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A Study of ψ-Hilfer Fractional Boundary Value Problem via Nonlinear Integral Conditions Describing Navier Model

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Abstract: This paper investigates existence, uniqueness, and Ulam's stability results for a nonlinear implicit ψ-Hilfer $\mathbb{FBVP}$ describing Navier model with $\mathbb{NIBC}$s. By Banach's fixed point theorem, the unique property is established. Meanwhile, existence results are proved by using the fixed point theory of Leray-Schauder's and Krasnoselskii's types. In addition, Ulam's stability results are analyzed. Furthermore, several instances are provided to demonstrate the efficacy of the main results.

Keywords: existence and uniqueness; ψ-Hilfer fractional derivative; fixed point theorem; Ulam-Hyers stability; nonlinear integral condition; ψ-Hilfer Navier problem

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1. Introduction

Hundreds of years ago, fractional calculus began and has been widely interested by researchers in branches of applied mathematics, science, engineering, and so on (see References [1–3]). It is also known as the non-integer order (fractional-order) of differential and integral operators. Various definitions of novel fractional integral and derivative operators are currently prominent tools in numerous publications. Normally, the real-world problems were simulated using differential equations and solved the difficulties using powerful techniques (see References [4,5]). The fractional calculus has been used to examine differential equations with non-integer order (fractional differential equations ($\mathbb{FDE}$s)). $\mathbb{FDE}$s via initial/boundary conditions have also been used to solve the problems since fractional-order has more additional degrees of freedom than integer-order, allowing for more precise and realistic solutions. Researchers have considered a variety of mathematical approaches in relation to $\mathbb{FDE}$s in a large number of papers (see References [6–19]).

Elastic beams are an essential element required in structural problems, including aircraft, ships, bridges, buildings, and so on (see References [20–36]). In the sense of mathematical analysis, the deformation of the beam can be analyzed using the fourth-order boundary value problem ($\mathbb{BVP}$) describing the Navier model [37]:

$$\begin{cases} u^{(4)}(\tau) = g(\tau, u(\tau), u''(\tau)), \quad \tau \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{1}$$

where $g \in \mathcal{C}([0,1] \times \mathbb{R}^2, \mathbb{R})$. Problem (1) has attracted the attention of many researchers due to its dominance in the field of mechanics. It simulates the bending equilibrium of a beam supported at both ends by an elastic basis. We will go through some important works on the subject shortly below. For instance, in 1986, Aftabizadeh [38] converted (1) into a second-order integro-differential equation with f is bounded on $[0,1] \times \mathbb{R}^2$. The existence results were analyzed by Schauder's fixed point theorem. In 1997, Ma et al. [39] examined the existence of a solution for (1) by applying the upper and lower solutions method. After that, in 2004, Bai et al. [40] developed upper and lower solutions of (1). Dang et al. [41] examined the problem (1) by reducing it to an operator equation and using some simply confirmed conditions. In recent years, many literature examples pay attention to $\mathbb{BVP}$s under many kinds of fractional derivatives; for instance, in 2020, Bachar and Eltayeb [42] studied the Navier $\mathbb{BVP}$ under Riemann-Liouville ($\mathbb{RL}$) fractional derivative type:

$$\begin{cases} {}^{\mathbb{RL}}\mathfrak{D}^{\alpha}_{0^+}\left({}^{\mathbb{RL}}\mathfrak{D}^{\beta}_{0^+}u\right)(\tau) = g(\tau, u(\tau), {}^{\mathbb{RL}}\mathfrak{D}^{\beta}_{0^+}u(\tau)), \quad \tau \in (0,1), \\ u(0) = {}^{\mathbb{RL}}\mathfrak{D}^{\beta}_{0^+}u(0) = u(1) = {}^{\mathbb{RL}}\mathfrak{D}^{\beta}_{0^+}u(1) = 0, \end{cases} \tag{2}$$

where ${}^{\mathbb{RL}}\mathfrak{D}^{q}_{0^+}$ denotes the $\mathbb{RL}$-fractional derivative of order $q = \{\alpha, \beta\} \in (1,2]$ and $g \in \mathcal{C}([0,1], \mathbb{R}^2)$. The Green properties and helpful inequality technique are used to establish the uniqueness result of positive solutions for (2). $\mathbb{FDE}$s have been discussed in depth by several researchers. Clearly, the existence, uniqueness, and stability analysis of solutions are some important properties of $\mathbb{FDE}$s. Because the exact solution to differential equations or $\mathbb{FDE}$s is quite difficult, several researchers have attempted to identify the best technique to access the existence results. To establish the existence and stability of solutions for $\mathbb{FDE}$s, several analytical techniques, including fixed-point theory, have been investigated. Ulam's stability is one of the most useful strategies which guarantee that there exists a close exact solution. Ulam's stability has four types, such as Ulam–Hyers ($\mathbb{UH}$), generalized Ulam–Hyers ($\mathbb{GUH}$), Ulam–Hyers–Rassias ($\mathbb{UHR}$), and generalized Ulam–Hyers–Rassias ($\mathbb{GUHR}$) stabilities; see References [43–54] and references cited therein. However, to the authors' knowledge, a few papers involving the Navier model in sense of ψ-Hilfer fractional operators have been concerned.

As a result of the preceding debates, we discuss a new class of nonlinear implicit ψ-Hilfer $\mathbb{FBVP}$ describing Navier model with nonlinear integral boundary conditions ($\mathbb{NIBC}$s):

$$\begin{cases} {}^{H}\mathfrak{D}^{\alpha,\rho;\psi}_{a^+}\left({}^{H}\mathfrak{D}^{\beta,\rho;\psi}_{a^+}u\right)(\tau) = f(\tau, u(\tau), (\mathcal{K}u)(\tau), (\mathcal{W}u)(\tau)), \quad \tau \in (a,b), \\ u(a) = 0, \qquad {}^{H}\mathfrak{D}^{\beta,\rho;\psi}_{a^+}u(a) = 0, \\ \sum_{i=1}^{m} \xi_i u(\eta_i) = \mathcal{I}^{\varphi;\psi}_{a^+}\mathcal{G}(\sigma, u(\sigma)), \qquad \sum_{j=1}^{n} \mu_j {}^{H}\mathfrak{D}^{\phi_j,\rho;\psi}_{a^+}u(\lambda_j) = \mathcal{I}^{\nu;\psi}_{a^+}\mathcal{H}(\zeta, u(\zeta)), \end{cases} \tag{3}$$

where ${}^{H}\mathfrak{D}^{q,\rho;\psi}_{a^+}$ denotes ψ-Hilfer fractional derivative of order $q = \{\alpha, \beta, \phi_j\}$, α, β, $\phi_j \in (1,2]$, $j = 1,2,\ldots,n$, $\rho \in [0,1]$, $\mathcal{I}^{v;\psi}_{a^+}$ denotes ψ-$\mathbb{RL}$-fractional integral of order $v = \{\varphi, \nu\} > 0$, $f \in \mathcal{C}(\mathcal{J} \times \mathbb{R}^3, \mathbb{R})$, $\mathcal{G}$, $\mathcal{H} \in \mathcal{C}(\mathcal{J}, \mathbb{R})$, $\mathcal{J} := [a,b]$, $b > a > 0$, σ, ζ, ξ_i, $\mu_j \in \mathbb{R}$, σ, ζ, η_i, $\lambda_j \in (a,b)$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$, and

$$(\mathcal{K}u)(\tau) = \frac{1}{\Gamma(\theta)}\int_a^\tau (\psi(\tau) - \psi(s))^{\theta-1}\psi'(s)k(\tau,s)u(s)ds, \quad \tau \in \mathcal{J}, \tag{4}$$

$$(\mathcal{W}u)(\tau) = \frac{1}{\Gamma(\delta)}\int_a^\tau (\psi(\tau) - \psi(s))^{\delta-1}\psi'(s)w(\tau,s)u(s)ds, \quad \tau \in \mathcal{J}, \tag{5}$$

where k, $w \in \mathcal{C}(\mathcal{J}^2, [a,\infty))$. The existence and uniqueness property is proved by using Banach's fixed point theorem (Lemma 5), and the existence properties are derived by applying Leray-Schauder's nonlinear alternative (Lemma 8) and Krasnoselskii's fixed point theorems (Lemma 9) for the ψ-Hilfer $\mathbb{FBVP}$ describing Navier model with $\mathbb{NIBC}$s (3). We employ $\mathbb{UH}$, $\mathbb{GUH}$, $\mathbb{UHR}$, and $\mathbb{GUHR}$ stables to investigate the stability of (3). Finally, we give some numerical examples of various functions that were explored in order to confirm the theoretical results. In addition, we give our findings on a broad platform that covers a wide area of specific situations for different values ρ and ψ. For example, ψ-Riemann-Liouville problem if $\rho = 0$, ψ-Caputo problem if $\rho = 1$, Riemann-Liouville $\mathbb{BVP}$ if $\rho = 0$, $\psi(\tau) = \tau$, Caputo problem if $\rho = 1$, $\psi(\tau) = \tau$, Hilfer problem if $\psi(\tau) = \tau$, Katugampola problem if $\psi(\tau) = \tau^q$, Hilfer-Hadamard problem if $\psi(\tau) = \log(\tau)$, and so on. The received results are improved: if $\alpha = 2$, $\beta = 2$, $\rho = 1$, and $\psi(t) = t$, then we obtained Reference [41]; if $\rho = 1$ and $\psi(t) = t$, then we obtained Reference [42].

This paper is structured the continuing parts of the paper as follows: In Section 2, we provide an essential system of symbols, definitions, and lemmas of ψ-Hilfer fractional calculus. Next, we state a lemma which is used in proving the main results. In Section 3, fixed point theorems are used to obtain the existence results of the proposed problem. By helping with the nonlinear analysis method, in Section 4, we analyze various of Ulam's stability for the problem. Examples illustrate to confirm the effectiveness of the acquired theoretical results in Section 5. Finally, the conclusion and discussion of this paper are presented in Section 6.

2. Preliminaries

We provide the basic concepts of ψ-Hilfer fractional calculus, as well as important crucial results that will be engaged in this paper. Assume that $\mathcal{E} = \mathcal{C}(\mathcal{J}, \mathbb{R})$ is the Banach space of continuous functions on $\mathcal{J}$ with $\|u\| = \sup_{\tau\in\mathcal{J}}\{|u(\tau)|\}$. Assume that $\mathcal{AC}^n(\mathcal{J}, \mathbb{R})$ is the space of n-times absolutely continuous functions with $\mathcal{AC}^n(\mathcal{J}, \mathbb{R}) = \{u : \mathcal{J} \to \mathbb{R}; u^{(n-1)} \in \mathcal{AC}(\mathcal{J}, \mathbb{R})\}$.

Definition 1. (*Reference [3]*). *Assume that $\psi(\tau) \in \mathcal{C}^1(\mathcal{J}, \mathbb{R})$ is an increasing function with $\psi'(\tau) \neq 0$ for each $\tau \in \mathcal{J}$. The ψ-$\mathbb{RL}$-fractional integral of order α of f depending on ψ on $\mathcal{J}$ is defined by*

$$\mathcal{I}_{a^+}^{\alpha;\psi} f(\tau) = \frac{1}{\Gamma(\alpha)} \int_a^\tau (\psi(\tau) - \psi(s))^{\alpha-1}\psi'(s)f(s)ds, \quad \tau > a > 0, \quad \alpha > 0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2. (*Reference [3]*). *Assume that $\psi(\tau)$ is defined as in Definition 1 with $\psi'(\tau) \neq 0$. The ψ-$\mathbb{RL}$-fractional derivative of f depending on ψ is defined as $\mathfrak{D}_{a^+}^{\alpha;\psi} f(\tau) = \left(\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\right)^n \mathcal{I}_{a^+}^{n-\alpha;\psi} f(\tau)$ or*

$$\mathfrak{D}_{a^+}^{\alpha;\psi} f(\tau) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\right)^n \int_a^\tau (\psi(\tau) - \psi(s))^{n-\alpha-1}\psi'(s)f(s)ds, \quad \alpha > 0,$$

where $n = [\alpha] + 1$, and $[\alpha]$ is an integer part of $Re(\alpha)$.

Definition 3. (*Reference [55]*). *Assume that $\gamma = \alpha + \rho(n - \alpha)$, $\alpha \in (n - 1, n)$ with $n \in \mathbb{N}$, $f \in \mathcal{C}^n(\mathcal{J}, \mathbb{R})$ and $\psi(\tau) \in \mathcal{C}^1(\mathcal{J}, \mathbb{R})$ is increasing with $\psi'(\tau) \neq 0$ for each $\tau \in \mathcal{J}$. Then, the ψ-Hilfer fractional derivative of type $\rho \in [0, 1]$ of f depending on ψ is defined as*

$${}^H\mathfrak{D}_{a^+}^{\alpha,\rho;\psi} f(\tau) = \mathcal{I}_{a^+}^{\rho(n-\alpha);\psi} \left(\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\right)^n \mathcal{I}_{a^+}^{(1-\rho)(n-\alpha);\psi} f(\tau) = \mathcal{I}_{a^+}^{\gamma-\alpha;\psi}\mathfrak{D}_{a^+}^{\gamma;\psi} f(\tau),$$

where $\mathfrak{D}_{a^+}^{\gamma;\psi} f(\tau) = \mathfrak{D}_{a^+}^{n;\psi}\mathcal{I}_{a^+}^{(1-\rho)(n-\alpha);\psi} f(\tau)$.

Lemma 1. (*Reference [3]*). *Assume that $\alpha, \beta > 0$. Then, $\mathcal{I}_{a^+}^{\alpha;\psi}\mathcal{I}_{a^+}^{\beta;\psi} f(\tau) = \mathcal{I}_{a^+}^{\alpha+\beta;\psi} f(\tau)$, $\tau > a$.*

Proposition 1. (*References [3,55]*). *Assume that $\tau > a$ and $\mathcal{G}^{v}(\tau) = (\psi(\tau) - \psi(a))^{v}$. Then, for $v > 0$, $\alpha \geq 0$, we have*

(i) $\mathcal{I}_{a^+}^{\alpha;\psi}\mathcal{G}^{v-1}(\tau) = \frac{\Gamma(v)}{\Gamma(v+\alpha)}\mathcal{G}^{v+\alpha-1}(\tau)$;

(ii) $\mathfrak{D}_{a^+}^{\alpha,\rho;\psi}\mathcal{G}^{v-1}(\tau) = \frac{\Gamma(v)}{\Gamma(v-\alpha)}\mathcal{G}^{v-\alpha-1}(\tau)$;

(iii) ${}^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi}\mathcal{G}^{v-1}(\tau) = \frac{\Gamma(v)}{\Gamma(v-\alpha)}\mathcal{G}^{v-\alpha-1}(\tau), \quad v > \gamma = \alpha + \rho(n-\alpha)$.

Lemma 2. (*Reference [55]*) *Assume that $f \in \mathcal{C}^{n}(\mathcal{J},\mathbb{R})$, $\alpha \in (n-1,n)$, $\rho \in [0,1]$, $\gamma = \alpha + \rho(n-\alpha)$. Then, we obtain*

$$\mathcal{I}_{a^+}^{\alpha;\psi}\,{}^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi}f(\tau) = f(\tau) - \sum_{k=1}^{n}\frac{(\psi(\tau)-\psi(a))^{q-k}}{\Gamma(q-k+1)}f_{\psi}^{[n-k]}\mathcal{I}_{a^+}^{(1-\rho)(n-\alpha);\psi}f(a),$$

for all $\tau \in \mathcal{J}$, where $f_{\psi}^{[n]}f(\tau) := \left(\frac{1}{\psi'(\tau)}\frac{d}{dt}\right)^{n}f(\tau)$.

Lemma 3. (*Reference [51]*) *Assuming that $\alpha \in (m-1,m)$, $\beta \in (n-1,n)$, $n, m \in \mathbb{N}$, $n \leq m$, $\rho \in [0,1]$, and $\alpha > \beta + \rho(n-\beta)$. If $f \in \mathcal{C}_{1-\gamma,\psi}(\mathcal{J},\mathbb{R})$, then ${}^{H}\mathfrak{D}_{a^+}^{\beta,\rho;\psi}\mathcal{I}_{a^+}^{\alpha;\psi}f(\varsigma) = \mathcal{I}_{0^+}^{\alpha-\beta;\psi}f(\varsigma)$.*

Lemma 4. *Let α, β, $\phi_j \in (1,2]$, $(j = 1,2,\ldots,n)$, $\rho \in [0,1]$, $\gamma_1 = \alpha + \rho(2-\alpha)$, $\gamma_2 = \beta + \rho(2-\beta)$, ω, φ, $\nu > 0$. Suppose that $h \in \mathcal{E}$ and $\Omega = \Omega_{11}\Omega_{22} - \Omega_{12}\Omega_{21} \neq 0$. Then, $u \in \mathcal{C}^{2}(\mathcal{J},\mathbb{R})$ is a solution of*

$$\begin{cases} {}^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi}\left({}^{H}\mathfrak{D}_{a^+}^{\beta,\rho;\psi}u\right)(\tau) = h(\tau), \quad \tau \in (a,b), \\ u(a) = 0, \qquad {}^{H}\mathfrak{D}_{a^+}^{\beta,\rho;\psi}u(a) = 0, \\ \sum_{i=1}^{m}\xi_i u(\eta_i) = \mathcal{I}_{a^+}^{\varphi;\psi}\mathcal{G}(\sigma,u(\sigma)), \qquad \sum_{j=1}^{n}\mu_j\,{}^{H}\mathfrak{D}_{a^+}^{\phi_j,\rho;\psi}u(\lambda_j) = \mathcal{I}_{a^+}^{\nu;\psi}\mathcal{H}(\zeta,u(\zeta)), \end{cases} \tag{6}$$

if and only if u verifies the integral equation

$$\begin{aligned} u(\tau) &= \mathcal{I}_{a^+}^{\alpha+\beta;\psi}h(\tau) + \frac{(\psi(\tau)-\psi(a))^{\gamma_1+\beta-1}}{\Omega\Gamma(\gamma_1+\beta)}\Bigg[\Omega_{22}\left(\mathcal{I}_{a^+}^{\varphi;\psi}\mathcal{G}(\sigma,u(\sigma)) - \textstyle\sum_{i=1}^{m}\xi_i\mathcal{I}_{a^+}^{\alpha+\beta;\psi}h(\eta_i)\right) \\ &\quad -\Omega_{12}\left(\mathcal{I}_{a^+}^{\nu;\psi}\mathcal{H}(\zeta,u(\zeta)) - \textstyle\sum_{j=1}^{n}\mu_j\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}h(\lambda_j)\right)\Bigg] \\ &\quad +\frac{(\psi(\tau)-\psi(a))^{\gamma_2-1}}{\Omega\Gamma(\gamma_2)}\Bigg[\Omega_{11}\left(\mathcal{I}_{a^+}^{\nu;\psi}\mathcal{H}(\zeta,u(\zeta)) - \textstyle\sum_{j=1}^{n}\mu_j\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}h(\lambda_j)\right) \\ &\quad -\Omega_{21}\left(\mathcal{I}_{a^+}^{\varphi;\psi}\mathcal{G}(\sigma,u(\sigma)) - \textstyle\sum_{i=1}^{m}\xi_i\mathcal{I}_{a^+}^{\alpha+\beta;\psi}h(\eta_i)\right)\Bigg], \end{aligned} \tag{7}$$

where

$$\Omega_{11} = \sum_{i=1}^{m}\frac{\xi_i(\psi(\eta_i)-\psi(a))^{\gamma_1+\beta-1}}{\Gamma(\gamma_1+\beta)}, \qquad \Omega_{12} = \sum_{i=1}^{m}\frac{\xi_i(\psi(\eta_i)-\psi(a))^{\gamma_2-1}}{\Gamma(\gamma_2)}, \tag{8}$$

$$\Omega_{21} = \sum_{j=1}^{n}\frac{\mu_j(\psi(\lambda_j)-\psi(a))^{\gamma_1+\beta-\phi_j-1}}{\Gamma(\gamma_1+\beta-\phi_j)}, \quad \Omega_{22} = \sum_{j=1}^{n}\frac{\mu_j(\psi(\lambda_j)-\psi(a))^{\gamma_2-\phi_j-1}}{\Gamma(\gamma_2-\phi_j)}. \tag{9}$$

Proof. Suppose that $u \in \mathcal{E}$ is the solution of (6). Taking $\mathcal{I}_{a^+}^{\alpha;\psi}$ into both sides of (6) via Lemma 2, we obtain

$$ {}^H\mathfrak{D}_{a^+}^{\beta,\rho;\psi}u(\tau) = \mathcal{I}_{a^+}^{\alpha;\psi}h(\tau) + \frac{(\psi(\tau)-\psi(a))^{\gamma_1-1}}{\Gamma(\gamma_1)}c_1 + \frac{(\psi(\tau)-\psi(a))^{\gamma_1-2}}{\Gamma(\gamma_1-1)}c_2, \tag{10} $$

where $c_1, c_2 \in \mathbb{R}$.

