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On the Local Convergence of Two-Step Newton Type Method in Banach Spaces under Generalized Lipschitz Conditions

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Abstract: The motive of this paper is to discuss the local convergence of a two-step Newton-type method of convergence rate three for solving nonlinear equations in Banach spaces. It is assumed that the first order derivative of nonlinear operator satisfies the generalized Lipschitz i.e., L-average condition. Also, some results on convergence of the same method in Banach spaces are established under the assumption that the derivative of the operators satisfies the radius or center Lipschitz condition with a weak L-average particularly it is assumed that L is positive integrable function but not necessarily non-decreasing. Our new idea gives a tighter convergence analysis without new conditions. The proposed technique is useful in expanding the applicability of iterative methods. Useful examples justify the theoretical conclusions.

Keywords: banach space; nonlinear problem; local convergence; lipschitz condition; L-average; convergence ball

MSC: 65H10

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1. Introduction

Consider a nonlinear operator $t : \Omega \subseteq X \to Y$ such that X and Y are two Banach spaces, Ω is a non-empty open convex subset and t is Fréchet differentiable nonlinear operator. Nonlinear problems has so many applications in the field of chemical engineering, transportation, operational research etc. which can be seen in the form of

$$t(x) = 0. \tag{1}$$

To find the solution of Equation (1), Newton's method defined as

$$x_{k+1} = x_k - [t'(x_k)]^{-1}t(x_k),\ k \geq 0, \tag{2}$$

is being preferred though its speed of convergence is low. Newton's method [1], is a well known iterative method which converges quadratically, which was initially studied by Kantorovich [2] and then scrutinized by Rall [3].

Some Newton-type methods with third-order convergence that do not require the computation of second order derivatives have been developed in the refs [4–7]. While the methods of higher R-order of convergence are generally not executed frequently despite having fast speed of convergence because the operational cost is high. However, the method of higher R-order of convergence can be used in the problems of stiff system [2] where fast convergence is required.

From the numerical point of view, the convergence domain plays a crucial role for the stable behaviour of an iterative scheme. Research about the convergence study of Newton methods involves two types: semilocal and local convergence analysis. The semilocal convergence study is based on the information around an initial point to give criteria ensuring the convergence of iterative methods; meanwhile, the local one is, based on the information around a solution, to find estimates for the radii of the convergence balls. Numerous researchers studied the local convergence analysis for Newton-type, Jarratt-type, Weerakoon-type, etc. in Banach space setting in the articles [8–14] and reference therein. In most of articles, the local convergence have been discussed using the hypotheses of Lipschitz, Hölder or w-continuity conditions but sometimes, we will come across that the nonlinear problems do not fulfilled any of these three conditions which limits the applicability of nonlinear equations, but satisfy the generalized Lipschitz condition. Also, the notable feature is that all these three are a particular case of the generalized Lipschitz or L-average condition.

Here, we discuss the local convergence of the classical third-order modification of two-step Newton's method [15] under the L-average condition which is expressed as:

$$\begin{aligned} y_k &= x_k - [t'(x_k)]^{-1}t(x_k), \\ x_{k+1} &= y_k - [t'(x_k)]^{-1}t(y_k),\ k \geq 0. \end{aligned} \tag{3}$$

The important characteristic of the method (3) is that: it is simplest and efficient third-order iterative method, per jth iteration it requires two evaluations of the function t_j, one of the first derivative t'_j and no evaluations of the second derivative t''_j hence makes it computationally efficient. We find, in the literature, several studies on the weakness and/or extension of the hypotheses made on the underlying operators.

For re-investigating the local convergence of Newton's method, generalized Lipschitz conditions was constructed by Wang [16], in which a non-decreasing positive integrable function was used instead of usual Lipschitz constant. Furthermore, Wang and Li [17] derived some results on convergence of Newton's method in Banach spaces when derivative of the operators satisfies the radius or center Lipschitz condition but with a weak L-average. Shakhno [18] have studied the local convergence of the two step Secant-type method [2], when the first-order divided differences satisfy the generalized Lipschitz conditions.

As a motivational example let $X = Y = R^3, D = \overline{V}(0,1)$ and $X^* = (0,0,0)^T$. Define function t on D for $w = (x,y,z)^T$ by

$$t(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet derivative is

$$t'(w) = \begin{pmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

Hence, $L = \frac{e}{2}, L_0 = \frac{e-1}{2}$ and $L_0 < L$ (see definitions (5) and (6)). Therefore, replacing L by L_0 at the denominator gives the benefits. If L and L_0 are not constants then we can take $L(u) = \frac{eu}{2}, L_0(u) = \frac{(e-1)u}{2}$ and $\overline{L}(u) = \frac{e^{\frac{1}{(e-1)}}u}{2}$ (see definitions (7), (8) and (110)).

Next, the intriguing question strikes out that whether the radius Lipschitz condition with L-average and the non-decreasing of L are necessary for the convergence of the third-order modification of Newton's method. Motivated and inspired by the above mentioned research works in this direction in the present paper, we derived some theorems for scheme (3). In the first result generalized Lipschitz conditions has been used to study the local convergence which is important to enlarge the convergence region without additional hypotheses along with an error estimate. In the second theorem, the domain of uniqueness of solution has been derived under center Lipschitz condition. In the last two theorems, weak L-average has been used to derive the convergence result of the considered third-order scheme. Also, few corollaries are stated.

The rest part of this paper is structured as follows: Section 2 contains the definitions related to L-average conditions. The local convergence and its domain of uniqueness is mentioned in Sections 3 and 4, respectively. Section 5 deals with the improvement in assumption that the derivative of t satisfies the radius and center Lipschitz condition with weak L-average namely L and L_0 is assumed to belong to some family of positive integrable functions that are not necessarily non-decreasing for convergence theorems. Numerical examples are presented to justify the significance of the results.

2. Generalized Lipschitz Conditions

Here, we denote by $V(x^*, r) = \{x : ||x - x^*|| < r\}$ a ball with radius r and center x^*. The condition imposed on the function t

$$||t'(x) - t'(y^\tau)|| \leq L(1-\tau)(||x - x^*|| + ||y - x^*||), \forall\, x,\, y \in V(x^*, r), \tag{5}$$

where $y^\tau = x^* + \tau(y - x^*), 0 \leq \tau \leq 1$, is usually called radius Lipschitz condition in the ball $V(x^*, r)$ with constant L. Sometimes, if it is only required to satisfy

$$||t'(x) - t'(x^*)|| \leq 2L_0||x - x^*||, \forall\, x \in V(x^*, r). \tag{6}$$

We call it the center Lipschitz condition in the ball $V(x^*, r)$ with constant L_0 where $L_0 \leq L$. Replacing L by L_0 in case $L_0 < L$ leads to wider choice of initial guesses (larger radius of convergence than in traditional studies) and fewer iterates to achieve an error tolerance and the uniqueness of the solution x^* is also extended in this case [8,12]. Furthermore, L and L_0 in the Lipschitz conditions do not necessarily have to be constant but can be a positive integrable function. In this case, conditions (5)–(6) are respectively, replaced by

$$||t'(x) - t'(y^\tau)|| \leq \int_{\tau(\rho(x)+\rho(y))}^{\rho(x)+\rho(y)} L(u)du, \forall\, x,\, y \in V(x^*, r), 0 \leq \tau \leq 1 \tag{7}$$

and

$$||t'(x) - t'(x^*)|| \leq \int_0^{2\rho(x)} L_0(u)du, \forall\, x \in V(x^*, r), \tag{8}$$

where $\rho(x) = ||x - x^*||$ and we have $L_0(u) \leq L(u)$. At the same time, the corresponding 'Lipschitz conditions' is referred as to as having the L-average or generalized Lipschitz conditions. Next, we start with the following lemmas, which will be used later in the main theorems.

