

Article

Repdigits as Product of Terms of k -Bonacci Sequences

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Abstract: For any integer $k \geq 2$, the sequence of the k -generalized Fibonacci numbers (or k -bonacci numbers) is defined by the k initial values $F_{-(k-2)}^{(k)} = \dots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$ and such that each term afterwards is the sum of the k preceding ones. In this paper, we search for repdigits (i.e., a number whose decimal expansion is of the form $aa \dots a$, with $a \in [1, 9]$) in the sequence $(F_n^{(k)} F_n^{(k+m)})_n$, for $m \in [1, 9]$. This result generalizes a recent work of Bednařík and Trojovská (the case in which $(k, m) = (2, 1)$). Our main tools are the transcendental method (for Diophantine equations) together with the theory of continued fractions (reduction method).

Keywords: k -generalized Fibonacci numbers; linear forms in logarithms; reduction method

MSC: 11A63; 11B37; 11B39; 11J86



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1. Introduction

We start by recalling that the Fibonacci sequence $(F_n)_n$ is defined by the recurrence

$$F_{n+1} = F_n + F_{n-1}, \quad (1)$$

with initial values $F_0 = 0$ and $F_1 = 1$ (see, e.g., [1–3]). This sequence admits many generalizations and one of the most known is its higher order version. The Fibonacci sequence is a binary (or second order) recurrence and then, for any integer $k \geq 2$, the sequence of the k -generalized Fibonacci numbers (or k -bonacci numbers) is defined by the k th order recurrence

$$F_n^{(k)} = F_{n-1}^{(k)} + \dots + F_{n-k}^{(k)}$$

with initial values $F_{-(k-2)}^{(k)} = \dots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$. Clearly, for $k = 2$, we have the Fibonacci numbers and for $k = 3$, we have Tribonacci numbers (which is one of the most well-studied generalizations of Fibonacci numbers).

On the other hand, a repdigit (short for “repeated digit”) is a number of the form

$$a \left(\frac{10^\ell - 1}{9} \right), \quad (2)$$

where $\ell \geq 1$ and $a \in [1, 9]$ (here, as usual, for integers $x < y$, we denote $[x, y] = \{x, x + 1, \dots, y\}$), that is, a number with only one distinct digit (in this case a) in its decimal expansion.

We point out that many authors have been interested in solving Diophantine equations involving repdigits (their sums, products, concatenations, etc.) and some special forms of linear recurrences (like their product, sums, etc.). For some works in this direction, we refer the reader to [4–33] and the references therein.

Luca [34], in 2000, and Marques [35], in 2012, proved that the largest repdigits in the Fibonacci and Tribonacci sequence are $F_{10} = 55$ and $T_8 = 44$, respectively. Recently, Bednařík and Trojovská [36] and Trojovský [37] found all repdigits of the form $F_n T_n$ and $F_n + T_n$, respectively.

The aim of this paper is to continue this program and generalize the main result of [36]. More precisely, we search for repdigits which are the product of the n th k -bonacci number by the n th $(k + m)$ -bonacci number, for $m \in [1, 9]$. Our main result is the following:

Theorem 1. *The Diophantine equation*

$$F_n^{(k)} F_n^{(k+m)} = a \left(\frac{10^\ell - 1}{9} \right), \quad (3)$$

does not have a solution in positive integers n, a, k, ℓ , with $k \geq 2$, $\ell > 1$ and $a, m \in [1, 9]$.

The main tools in the proof is the transcendental method (lower bounds for linear logarithm of real algebraic numbers) together with the theory of continued fractions (reduction method). It is important to stress that the method can be implemented for any given range of values for m . However, we chose $m \in [1, 9]$ in order to avoid too much time of computation (by using Mathematica software).

2. Auxiliary Results

The results of this section can be found in the classical literature about this kind of Diophantine equation (see, for example, [20] and the references therein). For this reason, we shall present here these tools as succinctly as possible.

The first useful result is due to Dresden and Du [38] (Theorem 1) who proved that

$$F_n^{(k)} = g(\alpha, k) \alpha^{n-1} + E_{n,k}, \quad (4)$$

with $|E_{n,k}| < 1/2$, where α is the dominant root of the polynomial $\psi_k(x) := x^k - \sum_{j=0}^{k-1} x^j$. Moreover, we have the notation $g(x, y) := (x - 1)/(2 + (y + 1)(x - 2))$. Furthermore, Bravo and Luca [39] showed that

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1}. \quad (5)$$

Another very useful ingredient is the following result à la Baker:

Lemma 1. *Let $\gamma_1, \dots, \gamma_t \in \mathbb{R}$ be algebraic numbers and let b_1, \dots, b_t be nonzero integer numbers. Let D be the degree of the number field extension $\mathbb{Q}(\gamma_1, \dots, \gamma_t)/\mathbb{Q}$ and let $A_j \in \mathbb{R}$ be any constant such that*

$$A_j \geq \max\{Dh(\gamma_j), |\log \gamma_j|, 0.16\}, \text{ for } j \in [1, t].$$

In addition, choose a constant B for which

$$B \geq \max\{|b_1|, \dots, |b_t|\}.$$

If $\gamma_1^{b_1} \cdots \gamma_t^{b_t} \neq 1$, then

$$|\gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1| \geq \exp(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2 (1 + \log D) (1 + \log B) A_1 \cdots A_t).$$

This result is a version of a Matveev theorem [40] due to Bugeaud et al. (see its proof in [41]). In the previous statement, $h(\gamma)$ denotes the logarithmic height of an ℓ -degree algebraic number γ . This function satisfies the following properties (the proof of these facts can be found in [42]):

Lemma 2. We have

- i. $h(xy) \leq h(x) + h(y)$;
- ii. $h(x + y) \leq h(x) + h(y) + \log 2$;
- iii. $h(\alpha^r) = |r| \cdot h(\alpha)$, for all $r \in \mathbb{Z}$.