From the boundary condition ${}^H\mathfrak{D}_{0^+}^{\beta,\rho;\psi}u(a) = 0$, we get $c_2 = 0$. Taking $\mathcal{I}_{a^+}^{\beta;\psi}$ into (10) via Lemma 2 again, it follows that

$$ \begin{aligned} u(\tau) &= \mathcal{I}_{a^+}^{\alpha+\beta;\psi}h(\tau) + \frac{(\psi(\tau)-\psi(a))^{\gamma_1+\beta-1}}{\Gamma(\gamma_1+\beta)}c_1 \\ &\quad + \frac{(\psi(\tau)-\psi(a))^{\gamma_2-1}}{\Gamma(\gamma_2)}c_3 + \frac{(\psi(\tau)-\psi(a))^{\gamma_2-2}}{\Gamma(\gamma_2-1)}c_4, \end{aligned} \tag{11} $$

where $c_3, c_4 \in \mathbb{R}$. From condition $x(a) = 0$, it is implied that $c_4 = 0$. So,

$$ u(\tau) = \mathcal{I}_{a^+}^{\alpha+\beta;\psi}h(\tau) + \frac{(\psi(\tau)-\psi(a))^{\gamma_1+\beta-1}}{\Gamma(\gamma_1+\beta)}c_1 + \frac{(\psi(\tau)-\psi(a))^{\gamma_2-1}}{\Gamma(\gamma_2)}c_3. \tag{12} $$

Taking ${}^H\mathfrak{D}_{a^+}^{\phi_j,\rho;\psi}$ into (12), then

$$ {}^H\mathfrak{D}_{a^+}^{\phi_j,\rho;\psi}u(\tau) = \mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}h(\tau) + \frac{(\psi(\tau)-\psi(a))^{\gamma_1+\beta-\phi_j-1}}{\Gamma(\gamma_1+\beta-\phi_j)}c_1 + \frac{(\psi(\tau)-\psi(a))^{\gamma_2-\phi_j-1}}{\Gamma(\gamma_2-\phi_j)}c_3. $$

By using boundary conditions in (6), we get

$$ \Omega_{11}c_1 + \Omega_{12}c_3 = \mathcal{I}_{a^+}^{\varphi;\psi}\mathcal{G}(\sigma,u(\sigma)) - \sum_{i=1}^{m}\xi_i\mathcal{I}_{a^+}^{\alpha+\beta;\psi}h(\eta_i), \tag{13} $$

$$ \Omega_{21}c_1 + \Omega_{22}c_3 = \mathcal{I}_{a^+}^{\nu;\psi}\mathcal{H}(\zeta,u(\zeta)) - \sum_{j=1}^{n}\mu_j\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}h(\lambda_j), \tag{14} $$

where Ω_{11}, Ω_{12}, Ω_{21}, and Ω_{22} are given by (8) and (9). Solving (13) and (14), we have

$$ \begin{aligned} c_1 &= \frac{1}{\Omega}\Bigg[\Omega_{22}\left(\mathcal{I}_{a^+}^{\varphi;\psi}\mathcal{G}(\sigma,x(\sigma)) - \sum_{i=1}^{m}\xi_i\mathcal{I}_{a^+}^{\alpha+\beta;\psi}h(\eta_i)\right) \\ &\quad -\Omega_{12}\left(\mathcal{I}_{a^+}^{\nu;\psi}\mathcal{H}(\zeta,u(\zeta)) - \sum_{j=1}^{n}\mu_j\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}h(\lambda_j)\right)\Bigg], \\ c_3 &= \frac{1}{\Omega}\Bigg[\Omega_{11}\left(\mathcal{I}_{a^+}^{\nu;\psi}\mathcal{H}(\zeta,u(\zeta)) - \sum_{j=1}^{n}\mu_j\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}h(\lambda_j)\right) \\ &\quad -\Omega_{21}\left(\mathcal{I}_{a^+}^{\varphi;\psi}\mathcal{G}(\sigma,u(\sigma)) - \sum_{i=1}^{m}\xi_i\mathcal{I}_{a^+}^{\alpha+\beta;\psi}h(\eta_i)\right)\Bigg]. \end{aligned} $$

Hence, the solution u follows by using c_1 and c_3 in (10). This yields that $u(\tau)$ verifies (7). On the other hand, in a direct way, we can show that $u(\tau)$ is defined by (7) satisfies (6) under nonlinear integral boundary conditions. □

3. Existence Results

Setting the symbol

$$ \mathcal{I}_{a^+}^{q;\psi}\mathcal{F}_u(c) = \frac{1}{\Gamma(q)}\int_a^c(\psi(c)-\psi(s))^{q-1}\psi'(s)\mathcal{F}_u(s)ds, $$

where $\mathcal{F}_u(\tau) = f(\tau, u(\tau), (\mathcal{K}u)(\tau), (\mathcal{W}u)(\tau))$ with $q \in \{\alpha+\beta, \varphi, \nu, \alpha+\beta-\phi_j\}$, $c = \{\tau, \sigma, \zeta, \eta_i, \lambda_j, b\}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. Thanks to Lemma 4, we determine $\mathcal{Q} : \mathcal{E} \to \mathcal{E}$

$$\begin{aligned}(\mathcal{Q}u)(\tau) &= \mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_u(\tau) + \frac{(\psi(\tau)-\psi(a))^{\gamma_1+\beta-1}}{\Omega\Gamma(\gamma_1+\beta)}\Bigg[\Omega_{22}\Big(\mathcal{I}_{a^+}^{\varphi;\psi}\mathcal{G}(\sigma, u(\sigma)) - \textstyle\sum_{i=1}^m \xi_i \mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_u(\eta_i)\Big) \\ &\quad -\Omega_{12}\Big(\mathcal{I}_{a^+}^{\nu;\psi}\mathcal{H}(\zeta, u(\zeta)) - \textstyle\sum_{j=1}^n \mu_j \mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}\mathcal{F}_u(\lambda_j)\Big)\Bigg] \\ &\quad + \frac{(\psi(\tau)-\psi(a))^{\gamma_2-1}}{\Omega\Gamma(\gamma_2)}\Bigg[\Omega_{11}\Big(\mathcal{I}_{a^+}^{\nu;\psi}\mathcal{H}(\zeta, u(\zeta)) - \textstyle\sum_{j=1}^n \mu_j \mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}\mathcal{F}_u(\lambda_j)\Big) \\ &\quad -\Omega_{21}\Big(\mathcal{I}_{a^+}^{\varphi;\psi}\mathcal{G}(\sigma, u(\sigma)) - \textstyle\sum_{i=1}^m \xi_i \mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_u(\eta_i)\Big)\Bigg].\end{aligned} \tag{15}$$

Clearly, $\mathcal{Q}$ has fixed points if and only if problem (3) has solutions. To simplify,

$$\Psi^q(a, v) = \frac{(\psi(v)-\psi(a))^q}{\Gamma(q+1)}, \tag{16}$$

$$\Phi(A, B) = \frac{1}{|\Omega|}\Big(|A|\Psi^{\gamma_1+\beta-1}(b) + |B|\Psi^{\gamma_2-1}(b)\Big), \tag{17}$$

$$\begin{aligned}\Lambda(q) &= \Psi^q(b) + \Phi(\Omega_{12}, \Omega_{11})\sum_{j=1}^n |\mu_j|\Psi^{q-\phi_j}(\lambda_j) \\ &\quad + \Phi(\Omega_{22}, \Omega_{21})\sum_{i=1}^m |\xi_i|\Psi^q(\eta_i).\end{aligned} \tag{18}$$

3.1. Uniqueness Property via Banach's Fixed Point Theorem

Lemma 5. (*Banach's fixed point theorem [56]*). *Assume that X is a non-empty closed subset of $\mathcal{E}$, where $\mathcal{E}$ is a Banach space. Then, any contraction mapping $\mathcal{Q}$ from $\mathcal{E}$ into itself has a unique fixed point.*

Theorem 1. *Assume that $f \in \mathcal{C}(\mathcal{J} \times \mathbb{R}^3, \mathbb{R})$ and $k \in \mathcal{C}(\mathcal{J}^2 \times \mathbb{R}, \mathbb{R})$ verifies the conditions:*
($\mathcal{P}_1$) There exist constants $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3 > 0$ with $\mathcal{L}_2 < 1$ such that

$$|f(\tau, u_1, v_1, w_1) - f(\tau, u_2, v_2, w_2)| \le \mathcal{L}_1|u_1 - u_2| + \mathcal{L}_2|v_1 - v_2| + \mathcal{L}_3|w_1 - w_2|,$$

for any u_i, v_i, $w_i \in \mathbb{R}$, $i = 1, 2$, $\tau \in \mathcal{J}$.
($\mathcal{P}_2$) There exist constants $\mathcal{H}_1^*$, $\mathcal{G}_1^* > 0$ such that

$$|\mathcal{H}(\tau, u_1) - \mathcal{H}(\tau, u_2)| \le \mathcal{H}_1^*|u_1 - u_2| \quad \text{and} \quad |\mathcal{G}(\tau, u_1) - \mathcal{G}(\tau, u_2)| \le \mathcal{G}_1^*|u_1 - u_2|,$$

for any $u_i \in \mathbb{R}$, $i = 1, 2$, $\tau \in \mathcal{J}$.
If

$$\Delta_1 + \Delta_2 < 1, \tag{19}$$

where

$$\Delta_1 = \Lambda(\alpha+\beta)\mathcal{L}_1 + \Lambda(\theta+\alpha+\beta)\mathcal{L}_2 k_1^* + \Lambda(\delta+\alpha+\beta)\mathcal{L}_3 w_1^*, \tag{20}$$

$$\Delta_2 = \Phi(\Omega_{22}, \Omega_{21})\Psi^{\varphi}(\sigma)\mathcal{G}_1^* + \Phi(\Omega_{12}, \Omega_{11})\Psi^{\nu}(\zeta)\mathcal{H}_1^*, \tag{21}$$

then the ψ-Hilfer $\mathbb{FBVP}$ describing Navier model with $\mathbb{NIBC}$s (3) has a unique solution $u \in \mathcal{E}$.

Proof. The problem (3) will transform to $u = \mathcal{Q}u$, where $\mathcal{Q}$ is given by (15). Clearly, the fixed points of $\mathcal{Q}$ are the possible solutions of (3). By applying Lemma 5, we will guarantee that $\mathcal{Q}$ has a unique fixed point, which implies that (3) has a unique solution. Define a bounded, closed, and convex subset $\mathcal{B}_{r_1} := \{u \in \mathcal{E} : \|u\| \le r_1\}$ with

$$r_1 \ge \frac{\Lambda(\alpha+\beta)\mathbb{F}_1 + \Phi(\Omega_{22},\Omega_{21})\Psi^{\varphi}(\sigma)\mathbb{G}_1 + \Phi(\Omega_{12},\Omega_{11})\Psi^{\nu}(\zeta)\mathbb{H}_1}{1-(\Delta_1+\Delta_2)}, \tag{22}$$

where Δ_i for $i=1,2$ are given by (20) and (21). Assume that $\sup_{\tau\in\mathcal{J}}|f(\tau,0,0,0)| := \mathbb{F}_1 < \infty$, $\sup_{\tau\in\mathcal{J}}|\mathcal{H}(\tau,0)| := \mathbb{H}_1 < \infty$, and $\sup_{\tau\in\mathcal{J}}|\mathcal{G}(\tau,0)| := \mathbb{G}_1 < \infty$.

Step I. $\mathcal{Q}\mathcal{B}_{r_1} \subset \mathcal{B}_{r_1}$.

Let $u \in \mathcal{B}_{r_1}$, $\tau \in \mathcal{J}$. Then,

$$\begin{aligned} |(\mathcal{Q}u)(\tau)| \le\; & \mathcal{I}_{a^+}^{\alpha+\beta;\psi}|\mathcal{F}_u(b)| + \tfrac{(\psi(b)-\psi(a))^{\gamma_1+\beta-1}}{|\Omega|\Gamma(\gamma_1+\beta)}\Bigg[|\Omega_{22}|\Bigg(\mathcal{I}_{a^+}^{\varphi;\psi}|\mathcal{G}(\sigma,u(\sigma))| \\ & + \textstyle\sum_{i=1}^m|\xi_i|\mathcal{I}_{a^+}^{\alpha+\beta;\psi}|\mathcal{F}_x(\eta_i)|\Bigg) + |\Omega_{12}|\Big(\mathcal{I}_{a^+}^{\nu;\psi}|\mathcal{H}(\zeta,u(\zeta))| + \textstyle\sum_{j=1}^n|\mu_j|\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}|\mathcal{F}_u(\lambda_j)|\Big)\Bigg] \\ & + \tfrac{(\psi(b)-\psi(a))^{\gamma_2-1}}{|\Omega|\Gamma(\gamma_2)}\Bigg[|\Omega_{11}|\Big(\mathcal{I}_{a^+}^{\nu;\psi}|\mathcal{H}(\zeta,u(\zeta))| + \textstyle\sum_{j=1}^n|\mu_j|\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}|\mathcal{F}_u(\lambda_j)|\Big) \\ & + |\Omega_{21}|\Big(\mathcal{I}_{a^+}^{\varphi;\psi}|\mathcal{G}(\sigma,u(\sigma))| + \textstyle\sum_{i=1}^m|\xi_i|\mathcal{I}_{a^+}^{\alpha+\beta;\psi}|\mathcal{F}_u(\eta_i)|\Big)\Bigg]. \end{aligned} \tag{23}$$

By using (i) in Proposition 1, we get

$$\begin{aligned} \mathcal{I}_{a^+}^{q;\psi}|u(\tau)| &= \frac{1}{\Gamma(q)}\int_a^\tau (\psi(\tau)-\psi(s))^{q-1}\psi'(s)|u(s)|ds \\ &\le \frac{(\psi(\tau)-\psi(a))^q}{\Gamma(q+1)}\|u\| = \Psi^q(\tau)\|u\|. \end{aligned} \tag{24}$$

Thanks to (24) with $k_1^* = \sup_{(\tau,s)\in\mathcal{J}\times\mathcal{J}}\{|k(\tau,s)|\}$ and $w_1^* = \sup_{(\tau,s)\in\mathcal{J}\times\mathcal{J}}\{|w(\tau,s)|\}$, this yields that

$$|(\mathcal{K}u)(\tau)| \le \frac{1}{\Gamma(\theta)}\int_a^\tau (\psi(\tau)-\psi(s))^{\theta-1}\psi'(s)|k(\tau,s)||u(s)|ds \le k_1^*\Psi^\theta(\tau)\|u\|, \tag{25}$$

$$|(\mathcal{W}u)(\tau)| \le \frac{1}{\Gamma(\delta)}\int_a^\tau (\psi(\tau)-\psi(s))^{\delta-1}\psi'(s)|w(\tau,s)||u(s)|ds \le w_1^*\Psi^\delta(\tau)\|u\|. \tag{26}$$

From the conditions $(\mathcal{P}_1)$, $(\mathcal{P}_2)$ and (24)–(26), we can estimate

$$\begin{aligned} |\mathcal{F}_u(\tau)| &\le |f(\tau,u(\tau),(\mathcal{K}u)(\tau),(\mathcal{W}u)(\tau)) - f(\tau,0,0,0)| + |f(\tau,0,0,0)| \\ &\le \mathcal{L}_1|u(\tau)| + \mathcal{L}_2|(\mathcal{K}u)(\tau)| + \mathcal{L}_3|(\mathcal{W}u)(\tau)| + \mathbb{F}_1 \\ &\le \mathcal{L}_1\|u\| + \mathcal{L}_2k_1^*\Psi^\theta(\tau)\|u\| + \mathcal{L}_3w_1^*\Psi^\delta(\tau)\|u\| + \mathbb{F}_1 \\ &= \big(\mathcal{L}_1 + \mathcal{L}_2k_1^*\Psi^\theta(\tau) + \mathcal{L}_3w_1^*\Psi^\delta(\tau)\big)\|u\| + \mathbb{F}_1, \end{aligned} \tag{27}$$

$$|\mathcal{H}(\tau,u(\tau))| \le |\mathcal{H}(\tau,u(\tau)) - \mathcal{H}(\tau,0)| + |\mathcal{H}(\tau,0)| \le \mathcal{H}_1^*\|u\| + \mathbb{H}_1, \tag{28}$$

$$|\mathcal{G}(\tau,u(\tau))| \le |\mathcal{G}(\tau,u(\tau)) - \mathcal{G}(\tau,0)| + |\mathcal{G}(\tau,0)| \le \mathcal{G}_1^*\|u\| + \mathbb{G}_1. \tag{29}$$

By (27)–(29) via (i) of Proposition 1, we have

$$\begin{aligned} \mathcal{I}_{a^+}^{\alpha+\beta;\psi}|\mathcal{F}_u(b)| \le\; & \big(\mathcal{L}_1\Psi^{\alpha+\beta}(b) + \mathcal{L}_2k_1^*\Psi^{\theta+\alpha+\beta}(b) + \mathcal{L}_3w_1^*\Psi^{\delta+\alpha+\beta}(b)\big)\|u\| \\ & + \mathbb{F}_1\Psi^{\alpha+\beta}(b), \end{aligned} \tag{30}$$

$$\begin{aligned} \mathcal{I}_{a^+}^{\alpha+\beta;\psi}|\mathcal{F}_u(\eta_i)| \le\; & \big(\mathcal{L}_1\Psi^{\alpha+\beta}(\eta_i) + \mathcal{L}_2k_1^*\Psi^{\theta+\alpha+\beta}(\eta_i) + \mathcal{L}_3w_1^*\Psi^{\delta+\alpha+\beta}(\eta_i)\big)\|u\| \\ & + \mathbb{F}_1\Psi^{\alpha+\beta}(\eta_i), \end{aligned} \tag{31}$$

$$\begin{aligned}\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}|\mathcal{F}_u(\lambda_j)| \leq & \Big(\mathcal{L}_1\Psi^{\alpha+\beta-\phi_j}(\lambda_j)+\mathcal{L}_2k_1^*\Psi^{\theta+\alpha+\beta-\phi_j}(\lambda_j) \\ & +\mathcal{L}_3w_1^*\Psi^{\delta+\alpha+\beta-\phi_j}(\lambda_j)\Big)\|u\|+\mathbb{F}_1\Psi^{\alpha+\beta-\phi_j}(\lambda_j),\end{aligned} \tag{32}$$

$$\mathcal{I}_{a^+}^{\nu;\psi}|\mathcal{H}(\zeta,u(\zeta))| \leq \mathcal{H}_1^*\|u\|\Psi^{\nu}(\zeta)+\mathbb{H}_1\Psi^{\nu}(\zeta), \tag{33}$$

$$\mathcal{I}_{a^+}^{\varphi;\psi}|\mathcal{G}(\sigma,u(\sigma))| \leq \mathcal{G}_1^*\|u\|\Psi^{\varphi}(\sigma)+\mathbb{G}_1\Psi^{\varphi}(\sigma). \tag{34}$$