Lemma 1. *Suppose that t has a continuous derivative in $V(x^*, r)$ and $[t'(x^*)]^{-1}$ exists.*
(i) If $[t'(x^*)]^{-1}t'$ satisfies the radius Lipschitz condition with the L-average:

$$||[t'(x^*)]^{-1}(t'(x) - t'(y^\tau))|| \leq \int_{\tau(\rho(x)+\rho(y))}^{\rho(x)+\rho(y)} L(u)du, \forall\, x,\, y \in V(x^*, r), 0 \leq \tau \leq 1, \tag{9}$$

where $y^\tau = x^* + \tau(y - x^*)$, $\rho(x) = ||x - x^*||$ and L is non-decreasing, then we have

$$\int_0^1 ||[t'(x^*)]^{-1}(t'(x) - t'(y^\tau))||\rho(y)d\tau \leq \int_0^{\rho(x)+\rho(y)} L(u)\frac{u}{\rho(x)+\rho(y)}\rho(y)du. \tag{10}$$

(ii) If $[t'(x^*)]^{-1}t'$ satisfies the center Lipschitz condition with the L_0-average:

$$||[t'(x^*)]^{-1}(t'(x^\tau) - t'(x^*))|| \leq \int_0^{2\tau\rho(x)} L_0(u)du, \forall\, x \in V(x^*, r), 0 \leq \tau \leq 1, \tag{11}$$

where $\rho(x) = ||x - x^*||$ and L_0 is non-decreasing, then we have

$$\int_0^1 ||[t'(x^*)]^{-1}(t'(x^\tau) - t'(x^*))||\rho(x)d\tau \le \int_0^{2\rho(x)} L_0(u)\Big(\rho(x) - \frac{u}{2}\Big)du. \tag{12}$$

Proof. The Lipschitz conditions (9) and (11), respectively, imply that

$$\begin{aligned}
\int_0^1 ||[t'(x^*)]^{-1}(t'(x) - t'(y^\tau))||\rho(y)d\tau &\le \int_0^1 \int_{\tau(\rho(x)+\rho(y))}^{\rho(x)+\rho(y)} L(u)du\rho(y)d\tau \\
&= \int_0^{\rho(x)+\rho(y)} L(u)\frac{u}{\rho(x)+\rho(y)}\rho(y)du. \\
\int_0^1 ||[t'(x^*)]^{-1}(t'(x^\tau) - t'(x^*))||\rho(x)d\tau &\le \int_0^1 \int_0^{2\tau\rho(x)} L_0(u)du\rho(x)d\tau \\
&= \int_0^{2\rho(x)} L_0(u)\Big(\rho(x) - \frac{u}{2}\Big)du.
\end{aligned}$$

where $x^\tau = x^* + \tau(x - x^*)$ and $y^\tau = x^* + \tau(y - x^*)$. □

Lemma 2. *[17] Suppose that L is positive integrable. Assume that the function L_a defined by relation (62) is non-decreasing for some a with $0 \le a \le 1$. Then, for each $b \ge 0$, the function $\varphi_{b,a}$ defined by*

$$\varphi_{b,a}(f) = \frac{1}{f^{a+b}} \int_0^f u^b L(u)du \tag{13}$$

is also non-decreasing.

3. Local Convergence of Newton Type Method (3)

In this section, we state existence theorem under radius Lipschitz condition for Newton-type method (3).

Theorem 1. *Suppose that $t(x^*) = 0$, t has a continuous derivative in $V(x^*, r)$, $[t'(x^*)]^{-1}$ exists and $[t'(x^*)]^{-1}t'$ satisfies (7) and (8), L_0 and L are non-decreasing. Let r satisfy the relation*

$$\int_0^{2r} L_0(u)du \le 1 \text{ and } \frac{\int_0^{2r} L(u)udu}{2r(1 - \int_0^{2r} L_0(u)du)} \le 1. \tag{14}$$

Then, the two-step Newton-type method (3) is convergent for all $x_0 \in V(x^*, r)$ and

$$||y_n - x^*|| \le \frac{\int_0^{2\rho(x_n)} L(u)udu}{2(1 - \int_0^{2\rho(x_n)} L_0(u)du)} \le \frac{q_1}{\rho(x_0)}\rho(x_n)^2, \tag{15}$$

$$||x_{n+1} - x^*|| \le \frac{\int_0^{\rho(x_n)+\rho(y_n)} L(u)udu}{(\rho(x_n)+\rho(y_n))(1 - \int_0^{2\rho(x_n)} L_0(u)du)}\rho(y_n) \le \frac{q_2 q_1}{\rho(x_0)\rho(y_0)}\rho(x_n)^3, \tag{16}$$

where the quantities

$$q_1 = \frac{\int_0^{2\rho(x_0)} L(u)udu}{2\rho(x_0)(1 - \int_0^{2\rho(x_0)} L_0(u)du)},\ q_2 = \frac{\int_0^{\rho(x_0)+\rho(y_0)} L(u)udu}{(\rho(x_0)+\rho(y_0))(1 - \int_0^{2\rho(x_0)} L_0(u)du)} \tag{17}$$

are less than 1. Furthermore,

$$||x_n - x^*|| \le C^{3^n - 1}||x_0 - x^*||,\ n = 1, 2, \cdots, C = q_1\frac{\rho(x_0)}{\rho(y_0)}. \tag{18}$$

Proof. Since $L(u)$ is a positive integrable function and non-decreasing monotonically in $[0,r]$, we achieve

$$\begin{aligned} \left(\frac{1}{t_2^2}\int_0^{t_2} - \frac{1}{t_1^2}\int_0^{t_1}\right)L(u)udu &= \left(\frac{1}{t_2^2}\int_{t_1}^{t_2} + \left(\frac{1}{t_2^2} - \frac{1}{t_1^2}\right)\int_0^{t_1}\right)L(u)udu \\ &\geq L(t_1)\left(\frac{1}{t_2^2}\int_{t_1}^{t_2} + \left(\frac{1}{t_2^2} - \frac{1}{t_1^2}\right)\int_0^{t_1}\right)udu \\ &= L(t_1)\left(\frac{1}{t_2^2}\int_0^{t_2} - \frac{1}{t_1^2}\int_0^{t_1}\right)udu = 0, \end{aligned}$$

for $0 < t_1 < t_2$. Thus, $\frac{1}{t^2}\int_0^t L(u)udu$ is non-decreasing with respect to t. Next, on arbitrarily choosing $x_0 \in V(x^*, r)$ and using the non-decreasing property of $\frac{1}{t^2}\int_0^t L(u)udu$ and the inequality (14), it follows that

$$\begin{aligned} q_1 &= \frac{\int_0^{2\rho(x_0)} L(u)udu}{2\rho(x_0)^2(1 - \int_0^{2\rho(x_0)} L_0(u)du)}\rho(x_0) \\ &\leq \frac{\int_0^{2r} L(u)udu}{2r^2(1 - \int_0^{2r} L_0(u)du)}\rho(x_0) \leq \frac{||x_0 - x^*||}{r} < 1, \end{aligned} \tag{19}$$

Similarly,

$$\begin{aligned} q_2 &= \frac{\int_0^{\rho(x_0)+\rho(y_0)} L(u)udu}{(\rho(x_0) + \rho(y_0))^2(1 - \int_0^{2\rho(x_0)} L_0(u)du)}(\rho(x_0 + \rho(y_0)) \\ &\leq \frac{\int_0^{2r} L(u)udu}{2r^2(1 - \int_0^{2r} L_0(u)du)}(\rho(x_0) + \rho(y_0)) \leq \frac{||x_0 - x^*|| + ||y_0 - x^*||}{2r} < 1, \end{aligned}$$

Thus, q_1 and q_2, defined according to Equation (17) are less than 1. Obviously, if $x \in V(x^*, r)$, then using center Lipschitz condition with the L-average (11), we have

$$||[t'(x^*)]^{-1}[t'(x) - t'(x^*)]|| \leq \int_0^{2\rho(x)} L_0(u)du \leq \int_0^{2r} L_0(u)du \leq 1, \tag{20}$$

then taking into account the Banach Lemma and the below equation

$$||I - ([t'(x^*)]^{-1}t'(x) - I)||^{-1} = ||[t'(x)]^{-1}t'(x^*)||,$$

we come to following inequality by using the relation (20)

$$||[t'(x)]^{-1}t'(x^*)|| \leq \frac{1}{1 - \int_0^{2\rho(x)} L_0(u)du}. \tag{21}$$

