial variables remain unaltered, because they coincide in both systems. Similarly, if we multiply and simultaneously divide (137)–(139) by c, the spherical orthonormal basis (10)–(12) is obtained by means of (140), while the gradient operator in spherical geometry (9) is instantly taken with a simple limit operation on (136) by virtue of (140). Beyond the reduction of the geometrical characteristics of the prolate spheroidal to the spherical system, it is necessary for our task to reduce the relevant prolate spheroidal eigenfunctions P_ℓ^m and Q_ℓ^m, depending on the τ variable, to the corresponding spherical ones. Hence by virtue of the definition of the associated Legendre functions of the first and the second kind [20], we straightforwardly derive the limits

$$\lim_{c\to 0^+} c^\ell P_\ell^m(\tau) = p_\ell\frac{\ell!}{(\ell-m)!}r^\ell \text{ and } \lim_{c\to 0^+} c^{-(\ell+1)}Q_\ell^m(\tau) = q_\ell(-1)^m\frac{(\ell+m)!}{\ell!}r^{-(\ell+1)} \tag{141}$$

for $\ell \geq 0$ and at $\tau \geq \tau_s$ (prolate spheroid) or $r \geq \alpha$ (sphere), where

$$p_\ell = \frac{(2\ell)!}{2^\ell(\ell!)^2} \text{ and } q_\ell = \frac{1}{2^\ell}\sum_{k=0}^{[\ell/2]}\frac{(-1)^k(2\ell-2k)!}{k!(\ell-k)!(\ell-2k)!(2\ell-2k+1)} \text{ with } (2\ell+1)p_\ell q_\ell = 1, \tag{142}$$

while the related limits of their derivatives follow from (141), i.e.,

$$\lim_{c\to 0^+} c^{\ell-1} P_\ell^{m\prime}(\tau) = p_\ell \frac{\ell!}{(\ell-m)!} \ell r^{\ell-1} \text{ and } \lim_{c\to 0^+} c^{-(\ell+2)} Q_\ell^{m\prime}(\tau) = -q_\ell (-1)^m \frac{(\ell+m)!}{\ell!} (\ell+1) r^{-(\ell+2)} \tag{143}$$

for $\ell \geq 0$ and at $\tau \geq \tau_s$ (prolate spheroid) or $r \geq \alpha$ (sphere), in which the prime refers to differentiation with respect to the τ variable.

The degeneration of the Rayleigh low-frequency magnetic field of order $n = 0$, in order to recover the case of a spherical metallic body of radius $c\tau_s \to \alpha$, embedded within a lossless medium, is then a straightforward sequence of steps, since any kind of indeterminacies are absent. Towards this direction, we work as follows. The azimuthal unit normal angular vector (139) remains unaltered in both the prolate spheroidal and the spherical coordinate systems (see also (12)), while by a simple limiting process based on (140) to the other two unit normal vectors of the prolate spheroidal geometry (137) and (138), yielding

$$\begin{aligned}
\lim_{c\to 0^+} \hat{\boldsymbol{\tau}} &= \lim_{c\to 0^+} \left\{ \tfrac{1}{\sqrt{\tau^2-\zeta^2}} \left(\zeta\sqrt{\tau^2-1}\hat{\boldsymbol{x}}_1 + \tau\sqrt{1-\zeta^2}\cos\varphi\hat{\boldsymbol{x}}_2 + \tau\sqrt{1-\zeta^2}\sin\varphi\hat{\boldsymbol{x}}_3 \right) \right\} \\
&= \lim_{c\to 0^+} \left\{ \tfrac{1}{\sqrt{(c\tau)^2-(c\zeta)^2}} \left(\zeta\sqrt{(c\tau)^2-c^2}\hat{\boldsymbol{x}}_1 + (c\tau)\sqrt{1-\zeta^2}\cos\varphi\hat{\boldsymbol{x}}_2 + (c\tau)\sqrt{1-\zeta^2}\sin\varphi\hat{\boldsymbol{x}}_3 \right) \right\} \\
&= \tfrac{1}{r}\left(r\zeta\hat{\boldsymbol{x}}_1 + r\sqrt{1-\zeta^2}\cos\varphi\hat{\boldsymbol{x}}_2 + r\sqrt{1-\zeta^2}\sin\varphi\hat{\boldsymbol{x}}_3 \right) \\
&= \zeta\hat{\boldsymbol{x}}_1 + \sqrt{1-\zeta^2}\cos\varphi\hat{\boldsymbol{x}}_2 + \sqrt{1-\zeta^2}\sin\varphi\hat{\boldsymbol{x}}_3 \equiv \hat{\boldsymbol{r}} \text{ for } \zeta \in [-1,1],\ \varphi \in [0,2\pi)
\end{aligned} \tag{144}$$