Substituting (30)–(34) into (23), we obtain

$$\begin{aligned}
& |(\mathcal{Q}u)(\tau)| \\
\leq & \Big(\mathcal{L}_1\Psi^{\alpha+\beta}(b)+\mathcal{L}_2k_1^*\Psi^{\theta+\alpha+\beta}(b)+\mathcal{L}_3w_1^*\Psi^{\delta+\alpha+\beta}(b)\Big)\|u\|+\mathbb{F}_1\Psi^{\alpha+\beta}(b) \\
& +\frac{\Psi^{\gamma_1+\beta-1}(b)}{|\Omega|}\Bigg[|\Omega_{22}|\Bigg(\mathcal{G}_1^*\|u\|\Psi^{\varphi}(\sigma)+\mathbb{G}_1\Psi^{\varphi}(\sigma)+\sum_{i=1}^{m}|\xi_i|\bigg\{\Big(\mathcal{L}_1\Psi^{\alpha+\beta}(\eta_i) \\
& +\mathcal{L}_2k_1^*\Psi^{\theta+\alpha+\beta}(\eta_i)+\mathcal{L}_3w_1^*\Psi^{\delta+\alpha+\beta}(\eta_i)\Big)\|u\|+\mathbb{F}_1\Psi^{\alpha+\beta}(\eta_i)\bigg\}\Bigg) \\
& +|\Omega_{12}|\Bigg(\mathbb{H}_1\Psi^{\nu}(\zeta)+\mathcal{H}_1^*\|u\|\Psi^{\nu}(\zeta)+\sum_{j=1}^{n}|\mu_j|\bigg\{\Big(\mathcal{L}_1\Psi^{\alpha+\beta-\phi_j}(\lambda_j) \\
& +\mathcal{L}_2k_1^*\Psi^{\theta+\alpha+\beta-\phi_j}(\lambda_j)+\mathcal{L}_3w_1^*\Psi^{\delta+\alpha+\beta-\phi_j}(\lambda_j)\Big)\|u\|+\mathbb{F}_1\Psi^{\alpha+\beta-\phi_j}(\lambda_j)\bigg\}\Bigg)\Bigg] \\
& +\frac{\Psi^{\gamma_2-1}(b)}{|\Omega|}\Bigg[|\Omega_{11}|\Bigg(\mathcal{H}_1^*\|u\|\Psi^{\nu}(\zeta)+\mathbb{H}_1\Psi^{\nu}(\zeta)+\sum_{j=1}^{n}|\mu_j|\bigg\{\Big(\mathcal{L}_1\Psi^{\alpha+\beta-\phi_j}(\lambda_j) \\
& +\mathcal{L}_2k_1^*\Psi^{\theta+\alpha+\beta-\phi_j}(\lambda_j)+\mathcal{L}_3w_1^*\Psi^{\delta+\alpha+\beta-\phi_j}(\lambda_j)\Big)\|u\|+\mathbb{F}_1\Psi^{\alpha+\beta-\phi_j}(\lambda_j)\bigg\}\Bigg) \\
& +|\Omega_{21}|\Bigg(\mathcal{G}_1^*\|u\|\Psi^{\varphi}(\sigma)+\mathbb{G}_1\Psi^{\varphi}(\sigma)+\sum_{i=1}^{m}|\xi_i|\bigg\{\Big(\mathcal{L}_1\Psi^{\alpha+\beta}(\eta_i) \\
& +\mathcal{L}_2k_1^*\Psi^{\theta+\alpha+\beta}(\eta_i)+\mathcal{L}_3w_1^*\Psi^{\delta+\alpha+\beta}(\eta_i)\Big)\|u\|+\mathbb{F}_1\Psi^{\alpha+\beta}(\eta_i)\bigg\}\Bigg)\Bigg] \\
\leq & \Bigg(\Psi^{\alpha+\beta}(b)+\frac{1}{|\Omega|}\Big(|\Omega_{12}|\Psi^{\gamma_1+\beta-1}(b)+|\Omega_{11}|\Psi^{\gamma_2-1}(b)\Big)\sum_{j=1}^{n}|\mu_j|\Psi^{\alpha+\beta-\phi_j}(\lambda_j) \\
& +\frac{1}{|\Omega|}\Big(|\Omega_{22}|\Psi^{\gamma_1+\beta-1}(b)+|\Omega_{21}|\Psi^{\gamma_2-1}(b)\Big)\sum_{i=1}^{m}|\xi_i|\Psi^{\alpha+\beta}(\eta_i)\Bigg)\mathcal{L}_1\|u\| \\
& +\Bigg(\Psi^{\theta+\alpha+\beta}(b)+\frac{1}{|\Omega|}\Big(|\Omega_{12}|\Psi^{\gamma_1+\beta-1}(b)+|\Omega_{11}|\Psi^{\gamma_2-1}(b)\Big)\sum_{j=1}^{n}|\mu_j|\Psi^{\theta+\alpha+\beta-\phi_j}(\lambda_j) \\
& +\frac{1}{|\Omega|}\Big(|\Omega_{22}|\Psi^{\gamma_1+\beta-1}(b)+|\Omega_{21}|\Psi^{\gamma_2-1}(b)\Big)\sum_{i=1}^{m}|\xi_i|\Psi^{\theta+\alpha+\beta}(\eta_i)\Bigg)\mathcal{L}_2k_1^*\|u\|
\end{aligned}$$

$$
\begin{aligned}
&+\Bigg(\Psi^{\delta+\alpha+\beta}(b)+\frac{1}{|\Omega|}\Big(|\Omega_{12}|\Psi^{\gamma_1+\beta-1}(b)+|\Omega_{11}|\Psi^{\gamma_2-1}(b)\Big)\sum_{j=1}^{n}|\mu_j|\Psi^{\delta+\alpha+\beta-\phi_j}(\lambda_j)\\
&+\frac{1}{|\Omega|}\Big(|\Omega_{22}|\Psi^{\gamma_1+\beta-1}(b)+|\Omega_{21}|\Psi^{\gamma_2-1}(b)\Big)\sum_{i=1}^{m}|\xi_i|\Psi^{\delta+\alpha+\beta}(\eta_i)\Bigg)\mathcal{L}_3 w_1^*\|u\|\\
&+\frac{1}{|\Omega|}\Big(|\Omega_{22}|\Psi^{\gamma_1+\beta-1}(b)+|\Omega_{21}|\Psi^{\gamma_2-1}(b)\Big)\Psi^{\varphi}(\sigma)\mathcal{G}_1^*\|u\|\\
&+\frac{1}{|\Omega|}\Big(|\Omega_{12}|\Psi^{\gamma_1+\beta-1}(b)+|\Omega_{11}|\Psi^{\gamma_2-1}(b)\Big)\Psi^{\nu}(\zeta)\mathcal{H}_1^*\|u\|\\
&+\Bigg(\Psi^{\alpha+\beta}(b)+\frac{1}{|\Omega|}\Big(|\Omega_{12}|\Psi^{\gamma_1+\beta-1}(b)+|\Omega_{11}|\Psi^{\gamma_2-1}(b)\Big)\sum_{j=1}^{n}|\mu_j|\Psi^{\alpha+\beta-\phi_j}(\lambda_j)\\
&+\frac{1}{|\Omega|}\Big(|\Omega_{22}|\Psi^{\gamma_1+\beta-1}(b)+|\Omega_{21}|\Psi^{\gamma_2-1}(b)\Big)\sum_{i=1}^{m}|\xi_i|\Psi^{\alpha+\beta}(\eta_i)\Bigg)\mathbb{F}_1\\
&+\frac{1}{|\Omega|}\Big(|\Omega_{22}|\Psi^{\gamma_1+\beta-1}(b)+|\Omega_{21}|\Psi^{\gamma_2-1}(b)\Big)\Psi^{\varphi}(\sigma)\mathbb{G}_1\\
&+\frac{1}{|\Omega|}\Big(|\Omega_{12}\Psi^{\gamma_1+\beta-1}(b)|+|\Omega_{11}\Psi^{\gamma_2-1}(b)|\Big)\Psi^{\nu}(\zeta)\mathbb{H}_1\\
\le\;&\Bigg\{\Bigg(\Psi^{\alpha+\beta}(b)+\Phi(\Omega_{12},\Omega_{11})\sum_{j=1}^{n}|\mu_j|\Psi^{\alpha+\beta-\phi_j}(\lambda_j)+\Phi(\Omega_{22},\Omega_{21})\sum_{i=1}^{m}|\xi_i|\Psi^{\alpha+\beta}(\eta_i)\Bigg)\mathcal{L}_1\\
&+\Bigg(\Psi^{\theta+\alpha+\beta}(b)+\Phi(\Omega_{12},\Omega_{11})\sum_{j=1}^{n}|\mu_j|\Psi^{\theta+\alpha+\beta-\phi_j}(\lambda_j)\\
&+\Phi(\Omega_{22},\Omega_{21})\sum_{i=1}^{m}|\xi_i|\Psi^{\theta+\alpha+\beta}(\eta_i)\Bigg)\mathcal{L}_2 k_1^*+\Bigg(\Psi^{\delta+\alpha+\beta}(b)\\
&+\Phi(\Omega_{12},\Omega_{11})\sum_{j=1}^{n}|\mu_j|\Psi^{\delta+\alpha+\beta-\phi_j}(\lambda_j)+\Phi(\Omega_{22},\Omega_{21})\sum_{i=1}^{m}|\xi_i|\Psi^{\delta+\alpha+\beta}(\eta_i)\Bigg)\mathcal{L}_3 w_1^*\\
&+\Phi(\Omega_{22},\Omega_{21})\Psi^{\varphi}(\sigma)\mathcal{G}_1^*+\Phi(\Omega_{12},\Omega_{11})\Psi^{\nu}(\zeta)\mathcal{H}_1^*\Bigg\}r_1+\Bigg(\Psi^{\alpha+\beta}(b)\\
&+\Phi(\Omega_{12},\Omega_{11})\sum_{j=1}^{n}|\mu_j|\Psi^{\alpha+\beta-\phi_j}(\lambda_j)+\Phi(\Omega_{22},\Omega_{21})\sum_{i=1}^{m}|\xi_i|\Psi^{\alpha+\beta}(\eta_i)\Bigg)\mathbb{F}_1\\
&+\Phi(\Omega_{22},\Omega_{21})\Psi^{\varphi}(\sigma)\mathbb{G}_1+\Phi(\Omega_{12},\Omega_{11})\Psi^{\nu}(\zeta)\mathbb{H}_1\\
=\;&\Big\{\Lambda(\alpha+\beta)\mathcal{L}_1+\Lambda(\theta+\alpha+\beta)\mathcal{L}_2 k_1^*+\Lambda(\delta+\alpha+\beta)\mathcal{L}_3 w_1^*+\Phi(\Omega_{22},\Omega_{21})\Psi^{\varphi}(\sigma)\mathcal{G}_1^*\\
&+\Phi(\Omega_{12},\Omega_{11})\Psi^{\nu}(\zeta)\mathcal{H}_1^*\Big\}r_1+\Lambda(\alpha+\beta)\mathbb{F}_1+\Phi(\Omega_{22},\Omega_{21})\Psi^{\varphi}(\sigma)\mathbb{G}_1\\
&+\Phi(\Omega_{12},\Omega_{11})\Psi^{\nu}(\zeta)\mathbb{H}_1.
\end{aligned}
$$

Then,

$$
|(\mathcal{Q}u)(\tau)|\le(\Delta_1+\Delta_2)r_1+\Lambda(\alpha+\beta)\mathbb{F}_1+\Phi(\Omega_{22},\Omega_{21})\Psi^{\varphi}(\sigma)\mathbb{G}_1+\Phi(\Omega_{12},\Omega_{11})\Psi^{\nu}(\zeta)\mathbb{H}_1,
$$

which implies that $\|\mathcal{Q}u\|\le r_1$. Thus, $\mathcal{Q}\mathcal{B}_{r_1}\subset\mathcal{B}_{r_1}$.

Step II. $\mathcal{Q} : \mathcal{E} \to \mathcal{E}$ is a contraction.
Assume that $u, v \in \mathcal{E}$, $\tau \in \mathcal{J}$. Then, we obtain

$$\begin{aligned}
& |(\mathcal{Q}u)(\tau) - (\mathcal{Q}v)(\tau)| \\
\leq\ & \mathcal{I}_{a^+}^{\alpha+\beta;\psi}|\mathcal{F}_u(s) - \mathcal{F}_v(s)|(b) + \frac{\Psi^{\gamma_1+\beta-1}}{|\Omega|}\Bigg[|\Omega_{22}|\Bigg(\mathcal{I}_{a^+}^{\varphi;\psi}|\mathcal{G}(s,u(s)) - \mathcal{G}(s,v(s))|(\sigma) \\
& + \textstyle\sum_{i=1}^{m}|\xi_i|\mathcal{I}_{a^+}^{\alpha+\beta;\psi}|\mathcal{F}_u(s) - \mathcal{F}_v(s)|(\eta_i)\Bigg) + |\Omega_{12}|\Bigg(\mathcal{I}_{a^+}^{\nu;\psi}|\mathcal{H}(s,u(s)) - \mathcal{H}(s,v(s))|(\zeta) \\
& + \textstyle\sum_{j=1}^{n}|\mu_j|\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}|\mathcal{F}_u(s) - \mathcal{F}_v(s)|(\lambda_j)\Bigg)\Bigg] + \frac{\Psi^{\gamma_2-1}}{|\Omega|}\Bigg[|\Omega_{11}|\Bigg(\mathcal{I}_{a^+}^{\nu;\psi}|\mathcal{H}(s,u(s)) \\
& - \mathcal{H}(s,v(s))|(\zeta) + \textstyle\sum_{j=1}^{n}|\mu_j|\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}|\mathcal{F}_u(s) - \mathcal{F}_v(s)|(\lambda_j)\Bigg) + |\Omega_{21}|\Bigg(\mathcal{I}_{a^+}^{\varphi;\psi}|\mathcal{G}(s,u(s)) \\
& - \mathcal{G}(s,v(s))|(\sigma) + \textstyle\sum_{i=1}^{m}|\xi_i|\mathcal{I}_{a^+}^{\alpha+\beta;\psi}|\mathcal{F}_u(s) - \mathcal{F}_v(s)|(\eta_i)\Bigg)\Bigg].
\end{aligned} \tag{35}$$

By helping $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$, it is implied that

$$|(\mathcal{K}u)(\tau) - (\mathcal{K}v)(\tau)| \leq k_1^*\Psi^{\theta}(\tau)\|u - v\|, \tag{36}$$
$$|(\mathcal{W}u)(\tau) - (\mathcal{W}v)(\tau)| \leq w_1^*\Psi^{\delta}(\tau)\|u - v\|, \tag{37}$$
$$|\mathcal{H}(\tau,u(\tau)) - \mathcal{H}(\tau,v(\tau))| \leq \mathcal{H}_1^*\|u - v\|, \tag{38}$$
$$|\mathcal{G}(\tau,u(\tau)) - \mathcal{G}(\tau,v(\tau))| \leq \mathcal{G}_1^*\|u - v\|, \tag{39}$$

and

$$\begin{aligned}
|\mathcal{F}_u(\tau) - \mathcal{F}_v(\tau)| &= |f(\tau,u(\tau),(\mathcal{K}u)(\tau),(\mathcal{W}u)(\tau)) - f(\tau,v(\tau),(\mathcal{K}v)(\tau),(\mathcal{W}v)(\tau))| \\
&\leq \mathcal{L}_1|u(\tau) - v(\tau)| + \mathcal{L}_2|(\mathcal{K}u)(\tau) - (\mathcal{K}v)(\tau)| \\
&\quad + \mathcal{L}_3|(\mathcal{W}u)(\tau) - (\mathcal{W}v)(\tau)| \\
&\leq \Big(\mathcal{L}_1 + \mathcal{L}_2k_1^*\Psi^{\theta}(\tau) + \mathcal{L}_3w_1^*\Psi^{\delta}(\tau)\Big)\|u - v\|.
\end{aligned} \tag{40}$$

Hence, by inserting (36)–(40) into (35) and using Proposition 1 (i), which yields that

$$\begin{aligned}
& |((\mathcal{Q}u)(\tau) - (\mathcal{Q}v)(\tau))| \\
\leq\ & \Big(\mathcal{L}_1\Psi^{\alpha+\beta}(b) + \mathcal{L}_2k_1^*\Psi^{\theta+\alpha+\beta}(b) + \mathcal{L}_3w_1^*\Psi^{\delta+\alpha+\beta}(b)\Big)\|u - v\| \\
& + \frac{\Psi^{\gamma_1+\beta-1}}{|\Omega|}\Bigg[|\Omega_{22}|\Bigg(\mathcal{G}_1^*\Psi^{\varphi}(\sigma)\|u - v\| + \sum_{i=1}^{m}|\xi_i|\Big(\mathcal{L}_1\Psi^{\alpha+\beta}(\eta_i) + \mathcal{L}_2k_1^*\Psi^{\theta+\alpha+\beta}(\eta_i) \\
& + \mathcal{L}_3w_1^*\Psi^{\delta+\alpha+\beta}(\eta_i)\Big)\|u - v\|\Bigg) + |\Omega_{12}|\Bigg(\mathcal{H}_1^*\Psi^{\nu}(\zeta)\|u - v\| \\
& + \sum_{j=1}^{n}|\mu_j|\Big(\mathcal{L}_1\Psi^{\alpha+\beta-\phi_j}((\lambda_j)) + \mathcal{L}_2k_1^*\Psi^{\theta+\alpha+\beta-\phi_j}((\lambda_j)) \\
& + \mathcal{L}_3w_1^*\Psi^{\delta+\alpha+\beta-\phi_j}((\lambda_j))\Big)\|u - v\|\Bigg)\Bigg] + \frac{\Psi^{\gamma_2-1}}{|\Omega|}\Bigg[|\Omega_{11}|\Bigg(\mathcal{H}_1^*\Psi^{\nu}(\zeta)\|u - v\| \\
& + \sum_{j=1}^{n}|\mu_j|\Big(\mathcal{L}_1\Psi^{\alpha+\beta-\phi_j}((\lambda_j)) + \mathcal{L}_2k_1^*\Psi^{\theta+\alpha+\beta-\phi_j}((\lambda_j)) \\
& + \mathcal{L}_3w_1^*\Psi^{\delta+\alpha+\beta-\phi_j}((\lambda_j))\Big)\|u - v\|\Bigg) + |\Omega_{21}|\Bigg(\mathcal{G}_1^*\Psi^{\varphi}(\sigma)\|u - v\| \\
& + \sum_{i=1}^{m}|\xi_i|\Big(\mathcal{L}_1\Psi^{\alpha+\beta}(\eta_i) + \mathcal{L}_2k_1^*\Psi^{\theta+\alpha+\beta}(\eta_i) + \mathcal{L}_3w_1^*\Psi^{\delta+\alpha+\beta}(\eta_i)\Big)\|u - v\|\Bigg)\Bigg] v
\end{aligned}$$

$$\leq \left\{ \left(\Psi^{\alpha+\beta}(b) + \Phi(\Omega_{12}, \Lambda_{11}) \sum_{j=1}^{n} |\mu_j| \Psi^{\alpha+\beta-\phi_j}(\lambda_j) + \Phi(\Omega_{22}, \Omega_{21}) \sum_{i=1}^{m} |\xi_i| \Psi^{\alpha+\beta}(\eta_i) \right) \mathcal{L}_1 \right.$$
$$+ \left(\Psi^{\theta+\alpha+\beta}(b) + \Phi(\Omega_{12}, \Omega_{11}) \sum_{j=1}^{n} |\mu_j| \Psi^{\theta+\alpha+\beta-\phi_j}(\lambda_j) \right.$$
$$\left. + \Phi(\Omega_{22}, \Omega_{21}) \sum_{i=1}^{m} |\xi_i| \Psi^{\theta+\alpha+\beta}(\eta_i) \right) \mathcal{L}_2 k_1^* + \left(\Psi^{\delta+\alpha+\beta}(b) \right.$$
$$\left. + \Phi(\Omega_{12}, \Omega_{11}) \sum_{j=1}^{n} |\mu_j| \Psi^{\delta+\alpha+\beta-\phi_j}(\lambda_j) + \Phi(\Omega_{22}, \Omega_{21}) \sum_{i=1}^{m} |\xi_i| \Psi^{\delta+\alpha+\beta}(\eta_i) \right) \mathcal{L}_3 w_1^*$$
$$\left. + \Phi(\Omega_{22}, \Omega_{21}) \Psi^{\varphi}(\sigma) \mathcal{G}_1^* + \Phi(\Omega_{12}, \Omega_{11}) \Psi^{\nu}(\zeta) \mathcal{H}_1^* \right\} \|u - v\|,$$

then, $\|\mathcal{Q}u - \mathcal{Q}v\| \leq (\Delta_1 + \Delta_2)\|u - v\|$. Thanks to (19), $\Delta_1 + \Delta_2 < 1$; thus, $\mathcal{Q}$ is a contraction. Hence, by applying Lemma 5, the problem (3) has a unique solution $x \in \mathcal{E}$. □

3.2. Existence Property via Leray-Schauder's Type

Lemma 6. *(Arzelá-Ascoli theorem [57]). A set of functions in $\mathcal{C}([a,b])$ is relatively compact if and only if it is uniformly bounded and equicontinuous on $[a,b]$.*

Lemma 7. *(Reference [57]). If a set is closed and relatively compact, then it is compact.*

Lemma 8. *(Leray-Schauder's nonlinear alternative [56]) Assume that $\mathcal{E}$ is a Banach space, C is a closed, convex subset of M, X is an open subset of C, and $0 \in X$. Assume that $\mathcal{Q} : \overline{X} \to C$ is a continuous, compact (that is, $\mathcal{Q}(\overline{X})$ is a relatively compact subset of C) map. Then, either (i) $\mathcal{Q}$ has a fixed point in $\overline{X}$, or (ii) there is $x \in \partial X$ (the boundary of X in C) and $\varrho \in (0,1)$ with $u = \varrho\mathcal{Q}(u)$.*

Theorem 2. *Suppose that $f \in \mathcal{C}(\mathcal{J} \times \mathbb{R}^3, \mathbb{R})$ satisfies the following conditions*

$(\mathcal{P}_3)$ There exist nondecreasing continuous functions $\mathbb{U}$, $\mathbb{V}$, $\mathbb{W} : \mathbb{R}^+ \to \mathbb{R}^+$, p_i, $q_j \in \mathcal{C}(\mathcal{J}, \mathbb{R}^+)$, for $i = 1,2,3$, $j = 1,2$, such that

$$\begin{aligned} |f(\tau,x,y,z)| &\leq p_1(\tau)\mathbb{U}(|u|) + p_2(t)|v| + p_3(\tau)|w|, \quad \forall(\tau,u,v,w) \in \mathcal{J} \times \mathbb{R}^3, \\ |\mathcal{H}(\tau,u)| &\leq q_1(\tau)\mathbb{V}(|u|), \quad \forall(\tau,u) \in \mathcal{J} \times \mathbb{R}, \\ |\mathcal{G}(\tau,u)| &\leq q_2(\tau)\mathbb{W}(|u|), \quad \forall(\tau,u) \in \mathcal{J} \times \mathbb{R}, \end{aligned}$$

with $p_i^* = \sup_{\tau \in \mathcal{J}}\{p_i(\tau)\}$, $q_j^* = \sup_{\tau \in \mathcal{J}}\{q_j(\tau)\}$, $i = 1,2,3$, $j = 1,2$.