Now, if $x_n \in V(x^*, r)$ then we may write from expression (3)

$$\begin{aligned} y_n - x^* &= x_n - x^* - [t'(x_n)]^{-1}t(x_n) \\ &= [t'(x_n)]^{-1}t'(x_n)(x_n - x^*) - [t'(x_n)]^{-1}[t(x_n) + t(x^*)] \\ &= [t'(x_n)]^{-1}[t'(x_n)(x_n - x^*) - t(x_n) + t(x^*)]. \end{aligned} \tag{22}$$

Expanding $t(x_n)$ along x^* from Taylor series expansion, we attain

$$\begin{aligned} t(x_n) &= t(x^*) + t'(x_n)(x_n - x^*) + t'(x^*) \int_0^1 [t'(x^*)]^{-1} t'(x_n^\tau) - [t'(x_n)] d\tau (x_n - x^*) \\ &= t'(x^*) \int_0^1 [t'(x^*)]^{-1} [t'(x_n) - t'(x_n^\tau)] d\tau (x_n - x^*). \end{aligned} \tag{23}$$

On substituting Equation (23) in (22), we get

$$y_n - x^* = [t'(x_n)]^{-1} t'(x^*). \int_0^1 [t'(x^*)]^{-1} [t'(x_n) - t'(x_n^\tau)] d\tau.(x_n - x^*). \tag{24}$$

Also, taking the norm on both the sides of Equation (24), we get

$$||y_n - x^*|| \leq ||[t'(x_n)]^{-1} t'(x^*)||.|| \int_0^1 [t'(x^*)]^{-1} [t'(x_n) - t'(x_n^\tau)] d\tau||.||(x_n - x^*)||. \tag{25}$$

Next, from the definition of radius Lipschitz given in the inequality (9) and using the inequality (21), it can written as

$$||y_n - x^*|| \leq \frac{1}{1 - \int_0^{2\rho(x_n)} L_0(u) du} \int_0^1 \int_{2\tau\rho(x_n)}^{2\rho(x_n)} L(u) du \rho(x_n) d\tau. \tag{26}$$

In view of Lemma 1 and the above inequality, we can obtain

$$||y_n - x^*|| \leq \frac{\int_0^{2\rho(x_n)} L(u) u du}{2(1 - \int_0^{2\rho(x_n)} L_0(u) du)}, \tag{27}$$

which is the first inequality of expression (15). By similar analogy and using the last sub-step of the scheme (3), we can write

$$\begin{aligned} x_{n+1} - x^* &= y_n - x^* - [t'(x_n)]^{-1} t(y_n) \\ &= [t'(x_n)]^{-1} t'(x_n)(y_n - x^*) - [t'(x_n)]^{-1} [t(y_n) + t(x^*)] \\ &= [t'(x_n)]^{-1} [t'(x_n)(y_n - x^*) - t(y_n) + t(x^*)]. \end{aligned} \tag{28}$$

Expanding $t(y_n)$ along x^* from Taylor series expansion, we attain

$$t(y_n) = t(x^*) + t'(x_n)(y_n - x^*) + t'(x^*) \int_0^1 [t'(x^*)]^{-1} t'(y_n^\tau) - [t'(x_n)] d\tau (y_n - x^*)$$

or

$$t(x^*) - t(y_n) + t'(x_n)(y_n - x^*) = t'(x^*) \int_0^1 [t'(x^*)]^{-1} [t'(x_n) - t'(x_y^\tau)] d\tau (y_n - x^*). \tag{29}$$

On substituting Equation (29) in (28), we get

$$x_{n+1} - x^* = [t'(x_n)]^{-1} t'(x^*). \int_0^1 [t'(x^*)]^{-1} [t'(x_n) - t'(y_n^\tau)] d\tau.(y_n - x^*). \tag{30}$$

Also, taking the norm on both the sides of Equation (30), we get

$$||x_{n+1} - x^*|| \leq ||[t'(x_n)]^{-1} t'(x^*)||.|| \int_0^1 [t'(x^*)]^{-1} [t'(x_n) - t'(y_n^\tau)] d\tau||.||(y_n - x^*)||. \tag{31}$$

Next, from the definition of radius Lipschitz given in the inequality (9) and using the inequality (21), it can written as

$$||x_{n+1} - x^*|| \quad \leq \quad \frac{1}{1 - \int_0^{2\rho(x_n)} L_0(u)du} \int_0^1 \int_{\tau(\rho(x_n)+\rho(y_n))}^{\rho(x_n)+\rho(y_n)} L(u)du\rho(y_n)d\tau. \tag{32}$$

Using Lemma 1 and the above expression, we can get

$$||x_{n+1} - x^*|| \quad \leq \quad \frac{\int_0^{\rho(x_n)+\rho(y_n)} L(u)udu}{(\rho(x_n)+\rho(y_n))(1 - \int_0^{2\rho(x_n)} L_0(u)du)}\rho(y_n), \tag{33}$$

which is the first inequality of expression (16). Furthermore, $\rho(x_n)$ and $\rho(y_n)$ are decreasing monotonically, therefore for all $n = 0, 1, ...,$ we have

$$\begin{aligned} ||y_n - x^*|| \quad &\leq \quad \frac{\int_0^{2\rho(x_n)} L(u)udu}{2(1 - \int_0^{2\rho(x_n)} L_0(u)du)} \\ &\leq \quad \frac{\int_0^{2\rho(x_0)} L(u)udu}{2\rho(x_0)^2(1 - \int_0^{2\rho(x_n)} L_0(u)du)} 2\rho(x_n)^2 \leq \frac{q_1}{\rho(x_0)}\rho(x_n)^2. \end{aligned} \tag{34}$$

Also, by using second inequality of expression (15), we have

$$\begin{aligned} ||x_{n+1} - x^*|| \quad &\leq \quad \frac{\int_0^{\rho(x_n)+\rho(y_n)} L(u)udu}{(\rho(x_n)+\rho(y_n))^2(1 - \int_0^{2\rho(x_n)} L_0(u)du)}\rho(y_n).[\rho(x_n)+\rho(y_n)] \\ &\leq \quad \frac{q_2}{\rho(x_0)+\rho(y_0)}[\rho(x_0)\rho(y_n)+\rho(y_n)^2] \leq \frac{q_2 q_1}{\rho(x_0)\rho(y_0)}\rho(x_n)^3. \end{aligned} \tag{35}$$

Hence, we have the complete inequalities of expressions (15) and (16). Also, it can be seen that inequality (18) may be easily derived from the expression (35). □

4. The Uniqueness Ball for the Solution of Equations

Here, we derived uniqueness theorem under center Lipschitz condition for Newton-type method (3).

Theorem 2. *Suppose that $t(x^*) = 0$, t has a continuous derivative in $V(x^*, r)$, $[t'(x^*)]^{-1}$ exists and $[t'(x^*)]^{-1}t'$ satisfies (8). Let r satisfy the relation*

$$\frac{\int_0^{2r} L_0(u)(2r-u)du}{2r} \leq 1. \tag{36}$$

Then, the equation $t(x) = 0$ has a unique solution x^* in $V(x^*, r)$.

Proof. On arbitrarily choosing $y^* \in V(x^*, r)$, $y^* \neq x^*$ and considering the iteration, we get

$$\begin{aligned} ||y^* - x^*|| \quad &= \quad ||y^* - x^* - [t'(x^*)]^{-1}t(y^*)|| \\ &= \quad ||[t'(x^*)]^{-1}[t'(x^*)(y^* - x^*) - t(y^*) + t(x^*)]||. \end{aligned} \tag{37}$$

Expanding $t(y^*)$ along x^* from Taylor's expansion, we have

$$t(x^*) - t(y^*) + t'(x^*)(y^* - x^*) = \int_0^1 [t'(x^*)]^{-1}[t'(y^{*\tau}) - t'(x^*)]d\tau(y^* - x^*). \tag{38}$$

Following the expression (11) and combining the inequalities (37) and (38), we can write

$$\begin{aligned} ||y^* - x^*|| &\leq ||[t'(x^*)]^{-1}t'(x^*)||.|| \int_0^1 [t'(x^*)]^{-1}[t'(y^{*\tau}) - t'(x^*)]d\tau||.||(y^* - x^*)||. \\ &\leq \int_0^1 \int_0^{2\tau\rho(y^*)} L_0(u)du\rho(y^*)d\tau. \end{aligned} \tag{39}$$

In view of Lemma (1) and expression (39), we obtain

$$\begin{aligned} ||y^* - x^*|| &\leq \frac{1}{2\rho(y^*)} \int_0^{2\rho(y^*)} L_0(u)[2\rho(y^*) - u]du(y^* - x^*) \\ &\leq \frac{\int_0^{2r} L_0(u)(2r - u)du}{2r}\rho(y^*) \leq ||y^* - x^*||. \end{aligned} \tag{40}$$

However, this contradicts our assumption. Thus, we see that $y^* = x^*$. This completes the proof of the theorem. □

In particular, assuming that L and L_0 are constants, we obtain the following Corollaries 1 and 2 from Theorems 1 and 2, respectively.