and

$$\begin{aligned}
\lim_{c\to 0^+} \hat{\boldsymbol{\zeta}} &= \lim_{c\to 0^+} \left\{ \tfrac{1}{\sqrt{\tau^2-\zeta^2}} \left(-\tau\sqrt{1-\zeta^2}\hat{\boldsymbol{x}}_1 + \zeta\sqrt{\tau^2-1}\cos\varphi\hat{\boldsymbol{x}}_2 + \zeta\sqrt{\tau^2-1}\sin\varphi\hat{\boldsymbol{x}}_3 \right) \right\} \\
&= \lim_{c\to 0^+} \left\{ \tfrac{1}{\sqrt{(c\tau)^2-(c\zeta)^2}} \left(-(c\tau)\sqrt{1-\zeta^2}\hat{\boldsymbol{x}}_1 + \zeta\sqrt{(c\tau)^2-c^2}\cos\varphi\hat{\boldsymbol{x}}_2 + \zeta\sqrt{(c\tau)^2-c^2}\sin\varphi\hat{\boldsymbol{x}}_3 \right) \right\} \\
&= \tfrac{1}{r}\left(-r\sqrt{1-\zeta^2}\hat{\boldsymbol{x}}_1 + r\zeta\cos\varphi\hat{\boldsymbol{x}}_2 + r\zeta\sin\varphi\hat{\boldsymbol{x}}_3 \right) \\
&= -\sqrt{1-\zeta^2}\hat{\boldsymbol{x}}_1 + \zeta\cos\varphi\hat{\boldsymbol{x}}_2 + \zeta\sin\varphi\hat{\boldsymbol{x}}_3 \equiv \hat{\boldsymbol{\zeta}} \text{ for } \zeta \in [-1,1],\ \varphi \in [0,2\pi),
\end{aligned} \tag{145}$$

provides us with the corresponding two spherical unit normal vectors (10) and (11), respectively. Hence, we are able to evaluate the relative limit of the prolate spheroidal gradient operator (136), wherein doing so, we obtain

$$\begin{aligned}
\lim_{c\to 0^+} \nabla_{sp} &= \lim_{c\to 0^+} \left\{ \tfrac{1}{c\sqrt{\tau^2-\zeta^2}} \left[\sqrt{\tau^2-1}\hat{\boldsymbol{\tau}}\tfrac{\partial}{\partial\tau} - \sqrt{1-\zeta^2}\hat{\boldsymbol{\zeta}}\tfrac{\partial}{\partial\zeta} \right] + \tfrac{1}{c\sqrt{\tau^2-1}\sqrt{1-\zeta^2}}\hat{\boldsymbol{\varphi}}\tfrac{\partial}{\partial\varphi} \right\} \\
&= \lim_{c\to 0^+} \left\{ \tfrac{1}{\sqrt{(c\tau)^2-(c\zeta)^2}} \left[\sqrt{(c\tau)^2-c^2}\hat{\boldsymbol{\tau}}\tfrac{\partial}{\partial(c\tau)} - \sqrt{1-\zeta^2}\hat{\boldsymbol{\zeta}}\tfrac{\partial}{\partial\zeta} \right] + \tfrac{1}{\sqrt{(c\tau)^2-c^2}\sqrt{1-\zeta^2}}\hat{\boldsymbol{\varphi}}\tfrac{\partial}{\partial\varphi} \right\} \\
&= \tfrac{1}{r}\left[r\left(\lim_{c\to 0^+}\hat{\boldsymbol{\tau}}\right)\tfrac{\partial}{\partial r} - \sqrt{1-\zeta^2}\left(\lim_{c\to 0^+}\hat{\boldsymbol{\zeta}}\right)\tfrac{\partial}{\partial\zeta} \right] + \tfrac{1}{r\sqrt{1-\zeta^2}}\left(\lim_{c\to 0^+}\hat{\boldsymbol{\varphi}}\right)\tfrac{\partial}{\partial\varphi} \\
&= \hat{\boldsymbol{r}}\tfrac{\partial}{\partial r} - \tfrac{\sqrt{1-\zeta^2}}{r}\hat{\boldsymbol{\zeta}}\tfrac{\partial}{\partial\zeta} + \tfrac{1}{r\sqrt{1-\zeta^2}}\hat{\boldsymbol{\varphi}}\tfrac{\partial}{\partial\varphi} \equiv \nabla,
\end{aligned} \tag{146}$$