$(\mathcal{P}_4)$ There exists a constant $\mathcal{M}^* > 0$ satisfy

$$\frac{\left[1 - \left(\Lambda(\theta+\alpha+\beta)p_2^* k_1^* + \Lambda(\delta+\alpha+\beta)p_3^* w_1^*\right)\right]\mathcal{M}^*}{\Lambda(\alpha+\beta)p_1^*\mathbb{U}(\mathcal{M}^*) + \Phi(\Omega_{12},\Omega_{11})q_1^*\Psi^{\nu}(\zeta)\mathbb{V}(\mathcal{M}^*) + \Phi(\Omega_{22},\Omega_{21})q_2^*\Psi^{\varphi}(\sigma)\mathbb{W}(\mathcal{M}^*)} > 1.$$

Then, the ψ-Hilfer $\mathbb{FBVP}$ describing Navier model with $\mathbb{NIBC}$s (3) has at least one solution $x \in \mathcal{E}$.

Proof. Assume that $\mathcal{Q}$ is given by (15). In the first step, we will prove that $\mathcal{Q}$ maps bounded sets (balls) into bounded sets in $\mathcal{E}$. For any a real constant $r_2 > 0$, given

$B_{r_2} := \{u \in \mathcal{E} : \|u\| \leq r_2\}$ is a bounded set (ball) in $\mathcal{E}$. From (23) in Theorem 1 with $(\mathcal{P}_4)$, we obtain

$$\begin{aligned}
& |(\mathcal{Q}u)(\tau)| \\
\leq\ & \mathcal{I}_{a^+}^{\alpha+\beta;\psi}\Big(p_1(\tau)\mathbb{U}(|u(\tau)|) + p_2(\tau)|(\mathcal{K}u)(\tau)| + p_3(\tau)|(\mathcal{W}u)(\tau)|\Big) \\
& + \frac{\Psi^{\gamma_1+\beta-1}(b)}{|\Omega|}\Bigg[|\Omega_{22}|\Bigg(\mathcal{I}_{a^+}^{\varphi;\psi}\Big(q_2(\sigma)\mathbb{W}(|u(\sigma)|)\Big) + \sum_{i=1}^{m}|\xi_i|\mathcal{I}_{a^+}^{\alpha+\beta;\psi}\Big(p_1(\eta_i)\mathbb{U}(|u(\eta_i)|) \\
& + p_2(\eta_i)|(\mathcal{K}u)(\eta_i)| + p_3(\eta_i)|(\mathcal{W}u)(\eta_i)|\Big)\Bigg) + |\Omega_{12}|\Bigg(\mathcal{I}_{a^+}^{\nu;\psi}\Big(q_1(\zeta)\mathbb{V}(|u(\zeta)|)\Big) \\
& + \sum_{j=1}^{n}|\mu_j|\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}\Big(p_1(\lambda_j)\mathbb{U}(|u(\lambda_j)|) + p_2(\lambda_j)|(\mathcal{K}u)(\lambda_j)| + p_3(\lambda_j)|(\mathcal{W}u)(\lambda_j)|\Big)\Bigg)\Bigg] \\
& + \frac{\Psi^{\gamma_2-1}(b)}{|\Omega|}\Bigg[|\Omega_{11}|\Bigg(\mathcal{I}_{a^+}^{\nu;\psi}\Big(q_1(\zeta)\mathbb{V}(|u(\zeta)|)\Big) + \sum_{j=1}^{n}|\mu_j|\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}\Big(p_1(\lambda_j)\mathbb{U}(|u(\lambda_j)|) \\
& + p_2(\lambda_j)|(\mathcal{K}u)(\lambda_j)| + p_3(\lambda_j)|(\mathcal{W}u)(\lambda_j)|\Big)\Bigg) + |\Omega_{21}|\Bigg(\mathcal{I}_{a^+}^{\varphi;\psi}\Big(q_2(\sigma)\mathbb{W}(|u(\sigma)|)\Big) \\
& + \sum_{i=1}^{m}|\xi_i|\mathcal{I}_{a^+}^{\alpha+\beta;\psi}\Big(p_1(\eta_i)\mathbb{U}(|u(\eta_i)|) + p_2(\eta_i)|(\mathcal{K}u)(\eta_i)| + p_3(\eta_i)|(\mathcal{W}u)(\eta_i)|\Big)\Bigg)\Bigg].
\end{aligned}$$

By the same process in Theorem 1, we can estimate

$$\begin{aligned}
\|\mathcal{Q}u\| \leq\ & \Lambda(\alpha+\beta)p_1^*\mathbb{U}(r_2) + \Phi(\Omega_{12},\Omega_{11})q_1^*\Psi^{\nu}(\zeta)\mathbb{V}(r_2) + \Phi(\Omega_{22},\Omega_{21})q_2^*\Psi^{\varphi}(\sigma)\mathbb{W}(r_2) \\
& + \Big(\Lambda(\theta+\alpha+\beta)p_2^*k_1^* + \Lambda(\delta+\alpha+\beta)p_3^*w_1^*\Big)r_2 := \mathcal{N}_2.
\end{aligned}$$

Next, we prove that $\mathcal{Q}$ maps bounded sets into equicontinuous sets of $\mathcal{E}$. Assuming that the point $\tau_1, \tau_2 \in \mathcal{J}$, where $\tau_1 < \tau_2$ and $u \in B_{r_2}$, where B_{r_2} is bounded set in $\mathcal{E}$, we have

$$\begin{aligned}
& |(\mathcal{Q}u)(\tau_2) - (\mathcal{Q}u)(\tau_1)| \\
\leq\ & \Big|\mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_u(\tau_2) - \mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_u(\tau_1)\Big| \\
& + \frac{\Big|(\psi(\tau_2)-\psi(a))^{\gamma_1+\beta-1} - (\psi(\tau_1)-\psi(a))^{\gamma_1+\beta-1}\Big|}{|\Omega|\Gamma(\gamma_1+\beta)}\Bigg[|\Omega_{22}|\Bigg(\mathcal{I}_{a^+}^{\varphi;\psi}|\mathcal{G}(\sigma,u(\sigma))| \\
& + \sum_{i=1}^{m}|\xi_i|\mathcal{I}_{a^+}^{\alpha+\beta;\psi}|\mathcal{F}_u(\eta_i)|\Bigg) + |\Omega_{12}|\Bigg(\mathcal{I}_{a^+}^{\nu;\psi}|\mathcal{H}(\zeta,u(\zeta))| + \sum_{j=1}^{n}|\mu_j|\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}|\mathcal{F}_u(\lambda_j)|\Bigg)\Bigg] \\
& + \frac{\Big|(\psi(\tau_2)-\psi(a))^{\gamma_2-1} - (\psi(\tau_1)-\psi(a))^{\gamma_2-1}\Big|}{|\Omega|\Gamma(\gamma_2)}\Bigg[|\Omega_{11}|\Bigg(\mathcal{I}_{a^+}^{\nu;\psi}|\mathcal{H}(\zeta,u(\zeta))| \\
& + \sum_{j=1}^{n}|\mu_j|\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}|\mathcal{F}_u(\lambda_j)|\Bigg) + |\Omega_{21}|\Bigg(\mathcal{I}_{a^+}^{\varphi;\psi}|\mathcal{G}(\sigma,u(\sigma))| + \sum_{i=1}^{m}|\xi_i|\mathcal{I}_{a^+}^{\alpha+\beta;\psi}|\mathcal{F}_u(\eta_i)|\Bigg)\Bigg] \\
\leq\ & \Big(p_1^*\mathbb{U}(r_2) + p_2^*k_1^*\Psi^{\theta}(b)r_2 + p_3^*w_1^*\Psi^{\delta}(b)r_2\Big)\Bigg(\frac{1}{\Gamma(\alpha+\beta)}\int_{\tau_1}^{t_2}\psi'(s)(\psi(\tau_2)-\psi(s))^{\alpha+\beta-1}ds \\
& + \frac{1}{\Gamma(\alpha+\beta)}\int_a^{\tau_1}\psi'(s)\Big|(\psi(\tau_2)-\psi(s))^{\alpha+\beta-1} - (\psi(\tau_1)-\psi(s))^{\alpha+\beta-1}\Big|ds\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\left|(\psi(\tau_2)-\psi(a))^{\gamma_1+\beta-1}-(\psi(\tau_1)-\psi(a))^{\gamma_1+\beta-1}\right|}{|\Omega|\Gamma(\gamma_1+\beta)}\Bigg[|\Omega_{22}|\Bigg(q_2^*\mathbb{W}(r_2)\Psi^{\varphi}(\sigma)\\
&+\sum_{i=1}^{m}|\xi_i|\Big(p_1^*\mathbb{U}(r_2)\Psi^{\alpha+\beta}(\eta_i)+p_2^*k_1^*\Psi^{\theta+\alpha+\beta}(\eta_i)r_2+p_3^*w_1^*\Psi^{\delta+\alpha+\beta}(\eta_i)r_2\Big)\Bigg)\\
&+|\Omega_{12}|\Bigg(q_1^*\mathbb{V}(r_2)\Psi^{\nu}(\zeta)+\sum_{j=1}^{n}|\mu_j|\Big(p_1^*\mathbb{U}(r_2)\Psi^{\alpha+\beta-\phi_j}(\lambda_j)+p_2^*k_1^*\Psi^{\theta+\alpha+\beta-\phi_j}(\lambda_j)r_2\\
&+p_3^*w_1^*\Psi^{\delta+\alpha+\beta-\phi_j}(\lambda_j)r_2\Big)\Bigg)\Bigg]+\frac{\left|(\psi(\tau_2)-\psi(a))^{\gamma_2-1}-(\psi(\tau_1)-\psi(a))^{\gamma_2-1}\right|}{|\Omega|\Gamma(\gamma_2)}\\
&\times\Bigg[|\Omega_{11}|\Bigg(q_1^*\mathbb{V}(r_2)\Psi^{\nu}(\zeta)|+\sum_{j=1}^{n}|\mu_j|\Big(p_1^*\mathbb{U}(r_2)\Psi^{\alpha+\beta-\phi_j}(\lambda_j)+p_2^*k_1^*\Psi^{\theta+\alpha+\beta-\phi_j}(\lambda_j)r_2\\
&+p_3^*w_1^*\Psi^{\delta+\alpha+\beta-\phi_j}(\lambda_j)r_2\Big)\Bigg)+|\Omega_{21}|\Bigg(q_2^*\mathbb{W}(r_2)\Psi^{\varphi}(\sigma)+\sum_{i=1}^{m}|\xi_i|\Big(p_1^*\mathbb{U}(r_2)\Psi^{\alpha+\beta}(\eta_i)\\
&+p_2^*k_1^*\Psi^{\theta+\alpha+\beta}(\eta_i)r_2+p_3^*w_1^*\Psi^{\delta+\alpha+\beta}(\eta_i)r_2\Big)\Bigg)\Bigg]\\
\leq\ &\Big(2(\psi(\tau_2)-\psi(\tau_1))^{\alpha+\beta}+\left|(\psi(\tau_2)-\psi(a))^{\alpha+\beta}-(\psi(\tau_1)-\psi(a))^{\alpha+\beta}\right|\Big)\\
&\times\frac{1}{\Gamma(\alpha+\beta+1)}\Big(p_1^*\mathbb{U}(r_2)+p_2^*k_1^*\Psi^{\theta}(b)r_2+p_3^*w_1^*\Psi^{\delta}(b)r_2\Big)\\
&+\frac{\left|(\psi(\tau_2)-\psi(a))^{\gamma_1+\beta-1}-(\psi(\tau_1)-\psi(a))^{\gamma_1+\beta-1}\right|}{|\Omega|\Gamma(\gamma_1+\beta)}\Bigg[|\Omega_{22}|\Bigg(q_2^*\mathbb{W}(r_2)\Psi^{\varphi}(\sigma)\\
&+\sum_{i=1}^{m}|\xi_i|\Big(p_1^*\mathbb{U}(r_2)\Psi^{\alpha+\beta}(\eta_i)+p_2^*k_1^*\Psi^{\theta+\alpha+\beta}(\eta_i)r_2+p_3^*w_1^*\Psi^{\delta+\alpha+\beta}(\eta_i)r_2\Big)\Bigg)\\
&+|\Omega_{12}|\Bigg(q_1^*\mathbb{V}(r_2)\Psi^{\nu}(\zeta)+\sum_{j=1}^{n}|\mu_j|\Big(p_1^*\mathbb{U}(r_2)\Psi^{\alpha+\beta-\phi_j}(\lambda_j)+p_2^*k_1^*\Psi^{\theta+\alpha+\beta-\phi_j}(\lambda_j)r_2\\
&+p_3^*w_1^*\Psi^{\delta+\alpha+\beta-\phi_j}(\lambda_j)r_2\Big)\Bigg)\Bigg]+\frac{\left|(\psi(\tau_2)-\psi(a))^{\gamma_2-1}-(\psi(\tau_1)-\psi(a))^{\gamma_2-1}\right|}{|\Omega|\Gamma(\gamma_2)}\\
&\times\Bigg[|\Omega_{11}|\Bigg(q_1^*\mathbb{V}(r_2)\Psi^{\nu}(\zeta)|+\sum_{j=1}^{n}|\mu_j|\Big(p_1^*\mathbb{U}(r_2)\Psi^{\alpha+\beta-\phi_j}(\lambda_j)+p_2^*k_1^*\Psi^{\theta+\alpha+\beta-\phi_j}(\lambda_j)r_2\\
&+p_3^*w_1^*\Psi^{\delta+\alpha+\beta-\phi_j}(\lambda_j)r_2\Big)\Bigg)+|\Omega_{21}|\Bigg(q_2^*\mathbb{W}(r_2)\Psi^{\varphi}(\sigma)+\sum_{i=1}^{m}|\xi_i|\Big(p_1^*\mathbb{U}(r_2)\Psi^{\alpha+\beta}(\eta_i)\\
&+p_2^*k_1^*\Psi^{\theta+\alpha+\beta}(\eta_i)r_2+p_3^*w_1^*\Psi^{\delta+\alpha+\beta}(\eta_i)r_2\Big)\Bigg)\Bigg].
\end{aligned}
$$

Clearly, the right-hand side of the above inequality tends to zero as $\tau_2-\tau_1\to 0$, which independent of $u\in B_{r_2}$. Then, by the Arzelá-Ascoli theorem (Lemma 6), $\mathcal{Q}:\mathcal{E}\to\mathcal{E}$ is completely continuous.

Next, we show that there is an open set $\mathcal{D}\subset\mathcal{E}$ with $u\neq\kappa\mathcal{Q}(u)$ for $\kappa\in(0,1)$ and $u\in\partial\mathcal{D}$. Assume that $u\in\mathcal{E}$ is a solution of $u=\kappa\mathcal{Q}u$ for each $\kappa\in(0,1)$. So, for $\tau\in\mathcal{J}$, we will show that $\mathcal{Q}$ is bounded, and then

$$\begin{aligned}
&|u(\tau)| \\
=\ & |\kappa(\mathcal{Q}u)(\tau)| \\
\leq\ & \Lambda(\alpha+\beta)p_1^*\mathbb{U}(\|u\|) + \Phi(\Omega_{12},\Omega_{11})q_1^*\Psi^{\nu}(\zeta)\mathbb{V}(\|u\|) \\
& +\Phi(\Omega_{22},\Omega_{21})q_2^*\Psi^{\varphi}(\sigma)\mathbb{W}(\|u\|) + \Big(\Lambda(\theta+\alpha+\beta)p_2^*k_1^* + \Lambda(\delta+\alpha+\beta)p_3^*w_1^*\Big)\|u\|.
\end{aligned}$$

Taking the norm for $\tau \in \mathcal{J}$, then

$$\begin{aligned}
\|u\| \leq\ & \Lambda(\alpha+\beta)p_1^*\mathbb{U}(\|u\|) + \Phi(\Omega_{12},\Omega_{11})q_1^*\Psi^{\nu}(\zeta)\mathbb{V}(\|u\|) \\
& +\Phi(\Omega_{22},\Omega_{21})q_2^*\Psi^{\varphi}(\sigma)\mathbb{W}(\|u\|) \\
& +\Big(\Lambda(\theta+\alpha+\beta)p_2^*k_1^* + \Lambda(\delta+\alpha+\beta)p_3^*w_1^*\Big)\|u\|.
\end{aligned}$$

Consequently, we obtain

$$\frac{\Big[1-\Big(\Lambda(\theta+\alpha+\beta)p_2^*k_1^* + \Lambda(\delta+\alpha+\beta)p_3^*w_1^*\Big)\Big]\|x\|}{\Lambda(\alpha+\beta)p_1^*\mathbb{U}(\|u\|) + \Phi(\Omega_{12},\Omega_{11})q_1^*\Psi^{\nu}(\zeta)\mathbb{V}(\|u\|) + \Phi(\Omega_{22},\Omega_{21})q_2^*\Psi^{\varphi}(\sigma)\mathbb{W}(\|u\|)} \leq 1.$$

Thanks to $(\mathcal{P}_5)$, there is a constant $\mathcal{M}^* > 0$ such that $\|u\| \neq \mathcal{M}^*$. Set

$$\mathcal{D} := \{u \in \mathcal{E} : \|u\| \leq \mathcal{M}^* + 1\}, \quad \text{and} \quad \mathcal{U} = \mathcal{D} \cup B_{r_2}.$$

Notice that $\mathcal{Q} : \overline{\mathcal{U}} \to \mathcal{E}$ is continuous and completely continuous. By the choice of $\mathcal{D}$, there exists no $u \in \partial\mathcal{D}$ so that $u = \kappa\mathcal{Q}u$ for some $\kappa \in (0,1)$.