Corollary 1. *Suppose that x^* satisfies $t(x^*) = 0$, t has a continuous derivative in $V(x^*, r)$, $[t'(x^*)]^{-1}$ exists and $[t'(x^*)]^{-1}t'$ satisfies (5) and (6). Let r satisfy the relation*

$$r = \frac{1}{2L_0 + L}. \tag{41}$$

Then, the two-step Newton-type method (3) is convergent for all $x_0 \in V(x^*, r)$ and

$$||y_n - x^*|| \leq \frac{q_1}{\rho(x_0)}\rho(x_n)^2, \tag{42}$$

$$||x_{n+1} - x^*|| \leq \frac{q_2 q_1}{\rho(x_0)\rho(y_0)}\rho(x_n)^3, \tag{43}$$

where the quantities

$$q_1 = \frac{L\rho(x_0)}{1 - 2L_0\rho(x_0)}, \; q_2 = \frac{L(\rho(x_0) + \rho(y_0))}{2(1 - 2L_0\rho(x_0))}, \tag{44}$$

are less than 1. *Moreover*

$$||x_n - x^*|| \leq C^{3^n - 1}||x_0 - x^*||, \; n = 1, 2, ...; C = q_1\frac{\rho(x_0)}{\rho(y_0)}. \tag{45}$$

Corollary 2. *Suppose that x^* satisfies $t(x^*) = 0$, t has a continuous derivative in $V(x^*, r)$, $[t'(x^*)]^{-1}$ exists and $[t'(x^*)]^{-1}t'$ satisfies the assumption (6). Let r fulfill the condition*

$$r = \frac{1}{L_0}. \tag{46}$$

Then, the equation $t(x) = 0$ has a unique solution x^* in $V(x^*, r)$. Moreover, the ball radius r depends only on L_0.

Next, we will apply our main theorems to some special function L and immediately obtain the following corollaries.

Corollary 3. *Suppose that x^* satisfies $t(x^*) = 0$, t has a continuous derivative in $V(x^*, r)$, $[t'(x^*)]^{-1}$ exists and $[t'(x^*)]^{-1}t'$ s satisfies (9), (11) where given fixed positive constants γ, $L > 0$ and $L_0 > 0$ with $L(u) = \gamma + Lu$ and $L_0(u) = \gamma + L_0 u$ i.e.,*

$$\begin{aligned} ||[t'(x^*)]^{-1}(t'(x) - t'(y^\tau))|| &\leq \gamma(1-\tau)(||x - x^*|| + ||y - x^*||) \\ &+ \frac{L}{2}(1-\tau^2)(||x - x^*|| + ||y - x^*||)^2 \end{aligned} \tag{47}$$

and

$$||[t'(x^*)]^{-1}(t'(x) - t'(x^*))|| \leq 2||x - x^*||(\gamma + L_0||x - x^*||), \tag{48}$$

$\forall\ x,\ y \in V(x^*, r), 0 \leq \tau \leq 1$, where $y^\tau = x^* + \tau(y - x^*)$, $\rho(x) = ||x - x^*||$. Let r satisfy the relation

$$r = \frac{-3\gamma + \sqrt{9\gamma^2 + (16/3)L + 8L_0}}{8/3L + 4L_0} \text{ and } 9\gamma^2 + (16/3)L + 8L_0 \geq 0. \tag{49}$$

Then, the two-step Newton-type method (3) is convergent for all $x_0 \in V(x^*, r)$ and

$$||y_n - x^*|| \leq \frac{q_1}{\rho(x_0)}\rho(x_n)^2, \tag{50}$$

$$||x_{n+1} - x^*|| \leq \frac{q_2 q_1}{\rho(x_0)\rho(y_0)}\rho(x_n)^3, \tag{51}$$

where the quantities

$$q_1 = \frac{\rho(x_0)[\gamma + 4/3L\rho(x_0)]}{[1 - 2\gamma\rho(x_0) - 2L_0\rho(x_0)^2]}, \tag{52}$$

$$q_2 = \frac{\rho(x_0) + \rho(y_0)[\gamma/2 + L/3(\rho(x_0) + \rho(y_0))]}{[1 - 2\gamma\rho(x_0) - 2L_0\rho(x_0)^2]} \tag{53}$$

are less than 1. Moreover

$$||x_n - x^*|| \leq C^{3^n - 1}||x_0 - x^*||,\ n = 1, 2, \ldots; C = q_1\frac{\rho(x_0)}{\rho(y_0)}. \tag{54}$$

Corollary 4. *Suppose that x^* satisfies $t(x^*) = 0$, t has a continuous derivative in $V(x^*, r)$, $[t'(x^*)]^{-1}$ exists and $[t'(x^*)]^{-1}t'$ satisfies (11) where given fixed positive constants γ and $L_0 > 0$ with $L_0(u) = \gamma + L_0 u$ i.e.,*

$$||[t'(x^*)]^{-1}(t'(x) - t'(x^*))|| \leq 2||x - x^*||(\gamma + L_0||x - x^*||), \forall\ x \in V(x^*, r), \tag{55}$$

where $\rho(x) = ||x - x^*||$. Let r satisfy the relation

$$r = \frac{2\gamma - \sqrt{4\gamma^2 - (16/3)L_0}}{(8/3)L_0} \text{ and } 4\gamma^2 - (16/3)L_0 \geq 0. \tag{56}$$

Then, the equation $t(x) = 0$ has a unique solution x^* in $V(x^*, r)$. Moreover, the ball radius r depends only on L_0 and γ.

5. Convergence under Weak L-Average

This section contains the results on re-investigation of the conditions and radius of convergence of considered scheme already presented in the first theorem but L is not taken as non-decreasing function. It has been noticed that the convergence order decreases. The second theorem of this section gives a similar result to Theorem 1 but under the assumption of center Lipschitz condition.

Theorem 3. *Suppose that $t(x^*) = 0$, t has a continuous derivative in $V(x^*, r)$, $[t'(x^*)]^{-1}$ exists and $[t'(x^*)]^{-1}t'$ satisfies the assumptions (7) and (8), L_0 and L are positive integrable. Let r satisfy*

$$\int_0^{2r} L_0(u)du \leq 1 \text{ and } \int_0^{2r} (L(u) + L_0(u))du \leq 1. \tag{57}$$

Then, the two-step Newton-type method (3) is convergent for all $x_0 \in V(x^*, r)$ and

$$||y_n - x^*|| \leq \frac{\int_0^{2\rho(x_n)} L(u)udu}{2(1 - \int_0^{2\rho(x_n)} L_0(u)du)} \leq q_1\rho(x_n), \tag{58}$$

$$||x_{n+1} - x^*|| \leq \frac{\int_0^{\rho(x_n)+\rho(y_n)} L(u)udu}{(\rho(x_n) + \rho(y_n))(1 - \int_0^{2\rho(x_n)} L_0(u)du)}\rho(y_n) \leq q_2 q_1\rho(x_n), \tag{59}$$

where the quantities

$$q_1 = \frac{\int_0^{2\rho(x_0)} L(u)du}{1 - \int_0^{2\rho(x_0)} L_0(u)du}, \ q_2 = \frac{\int_0^{\rho(x_0)+\rho(y_0)} L(u)du}{1 - \int_0^{2\rho(x_0} L_0(u)du} \tag{60}$$

are less than 1. *Moreover,*

$$||x_n - x^*|| \leq (q_1q_2)^n||x_0 - x^*||, \ n = 1, 2, ... \tag{61}$$

Furthermore, suppose that the function L_a is defined by

$$L_a(f) = f^{1-a}L(f) \tag{62}$$

is non-decreasing for some a with $0 \leq a \leq 1$ and r satisfies

$$\frac{1}{2r}\int_0^{2r} (2rL_0(u) + uL(u))du \leq 1. \tag{63}$$

Then, the two-step Newton-type method (3) is convergent for all $x_0 \in V(x^*, r)$ and

$$||x_n - x^*|| \leq C^{(1+2a)^n - 1}||x_0 - x^*||, \ n = 1, 2, \cdots, C = Q_1\frac{\rho(x_0)}{\rho(y_0)}, \tag{64}$$

where the quantity

$$Q_1 = \frac{\int_0^{2\rho(x_0)} L(u)udu}{2\rho(x_0)(1 - \int_0^{2\rho(x_0)} L_0(u)du)}, \tag{65}$$

is less than 1.