due to (140) and using (144) and (145), as well as either (12) or (139), recovering in that way the gradient in the spherical geometry that is given by (9). Obviously, it similarly holds

$$\lim_{c\to 0^+} \nabla_{sp,\mathbf{r}_0} = \nabla_{\mathbf{r}_0}, \tag{147}$$

which is a consequence of (146) and leads to the spherical gradient, operating at the position of the dipole source. Gathering all above information of (140)–(147), we apply the limit, as the semifocal distance of the prolate spheroid tends to zero onto (135), and according to classical properties of a standard limiting procedure, we obtain

$$
\begin{aligned}
\lim_{c\to 0^+}\mathbf{H}^s_{0,sp}(\mathbf{r}) &= \lim_{c\to 0^+}\sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o}(2\ell+1)\left[\tfrac{(\ell-m)!}{(\ell+m)!}\right]^2(-1)^m\varepsilon_m\tfrac{P_\ell^{m\prime}(\tau_s)}{Q_\ell^{m\prime}(\tau_s)} \\
&\left[\tfrac{\mathbf{m}}{4\pi c}\cdot\nabla_{sp,\mathbf{r}_0}\left(Q_\ell^m(\tau_0)P_\ell^m(\zeta_0)f_m^q(\varphi_0)\right)\right]\nabla_{sp}\left(Q_\ell^m(\tau)P_\ell^m(\zeta)f_m^q(\varphi)\right) \\
&= \sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o}(2\ell+1)\left[\tfrac{(\ell-m)!}{(\ell+m)!}\right]^2(-1)^m\varepsilon_m\lim_{c\to 0^+}\left(\tfrac{P_\ell^{m\prime}(\tau_s)}{Q_\ell^{m\prime}(\tau_s)}\right) \\
&\tfrac{\mathbf{m}}{4\pi c}\cdot\lim_{c\to 0^+}\left[\nabla_{sp,\mathbf{r}_0}\left(Q_\ell^m(\tau_0)P_\ell^m(\zeta_0)f_m^q(\varphi_0)\right)\right] \\
&\lim_{c\to 0^+}\left[\nabla_{sp}\left(Q_\ell^m(\tau)P_\ell^m(\zeta)f_m^q(\varphi)\right)\right] \\
&= \sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o}(2\ell+1)\left[\tfrac{(\ell-m)!}{(\ell+m)!}\right]^2(-1)^m\varepsilon_m\tfrac{1}{c^{2\ell+1}}\lim_{c\to 0^+}\left(\tfrac{c^{\ell-1}P_\ell^{m\prime}(\tau_s)}{c^{-(\ell+2)}Q_\ell^{m\prime}(\tau_s)}\right) \\
&\tfrac{\mathbf{m}}{4\pi c}\cdot c^{\ell+1}\left[\lim_{c\to 0^+}\nabla_{sp,\mathbf{r}_0}\lim_{c\to 0^+}\left(c^{-(\ell+1)}Q_\ell^m(\tau_0)\right)P_\ell^m(\zeta_0)f_m^q(\varphi_0)\right] \\
&c^{\ell+1}\left[\lim_{c\to 0^+}\nabla_{sp}\lim_{c\to 0^+}\left(c^{-(\ell+1)}Q_\ell^m(\tau)\right)P_\ell^m(\zeta)f_m^q(\varphi)\right] \\
&= \sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o}(2\ell+1)\left[\tfrac{(\ell-m)!}{(\ell+m)!}\right]^2(-1)^m\varepsilon_m\frac{p_\ell\frac{\ell!}{(\ell-m)!}\ell\alpha^{\ell-1}}{-q_\ell(-1)^m\frac{(\ell+m)!}{\ell!}(\ell+1)\alpha^{-(\ell+2)}} \\
&\tfrac{\mathbf{m}}{4\pi}\cdot\nabla_{\mathbf{r}_0}\left[q_\ell(-1)^m\tfrac{(\ell+m)!}{\ell!}r_0^{-(\ell+1)}P_\ell^m(\zeta_0)f_m^q(\varphi_0)\right] \\
&\nabla\left[q_\ell(-1)^m\tfrac{(\ell+m)!}{\ell!}r^{-(\ell+1)}P_\ell^m(\zeta)f_m^q(\varphi)\right] \\
&= -\sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o}(2\ell+1)p_\ell q_\ell\tfrac{(\ell-m)!}{(\ell+m)!}\varepsilon_m\tfrac{\ell\alpha^{\ell-1}}{(\ell+1)\alpha^{-(\ell+2)}} \\
&\left\{\tfrac{\mathbf{m}}{4\pi}\cdot\nabla_{\mathbf{r}_0}\left[r_0^{-(\ell+1)}P_\ell^m(\zeta_0)f_m^q(\varphi_0)\right]\right\}\nabla\left[r^{-(\ell+1)}P_\ell^m(\zeta)f_m^q(\varphi)\right] \\
&= -\sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o}\tfrac{\ell\alpha^{2\ell+1}}{(\ell+1)}\tfrac{(\ell-m)!}{(\ell+m)!}\varepsilon_m \\
&\left\{\tfrac{\mathbf{m}}{4\pi}\cdot\nabla_{\mathbf{r}_0}\left[r_0^{-(\ell+1)}P_\ell^m(\zeta_0)f_m^q(\varphi_0)\right]\right\}\nabla\left[r^{-(\ell+1)}P_\ell^m(\zeta)f_m^q(\varphi)\right]
\end{aligned}
\tag{148}
$$