Therefore, by Lemma 8, we summarize that $\mathcal{Q}$ has fixed point $x \in \overline{\mathcal{U}}$, which suggests that the problem (3) has at least one solution on $\mathcal{J}$. □

3.3. Existence Property via Krasnoselskii's Fixed Point Theorem

Lemma 9. *(Krasnoselskii's fixed point theorem [58]) Let $\mathcal{B}$ be a closed, bounded, convex, and non-empty subset of a Banach space. Let $\mathcal{Q}_1$, $\mathcal{Q}_2$ be the operators such that (i) $\mathcal{Q}_1u + \mathcal{Q}_2v \in \mathcal{B}$ whenever u, $v \in \mathcal{B}$; (ii) $\mathcal{Q}_1$ is compact and continuous; (iii) $\mathcal{Q}_2$ is contraction mapping. Then, there exists $w \in \mathcal{B}$ such that $z = \mathcal{Q}_1w + \mathcal{Q}_2w$.*

Theorem 3. *Suppose that $f \in \mathcal{C}(\mathcal{J}\times\mathbb{R}^3,\mathbb{R})$ satisfies $(\mathcal{P}_1)$, $(\mathcal{P}_2)$, and*

$(\mathcal{P}_5)$ There exist f_i, g_j, h_k $\in$ $\mathcal{J},\mathbb{R}^+$, $i = 1,2,3,4$, $j = 1,2$, $k = 1,2$, such that $\forall(\tau,u,v,w) \in \mathcal{J}\times\mathbb{R}^3$,

$$\begin{aligned}
|f(\tau,u,v,w)| &\leq |f_1(\tau)| + |f_2(\tau)||u| + |f_3(\tau)||v| + |f_4(\tau)||w|, \\
|\mathcal{G}(\tau,u)| &\leq |g_1(\tau)| + |g_2(\tau)||u|, \qquad \forall(\tau,u) \in \mathcal{J}\times\mathbb{R}, \\
|\mathcal{H}(\tau,u)| &\leq |h_1(\tau)| + |h_2(\tau)||u|, \qquad \forall(\tau,u) \in \mathcal{J}\times\mathbb{R}.
\end{aligned}$$

If

$$\Big(\Delta_1 + \Delta_2 - \Psi^{\alpha+\beta}(b)\mathcal{L}_1 - \Psi^{\theta+\alpha+\beta}(b)k_1^*\mathcal{L}_2 - \Psi^{\delta+\alpha+\beta}(b)w_1^*\mathcal{L}_3\Big) < 1, \tag{41}$$

then the ψ-Hilfer $\mathbb{FBVP}$ describing Navier model with $\mathbb{NIBC}$s (3) has at least one solution $x \in \mathcal{E}$.

Proof. By setting $\sup_{\tau\in\mathcal{J}}|f_i(\tau)| = \|f_i\|$, $i = 1,2,3,4$, $\sup_{t\in\mathcal{J}}|g_j(\tau)| = \|g_j\|$, and $\sup_{\tau\in\mathcal{J}}|h_j(\tau)| = \|h_j\|$, $j = 1,2$, we consider $B_{r_3} = \{u \in \mathcal{E} : \|u\| \leq r_3\}$, where

$$r_3 \geq \frac{\Lambda(\alpha+\beta)\|f_1\| + \Phi(\Omega_{22},\Omega_{21})\Psi^{\varphi}(\sigma)\|g_1\| + \Phi(\Omega_{12},\Omega_{11})\Psi^{\nu}(\zeta)\|h_1\|}{1-\Delta_3},$$

with

$$\begin{aligned}\Delta_3 &= \Lambda(\alpha+\beta)\|f_2\| + \Lambda(\theta+\alpha+\beta)k_1^*\|f_3\| + \Lambda(\delta+\alpha+\beta)w_1^*\|f_4\| \\ &\quad + \Phi(\Omega_{22},\Omega_{21})\Psi^{\varphi}(\sigma)\|g_2\| + \Phi(\Omega_{12},\Omega_{11})\Psi^{\nu}(\zeta)\|h_2\|.\end{aligned}$$

Define $\mathcal{Q}_1$ and $\mathcal{Q}_2 : B_{r_3} \to \mathcal{E}$ as

$$(\mathcal{Q}_1 u)(\tau) = \mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_u(\tau), \tag{42}$$

$$\begin{aligned}(\mathcal{Q}_2 u)(\tau) &= \frac{\Psi^{\gamma_1+\beta-1}(t)}{\Omega}\Bigg[\Omega_{22}\Big(\mathcal{I}_{a^+}^{\varphi;\psi}\mathcal{G}(\sigma,u(\sigma)) - \textstyle\sum_{i=1}^m \xi_i \mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_u(\eta_i)\Big) \\ &\quad -\Omega_{12}\Big(\mathcal{I}_{a^+}^{\nu;\psi}\mathcal{H}(\zeta,u(\zeta)) - \textstyle\sum_{j=1}^n \mu_j \mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}\mathcal{F}_u(\lambda_j)\Big)\Bigg] \\ &\quad + \frac{\Psi^{\gamma_2-1}(t)}{\Omega}\Bigg[\Omega_{11}\Big(\mathcal{I}_{a^+}^{\nu;\psi}\mathcal{H}(\zeta,u(\zeta)) - \textstyle\sum_{j=1}^n \mu_j \mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}\mathcal{F}_u(\lambda_j)\Big) \\ &\quad -\Omega_{21}\Big(\mathcal{I}_{a^+}^{\varphi;\psi}\mathcal{G}(\sigma,u(\sigma)) - \textstyle\sum_{i=1}^m \xi_i \mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_u(\eta_i)\Big)\Bigg].\end{aligned} \tag{43}$$

Note that $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$. For any $u, v \in B_{r_3}$, it follows that

$$\begin{aligned}&|(\mathcal{Q}_1 u)(\tau) + (\mathcal{Q}_2 v)(\tau)| \\ \leq\; & \mathcal{I}_{a^+}^{\alpha+\beta;\psi}|\mathcal{F}_u(\tau)| + \frac{\Psi^{\gamma_1+\beta-1}(b)}{|\Omega|}\Bigg[|\Omega_{22}|\Bigg(\mathcal{I}_{a^+}^{\varphi;\psi}|\mathcal{G}(\sigma,v(\sigma))| + \sum_{i=1}^m |\xi_i|\mathcal{I}_{a^+}^{\alpha+\beta;\psi}|\mathcal{F}_v(\eta_i)|\Bigg) \\ & +|\Omega_{12}|\Bigg(\mathcal{I}_{a^+}^{\nu;\psi}|\mathcal{H}(\zeta,v(\zeta))| + \sum_{j=1}^n |\mu_j|\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}|\mathcal{F}_v(\lambda_j)|\Bigg)\Bigg] \\ & + \frac{\Psi^{\gamma_2-1}(b)}{|\Omega|}\Bigg[|\Omega_{11}|\Bigg(\mathcal{I}_{a^+}^{\nu;\psi}|\mathcal{H}(\zeta,v(\zeta))| + \sum_{j=1}^n |\mu_j|\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}|\mathcal{F}_v(\lambda_j)|\Bigg) \\ & +|\Omega_{21}|\Bigg(\mathcal{I}_{a^+}^{\varphi;\psi}|\mathcal{G}(\sigma,v(\sigma))| + \sum_{i=1}^m |\xi_i|\mathcal{I}_{a^+}^{\alpha+\beta;\psi}|\mathcal{F}_v(\eta_i)|\Bigg)\Bigg] \\ \leq\; & \mathcal{I}_{a^+}^{\alpha+\beta;\psi}\Big(|f_1(b)| + |f_2(b)||u(b)| + |f_3(b)||(\mathcal{K}u)(b)| + |f_4(b)||(\mathcal{W}u)(b)|\Big) \\ & + \frac{\Psi^{\gamma_1+\beta-1}(b)}{|\Omega|}\Bigg[|\Omega_{22}|\Bigg(\mathcal{I}_{a^+}^{\varphi;\psi}\Big(|g_1(\sigma)| + |g_2(\sigma)||v(\sigma)|\Big) + \sum_{i=1}^m |\xi_i|\mathcal{I}_{a^+}^{\alpha+\beta;\psi}\Big(|f_1(\eta_i)| \\ & +|f_2(\eta_i)||v(\eta_i)| + |f_3(\eta_i)||(\mathcal{K}v)(\eta_i)| + |f_4(\eta_i)||(\mathcal{W}v)(\eta_i)|\Big)\Bigg) \\ & +|\Omega_{12}|\Bigg(\mathcal{I}_{a^+}^{\nu;\psi}\Big(|h_1(\zeta)| + |h_2(\zeta)||v(\zeta)|\Big) + \sum_{j=1}^n |\mu_j|\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}\Big(|f_1(\lambda_j)| \\ & +|f_2(\lambda_j)||v(\lambda_j)| + |f_3(\lambda_j)||(\mathcal{K}v)(\lambda_j)| + |f_4(\lambda_j)||(\mathcal{W}v)(\lambda_j)|\Big)\Bigg)\Bigg] \\ & + \frac{\Psi^{\gamma_2-1}(b)}{|\Omega|}\Bigg[|\Omega_{11}|\Bigg(\mathcal{I}_{a^+}^{\nu;\psi}\Big(|h_1(\zeta)| + |h_2(\zeta)||v(\zeta)|\Big) + \sum_{j=1}^n |\mu_j|\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}\Big(|f_1(\lambda_j)| \\ & +|f_2(\lambda_j)||v(\lambda_j)| + |f_3(\lambda_j)||(\mathcal{K}v)(\lambda_j)| + |f_4(\lambda_j)||(\mathcal{W}v)(\lambda_j)|\Big)\Bigg) \\ & +|\Omega_{21}|\Bigg(\mathcal{I}_{a^+}^{\varphi;\psi}\Big(|g_1(\sigma)| + |g_2(\sigma)||v(\sigma)|\Big) + \sum_{i=1}^m |\xi_i|\mathcal{I}_{a^+}^{\alpha+\beta;\psi}\Big(|f_1(\eta_i)| + |f_2(\eta_i)||y(\eta_i)|\end{aligned}$$

$$
\begin{aligned}
&+|f_3(\eta_i)||(\mathcal{K}y)(\eta_i)|+|f_4(\eta_i)||(\mathcal{W}v)(\eta_i)|\Big)\Big)\Big]\\
\leq\ & \Big\{\Psi^{\alpha+\beta}(b)+\Phi(\Omega_{22},\Omega_{21})\sum_{i=1}^{m}|\xi_i|\Psi^{\alpha+\beta}(\eta_i)+\Phi(\Omega_{12},\Omega_{11})\sum_{j=1}^{n}|\mu_j|\Psi^{\alpha+\beta-\phi_j}(\lambda_j)\Big\}\|f_1\|\\
&+\Big\{\Psi^{\alpha+\beta}(b)\|u\|+\Phi(\Omega_{22},\Omega_{21})\sum_{i=1}^{m}|\xi_i|\Psi^{\alpha+\beta}(\eta_i)\|v\|\\
&+\Phi(\Omega_{12},\Omega_{11})\sum_{j=1}^{n}|\mu_j|\Psi^{\alpha+\beta-\phi_j}(\lambda_j)\|v\|\Big\}\|f_2\|+\Big\{\Psi^{\theta+\alpha+\beta}(b)\|u\|\\
&+\Phi(\Omega_{22},\Omega_{21})\sum_{i=1}^{m}|\xi_i|\Psi^{\theta+\alpha+\beta}(\eta_i)\|v\|+\Phi(\Omega_{12},\Omega_{11})\sum_{j=1}^{n}|\mu_j|\Psi^{\theta+\alpha+\beta-\phi_j}(\lambda_j)\|v\|\Big\}k_1^*\|f_3\|\\
&+\Big\{\Psi^{\delta+\alpha+\beta}(b)\|u\|+\Phi(\Omega_{22},\Omega_{21})\sum_{i=1}^{m}|\xi_i|\Psi^{\delta+\alpha+\beta}(\eta_i)\|v\|\\
&+\Phi(\Omega_{12},\Omega_{11})\sum_{j=1}^{n}|\mu_j|\Psi^{\delta+\alpha+\beta-\phi_j}(\lambda_j)\|v\|\Big\}w_1^*\|f_4\|+\Phi(\Omega_{22},\Omega_{21})\Big(\Psi^{\varphi}(\sigma)\|g_1\|\\
&+\Psi^{\varphi}(\sigma)\|g_2\|\|v\|\Big)+\Phi(\Omega_{12},\Omega_{11})\Big(\Psi^{\nu}(\zeta)\|h_1\|+\Psi^{\nu}(\zeta)\|h_2\|\|v\|\Big)\\
\leq\ & \Lambda(\alpha+\beta)\|f_1\|+\Phi(\Omega_{22},\Omega_{21})\Psi^{\varphi}(\sigma)\|g_1\|+\Phi(\Omega_{12},\Omega_{11})\Psi^{\nu}(\zeta)\|h_1\|\\
&+\Big(\Lambda(\alpha+\beta)\|f_2\|+\Lambda(\theta+\alpha+\beta)k_1^*\|f_3\|+\Lambda(\delta+\alpha+\beta)w_1^*\|f_4\|\\
&+\Phi(\Omega_{22},\Omega_{21})\Psi^{\varphi}(\sigma)\|g_2\|+\Phi(\Omega_{12},\Omega_{11})\Psi^{\nu}(\zeta)\|h_2\|\Big)r_3\leq r_3,
\end{aligned}
$$

which implies that $\mathcal{Q}_1u+\mathcal{Q}_2v\in B_{r_3}$ that assumption (i) of Lemma 9 is verified.

Now, we are going to prove that Lemma 9 (ii) is fulfilled. Assume that a sequence u_n so that $u_n\to u\in\mathcal{E}$ as $n\to\infty$. For $\tau\in\mathcal{J}$, we obtain

$$|(\mathcal{Q}_1u_n)(\tau)-(\mathcal{Q}_1u)(\tau)|\leq\mathcal{I}_{a^+}^{\alpha+\beta;\psi}|\mathcal{F}_{u_n}(b)-\mathcal{F}_u(b)|\leq\Psi^{\alpha+\beta}(b)\|\mathcal{F}_{u_n}(\cdot)-\mathcal{F}_u(\cdot)\|.$$

By continuity of f, we get that $\mathcal{F}_u$ is continuous. Hence, by the Lebergue dominated convergent theorem, this yields that $|(\mathcal{Q}_1u_n)(\tau)-(\mathcal{Q}_1u)(\tau)|\to0$ as $\tau\to\infty$. Then,

$$\|\mathcal{Q}_1u_n-\mathcal{Q}_1u)\|\to0\quad\text{as}\quad\tau\to\infty.$$

Therefore, $\mathcal{Q}_1u$ is continuous. So, $\mathcal{Q}_1B_{r_3}$ is uniformly bounded as

$$\|\mathcal{Q}_1u\|\leq\Psi^{\alpha+\beta}(b)\|f_1\|+\Psi^{\alpha+\beta}(b)r_2\|f_2\|+k_1^*\Psi^{\theta+\alpha+\beta}(b)r_3\|f_3\|+w_1^*\Psi^{\delta+\alpha+\beta}(b)r_3\|f_4\|.$$

Afterward, we show compactness of $\mathcal{Q}_1$. Define $\sup_{(\tau,u,v,w)\in\mathcal{J}\times\mathbb{R}^3}|f(\tau,u,v,w)|=f^*<+\infty$, for each $\tau_1,\tau_2\in\mathcal{J}$, where $\tau_1<\tau_2$, we have,

$$
\begin{aligned}
|(\mathcal{Q}_1u)(\tau_2)-(\mathcal{Q}_1u)(\tau_1)| &= \Big|\mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_u(\tau_2)-\mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_u(\tau_1)\Big|\\
&\leq \tfrac{f^*}{\Gamma(\alpha+\beta+1)}\Big(2(\psi(\tau_2)-\psi(\tau_1))^{\alpha+\beta}\\
&\quad+\big|(\psi(\tau_2)-\psi(a))^{\alpha+\beta}-(\psi(\tau_1)-\psi(a))^{\alpha+\beta}\big|\Big).
\end{aligned}\tag{44}
$$

Clearly, the right-hand side of (44) is independent of u and $|(\mathcal{Q}_1u)(\tau_2)-(\mathcal{Q}_1u)(\tau_1)|\to0$ as $\tau_2\to\tau_1$. Hence, this implies that $\mathcal{Q}_1B_{r_3}$ is equicontinuous, and $\mathcal{Q}_1$ maps bounded subsets into relatively compact subsets, and this yields that $\mathcal{Q}_1 B_{r_3}$ is relatively compact. Therefore, we summarize that $\mathcal{Q}_1$ is compact on B_{r_3} by the Arzelá-Ascoli theorem.

Next, we show that $\mathcal{Q}_2$ is contraction. For each $u, v \in B_{r_3}$, $\tau \in \mathcal{J}$, then

$$\begin{aligned}
& |(\mathcal{Q}_2 u)(\tau) - (\mathcal{Q}_2 v)(\tau)| \\
\leq\ & \frac{\Psi^{\gamma_1+\beta-1}(b)}{|\Omega|}\Bigg[|\Omega_{22}|\Bigg(\mathcal{I}^{\varphi;\psi}_{a^+}|\mathcal{G}(\sigma,u(\sigma)) - \mathcal{G}(\sigma,v(\sigma))| \\
& + \sum_{i=1}^{m}|\xi_i|\mathcal{I}^{\alpha+\beta;\psi}_{a^+}|\mathcal{F}_u(\eta_i) - \mathcal{F}_v(\eta_i)|\Bigg) + |\Omega_{12}|\Bigg(\mathcal{I}^{\nu;\psi}_{a^+}|\mathcal{H}(\zeta,u(\zeta)) - \mathcal{H}(\zeta,v(\zeta))| \\
& + \sum_{j=1}^{n}|\mu_j|\mathcal{I}^{\alpha+\beta-\phi_j;\psi}_{a^+}|\mathcal{F}_u(\lambda_j) - \mathcal{F}_v(\lambda_j)|\Bigg)\Bigg] + \frac{\Psi^{\gamma_2-1}(b)}{|\Omega|}\Bigg[|\Omega_{11}|\Bigg(\mathcal{I}^{\nu;\psi}_{a^+}|\mathcal{H}(\zeta,u(\zeta)) \\
& - \mathcal{H}(\zeta,v(\zeta))| + \sum_{j=1}^{n}|\mu_j|\mathcal{I}^{\alpha+\beta-\phi_j;\psi}_{a^+}|\mathcal{F}_u(\lambda_j) - \mathcal{F}_v(\lambda_j)|\Bigg) \\
& + |\Omega_{21}|\Bigg(\mathcal{I}^{\varphi;\psi}_{a^+}|\mathcal{G}(\sigma,u(\sigma)) - \mathcal{G}(\sigma,v(\sigma))| + \sum_{i=1}^{m}|\xi_i|\mathcal{I}^{\alpha+\beta;\psi}_{a^+}|\mathcal{F}_u(\eta_i) - \mathcal{F}_v(\eta_i)|\Bigg)\Bigg] \\
\leq\ & \Bigg\{\frac{\Psi^{\gamma_1+\beta-1}(b)}{|\Omega|}\Bigg[|\Omega_{22}|\Bigg(\Psi^{\varphi}(\sigma)\mathcal{G}^*_1 + \sum_{i=1}^{m}|\xi_i|\Big(\mathcal{L}_1\Psi^{\alpha+\beta}(\eta_i) + \mathcal{L}_2 k^*_1\Psi^{\theta+\alpha+\beta}(\eta_i) \\
& + \mathcal{L}_3 w^*_1\Psi^{\delta+\alpha+\beta}(\eta_i)\Big)\Bigg) + |\Omega_{12}|\Bigg(\Psi^{\nu}(\zeta)\mathcal{H}^*_1 + \sum_{j=1}^{n}|\mu_j|\Big(\mathcal{L}_1\Psi^{\alpha+\beta-\phi_j}(\lambda_j) \\
& + \mathcal{L}_2 k^*_1\Psi^{\theta+\alpha+\beta-\phi_j}(\lambda_j) + \mathcal{L}_3 w^*_1\Psi^{\delta+\alpha+\beta-\phi_j}(\lambda_j)\Big)\Bigg)\Bigg] + \frac{\Psi^{\gamma_2-1}(b)}{|\Omega|}\Bigg[|\Lambda_{11}|\Bigg(\Psi^{\nu}(\zeta)\mathcal{H}^*_1 \\
& + \sum_{j=1}^{n}|\mu_j|\Big(\mathcal{L}_1\Psi^{\alpha+\beta-\phi_j}(\lambda_j) + \mathcal{L}_2 k^*_1\Psi^{\theta+\alpha+\beta-\phi_j}(\lambda_j) + \mathcal{L}_3 w^*_1\Psi^{\delta+\alpha+\beta-\phi_j}(\lambda_j)\Big)\Bigg) \\
& + |\Omega_{21}|\Bigg(\Psi^{\varphi}(\sigma)\mathcal{G}^*_1 + \sum_{i=1}^{m}|\xi_i|\Big(\mathcal{L}_1\Psi^{\alpha+\beta}(\eta_i) + \mathcal{L}_2 k^*_1\Psi^{\theta+\alpha+\beta}(\eta_i) \\
& + \mathcal{L}_3 w^*_1\Psi^{\delta+\alpha+\beta}(\eta_i)\Big)\Bigg)\Bigg]\Bigg\}\|u - v\| \\
\leq\ & \Big(\Delta_1 + \Delta_2 - \Psi^{\alpha+\beta}(b)\mathcal{L}_1 - \Psi^{\theta+\alpha+\beta}(b)k^*_1\mathcal{L}_2 - \Psi^{\delta+\alpha+\beta}(b)w^*_1\mathcal{L}_3\Big)\|u - v\|.
\end{aligned}$$

Hence, by (41), $\mathcal{Q}_2$ is a contraction.