Proof. On arbitrarily choosing $x_0 \in V(x^*, r)$, using the property of $L(u)$ as a positive integrable function and the inequality (57), it follows that

$$q_1 = \frac{\int_0^{2\rho(x_0)} L(u)du}{1 - \int_0^{2\rho(x_0)} L_0(u)du} \leq \frac{\int_0^{2r} L(u)du}{1 - \int_0^{2r} L_0(u)du} < 1. \tag{66}$$

Similarly,

$$q_2 = \frac{\int_0^{\rho(x_0)+\rho(y_0)} L(u)du}{1 - \int_0^{2\rho(x_0)} L_0(u)du} \leq \frac{\int_0^{2r} L(u)du}{1 - \int_0^{2r} L_0(u)du} < 1,$$

which proves that the quantities q_1 and q_2 defined by Equation (60) are less than 1.

Obviously, if $x \in V(x^*, r)$, then using center Lipschitz condition with the L average, we have

$$||[t'(x^*)]^{-1}[t'(x) - t'(x^*)]|| \leq \int_0^{2\rho(x)} L_0(u)du \leq \int_0^{2r} L_0(u)du \leq 1, \tag{67}$$

then taking into account the Banach Lemma and the below equation

$$||I - ([t'(x^*)]^{-1}t'(x) - I)||^{-1} = ||[t'(x)]^{-1}t'(x^*)||,$$

we come to following inequality using the relation (67)

$$||[t'(x)]^{-1}t'(x^*)|| \quad \leq \quad \frac{1}{1 - \int_0^{2\rho(x)} L_0(u)du}. \tag{68}$$

Hence, if $x_n \in V(x^*, r)$, then we may write from first sub-step of scheme (3)

$$\begin{aligned} y_n - x^* &= x_n - x^* - [t'(x_n)]^{-1}t(x_n) \\ &= [t'(x_n)]^{-1}t'(x_n)(x_n - x^*) - [t'(x_n)]^{-1}[t(x_n) + t(x^*)] \\ &= [t'(x_n)]^{-1}[t'(x_n)(x_n - x^*) - t(x_n) + t(x^*)]. \end{aligned} \tag{69}$$

Expanding $t(x_n)$ along x^* from Taylor series expansion, we attain

$$t(x_n) = t(x^*) + t'(x_n)(x_n - x^*) + t'(x^*)\int_0^1 [t'(x^*)]^{-1}t'(x_n^\tau) - [t'(x_n)]d\tau(x_n - x^*)$$

or

$$t(x^*) - t(x_n) + t'(x_n)(x_n - x^*) = t'(x^*)\int_0^1 [t'(x^*)]^{-1}[t'(x_n) - t'(x_n^\tau)]d\tau(x_n - x^*). \tag{70}$$

On substituting Equation (70) in (69), we get

$$y_n - x^* = [t'(x_n)]^{-1}t'(x^*).\int_0^1 [t'(x^*)]^{-1}[t'(x_n) - t'(x_n^\tau)]d\tau.(x_n - x^*). \tag{71}$$

Also, taking the norm on both the sides of Equation (71), we get

$$||y_n - x^*|| \quad \leq \quad ||[t'(x_n)]^{-1}t'(x^*)||.||\int_0^1 [t'(x^*)]^{-1}[t'(x_n) - t'(x_n^\tau)]d\tau||.||(x_n - x^*)||. \tag{72}$$

Next, from the definition of radius Lipschitz given in the inequality (9) and using the inequality (68), it can written as

$$||y_n - x^*|| \quad \leq \quad \frac{1}{1 - \int_0^{2\rho(x_n)} L_0(u)du} \int_0^1 \int_{2\tau\rho(x_n)}^{2\rho(x_n)} L(u)du\rho(x_n)d\tau. \tag{73}$$

In view of Lemma (1) and the above inequality, we can obtain

$$||y_n - x^*|| \quad \leq \quad \frac{\int_0^{2\rho(x_n)} L(u)udu}{2(1 - \int_0^{2\rho(x_n)} L_0(u)du)}, \tag{74}$$

which is the first inequality of expression (58). By similar analogy and using the last sub-step of the scheme (3), we can write

$$\begin{aligned} x_{n+1} - x^* &= y_n - x^* - [t'(x_n)]^{-1} t(y_n) \\ &= [t'(x_n)]^{-1} t'(x_n)(y_n - x^*) - [t'(x_n)]^{-1}[t(y_n) + t(x^*)] \\ &= [t'(x_n)]^{-1}[t'(x_n)(y_n - x^*) - t(y_n) + t(x^*)]. \end{aligned} \tag{75}$$

Expanding $t(y_n)$ along x^* from Taylor series expansion, we attain

$$(y_n) = t(x^*) + t'(x_n)(y_n - x^*) + t'(x^*) \int_0^1 [t'(x^*)]^{-1} t'(y_n^\tau) - [t'(x_n)] d\tau (y_n - x^*)$$

or

$$t(x^*) - t(y_n) + t'(x_n)(y_n - x^*) = t'(x^*) \int_0^1 [t'(x^*)]^{-1}[t'(x_n) - t'(x_y^\tau)] d\tau (y_n - x^*). \tag{76}$$

On substituting Equation (76) in (75), we get

$$x_{n+1} - x^* = [t'(x_n)]^{-1} t'(x^*). \int_0^1 [t'(x^*)]^{-1}[t'(x_n) - t'(y_n^\tau)] d\tau.(y_n - x^*). \tag{77}$$

Also, taking the norm on both the sides of Equation (77), we get

$$||x_{n+1} - x^*|| \leq ||[t'(x_n)]^{-1} t'(x^*)||.|| \int_0^1 [t'(x^*)]^{-1}[t'(x_n) - t'(y_n^\tau)] d\tau||.||(y_n - x^*)||. \tag{78}$$

Next, from the definition of radius Lipschitz given in the inequality (9) and using the inequality (68), it can written as

$$||x_{n+1} - x^*|| \leq \frac{1}{1 - \int_0^{2\rho(x_n)} L_0(u) du} \int_0^1 \int_{\tau(\rho(x_n)+\rho(y_n))}^{\rho(x_n)+\rho(y_n)} L(u) du \rho(y_n) d\tau. \tag{79}$$

Using Lemma 1 and the above expression, we can get

$$||x_{n+1} - x^*|| \leq \frac{\int_0^{\rho(x_n)+\rho(y_n)} L(u) u du}{(\rho(x_n) + \rho(y_n))(1 - \int_0^{2\rho(x_n)} L_0(u) du)} \rho(y_n), \tag{80}$$

which is the first inequality of expression (59). Furthermore, $\rho(x_n)$ and $\rho(y_n)$ are decreasing monotonically, therefore for all $n = 0, 1, ...,$ we have

$$||y_n - x^*|| \leq \frac{\int_0^{2\rho(x_n)} L(u) u du}{2(1 - \int_0^{2\rho(x_n)} L_0(u) du)} \leq \frac{\int_0^{2\rho(x_0)} L(u) du}{(1 - \int_0^{2\rho(x_n)} L_0(u) du)} \rho(x_n) \leq q_1 \rho(x_n).$$