or by means of (48) and (56), we have

$$
\lim_{c\to 0^+}\mathbf{H}^s_{0,sp}(\mathbf{r}) = -\sum_{\ell=0}^{+\infty}\sum_{m=0}^{\ell}\sum_{q=e,o}\frac{\ell\alpha^{2\ell+1}}{\ell+1}\left(\frac{\mathbf{m}}{4\pi}\cdot\nabla_{\mathbf{r}_0}\rho_{\ell,ex}^{m/q}(r_0)\right)\nabla\left[r^{-(\ell+1)}Y_\ell^{m/q}(\zeta,\varphi)\right] \equiv \mathbf{H}^s_0(\mathbf{r}) \tag{149}
$$

for every $\mathbf{r}\in\Omega$, which coincides with the spherical Rayleigh magnetic static term (62). The entire procedure evidently verified, in an analytical fashion, the effectiveness of the presented methodology independently of the implied geometry.

Obviously, the rest of the results of this paper are recovered in a similar way, wherein we avoid writing down at this stage the cumbersome expressions for the obtained reduced relations, since the purpose of this work is not the reduction limiting procedure. This application shows the flexibility of our method, which is unique and permits general manipulation of the low-frequency equations.

6. Conclusions and Discussion

A rigorous low-frequency approximation of the fields scattered by a perfectly conductive spherical body within a lossless medium under the action of an arbitrarily oriented magnetic dipole was investigated. The source, representing a spatial singularity, was set far from the object and produced the time-harmonic incident fields at low frequencies as for the present research. The developed analytical method was based on the introduction of handy power series expansions of the electromagnetic fields in terms of the wave number of the ambient, which, in the low frequency realm, permits the restriction of the manipulation to the first four in-phase and quadrature orders, where the terms of higher orders are negligible. The classical Maxwell-type problem was converted to a sequence of interconnected Laplace and Poisson relationships, which were assigned to the impenetrable boundary conditions on the surface of the target, while the limiting behavior at an infinite distance was readily secured. Upon the adjustment of a suitable spherical geometry, the resultant boundary value problems were solved incrementally and led to three-dimensional closed-form solutions, obtained as infinite series expansions in terms of spherical harmonic eigenfunctions.

An application of the methodology also complemented the developed analysis at hand via an example of the limiting degeneration of the corresponding known spheroidal problem from the bibliography to our case, a fact that validated our approach. Indeed, the present work demonstrates the production of explicit ready-to apply formulae for the corresponding spherical electromagnetic problem as obtained through the standard analytical methodology, which are new in the corresponding literature. Moreover, these explicit formulae for the electric and the magnetic field provide a reliable benchmark, where the spherical and the spheroidal results meet, validating each other analytically for the specific statement of the problem.

It is obvious that in this work we provided formulations amenable to fast yet accurate computations towards the direction of a possible construction of an inversion scheme. Under this aim, the important contributions of numerical techniques in solving scattering problems are adequate and at the same time convenient. However, we should not overlook the fact that mathematical analysis is the backbone of numerical analysis. Therefore, the purely analytical methods, in addition to the insight they offer to the understanding of the physical background and their significance in checking the credibility of numerical methods or other more sophisticated analytical models, support the mathematical community as the bases of mathematics. This provides our motivation to perform mathematical analysis of the aforementioned physical problem.