Then, due to Lemma 9 being verified, this yields that the problem (3) has at least one solution on $\mathcal{J}$. $\square$

4. Ulam's Stability

This part analyzes a variety of Ulam's stability of solutions to the problem (3).

Definition 4. *The problem* (3) *is said to be* $\mathbb{UH}$ *stable if there is a constant* $\mathfrak{C}_f > 0$ *such that for any* $\epsilon > 0$ *and for each solution* $z \in \mathcal{E}$ *of*

$$\left|{}^{H}\mathfrak{D}^{\alpha,\rho;\psi}_{a^+}\left({}^{H}\mathfrak{D}^{\beta,\rho;\psi}_{a^+} z\right)(\tau) - \mathcal{F}_z(\tau)\right| \leq \epsilon, \tag{45}$$

there is a solution $u \in \mathcal{E}$ *of* (3) *such that*

$$|z(\tau) - u(\tau)| \leq \mathfrak{C}_f \epsilon, \quad \tau \in \mathcal{J}. \tag{46}$$

Definition 5. *The problem* (3) *is said to be* $\mathbb{GUH}$ *stable if there is a function* $\mathcal{T} \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$ *with* $\mathcal{T}(0) = 0$ *such that, for any solution* $z \in \mathcal{E}$ *of*

$$\left| {}^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi}\left({}^{H}\mathfrak{D}_{a^+}^{\beta,\rho;\psi} z\right)(\tau) - \mathcal{F}_z(\tau) \right| \leq \epsilon \mathcal{T}(\tau), \tag{47}$$

there is a solution $u \in \mathcal{E}$ *of* (3) *such that*

$$|z(\tau) - u(\tau)| \leq \mathcal{T}(\epsilon), \quad \tau \in \mathcal{J}. \tag{48}$$

Definition 6. *The problem* (3) *is said to be* $\mathbb{UHR}$ *stable with respect to* $\mathcal{T} \in \mathcal{C}(\mathcal{J}, \mathbb{R}^+)$ *if there is a constant* $\mathfrak{C}_{f,\mathcal{T}} > 0$ *such that for any* $\epsilon > 0$ *and for each a solution* $z \in \mathcal{E}$ *of* (47) *there is a solution* $u \in \mathcal{E}$ *of* (3) *such that*

$$|z(\tau) - u(\tau)| \leq \mathfrak{C}_{f,\mathcal{T}} \epsilon \mathcal{T}(\tau), \quad \tau \in \mathcal{J}. \tag{49}$$

Definition 7. *The problem* (3) *is said to be* $\mathbb{GUHR}$ *stable with respect to* $\mathcal{T} \in \mathcal{C}(\mathcal{J}, \mathbb{R}^+)$ *if there is a constant* $\mathfrak{C}_{f,\mathcal{T}} > 0$ *such that for any a solution* $z \in \mathcal{E}$ *of*

$$\left| {}^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi}\left({}^{H}\mathfrak{D}_{a^+}^{\beta,\rho;\psi} z\right)(\tau) - \mathcal{F}_z(\tau) \right| \leq \mathcal{T}(\tau), \tag{50}$$

there is a solution $u \in \mathcal{E}$ *of* (3) *such that*

$$|z(\tau) - u(\tau)| \leq \mathfrak{C}_{f,\mathcal{T}} \mathcal{T}(\tau), \quad \tau \in \mathcal{J}. \tag{51}$$

Remark 1. *It is clear that*

(i) *Definition* 4 $\Rightarrow$ *Definition* 5*;*
(ii) *Definition* 6 $\Rightarrow$ *Definition* 7*;*
(iii) *Definition* 6 *for* $\mathcal{T}(\tau) = 1 \Rightarrow$ *Definition* 4.

Remark 2. *A function* $z \in \mathcal{E}$ *is a solution of the inequality* (45) *if and only if there is a function* $v \in \mathcal{E}$ *(where* v *depends on* z*) such that:*

(i) $|v(\tau)| \leq \epsilon, \quad \forall \tau \in \mathcal{J};$
(ii) ${}^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi}\left({}^{H}\mathfrak{D}_{a^+}^{\beta,\rho;\psi} z\right)(\tau) = \mathcal{F}_z(\tau) + v(\tau), \quad \tau \in \mathcal{J}.$

Remark 3. *A function* $z \in \mathcal{E}$ *is a solution of the inequality* (47) *if and only if there is a function* $w \in \mathcal{E}$ *(where* w *depends on* z*) such that:*

(i) $|w(\tau)| \leq \epsilon \mathcal{T}(\tau), \quad \forall \tau \in \mathcal{J};$
(ii) ${}^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi}\left({}^{H}\mathfrak{D}_{a^+}^{\beta,\rho;\psi} z\right)(\tau) = \mathcal{F}_z(\tau) + w(\tau), \quad \tau \in \mathcal{J}.$

Remark 4. *For the analysis of* $\mathbb{UHR}$ *stability and* $\mathbb{GUHR}$ *stability, we assume the following assumption:*

(H_1) *There is an increasing function* $\mathcal{T} \in \mathcal{C}(\mathcal{J}, \mathbb{R}^+)$ *and there is a constant* $\lambda_{\mathcal{T}} > 0$*, such that, for any* $\tau \in \mathcal{J}$*, we obtain*

$$\mathcal{I}_{a^+}^{\alpha+\beta;\psi} \mathcal{T}(\tau) \leq \lambda_{\mathcal{T}} \mathcal{T}(\tau). \tag{52}$$

4.1. $\mathbb{UH}$ *Stability and* $\mathbb{GUH}$ *Stability*

In this subsection, we construct an essential lemma that will be used in proves on $\mathbb{UH}$ and $\mathbb{GUH}$ stables of the problem (3).

Lemma 10. *Assume that* $\alpha \in (3,4]$, $\rho \in [0,1]$, *and* $z \in \mathcal{E}$ *is a solution of* (45). *Then,* $z \in \mathcal{E}$ *verifies*

$$\left| z(\tau) - \mathcal{Z}(\tau) - \mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_z(\tau) \right| \leq \Lambda(\alpha+\beta)\,\epsilon, \tag{53}$$

where

$$\begin{aligned} \mathcal{Z}(\tau) \;=\; & \frac{\Psi^{\gamma_1+\beta-1}(\tau)}{\Omega}\Bigg[\Omega_{22}\Big(\mathcal{I}_{a^+}^{\varphi;\psi}\mathcal{G}(\sigma,z(\sigma)) - \textstyle\sum_{i=1}^{m}\xi_i\mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_z(\eta_i)\Big) \\ & -\Omega_{12}\Big(\mathcal{I}_{a^+}^{\nu;\psi}\mathcal{H}(\zeta,z(\zeta)) - \textstyle\sum_{j=1}^{n}\mu_j\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}\mathcal{F}_z(\lambda_j)\Big)\Bigg] \\ & +\frac{\Psi^{\gamma_2-1}(t)}{\Omega}\Bigg[\Omega_{11}\Big(\mathcal{I}_{a^+}^{\nu;\psi}\mathcal{H}(\zeta,z(\zeta)) - \textstyle\sum_{j=1}^{n}\mu_j\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}\mathcal{F}_z(\lambda_j)\Big) \\ & -\Omega_{21}\Big(\mathcal{I}_{a^+}^{\varphi;\psi}\mathcal{G}(\sigma,z(\sigma)) - \textstyle\sum_{i=1}^{m}\xi_i\mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_z(\eta_i)\Big)\Bigg], \end{aligned} \tag{54}$$

with Ω, Ω_{ij}, $i,j \in \{1,2\}$, *and* $\Lambda(\alpha+\beta)$ *are given in Lemma* 4 *and* (18).

Proof. Assume that z is a solution of (45). By Lemma 4 and (ii) of Remark 2, we get

$$\begin{cases} {}^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi}\Big({}^{H}\mathfrak{D}_{a^+}^{\beta,\rho;\psi}z\Big)(\tau) = \mathcal{F}_z(\tau) + v(\tau) \quad \tau \in (a,b), \\ z(a) = 0, \qquad {}^{H}\mathfrak{D}_{a^+}^{\beta,\rho;\psi}z(a) = 0, \\ \displaystyle\sum_{i=1}^{m}\xi_i z(\eta_i) = \mathcal{I}_{a^+}^{\varphi;\psi}\mathcal{G}(\sigma,z(\sigma)), \qquad \sum_{j=1}^{n}\mu_j\,{}^{H}\mathfrak{D}_{a^+}^{\phi_j,\rho;\psi}z(\lambda_j) = \mathcal{I}_{a^+}^{\nu;\psi}\mathcal{H}(\zeta,z(\zeta)), \end{cases} \tag{55}$$

and then the solution of (55) can be given as

$$\begin{aligned} z(\tau) \;=\; & \mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_z(\tau) + \frac{\Psi^{\gamma_1+\beta-1}(t)}{\Omega}\left[\Omega_{22}\left(\mathcal{I}_{a^+}^{\varphi;\psi}\mathcal{G}(\sigma,z(\sigma)) - \sum_{i=1}^{m}\xi_i\mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_z(\eta_i)\right)\right. \\ & \left.-\Omega_{12}\left(\mathcal{I}_{a^+}^{\nu;\psi}\mathcal{H}(\zeta,z(\zeta)) - \sum_{j=1}^{n}\mu_j\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}\mathcal{F}_z(\lambda_j)\right)\right] \\ & +\frac{\Psi^{\gamma_2-1}(\tau)}{\Omega}\left[\Omega_{11}\left(\mathcal{I}_{a^+}^{\nu;\psi}\mathcal{H}(\zeta,z(\zeta)) - \sum_{j=1}^{n}\mu_j\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}\mathcal{F}_z(\lambda_j)\right)\right. \\ & \left.-\Omega_{21}\left(\mathcal{I}_{a^+}^{\varphi;\psi}\mathcal{G}(\sigma,z(\sigma)) - \sum_{i=1}^{m}\xi_i\mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_z(\eta_i)\right)\right] + \mathcal{I}_{a^+}^{\alpha+\beta;\psi}v(\tau) \\ & \\ & +\frac{\Psi^{\gamma_1+\beta-1}(\tau)}{\Omega}\left(-\Omega_{22}\sum_{i=1}^{m}\xi_i\mathcal{I}_{a^+}^{\alpha+\beta;\psi}v(\eta_i) + \Omega_{12}\sum_{j=1}^{n}\mu_j\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}v(\lambda_j)\right) \\ & +\frac{\Psi^{\gamma_2-1}(\tau)}{\Omega}\left(-\Omega_{11}\sum_{j=1}^{n}\mu_j\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}v(\lambda_j) + \Omega_{21}\sum_{i=1}^{m}\xi_i\mathcal{I}_{a^+}^{\alpha+\beta;\psi}v(\eta_i)\right). \end{aligned}$$

Thanks to (i) of Remark 2, it is implied that

$$
\begin{aligned}
& \left|z(\tau)-\mathcal{Z}(\tau)-\mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_z(\tau)\right| \\
= & \left|\mathcal{I}_{a^+}^{\alpha+\beta;\psi}v(\tau)+\frac{\Psi^{\gamma_1+\beta-1}(\tau)}{\Omega}\left(-\Omega_{22}\sum_{i=1}^{m}\xi_i\mathcal{I}_{a^+}^{\alpha+\beta;\psi}v(\eta_i)+\Omega_{12}\sum_{j=1}^{n}\mu_j\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}\mathcal{F}_z(\lambda_j)\right)\right. \\
& \left.+\frac{\Psi^{\gamma_2-1}(t)}{\Omega}\left(-\Lambda_{11}\sum_{j=1}^{n}\mu_j\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}v(\lambda_j)+\Omega_{21}\sum_{i=1}^{m}\xi_i\mathcal{I}_{a^+}^{\alpha+\beta;\psi}v(\eta_i)\right)\right| \\
\le & \left\{\Psi^{\alpha+\beta}(b)+\frac{1}{|\Omega|}\left(|\Omega_{22}|\Psi^{\gamma_1+\beta-1}(b)+|\Omega_{21}|\Psi^{\gamma_2-1}(b)\right)\sum_{i=1}^{m}|\xi_i|\Psi^{\alpha+\beta}(\eta_i)\right. \\
& \left.+\frac{1}{|\Omega|}\left(|\Omega_{12}|\Psi^{\gamma_1+\beta-1}(b)+|\Omega_{11}|\Psi^{\gamma_2-1}(b)\right)\sum_{j=1}^{n}|\mu_j|\Psi^{\alpha+\beta-\phi_j}(\lambda_j)\right\}\epsilon \\
= & \left\{\Psi^{\alpha+\beta}(b)+\Phi(\Omega_{22},\Omega_{21})\sum_{i=1}^{m}|\xi_i|\Psi^{\alpha+\beta}(\eta_i)+\Phi(\Omega_{12},\Lambda_{11})\sum_{j=1}^{n}|\mu_j|\Psi^{\alpha+\beta-\phi_j}(\lambda_j)\right\}\epsilon.
\end{aligned}
$$

The proof of (53) is done. □

Next, we establish $\mathbb{UH}$ and $\mathbb{GUH}$ stables of solutions to the problem (3).

Theorem 4. *Assume that $f:\mathcal{J}\times\mathbb{R}^3\to\mathbb{R}$ is continuous. Suppose that assumptions $(\mathcal{P}_1)$–$(\mathcal{P}_3)$ and*

$$\mathcal{L}_1\Psi^{\alpha+\beta}(b)+\mathcal{L}_2k_1^*\Psi^{\theta+\alpha+\beta}(b)+\mathcal{L}_3w_1^*\Psi^{\delta+\alpha+\beta}(b)<1. \tag{56}$$

Then, the ψ-Hilfer $\mathbb{FBVP}$ describing Navier model with $\mathbb{NIBC}$s (3) is $\mathbb{UH}$ and $\mathbb{GUH}$ stables.

Proof. Assume that $z\in\mathcal{E}$ is a solution of (45), and $u\in\mathcal{E}$ is a unique solution of (3). From Lemma 4, which implies that $u(\tau)=\mathcal{X}(\tau)+\mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_u(\tau)$, where

$$
\begin{aligned}
\mathcal{X}(\tau) = & \tfrac{\Psi^{\gamma_1+\beta-1}(t)}{\Omega}\Big[\Omega_{22}\Big(\mathcal{I}_{a^+}^{\varphi;\psi}\mathcal{G}(\sigma,u(\sigma))-\textstyle\sum_{i=1}^{m}\xi_i\mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_u(\eta_i)\Big) \\
& -\Omega_{12}\Big(\mathcal{I}_{a^+}^{\nu;\psi}\mathcal{H}(\zeta,u(\zeta))-\textstyle\sum_{j=1}^{n}\mu_j\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}\mathcal{F}_u(\lambda_j)\Big)\Big] \\
& +\tfrac{\Psi^{\gamma_2-1}(t)}{\Omega}\Big[\Omega_{11}\Big(\mathcal{I}_{a^+}^{\nu;\psi}\mathcal{H}(\zeta,u(\zeta))-\textstyle\sum_{j=1}^{n}\mu_j\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}\mathcal{F}_u(\lambda_j)\Big) \\
& -\Omega_{21}\Big(\mathcal{I}_{a^+}^{\varphi;\psi}\mathcal{G}(\sigma,u(\sigma))-\textstyle\sum_{i=1}^{m}\xi_i\mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_u(\eta_i)\Big)\Big].
\end{aligned} \tag{57}
$$

Clearly, if $u(a)=z(a)$, ${}^H\mathfrak{D}_{a^+}^{\beta,\rho;\psi}u(a)={}^H\mathfrak{D}_{a^+}^{\beta,\rho;\psi}z(a)$, $\sum_{i=1}^{m}\xi_iu(\eta_i)=\sum_{i=1}^{m}\xi_iz(\eta_i)$, $\mathcal{I}_{a^+}^{\varphi;\psi}\mathcal{G}(\sigma,u(\sigma))=\mathcal{I}_{a^+}^{\varphi;\psi}\mathcal{G}(\sigma,z(\sigma))$, $\sum_{j=1}^{n}\mu_j{}^H\mathfrak{D}_{a^+}^{\phi_j,\rho;\psi}u(\lambda_j)=\sum_{j=1}^{n}\mu_j{}^H\mathfrak{D}_{a^+}^{\phi_j,\rho;\psi}z(\lambda_j)$, and $\mathcal{I}_{a^+}^{\nu;\psi}\mathcal{H}(\zeta,u(\zeta))=\mathcal{I}_{a^+}^{\nu;\psi}\mathcal{H}(\zeta,z(\zeta))$, then, we get that $\mathcal{X}(\tau)=\mathcal{Z}(\tau)$.

By using Lemma 10 and $|u+v|\le|u|+|v|$, for any $\tau\in\mathcal{J}$, yields that

$$
\begin{aligned}
& |z(\tau)-u(\tau)| \\
= & |z(\tau)-\mathcal{X}(\tau)-\mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_u(\tau)| \\
\le & |z(\tau)-\mathcal{Z}(\tau)-\mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_z(\tau)|+\mathcal{I}_{a^+}^{\alpha+\beta;\psi}|\mathcal{F}_z(\tau)-\mathcal{F}_u(\tau)|+|\mathcal{Z}(\tau)-\mathcal{X}(\tau)| \\
\le & \Lambda(\alpha+\beta)\,\epsilon+\left(\mathcal{L}_1\Psi^{\alpha+\beta}(b)+\mathcal{L}_2k_1^*\Psi^{\theta+\alpha+\beta}(b)+\mathcal{L}_3w_1^*\Psi^{\delta+\alpha+\beta}(b)\right)|z(\tau)-u(\tau)|,
\end{aligned}
$$

that is $|z(\tau) - u(\tau)| \leq \mathfrak{C}_f\, \epsilon$, where

$$\mathfrak{C}_f := \frac{\Lambda(\alpha+\beta)}{1 - \left(\mathcal{L}_1 \Psi^{\alpha+\beta}(b) + \mathcal{L}_2 k_1^* \Psi^{\theta+\alpha+\beta}(b) + \mathcal{L}_3 w_1^* \Psi^{\delta+\alpha+\beta}(b)\right)}.$$

Hence, the problem (3) is $\mathbb{UH}$ stable in $\mathcal{E}$. Moreover, if we take $\mathcal{T}(\epsilon) = \mathfrak{C}_f \epsilon$ with $\mathcal{T}(0) = 0$, thus, (3) is $\mathbb{GUH}$ stable in $\mathcal{E}$. $\square$

4.2. $\mathbb{UHR}$ Stability and $\mathbb{GUHR}$ Stability

Next, the result will be applied in the investigation results of $\mathbb{UHR}$ and $\mathbb{GUHR}$ stables.

Lemma 11. *Assume that $\alpha \in (3,4]$, $\rho \in [0,1]$, and $z \in \mathcal{E}$ is a solution of (47). Then, $z \in \mathcal{E}$ verifies*

$$\left| z(\tau) - \mathcal{Z}(\tau) - \mathcal{I}_{a^+}^{\alpha+\beta;\psi} \mathcal{F}_z(\tau) \right| \leq \Theta \epsilon \lambda_{\mathcal{T}} \mathcal{T}(\tau),$$

where

$$\Theta = 1 + \Phi(\Omega_{22}, \Omega_{21}) \sum_{i=1}^{m} |\xi_i| + \Phi(\Omega_{12}, \Omega_{11}) \sum_{j=1}^{n} |\mu_j|, \tag{58}$$

and $\mathcal{Z}(\tau)$ is given by (54).