Using the second inequality of expression (58), we arrive at

$$\begin{aligned} ||x_{n+1} - x^*|| &\leq \frac{\int_0^{\rho(x_n)+\rho(y_n)} L(u) u du}{(\rho(x_n) + \rho(y_n))(1 - \int_0^{2\rho(x_n)} L_0(u) du)} \rho(y_n) \\ &\leq \frac{\int_0^{\rho(x_0)+\rho(y_0)} L(u) du}{(1 - \int_0^{2\rho(x_0)} L_0(u) du)} \rho(y_n) \leq q_2 q_1 \rho(x_n). \end{aligned} \tag{81}$$

Also, the inequality (61) may be easily derived from the expression (81). Furthermore, if the function L_a defined by the relation (62) is non-decreasing for some a with $0 \leq a \leq 1$

and r is determined by inequality (63), it follows from the first inequality of expression (58) and Lemma (2) that

$$\begin{aligned} ||y_n - x^*|| &\leq \frac{\varphi_{1,a}(2\rho(x_n))2^a}{(1 - \int_0^{2\rho(x_n)} L_0(u)du)}\rho(x_n)^{a+1} \\ &\leq \frac{\varphi_{1,a}(2\rho(x_0))2^a}{(1 - \int_0^{2\rho(x_n)} L_0(u)du)}\rho(x_n)^{a+1} = \frac{Q_1}{\rho(x_0)^a}\rho(x_n)^{a+1}. \end{aligned}$$

Moreover, from the first inequality of (59) and Lemma 2, we can write

$$\begin{aligned} ||x_{n+1} - x^*|| &\leq \frac{\varphi_{1,a}(\rho(x_n)+\rho(y_n))(\rho(x_n)+\rho(y_n))^a}{(1 - \int_0^{2\rho(x_n)} L_0(u)du)}\rho(y_n) \\ &\leq \frac{\varphi_{1,a}(\rho(x_0)+\rho(y_0))(\rho(x_n)+\rho(y_n))^a}{(1 - \int_0^{2\rho(x_n)} L_0(u)du)}\rho(y_n) \\ &= \frac{Q_2Q_1}{\rho(x_0)^a\rho(y_0)^a}\rho(x_n)^{2a+1}. \end{aligned}$$

Next, using the nondecreasing property of $\frac{1}{t^2}\int_0^t L(u)udu$ and from the definition of r in the relation (63), it follows that

$$Q_1 = \frac{\int_0^{2\rho(x_0)} L(u)udu}{2\rho(x_0)(1 - \int_0^{2\rho(x_0)} L_0(u)du)} \leq \frac{\int_0^{2r} L(u)udu}{2r^2(1 - \int_0^{2r} L_0(u)du)}\rho(x_0) \leq \frac{||x_0 - x^*||}{r} < 1, \tag{82}$$

which shows $Q_1 < 1$ and by the same reason we can say $Q_2 = \frac{\int_0^{\rho(x_0)+\rho(y_0)} L(u)udu}{(\rho(x_0)+\rho(y_0))(1-\int_0^{2\rho(x_0)} L_0(u)du)} < 1$. Also, the inequality (64) may be easily derived and hence x_n converges to x^*. Thus, the proof is completed. □

Theorem 4. *Suppose that $t(x^*) = 0$, t has a continuous derivative in $V(x^*, r)$, $[t'(x^*)]^{-1}$ exists and $[t'(x^*)]^{-1}t'$ satisfies the assumption (8) and L_0 is positive integrable function. Let r satisfy*

$$\int_0^{2r} L_0(u)du \leq \frac{1}{3}. \tag{83}$$

Then, the two-step Newton-type method (3) is convergent for all $x_0 \in V(x^*, r)$ and

$$\begin{aligned} ||y_n - x^*|| &\leq \frac{2\int_0^{2\rho(x_n)} L_0(u)du}{1 - \int_0^{2\rho(x_n)} L_0(u)du}\rho(x_n) \leq q_1\rho(x_n), \\ ||x_{n+1} - x^*|| &\leq \frac{\int_0^{2\rho(x_n)} L_0(u)du + \int_0^{2\rho(y_n)} L_0(u)du}{1 - \int_0^{2\rho(x_n)} L_0(u)du}\rho(y_n) \leq q_2q_1\rho(x_n), \end{aligned} \tag{84}$$

where the quantities

$$q_1 = \frac{2\int_0^{2\rho(x_0)} L_0(u)du}{(1 - \int_0^{2\rho(x_0)} L_0(u)du)},\ q_2 = \frac{\int_0^{2\rho(x_0)} L_0(u)du + \int_0^{2\rho(y_0)} L_0(u)du}{(1 - \int_0^{2\rho(x_0)} L_0(u)du)} \tag{85}$$

are less than 1. Moreover,

$$||x_n - x^*|| \leq (q_1q_2)^n||x_0 - x^*||,\ n = 1, 2, ... \tag{86}$$

Furthermore, suppose that the function L_a defined by the relation (62) is non-decreasing for some a with $0 \le a \le 1$, then

$$||x_n - x^*|| \le C^{(1+2a)^n - 1}||x_0 - x^*||, \; n = 1, 2, \cdots, C = q_1 \frac{\rho(x_0)}{\rho(y_0)}. \tag{87}$$

and q_1 is given by the first expression of Equation (85).

Proof. Let $x_0 \in V(x^*, r)$ and x_n be the sequence generated by two-step Newton-type method given in (3). Next, on arbitrarily choosing $x_0 \in V(x^*, r)$, using the property of $L(u)$ as a positive integrable function and the inequality (83), it follows that

$$q_1 = \frac{2\int_0^{2\rho(x_0)} L_0(u)du}{1 - \int_0^{2\rho(x_0)} L_0(u)du} \le \frac{2\int_0^{2r} L_0(u)du}{1 - \int_0^{2r} L_0(u)du} < 1.$$

Similarly,

$$q_2 = \frac{\int_0^{2\rho(x_0)} L_0(u)du + \int_0^{2\rho(y_0)} L_0(u)du}{1 - \int_0^{2\rho(x_0)} L_0(u)du} \le \frac{2\int_0^{2r} L_0(u)du}{1 - \int_0^{2r} L_0(u)du} < 1.$$

Assume that $x_n \in V(x^*, r)$, then

$$\begin{aligned} ||y_n - x^*|| &= ||y_n - x^* - [t'(x_n)]^{-1}t(x_n)|| \\ &= ||[t'(x_n)]^{-1}[t'(x_n)(x_n - x^*) - t(x_n) + t(x^*)]||. \end{aligned} \tag{88}$$

Expanding $t(x_n)$ along x^* from Taylor series expansion, we have

$$t(x^*) - t(x_n) + t'(x_n)(x_n - x^*) = t'(x^*)\int_0^1 [t'(x^*)]^{-1}[t'(x_n) - t'(x_n^\tau)]d\tau(x_n - x^*). \tag{89}$$

Following the hypothesis (11) of the theorem and using Equations (88) and (89), it can be written as

$$\begin{aligned} ||y_n - x^*|| \le\; & ||[t'(x_n)]^{-1}t'(x^*)||.||\int_0^1 [t'(x^*)]^{-1}[t'(x_n) - t'(x^*) + t'(x^*) - t'(x_n^\tau)]d\tau|| \\ & .||(x_n - x^*)|| \\ \le\; & \frac{1}{1 - \int_0^{2\rho(x_n)} L_0(u)du}\left\{\int_0^1 \int_0^{2\tau\rho(x_n)} L_0(u)du\rho(x_n)d\tau \right. \\ & \left. + \int_0^1 \int_0^{2\rho(x_n)} L_0(u)du\rho(x_n)d\tau\right\}. \end{aligned} \tag{90}$$