Author Contributions: Formal analysis, E.S.; Supervision, P.V. and F.K.; Writing—original draft, E.S. and P.V.; Writing—review & editing, F.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Stratton, J.A. *Electromagnetic Theory*; McGraw-Hill: New York, NY, USA, 1941.
2. Dassios, G.; Kleinman, R.E. *Low Frequency Scattering*; Oxford University Press: Oxford, UK, 2000.
3. Ammari, H.; Kang, H. *Polarization and Moment Tensors: With Applications to Inverse Problems and Effective Medium Theory*; Applied Mathematical Sciences Series; Springer: New York, NY, USA, 2007; Volume 162.
4. Ammari, H.; Garnier, J.; Jing, W.; Kang, H.; Lim, M.; Solna, K.; Wang, H. *Mathematical and Statistical Methods for Multistatic Imaging*; Lecture Notes in Mathematics; Springer: Cham, Switzerland, 2013; Volume 2098.
5. Ammari, H.; Kang, H.; Lee, H.; Lim, M.; Zribi, H. Decomposition theorems and fine estimates for electrical fields in the presence of closely located circular inclusions. *J. Differ. Equ.* **2009**, *247*, 2897–2912. [CrossRef]
6. Björkberg, J.; Kristenson, G. Three-dimensional subterranean target identification by use of optimization techniques. *Prog. Electromagn. Res.* **1997**, *15*, 141–164. [CrossRef]
7. Yu, T.; Carin, L. Analysis of the electromagnetic inductive response of a void in a conducting-soil background. *IEEE Trans. Geosci. Remote Sens.* **2000**, *38*, 1320–1327.
8. Huang, H.; Won, I.J. Detecting metal objects in magnetic environments using a broadband electromagnetic method. *Geophysics* **2003**, *68*, 1877–1887. [CrossRef]
9. Cui, T.J.; Chew, W.C.; Wright, D.L.; Smith, D.V. Three dimensional imaging for buried objects in a very lossy earth by inversion of VETEM data. *IEEE Trans. Geosci. Remote Sens.* **2003**, *4*, 2197–2210.
10. Athanasiadis, C.; Costakis, G.; Stratis, I.G. Electromagnetic scattering by a perfectly conducting obstacle in a homogeneous chiral environment: Solvability and low-frequency theory. *Math. Methods Appl. Sci.* **2002**, *25*, 927–944. [CrossRef]
11. Vafeas, P.; Perrusson, G.; Lesselier, D. Low-frequency solution for a perfectly conducting sphere in a conductive medium with dipolar excitation. *Prog. Electromagn. Res.* **2004**, *49*, 87–111. [CrossRef]
12. March, H.W. The field of a magnetic dipole in the presence of a conducting sphere. *Geophysics* **1953**, *16*, 671–684. [CrossRef]

13. Perrusson, G.; Vafeas, P.; Chatjigeorgiou, I.K.; Lesselier, D. Low-frequency on-site identification of a highly conductive body buried in Earth from a model ellipsoid. *IMA J. Appl. Math.* **2015**, *80*, 963–980. [CrossRef]
14. Vafeas, P.; Papadopoulos, P.K.; Ding, P.-P.; Lesselier, D. Mathematical and numerical analysis of low-frequency scattering from a PEC ring torus in a conductive medium. *Appl. Math. Model.* **2016**, *40*, 6477–6500. [CrossRef]
15. Wait, J.R. On the electromagnetic response of a conducting sphere to a dipole field. *Geophysics* **1960**, *25*, 649–658. [CrossRef]
16. Vafeas, P. Dipolar excitation of a perfectly electrically conducting spheroid in a lossless medium at the low-frequency regime. *Adv. Math. Phys.* **2018**, *2018*, 9587972. [CrossRef]
17. Chan, E.; Ney, M.M. Near-field scattering of a conducting sphere irradiated by a dipole. In Proceedings of the 1988 Symposium on Antenna Technology and Applied Electromagnetics, Winnipeg, MB, Canada, 10–12 August 1988.
18. Valagiannopoulos, C. Single-series solution to the radiation of loop antenna in the presence of a conducting sphere. *Prog. Electromagn. Res.* **2007**, *71*, 277–294. [CrossRef]
19. Moon, P.; Spencer, E. *Field Theory Handbook*; Springer: Berlin, Germany, 1971.
20. Hobson, E.W. *The Theory of Spherical and Ellipsoidal Harmonics*; Chelsea Publishing Company: New York, NY, USA, 1965.