Proof. Assume that z is a solution of (47). By applying Lemma 4 and (ii) of Remark 3, then, the solution of the problem

$$\begin{cases} {}^H\mathfrak{D}_{a^+}^{\alpha,\rho;\psi}\left({}^H\mathfrak{D}_{a^+}^{\beta,\rho;\psi} z\right)(\tau) = \mathcal{F}_z(\tau) + w(\tau) \quad \tau \in (a,b), \\ z(a) = 0, \qquad {}^H\mathfrak{D}_{a^+}^{\beta,\rho;\psi} z(a) = 0, \\ \sum_{i=1}^{m} \xi_i z(\eta_i) = \mathcal{I}_{a^+}^{\varphi;\psi} \mathcal{G}(\sigma, z(\sigma)), \qquad \sum_{j=1}^{n} \mu_j {}^H\mathfrak{D}_{a^+}^{\phi_j,\rho;\psi} z(\lambda_j) = \mathcal{I}_{a^+}^{\nu;\psi} \mathcal{H}(\zeta, z(\zeta)), \end{cases} \tag{59}$$

is given by

$$\begin{aligned} z(\tau) \;=\; & \mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_z(\tau) + \frac{\Psi^{\gamma_1+\beta-1}(\tau)}{\Omega}\Bigg[\Omega_{22}\left(\mathcal{I}_{a^+}^{\varphi;\psi}\mathcal{G}(\sigma,z(\sigma)) - \sum_{i=1}^{m}\xi_i \mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_z(\eta_i)\right) \\ & -\Omega_{12}\left(\mathcal{I}_{a^+}^{\nu;\psi}\mathcal{H}(\zeta,z(\zeta)) - \sum_{j=1}^{n}\mu_j \mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}\mathcal{F}_z(\lambda_j)\right)\Bigg] \\ & + \frac{\Psi^{\gamma_2-1}(\tau)}{\Omega}\Bigg[\Omega_{11}\left(\mathcal{I}_{a^+}^{\nu;\psi}\mathcal{H}(\zeta,z(\zeta)) - \sum_{j=1}^{n}\mu_j \mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}\mathcal{F}_z(\lambda_j)\right) \\ & -\Omega_{21}\left(\mathcal{I}_{a^+}^{\varphi;\psi}\mathcal{G}(\sigma,z(\sigma)) - \sum_{i=1}^{m}\xi_i \mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_z(\eta_i)\right)\Bigg] + \mathcal{I}_{a^+}^{\alpha+\beta;\psi} w(\tau) \\ & + \frac{\Psi^{\gamma_1+\beta-1}(\tau)}{\Omega}\left(-\Omega_{22}\sum_{i=1}^{m}\xi_i \mathcal{I}_{a^+}^{\alpha+\beta;\psi} w(\eta_i) + \Omega_{12}\sum_{j=1}^{n}\mu_j \mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi} w(\lambda_j)\right) \\ & + \frac{\Psi^{\gamma_2-1}(\tau)}{\Omega}\left(-\Omega_{11}\sum_{j=1}^{n}\mu_j \mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi} w(\lambda_j) + \Omega_{21}\sum_{i=1}^{m}\xi_i \mathcal{I}_{a^+}^{\alpha+\beta;\psi} w(\eta_i)\right). \end{aligned}$$

Thanks to (i) of Remarks 3 and 4, one has

$$
\begin{aligned}
&\left|z(\tau)-\mathcal{Z}(\tau)-\mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_z(\tau)\right|\\
=\ &\Bigg|\mathcal{I}_{a^+}^{\alpha+\beta;\psi}w(\tau)+\frac{\Psi^{\gamma_1+\beta-1}(\tau)}{\Omega}\left(-\Omega_{22}\sum_{i=1}^{m}\xi_i\mathcal{I}_{a^+}^{\alpha+\beta;\psi}w(\eta_i)+\Omega_{12}\sum_{j=1}^{n}\mu_j\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}w(\lambda_j)\right)\\
&+\frac{\Psi^{\gamma_2-1}(\tau)}{\Omega}\left(-\Omega_{11}\sum_{j=1}^{n}\mu_j\mathcal{I}_{a^+}^{\alpha+\beta-\phi_j;\psi}w(\lambda_j)+\Omega_{21}\sum_{i=1}^{m}\xi_i\mathcal{I}_{a^+}^{\alpha+\beta;\psi}w(\eta_i)\right)\Bigg|\\
\leq\ &\Bigg\{1+\frac{1}{|\Omega|}\left(|\Omega_{22}|\Psi^{\gamma_1+\beta-1}(b)+|\Omega_{21}|\Psi^{\gamma_2-1}(b)\right)\sum_{i=1}^{m}|\xi_i|\\
&+\frac{1}{\Omega}\left(|\Omega_{12}|\Psi^{\gamma_1+\beta-1}(b)+|\Omega_{11}|\Psi^{\gamma_2-1}(b)\right)\sum_{j=1}^{n}|\mu_j|\Bigg\}\epsilon\lambda_{\mathcal{T}}\mathcal{T}(\tau)\\
=\ &\left\{1+\Phi(\Omega_{22},\Omega_{21})\sum_{i=1}^{m}|\xi_i|+\Phi(\Omega_{12},\Omega_{11})\sum_{j=1}^{n}|\mu_j|\right\}\epsilon\lambda_{\mathcal{T}}\mathcal{T}(\tau).
\end{aligned}
$$

The proof is done. □

This result studies $\mathbb{UHR}$ and $\mathbb{GUHR}$ stables of solutions to the problem (3).

Theorem 5. *Let $f:\mathcal{J}\times\mathbb{R}^3\to\mathbb{R}$ be continuous. Assume that $(\mathcal{P}_1)$-$(\mathcal{P}_3)$, (52) is fulfilled, and*

$$
\mathcal{L}_1\Psi^{\alpha+\beta}(b)+\mathcal{L}_2k_1^*\Psi^{\theta+\alpha+\beta}(b)+\mathcal{L}_3w_1^*\Psi^{\delta+\alpha+\beta}(b)<1. \tag{60}
$$

Then, the ψ-Hilfer $\mathbb{FBVP}$ describing Navier model with $\mathbb{NIBC}$s (3) is $\mathbb{UHR}$ and $\mathbb{GUHR}$ stables.

Proof. Assume that $z\in\mathcal{E}$ is a solution of (47), and x is a unique solution of (3). By applying Lemma 11, this yields that $u(\tau)=\mathcal{X}(\tau)+\mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_u(\tau)$, where $\mathcal{X}(\tau)$ is given by (57). Similarly, if $u(a)=z(a)$, ${}^{H}\mathfrak{D}_{a^+}^{\beta,\rho;\psi}u(a)={}^{H}\mathfrak{D}_{a^+}^{\beta,\rho;\psi}z(a)$, $\sum_{i=1}^{m}\xi_i u(\eta_i)=\sum_{i=1}^{m}\xi_i z(\eta_i)$, $\mathcal{I}_{a^+}^{\varphi;\psi}\mathcal{G}(\sigma,u(\sigma))=\mathcal{I}_{a^+}^{\varphi;\psi}\mathcal{G}(\sigma,z(\sigma))$, $\sum_{j=1}^{n}\mu_j{}^{H}\mathfrak{D}_{a^+}^{\phi_j,\rho;\psi}u(\lambda_j)=\sum_{j=1}^{n}\mu_j{}^{H}\mathfrak{D}_{a^+}^{\phi_j,\rho;\psi}z(\lambda_j)$, and $\mathcal{I}_{a^+}^{\nu;\psi}\mathcal{H}(\zeta,u(\zeta))=\mathcal{I}_{a^+}^{\nu;\psi}\mathcal{H}(\zeta,z(\zeta))$, then $\mathcal{X}(\tau)=\mathcal{Z}(\tau)$.

Applying Lemma 11 with triangle inequality, for any $\tau\in\mathcal{J}$, it follows that

$$
\begin{aligned}
&|z(\tau)-u(\tau)|\\
=\ &|z(\tau)-\mathcal{X}(\tau)-\mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_u(\tau)|\\
\leq\ &|z(\tau)-\mathcal{Z}(\tau)-\mathcal{I}_{a^+}^{\alpha+\beta;\psi}\mathcal{F}_z(\tau)|+\mathcal{I}_{a^+}^{\alpha+\beta;\psi}|\mathcal{F}_z(\tau)-\mathcal{F}_u(\tau)|+|\mathcal{Z}(\tau)-\mathcal{X}(\tau)|\\
\leq\ &\Theta\epsilon\lambda_{\mathcal{T}}\mathcal{T}(\tau)+\left(\mathcal{L}_1\Psi^{\alpha+\beta}(b)+\mathcal{L}_2k_1^*\Psi^{\theta+\alpha+\beta}(b)+\mathcal{L}_3w_1^*\Psi^{\delta+\alpha+\beta}(b)\right)|z(\tau)-u(\tau)|,
\end{aligned}
$$

where Θ is given as in (58); thus, $|z(\tau)-x(\tau)|\leq\mathfrak{C}_{f,\mathcal{T}}\mathcal{T}(\tau)\epsilon$ such that

$$
\mathfrak{C}_{f,\mathcal{T}}=\frac{\Theta\lambda_{\mathcal{T}}}{1-\left(\mathcal{L}_1\Psi^{\alpha+\beta}(b)+\mathcal{L}_2k_1^*\Psi^{\theta+\alpha+\beta}(b)+\mathcal{L}_3w_1^*\Psi^{\delta+\alpha+\beta}(b)\right)}.
$$

Hence, the problem (3) is $\mathbb{UHR}$ stable in $\mathcal{E}$.

Additionally, if we take $\epsilon=1$, in $|z(\tau)-x(\tau)|\leq\mathfrak{C}_{f,\mathcal{T}}\mathcal{T}(\tau)\epsilon$, with $\mathcal{T}(0)=0$, hence (3) is $\mathbb{GUHR}$ stable in $\mathcal{E}$. □

5. Examples

This section shows some illustrative examples of the exactness and applicability of the main results.

Example 1. *The ψ-Hilfer $\mathbb{FBVP}$ describing Navier model with $\mathbb{NIBC}$s:*

$$\begin{cases} {}^{H}\mathfrak{D}_{0^+}^{\frac{3}{2},\frac{4}{5};\frac{1}{2}\tan\left(\frac{\pi\tau}{\tau+3}\right)}\left({}^{H}\mathfrak{D}_{0^+}^{\frac{6}{5},\frac{4}{5};\frac{1}{2}\tan\left(\frac{\pi\tau}{\tau+3}\right)}x\right)(t) = f(\tau,u(\tau),(\mathcal{K}u)(\tau),(\mathcal{W}u)(\tau)),\ \tau\in(0,3/2), \\ u(0)=0, \qquad {}^{H}\mathfrak{D}_{0^+}^{\frac{6}{5},\frac{4}{5};\frac{1}{2}\tan\left(\frac{\pi\tau}{\tau+3}\right)}x(0)=0, \\ \sum_{i=1}^{2}\left(\frac{i}{i+1}\right)u\left(\frac{9i-6}{10}\right) = \mathcal{I}_{0^+}^{\frac{6}{5};\frac{1}{2}\tan\left(\frac{\pi\tau}{\tau+3}\right)}\mathcal{G}\left(\frac{6}{5},u\left(\frac{6}{5}\right)\right), \\ \sum_{j=1}^{3}\left(\frac{4-j}{6-j}\right){}^{H}\mathfrak{D}_{0^+}^{\frac{10-j}{5},\frac{4}{5};\frac{1}{2}\tan\left(\frac{\pi\tau}{\tau+3}\right)}u\left(\frac{4j-1}{10}\right) = \mathcal{I}_{0^+}^{\frac{3}{2};\frac{1}{2}\tan\left(\frac{\pi\tau}{\tau+3}\right)}\mathcal{H}\left(\frac{11}{10},u\left(\frac{11}{10}\right)\right). \end{cases} \tag{61}$$

Setting $\alpha=3/2$, $\rho=4/5$, $\psi(\tau)=0.5\tan(\pi\tau/(\tau+3))$,$\beta=6/5$, $a=0$, $b=3/2$, $\xi_i=i/(i+1)$, $\eta_i=(9i-6)/10$, $\varphi=6/5$, $\sigma=6/5$, $\mu_j=(4-j)/(6-j)$, $\phi_j=(10-j)/5$, $\lambda_j=(4j-1)/10$, $\nu=3/2$, $\zeta=11/10$, $i=1,2$, and $j=1,2,3$. From the given all datas, we obtain $\Omega_{11}\approx 0.11786018$, $\Omega_{12}\approx 0.5836819$, $\Omega_{21}\approx 0.94874011$, $\Omega_{22}\approx 0.68483544$, and $\Omega\approx -0.47304769\neq 0$. We consider $f(\tau,u(\tau),(\mathcal{K}u)(\tau),(\mathcal{W}u)(\tau))$, $\mathcal{H}(\zeta,u(\zeta))$, and $\mathcal{G}(\sigma,u(\sigma))$ as follows:

(i) Given the function

$$\begin{aligned} f(\tau,u(\tau),(\mathcal{K}u)(\tau),(\mathcal{W}u)(\tau)) &= \frac{\tau^4+2\tau^2-6}{5\tau^2+3t+4}+\frac{5e^{-2\tau-1}}{4-\sin^2\pi\tau}\cdot\frac{|u(\tau)|}{5+|u(\tau)|} \\ &\quad +\frac{2\tau+6}{3^{5\tau+2}}\left[\frac{|(\mathcal{K}u)(\tau)|}{4+|(\mathcal{K}u)(\tau)|}+\frac{|(\mathcal{W}u)(\tau))|}{5+|(\mathcal{W}u)(\tau))|}\right], \end{aligned}$$

$$\mathcal{H}(\zeta,u(\zeta))=\frac{2}{6+|u(\xi)|}, \qquad \mathcal{G}(\sigma,u(\sigma))=\frac{2|u(\sigma)|}{4+5|u(\sigma)|},$$

where

$$(\mathcal{K}u)(\tau) = \frac{1}{\Gamma(0.8)}\int_0^\tau(\psi(\tau)-\psi(s))^{0.8-1}\psi'(s)k(\tau,s)u(s)ds, \tag{62}$$

$$(\mathcal{W}u)(\tau) = \frac{1}{\Gamma(1.7)}\int_0^\tau(\psi(\tau)-\psi(s))^{1.7-1}\psi'(s)w(\tau,s)u(s)ds, \tag{63}$$

with

$$k(\tau,s)=\frac{2\sin(\pi\tau)}{s^2+2\tau+5}, \qquad w(\tau,s)=\frac{3(2\tau-2)}{5+\cos(\pi s)}.$$

For u_i, v_i, $w_i\in\mathbb{R}$, $i=1,2$, $\tau\in[0,3/2]$, we obtain

$$|f(\tau,u_1,v_1,w_1)-f(\tau,u_2,v_2,w_2)|\le\frac{1}{3}|u_1-u_2|+\frac{1}{4}|v_1-v_2|+\frac{1}{5}|w_1-w_2|,$$

$$|\mathcal{H}(\tau,u_1)-\mathcal{H}(\tau,u_2)|\le\frac{1}{3}|u_1-u_2| \quad \text{and} \quad |\mathcal{G}(\tau,u_1)-\mathcal{G}(\tau,u_2)|\le\frac{1}{2}|u_1-u_2|.$$

The conditions $(\mathcal{P}_1)$–$(\mathcal{P}_2)$ are satisfied with $\mathcal{L}_1=1/3$, $\mathcal{L}_2=1/4$, $\mathcal{L}_3=1/5$, $\mathcal{H}_1^*=1/3$, $\mathcal{G}_1^*=1/2$, $k_1^*=2/5$, and $w_1^*=3/5$. Hence, $\Delta_1+\Delta_2\approx 0.6077104343<1$. Since, Theorem 1 are fulfilled. Then, the problem (61) has a unique solution on $[0,3/2]$. Moreover, we have

$$\mathfrak{C}_f := \frac{\Lambda(\alpha+\beta)}{1-\left(\mathcal{L}_1\Psi^{\alpha+\beta}(b)+\mathcal{L}_2k_1^*\Psi^{\theta+\alpha+\beta}(b)+\mathcal{L}_3w_1^*\Psi^{\delta+\alpha+\beta}(b)\right)}\approx 0.55937654>0.$$

From Theorem 4, the problem (61) is $\mathbb{UH}$ and $\mathbb{GUH}$ stables on $[0,3/2]$. Take $\mathcal{T}(\tau)=(\psi(\tau)-\psi(0))^{1/3}$, and we have

$$\mathcal{I}_{a^+}^{\alpha;\psi}\mathcal{T}(\tau)=\frac{\Gamma(4/3)}{\Gamma(17/6)}(\psi(\tau)-\psi(0))^{3/2}\mathcal{T}(\tau)\le\frac{\Gamma(4/3)\tan^{3/2}(\pi/3)}{2\sqrt{2}\Gamma(17/6)}\,\mathcal{T}(\tau).$$

The inequality (52) *is fulfilled with* $\lambda_{\mathcal{T}} = \frac{\Gamma(4/3)\tan^{3/2}(\pi/3)}{2\sqrt{2}\Gamma(17/6)} > 0$ *and* $\Theta \approx 4.6986661$. *Then,*

$$\mathfrak{C}_{f,\mathcal{T}} = \frac{\Theta\lambda_{\mathcal{T}}}{1-\left(\mathcal{L}_1\Psi^{\alpha+\beta}(b)+\mathcal{L}_2k_1^*\Psi^{\theta+\alpha+\beta}(b)+\mathcal{L}_3w_1^*\Psi^{\delta+\alpha+\beta}(b)\right)} \approx 2.08782437 > 0.$$

Therefore, from Theorem 5, *the problem* (61) *is* $\mathbb{UHR}$ *and* $\mathbb{GUHR}$ *stables on* $[0,3/2]$.

(ii) Given the function

$$\begin{aligned} f(\tau,u(\tau),(\mathcal{K}u)(\tau),(\mathcal{W}u)(\tau)) &= \frac{2\cos(\pi\tau)+1}{3^{\sin(\pi\tau)+2}}\cdot\frac{|u(\tau)|+3}{2+|u(\tau)|} \\ &\quad+\frac{4\tau-1}{5^{3\tau+2}}\cdot\left[\frac{|(\mathcal{K}u)(\tau)|+|(\mathcal{W}u)(\tau)|}{4+|(\mathcal{K}u)(\tau)|+|(\mathcal{W}u)(\tau)|}\right], \\ \mathcal{H}(\tau,u(\tau)) = \frac{4\tau+2}{5-2\tau}\cdot\frac{|u(\tau)|}{8+5|u(\tau)|}, &\quad \mathcal{G}(\tau,u(\tau)) = \frac{\tau\sin(u(\tau))}{9}, \end{aligned}$$

with (62) *and* (63), *where* $k(\tau,s) = 2/(5+s^{\tau+1})$ *and* $w(\tau,s) = 3/(7-2\sin(\pi s\tau))$.

For u, v, $w \in \mathbb{R}$, *and* $\tau \in [0,3/2]$, *we estimate that*

$$\begin{aligned} |f(\tau,u,v,w)| &\le \frac{1+2\cos(\pi\tau)}{3^{\sin(\pi\tau)+2}}\cdot\frac{|u(\tau)|+3}{2}+\frac{4\tau-1}{5^{3\tau+2}}\cdot\frac{|v|+|w|}{4} \\ |\mathcal{H}(\tau,u)| &\le \frac{4\tau+2}{5-2\tau}\cdot\frac{|u|}{8}, \qquad |\mathcal{G}(\tau,u)| \le \frac{\tau}{9}\cdot|u|. \end{aligned}$$

The assumption $(\mathcal{P}_4)$ *is also valid with* $p_1(\tau) = (1+2\cos(\pi\tau))/(3^{\sin(\pi\tau)+2})$, $p_2(\tau) = (4\tau-1)/(4\cdot 5^{3\tau+2}) = p_3(\tau)$, $q_1(\tau) = (4\tau+2)/(5-2\tau)$, $q_2(\tau) = \tau/9$, $\mathbb{U}(|u|) = (|u|+3)/2$, $\mathbb{V}(|u|) = |u|/8$ *and* $\mathbb{W}(|u|) = |u|$. *Thus,* $p_1^* = 1/3$, $p_2^* = 1/20 = p_3^*$, $q_1^* = 4$, $q_2^* = 1/6$, $k_1^* = 2/5$, *and* $w_1^* = 3/5$. *There is a positive constant* $\mathcal{M}^* > 0.37949755$ *verifying* $(\mathcal{P}_5)$. *Then, Theorem* 2 *is fulfilled, and we can summarize that the problem* (61) *has at least one solution on* $[0,3/2]$.