In view of Lemma (1), the above inequality becomes

$$\begin{aligned} ||y_n - x^*|| &\le \frac{2\int_0^{2\rho(x_n)} L_0(u)du\rho(x_n) - \frac{1}{2}\int_0^{2\rho(x_n)} L_0(u)udu}{1 - \int_0^{2\rho(x_n)} L_0(u)du} \\ &\le \frac{2\int_0^{2\rho(x_n)} L_0(u)du}{1 - \int_0^{2\rho(x_n)} L_0(u)du}\rho(x_n) = q_1\rho(x_n), \end{aligned}$$

which is same as first inequality of (84). By similar analogy and form the final sub-step of the scheme (3), we can write

$$\begin{aligned} ||x_{n+1}-x^*|| &\le ||[t'(x_n)]^{-1}t'(x^*)||\Big\{||\int_0^1[t'(x^*)]^{-1}[t'(x_n)-t'(x^*)]d\tau||.||(y_n-x^*)|| \\ &\quad +||\int_0^1[t'(x^*)]^{-1}[t'(x^*)-t'(y_n^\tau)]d\tau||.||(y_n-x^*)||\Big\} \\ &\le \frac{1}{1-\int_0^{2\rho(x_n)}L_0(u)du}\Big\{\int_0^1\int_0^{2\tau\rho(y_n)}L_0(u)du\rho(y_n)d\tau \\ &\quad +\int_0^1\int_0^{2\rho(x_n)}L_0(u)du\rho(y_n)d\tau\Big\}. \end{aligned} \tag{91}$$

By virtue of Lemma 1, the above expression becomes

$$\begin{aligned} ||x_{n+1}-x^*|| &\le \frac{\int_0^{2\rho(x_n)}L_0(u)du\rho(y_n)+\int_0^{2\rho(y_n)}L_0(u)du\rho(y_n)-\frac{1}{2}\int_0^{2\rho(y_n)}L_0(u)udu}{1-\int_0^{2\rho(x_n)}L_0(u)du} \\ &\le \frac{\int_0^{2\rho(x_n)}L_0(u)du\rho(y_n)+\int_0^{2\rho(y_n)}L_0(u)du\rho(y_n)}{1-\int_0^{2\rho(x_n)}L_0(u)du}=q_2q_1\rho(x_n), \end{aligned}$$

where $q_1<1$ and $q_2<1$ are determined by the relation (83). Also, it can be seen that inequality (86) may be easily derived from the second expression (84) and hence x_n converges to x^*.

Furthermore, if the function L_a defined by the relation (62) is non-decreasing for some a with $0\le a\le 1$ and r is determined by the inequality (83), it follows from the first inequality of the expression (84) and Lemma 2 that

$$\begin{aligned} ||y_n-x^*|| &\le \frac{2\varphi_{0,a}(2\rho(x_n))2^a}{(1-\int_0^{2\rho(x_n)}L_0(u)du)}\rho(x_n)^{a+1} \\ &\le \frac{2\varphi_{0,a}(2\rho(x_0))2^a}{(1-\int_0^{2\rho(x_0)}L_0(u)du)}\rho(x_n)^{a+1}=\frac{q_1}{\rho(x_0)^a}\rho(x_n)^{a+1}. \end{aligned}$$

Moreover, from the second inequality of expression (84) and Lemma 2, we get

$$\begin{aligned} ||x_{n+1}-x^*|| &\le \frac{\varphi_{0,a}(2\rho(x_n))+\varphi_{0,a}(2\rho(y_n)).(2\rho(x_n))^a.\rho(y_n)}{(1-\int_0^{2\rho(x_n)}L_0(u)du)}\rho(y_n) \\ &\le \frac{\varphi_{0,a}(2\rho(x_0))+\varphi_{0,a}(2\rho(y_0)).(2\rho(x_n))^a.\rho(y_n)}{(1-\int_0^{2\rho(x_n)}L_0(u)du)}\rho(y_n) \\ &= \frac{q_2.q_1}{\rho(x_0)^a\rho(y_0)^a}\rho(x_n)^{2a+1}. \end{aligned}$$

Hence, it can be seen that inequality (87) may be easily derived and hence x_n converges to x^*. □

Next, we will apply our newly improved theorems to some special functions L and results from Theorems 3 and 4 are recaptured.

Corollary 5. *Suppose that x^* satisfies $t(x^*)=0$, t has a continuous derivative in $V(x^*,r)$, $[t'(x^*)]^{-1}$ exists and $[t'(x^*)]^{-1}t'$ satisfies (9), (11) with $L(u)=cau^{a-1}$ and $L_0(u)=c_0au^{a-1}$ i.e.,*

$$||[t'(x^*)]^{-1}(t'(x)-t'(y^\tau))||\le c.(1-\tau^a)(||x-x^*||+||y-x^*||)^a \tag{92}$$

and

$$||[t'(x^*)]^{-1}(t'(x)-t'(x^*))|| \leq c_0 2^a ||x-x^*||^a, \tag{93}$$

$\forall\, x,\ y \in V(x^*, r), 0 \leq \tau \leq 1$, *where* $y^\tau = x^* + \tau(y - x^*)$, $\rho(x) = ||x - x^*||$, $0 < a < 1$, $c > 0$ *and* $c_0 > 0$. *Let r satisfy*

$$r = \left(\frac{a+1}{2^a(c_0(a+1)+ca)}\right)^{\frac{1}{a}}. \tag{94}$$

Then, the two-step Newton-type method (3) *is convergent for all* $x_0 \in V(x^*, r)$ *and*

$$||y_n - x^*|| \quad \leq \quad \frac{\int_0^{2\rho(x_n)} L(u)udu}{2(1-\int_0^{2\rho(x_n)} L_0(u)du)} \leq q_1\rho(x_n), \tag{95}$$

$$||x_{n+1} - x^*|| \quad \leq \quad \frac{\int_0^{\rho(x_n)+\rho(y_n)} L(u)udu}{(\rho(x_n)+\rho(y_n))(1-\int_0^{2\rho(x_n)} L_0(u)du)}\rho(y_n) \leq q_2 q_1 \rho(x_n), \tag{96}$$

where the quantities

$$q_1 = \frac{ca2^a\rho(x_0)^a}{(1+a)[1-2^a c_0\rho(x_0)^a]},\ q_2 = \frac{ca(\rho(x_0)+\rho(y_0))^a}{(a+1)(1-2^a c_0 \rho(x_0)^a)} \tag{97}$$

are less than 1. *Furthermore,*

$$||x_n - x^*|| \leq C^{3^n-1}||x_0 - x^*||,\ n = 1, 2, ..., C = q_1\frac{\rho(x_0)}{\rho(y_0)}. \tag{98}$$

Corollary 6. *Suppose that* x^* *satisfies* $t(x^*) = 0$, t *has a continuous derivative in* $V(x^*, r)$, $[t'(x^*)]^{-1}$ *exists and* $[t'(x^*)]^{-1}t'$ *satisfies* (11) *with* $L_0(u) = c_0 a u^{a-1}$ *i.e.,*

$$||[t'(x^*)]^{-1}(t'(x)-t'(x^*))|| \leq c_0 2^a||x-x^*||^a, \forall\, x \in V(x^*, r), \tag{99}$$

where $\rho(x) = ||x - x^*||$, $0 < a < 1$ *and* $c_0 > 0$. *Let r satisfy*

$$r = \left(\frac{1}{3c_0 2^a}\right)^{\frac{1}{a}}. \tag{100}$$

Then, the two-step Newton-type method (3) *is convergent for all* $x_0 \in V(x^*, r)$ *and*

$$\begin{aligned} ||y_n - x^*|| &\leq \frac{2\int_0^{2\rho(x_n)} L_0(u)du}{1-\int_0^{2\rho(x_n)} L_0(u)du}\rho(x_n) \leq q_1\rho(x_n), \\ ||x_{n+1} - x^*|| &\leq \frac{\int_0^{2\rho(x_n)} L_0(u)du + \int_0^{2\rho(y_n)} L_0(u)du}{1-\int_0^{2\rho(x_n)} L_0(u)du}\rho(y_n) \leq q_2q_1\rho(x_n), \end{aligned} \tag{101}$$

where the quantities

$$q_1 = \frac{c_0 2^{a+1}\rho(x_0)^a}{[1-2^a c_0\rho(x_0)^a]},\ q_2 = \frac{c_0 2^a(\rho(x_0)^a + \rho(y_0)^a)}{(1-2^a c_0\rho(x_0)^a)} \tag{102}$$

are less than 1. Furthermore,

$$||x_n - x^*|| \leq C^{3^n-1}||x_0 - x^*||,\ n = 1, 2, ..., C = q_1\frac{\rho(x_0)}{\rho(y_0)}. \tag{103}$$

Corollary 7. *Suppose that x^* satisfies $t(x^*) = 0$, t has a continuous derivative in $V(x^*, r)$, $[t'(x^*)]^{-1}$ exists and $[t'(x^*)]^{-1}t'$ satisfies (11) with $L_0(u) = \frac{2\gamma c_0}{(1-\gamma u)^3}$ i.e.,*

$$||[t'(x^*)]^{-1}(t'(x) - t'(x^*))|| \leq \frac{c_0}{(1 - 2\gamma\rho(x))^2} - c_0, \forall\, x \in V(x^*, r) \tag{104}$$

where $\rho(x) = ||x - x^*||$, $\gamma > 0$ and $c_0 > 0$. Let r satisfy

$$r = \frac{3c_0 + 1 - \sqrt{3c_0(3c_0 + 1)}}{2\gamma(3c_0 + 1)}. \tag{105}$$

Then, two-step Newton-type method (3) is convergent for all $x_0 \in V(x^*, r)$ and

$$\begin{aligned} ||y_n - x^*|| &\leq \frac{2\int_0^{2\rho(x_n)} L_0(u)du}{1 - \int_0^{2\rho(x_n)} L_0(u)du}\rho(x_n) \leq q_1\rho(x_n), \\ ||x_{n+1} - x^*|| &\leq \frac{\int_0^{2\rho(x_n)} L_0(u)du + \int_0^{2\rho(y_n)} L_0(u)du}{1 - \int_0^{2\rho(x_n)} L_0(u)du}\rho(y_n) \leq q_2 q_1 \rho(x_n), \end{aligned} \tag{106}$$

where the quantities

$$q_1 = \frac{2c_0 - 2c_0(1 - 2\gamma\rho(x_0))^2}{[1 - 2\gamma\rho(x_0)]^2(1 + c_0) - c_0}, \tag{107}$$

$$q_2 = \frac{[c_0 - c_0(1 - 2\gamma\rho(x_0))^2](1 - 2\gamma\rho(y_0))^2) + [c_0 - c_0(1 - 2\gamma\rho(y_0))^2](1 - 2\gamma\rho(x_0))^2)}{([1 - 2\gamma\rho(x_0)]^2(1 + c_0) - c_0)(1 - 2\gamma\rho(y_0))^2)} \tag{108}$$

are less than 1. Furthermore,

$$||x_n - x^*|| \leq C^{3^n - 1}||x_0 - x^*||,\ n = 1, 2, ..., C = q_1\frac{\rho(x_0)}{\rho(y_0)}. \tag{109}$$

Remark 1. *(a) If $L_0 = L$, then our results specialize to earlier ones [5,10,15–17]. However, if $L_0 < L$, then the benefits stated in the abstract and the introduction are obtained (see also Example 1 and Example 2).*
(b) A further extension can be achieved as follows. Suppose (6) holds and equation $2L_0(u)u - 1 = 0$ has a minimal positive zero $\bar{r}$. Define $\tilde{V} = V(x^*, r) \cap V(x^*, \bar{r})$. Moreover, suppose

$$||t(x) - t(y^\tau)|| \leq \int_{\tau(\rho(x)+\rho(y))}^{\rho(x)+\rho(y)} \overline{L}(u)du, \tag{110}$$

where $\forall\, x,\ y \in \tilde{V}$, $0 \leq \tau \leq 1$, and $\overline{L}$ is as L. Then, we have

$$\overline{L}(u) \leq L(u) \text{ for all } u \in [0, min\{r, \bar{r}\}].$$

Then, in view of the proofs $\overline{L}$ can replace L in all results with L. However, if

$$\overline{L}(u) < L(u)$$

the benefits stated in the introduction are extended even further. In the case of the motivational example, we have

$$L_0 < \overline{L} = \frac{e^{\frac{1}{(e-1)}}}{2} < L.$$

6. Numerical Examples

Example 1. *Let $X = Y = R$, the reals. Define*

$$t(x) = \int_0^x \Big(1 + 2x \sin\frac{\pi}{x}\Big)dx, \ \forall x \in R.$$

Then

$$t'(x) = \begin{cases} 1 + 2x\sin\frac{\pi}{x}, & x \neq 0, \\ 1, & x = 0, \end{cases}$$

Obviously, $x^* = 0$ is a zero of t and t' satisfies that

$$||[t'(x^*)]^{-1}(t'(x) - t'(x^*))|| = \Big|2x\sin\frac{\pi}{x}\Big| \leq 2|x - x^*|, \forall\ x \in R.$$

It follows from Theorem 4 that for any $x_0 \in V(x^*, 1/6)$

$$||x_n - x^*|| \leq C^{3^n - 1}||x_0 - x^*||, \ n = 1, 2, \cdots, C = \left(\frac{4|x_0|^2}{1 - 2|x_0||y_0|}\right).$$

However, there is no positive integrable function L such that the inequality (7) is satisfied. In fact, notice that

$$||[t'(x^*)]^{-1}(t'(x) - t'(y^\tau))|| = \left|2x\sin\frac{\pi}{x} - 2y\tau\sin\frac{\pi}{y\tau}\right| = \frac{4}{2k+1},$$

for $x = 1/k, y = 1/k, \tau = \frac{2k}{2k+1}$ and $k = 1, 2, \cdots$ Thus, if there was a positive integrable function L such that the inequality (7) holds on $V(x^*, r)$ for some $r > 0$, it follows that there exists some $n_0 > 1$ such that

$$\int_0^{2r} L(u)du \geq \sum_{k=n_0}^{+\infty} \int_{\frac{4}{2k+1}}^{\frac{2}{k}} L(u)du \geq \sum_{k=n_0}^{+\infty} \frac{4}{2k+1} = +\infty,$$

which is a contradiction. This example shows that Theorem 4 is a crucial improvement of Theorem 3 if the radius of the convergence ball is ignored.

Example 2. *Let $X = Y = R^3, D = \overline{V}(0, 1)$ and $X^* = (0, 0, 0)^T$. Define a function t on D for $w = (x, y, z)^T$ by*

$$t(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet derivative is

$$t'(w) = \begin{pmatrix} e^x & 0 & 0 \\ 0 & (e-1)y + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{111}$$

Using (14) and $t'(x^*) = (1, 1, 1)^T$, we have:
Old case $L_0(u) = L(u) = \frac{e}{2}$ gives

$$r_0 = 0.245253.$$

Case $L_0(u) = \frac{e-1}{2}$ and $L(u) = \frac{e}{2}$ gives

$$r_1 = 0.324947.$$

Case $L_0(u) = \frac{e-1}{2}$ *and* $\overline{L}(u) = \frac{e^{\frac{1}{(e-1)}}}{2}$ *gives*

$$r_2 = 0.382692.$$

Notice that $r_0 < r_1 < r_2$.

Example 3. *Choose* $X = Y = C[0,1], \Omega = \overline{V}(0,1)$ *and* $x^* = 0$. *Then, define* t *on* Ω *as*

$$t(h)(x) = h(x) - \int_0^1 x\tau h(\tau)^3 d\tau.$$

Therefore,

$$t'(h(p))(x) = p(x) - 3\int_0^1 x\tau h(\tau)^2 p(\tau) d\tau \; for \; all \; p \in \Omega.$$

Then, we get

$$L_0(u) = 1.5u < L(u) = \overline{L}(u) = 3u.$$

Hence, again we obtain the same benefits as in Example 2 by solving (14).

7. Conclusions

A new technique is developed in view of which we achieve a tighter local convergence analysis compared with earlier studies, without additional hypothesis. The technique is quite general. That means that the same benefits appear on the study of other iterative methods. The third and fourth sections in this paper analyzed the local convergence of a two-step Newton-type method of order three when applied under generalized Lipschitz conditions, in which instead of Lipschitz constants some non-decreasing integrable functions are being used. It turns out that although the conditions are more general, they are also more flexible, leading to some advantages, without any additional computational effort. The examples also demonstrate our benefits. All results are obtained without additional requirements. Hence, we have extended the applicability of modified Newton's method in cases not covered before. Our approach paves the way for future research to improve local results for Newton-type methods, and other iterative procedures.

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