For any u_i, v_i, $w_i \in \mathbb{R}$, $i = 1,2$, *and* $\tau \in [0,3/2]$, *one has*

$$|f(\tau,u_1,v_1,w_1)-f(\tau,u_2,v_2,w_2)| \le \frac{5}{12}|u_1-u_2|+\frac{1}{20}|v_1-v_2|+\frac{1}{20}|w_1-w_2|.$$

The conditions $(\mathcal{P}_1)$-$(\mathcal{P}_2)$ *are verified with* $\mathcal{L}_1 = 5/12$, $\mathcal{L}_2 = \mathcal{L}_3 = 1/20$, $\mathcal{H}_1^* = 1/2$, $\mathcal{G}_1^* = 1/6$, $k_1^* = 2/5$ *and* $w_1^* = 3/5$. *Thus,* $\Delta_1+\Delta_2 \approx 0.44078217 < 1$. *Hence, the problem* (61) *has a unique solution on* $[0,3/2]$. *Moreover, we obtain* $\mathfrak{C}_f := 0.56437711 > 0$. *Theorem* 4 *is satisfied, the problem* (61) *is* $\mathbb{UH}$ *and* $\mathbb{GUH}$ *stables on* $[0,3/2]$. *Take* $\mathcal{T}(\tau) = (\psi(\tau)-\psi(0))^{1/2}$, *and we have*

$$\mathcal{I}_{a^+}^{\alpha;\psi}\mathcal{T}(\tau) = \frac{1}{4\sqrt{\pi}}(\psi(t)-\psi(0))^{3/2}\mathcal{T}(\tau) \le \frac{\tan^{3/2}(\pi/3)}{8\sqrt{2\pi}}\mathcal{T}(\tau).$$

The inequality (52) *is satisfied with* $\lambda_{\mathcal{T}} = \frac{\tan^{3/2}(\pi/3)}{8\sqrt{2\pi}} > 0$ *and* $\Theta \approx 4.69866612$. *Then,*

$$\mathfrak{C}_{f,\mathcal{T}} = \frac{\Theta\lambda_{\mathcal{T}}}{1-\left(\mathcal{L}_1\Psi^{\alpha+\beta}(b)+\mathcal{L}_2k_1^*\Psi^{\theta+\alpha+\beta}(b)+\mathcal{L}_3w_1^*\Psi^{\delta+\alpha+\beta}(b)\right)} \approx 2.10648855 > 0.$$

Hence, Theorem 5 *is true, and the problem* (61) *is* $\mathbb{UHR}$ *and* $\mathbb{GUHR}$ *stables on* $[0,3/2]$.

(iii) Given the function

$$
\begin{aligned}
f(\tau,u(\tau),(\mathcal{K}u)(\tau),(\mathcal{W}u)(\tau)) &= \frac{e^{3\tau-5}}{\ln(5-2\tau)}+\frac{\sqrt{4\tau^2+2\tau+4}}{\tau\sin(\tau)+6}\sin(u(\tau))\\
&\quad+\frac{2\tau}{9-2\tau}\cdot\frac{|(\mathcal{K}u)(\tau)|}{1+2|(\mathcal{K}u)(\tau)|}\\
&\quad+\frac{4\cos\tau}{3^{6\tau+2}}\arctan\big((\mathcal{W}u)(\tau)\big),\\
\mathcal{H}(\tau,u(\tau)) = \frac{|2u(\tau)+5|}{\tau^2+5}, &\qquad \mathcal{G}(\tau,u(\tau)) = \frac{2\tau\sin(u(\tau))+3}{5+\ln(2\tau+5)},
\end{aligned}
$$

with (62) and (63), where $k(\tau,s)=4/(7+s\tau)$ and $w(\tau,s)=2/(7+2s\cos^2(\pi\tau))$.

For any u_i, v_i, $w_i\in\mathbb{R}$, $i=1,2$, and $\tau\in[0,3/2]$, we obtain

$$
\begin{aligned}
&|f(\tau,u_1,v_1,w_1)-f(\tau,u_2,v_2,w_2)|\le\frac{2}{3}|u_1-u_2|+\frac{1}{2}|v_1-v_2|+\frac{4}{9}|w_1-w_2|,\\
&|\mathcal{H}(\tau,u_1)-\mathcal{H}(\tau,u_2)|\le\frac{2}{5}|u_1-u_2| \quad \text{and} \quad |\mathcal{G}(\tau,u_1)-\mathcal{G}(\tau,u_2)|\le\frac{3}{5}|u_1-u_2|.
\end{aligned}
$$

The conditions $(\mathcal{P}_1)$-$(\mathcal{P}_2)$ are satisfied with $\mathcal{L}_1=2/3$, $\mathcal{L}_2=1/2$, $\mathcal{L}_3=4/9$, $\mathcal{H}_1^*=2/5$, $\mathcal{G}_1^*=3/5$, $k_1^*=4/7$, and $w_1^*=2/7$. Hence, we have

$$
\Big(\Delta_1+\Delta_2-\Psi^{\alpha+\beta}(b)\mathcal{L}_1-\Psi^{\theta+\alpha+\beta}(b)k_1^*\mathcal{L}_2-\Psi^{\delta+\alpha+\beta}(b)w_1^*\mathcal{L}_3\Big)\approx 0.76735591<1.
$$

For u, v, $w\in\mathbb{R}$, and $\tau\in[0,3/2]$, we have

$$
\begin{aligned}
|f(\tau,u,v,w)| &\le \frac{e^{3\tau-5}}{\ln(5-2\tau)}+\frac{\sqrt{4\tau^2+2\tau+4}}{\tau\sin(\tau)+6}|u(\tau)|\\
&\quad+\frac{2\tau}{9-2\tau}|(\mathcal{K}u)(\tau)|+\frac{4\cos\tau}{3^{6\tau+2}}|(\mathcal{W}u)(\tau)|,\\
|\mathcal{H}(\tau,u)| &\le \frac{2|u|}{\tau^2+5}+\frac{5}{\tau^2+5}, \quad |\mathcal{G}(\tau,u)|\le\frac{2\tau|u|}{5+\ln(2\tau+5)}+\frac{3}{5+\ln(2\tau+5)}.
\end{aligned}
$$

The condition $(\mathcal{P}_5)$ is verified with $f_1(\tau)=e^{3\tau-5}/\ln(5-2\tau)$, $f_2(\tau)=\sqrt{4\tau^2+2\tau+4}/(\tau\sin(\tau)+6)$, $f_3(\tau)=2\tau/(9-2\tau)$, $f_4(\tau)=4\cos\tau/3^{6\tau+2}$, $h_1(\tau)=5/(\tau^2+5)$, $h_2(\tau)=2/(\tau^2+5)$, $g_1(\tau)=3/(5+\ln(2\tau+5)$, and $g_2(\tau)=2\tau/(5+\ln(2\tau+5))$. So, Theorem 3 is verified, and we can summarize that the problem (61) has at least one solution on $[0,3/2]$.

In addition, the problem (61) has a unique solution on $[0,3/2]$ with $\Delta_1+\Delta_2\approx 0.89211721<1$. Moreover, we have that $\mathfrak{C}_f:=0.60023816>0$. Then, Theorem 4 is true, and the problem (61) is $\mathbb{UH}$ and $\mathbb{GUH}$ stables on $[0,3/2]$. Take $\mathcal{T}(\tau)=(\psi(\tau)-\psi(0))^{1/4}$, and we get $\lambda_{\mathcal{T}}=\frac{\Gamma(5/4)\tan^{3/2}(\pi/3)}{2\sqrt{2}\Gamma(11/4)}\approx 0.45418619>0$ and $\Theta\approx 4.69866612$. Then, we obtain $\mathfrak{C}_{f,\mathcal{T}}\approx 2.43827113>0$. From Theorem 5, then, the problem (61), is $\mathbb{UHR}$ and $\mathbb{GUHR}$ stables on $[0,3/2]$.

(iv) *Consider $f(\tau,u(\tau),(\mathcal{K}u)(\tau),(\mathcal{W}u)(\tau))=\varrho_1(\psi(\tau)-\psi(0))^{\varpi_1}$ and*

$$
\mathcal{G}(\tau,u(\tau))=\varrho_2(\psi(\tau)-\psi(0))^{\varpi_2}, \qquad \mathcal{H}(\tau,u(\tau))=\varrho_3(\psi(\tau)-\psi(0))^{\varpi_3}.
$$

By Lemma 4 with $\varrho_1=2$, $\varrho_2=3$, $\varrho_3=4$, $\varpi_1=1/7$, $\varpi_2=1/5$, and $\varpi_3=1/3$, the solution of the problem (61) is given by

$$
\begin{aligned}
& x(\tau) \\
= \; & \frac{2\Gamma(8/7)(\psi(\tau)-\psi(0))^{1/7+\alpha+\beta}}{\Gamma(8/7+\alpha+\beta)} + \frac{(\psi(\tau)-\psi(0))^{\gamma_1+\beta-1}}{\Omega\Gamma(\gamma_1+\beta)} \\
& \times \Bigg[\Omega_{22} \left(\frac{3\Gamma(6/5)(\psi(\sigma)-\psi(0))^{7/5}}{\Gamma(12/5)} - \sum_{i=1}^{m} \frac{2\Gamma(8/7)\xi_i(\psi(\eta_i)-\psi(0))^{1/7+\alpha+\beta}}{\Gamma(8/7+\alpha+\beta)} \right) \\
& -\Omega_{12} \left(\frac{4\Gamma(4/3)(\psi(\zeta)-\psi(0))^{11/6}}{\Gamma(17/6)} - \sum_{j=1}^{n} \frac{2\Gamma(8/7)\mu_j(\psi(\lambda_j)-\psi(0))^{1/7+\alpha+\beta-\phi_j}}{\Gamma(8/7+\alpha+\beta-\phi_j)} \right) \Bigg] \\
& + \frac{(\psi(t)-\psi(0))^{\gamma_2-1}}{\Omega\Gamma(\gamma_2)} \Bigg[\Omega_{11} \left(\frac{4\Gamma(4/3)(\psi(\zeta)-\psi(0))^{11/6}}{\Gamma(17/6)} \right. \\
& \left. - \sum_{j=1}^{n} \frac{2\Gamma(8/7)\mu_j(\psi(\lambda_j)-\psi(0))^{1/7+\alpha+\beta-\phi_j}}{\Gamma(8/7+\alpha+\beta-\phi_j)} \right) \\
& -\Omega_{21} \left(\frac{3\Gamma(6/5)(\psi(\sigma)-\psi(0))^{7/5}}{\Gamma(12/5)} - \sum_{i=1}^{m} \frac{2\Gamma(8/7)\xi_i(\psi(\eta_i)-\psi(0))^{1/7+\alpha+\beta}}{\Gamma(8/7+\alpha+\beta)} \right) \Bigg].
\end{aligned}
$$

A graph displaying of $u(\tau)$ for the problem (61) under $\alpha = 1.65, 1.70, \ldots, 2.00$ and $\beta = 1.86, 1.88, \ldots, 2.00$ with $\psi(\tau) = \tau^{\sqrt{\alpha}+\sqrt{\beta}}$, $(\sin\tau)^{\sqrt{\alpha}+\sqrt{\beta}}$, $(\alpha+\beta)^{\frac{\tau}{2}}$, $(\ln(\tau+1))^{\sqrt{\alpha+\beta}}$, is shown in Figures 1–4.

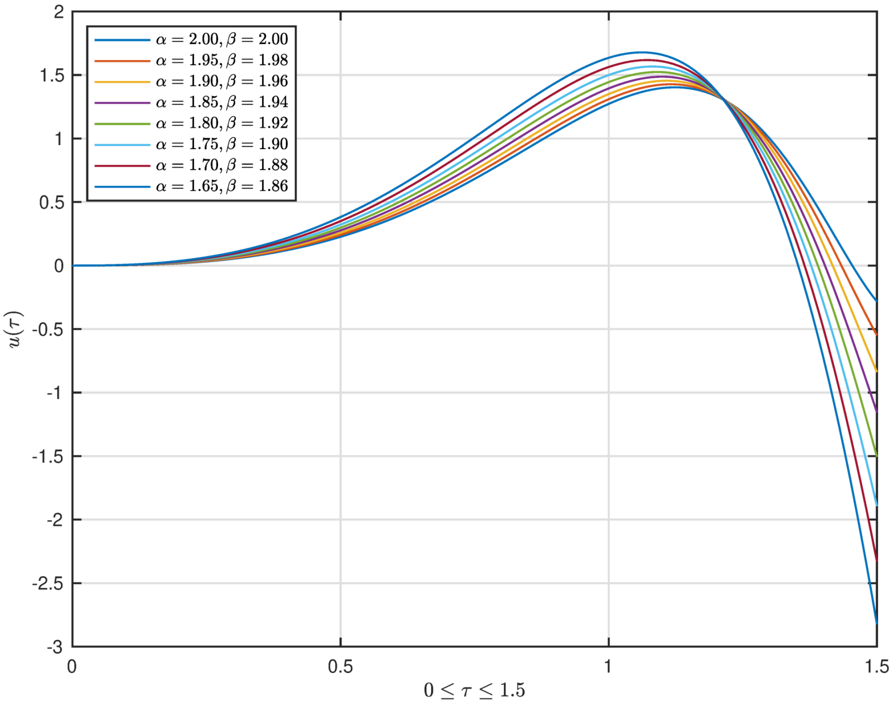

Figure 1. The graph displaying of $u(\tau)$ for (61) with $\psi(\tau) = \tau^{\sqrt{\alpha}+\sqrt{\beta}}$ for $\tau \in [0, 3/2]$.

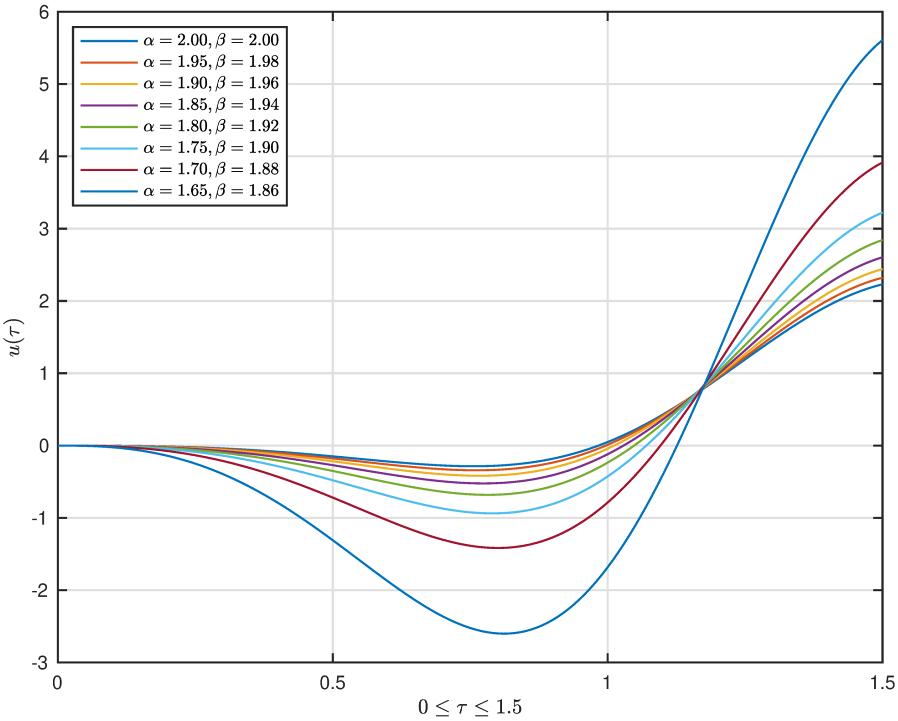

Figure 2. The graph displaying of $u(\tau)$ for (61) with $\psi(\tau) = (\sin\tau)^{\sqrt{\alpha}+\sqrt{\beta}}$ for $\tau \in [0, 3/2]$.

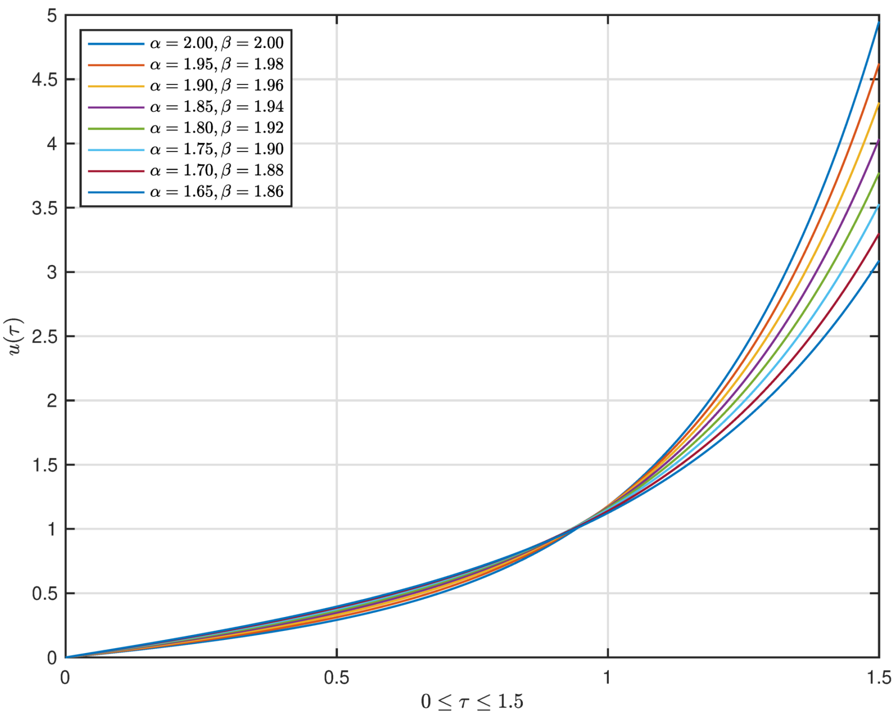

Figure 3. The graph displaying of $u(\tau)$ for (61) with $\psi(\tau) = (\alpha + \beta)^{\frac{\tau}{2}}$ for $\tau \in [0, 3/2]$.

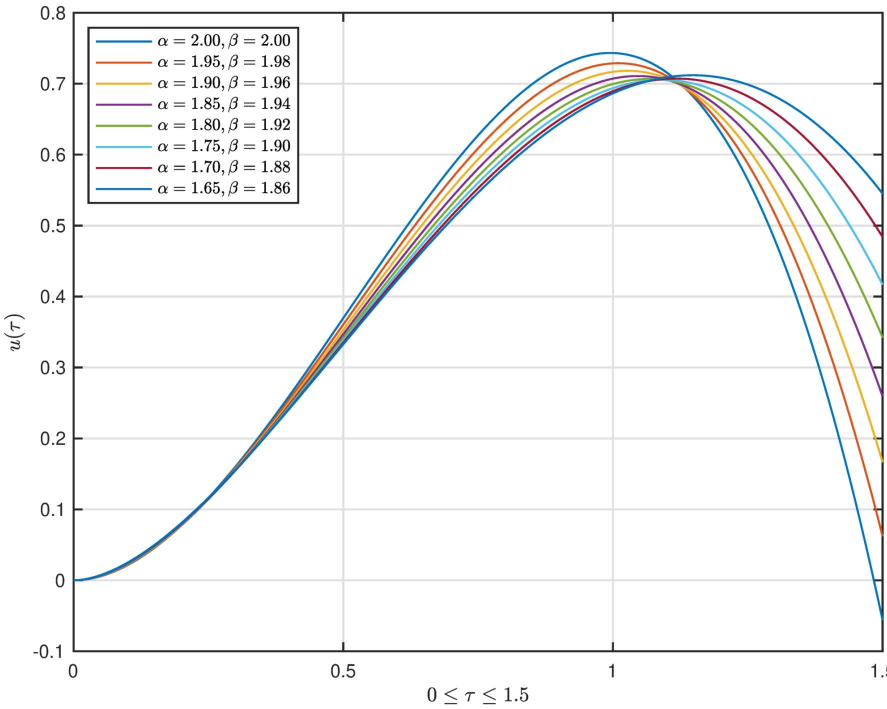

Figure 4. The graph displaying of $u(\tau)$ for (61) with $\psi(\tau) = (\ln(\tau+1))^{\sqrt{\alpha+\beta}}$ for $\tau \in [0, 3/2]$.

6. Conclusions

The main aims of this study have been accomplished. Firstly, the uniqueness result for a nonlinear ψ-Hilfer $\mathbb{FBVP}$ describing Navier model with $\mathbb{NIBC}$s was analyzed by helping Banach's fixed point theorem. Afterward, the existence results were established by applying fixed point theory of Leray-Schauder's and Kransnoselskii's types, while the guarantee of the existence of solutions was shown by the powerful techniques, such as Ulam's stability, including $\mathbb{UH}$, $\mathbb{GUH}$, $\mathbb{UHR}$, and $\mathbb{GUHR}$ stables. Finally, we ensured the theoretical results via some illustrates in the special cases of ψ are polynomial, trigonometry, exponential, and logarithm functions. This paper has considered different methods and is attractive for researchers who are interested in the work of the integro-differential equation describing Navier model under ψ-Hilfer fractional operators. We will concentrate on examining the qualitative theories of solutions to nonlinear equations or systems of real-world models with boundary conditions in the context of other fractional calculus in the future. It also remains to extend the results obtained to new Hilfer-type operators; see, for example, Reference [59].

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