Finally, the last tool was proved by Dujella and Pethő [43] (Lemma 5(a)):

Lemma 3. Let M be a positive integer and let p/q be a convergent of the continued fraction expansion of the irrational number γ such that $q > 6M$. Let $(A, B) \in \mathbb{R}_{>0} \times \mathbb{R}_{>1}$ and define $\epsilon := \|\mu q\| - M \|\gamma q\|$, where μ is a given real number. If $\epsilon > 0$, then there is no solution to the Diophantine inequality

$$0 < m\gamma - n + \mu < A \cdot B^{-k}$$

in positive integers m, n and k satisfying

$$m \leq M \text{ and } k \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

In the previous statement, we used the notation $\|x\| = \min\{|x - n| : n \in \mathbb{Z}\}$. Now, we are ready to deal with the proof of the theorem.

3. The Proof of Theorem 1

3.1. Upper Bounds for n and ℓ in Terms of k

Let α and β be the dominant roots of the sequences $(F^{(k)})_n$ and $(F^{(k+m)})_n$, respectively. Moreover, set $g := g(\alpha, k)$, $h := g(\beta, k + m)$, $e_1 := E_{n,k}$ and $e_2 := E_{n,k+m}$. Thus, by using (4) in Equation (3), we obtain

$$(g\alpha^{n-1} + e_1)(h\beta^{n-1} + e_2) = a \left(\frac{10^\ell - 1}{9} \right). \quad (6)$$

After some computations, we get

$$\left| gh(\alpha\beta)^{n-1} - a \frac{10^\ell}{9} \right| < 3\beta^{n-1}, \quad (7)$$

where we used that $\max\{|e_1|, |e_2|\} < 1/2$ and $\alpha < \beta$. Now, we divide the inequality by $gh(\alpha\beta)^{n-1}$, to derive

$$\left| 1 - a(9gh)^{-1}(\alpha\beta)^{n-1}10^\ell \right| < \frac{96}{\alpha^n}, \quad (8)$$

where we applied the inequality $\min\{|g|, |h|\} > 1/4$ (see [44]). Let us define

$$\Lambda = \ell \log 10 - (n - 1) \log(\alpha\beta) + \log \theta_{a,m}, \quad (9)$$

where $\theta_{a,m} := a(9gh)^{-1}$ (for $a, m \in [1, 9]$). Then, Equation (8) can be rewritten as

$$|e^\Lambda - 1| < \frac{96}{\alpha^n}. \quad (10)$$

First, we claim that $\Lambda \neq 0$ (for $\ell > 1$). On the contrary, we would have $a10^\ell/9 = gh(\alpha\beta)^{n-1}$. Now, we can apply to the previous equality, a nontrivial automorphism $\sigma : \mathbb{K} \rightarrow \mathbb{K}$ ($\alpha \mapsto \alpha_2$ and $\beta \mapsto \beta_2$) of the Galois group of $\mathbb{K} := \mathbb{Q}(\alpha, \beta)$ to obtain (after applying absolute values):

$$10^\ell/9 \leq a \cdot 10^\ell/9 = \left| \sigma(a \cdot 10^\ell/9) \right| = |\sigma(gh(\alpha\beta)^{n-1})| = |g_2 h_2(\alpha_2 \beta_2)^{n-1}| < 4,$$

where $g_2 = g(\alpha_2, k)$ and $h_2 = g(\beta_2, k+m)$ (both are smaller than 2). The previous inequality implies $10^\ell \leq 36$ and so $\ell = 1$ yielding a contradiction. Therefore, we have that $\Lambda \neq 0$ as desired.

In order to use Lemma 1, we take $t := 3$,

$$\gamma_1 := 10, \gamma_2 := \alpha\beta, \gamma_3 := \theta_{a,m}$$

and

$$b_1 := \ell, b_2 := -n+1, b_3 := 1.$$

For this choice, we have $D = [\mathbb{Q}(\alpha, \beta) : \mathbb{Q}] \leq k(k+m) \leq k(k+9) \leq 6k^2$, for $k \geq 2$. Note that $h(\gamma_1) = \log 10$, and by Lemma 2

$$h(\gamma_2) \leq h(\alpha) + h(\beta) = \frac{\log \alpha}{k} + \frac{\log \beta}{k+m} \leq \frac{2 \log 2}{k}.$$

Moreover, in [39] (p. 73), an estimate for $h(g)$ was given. More precisely, it was proved that $h(g) < \log(4k+4)$. Thus, by using again Lemma 2, we have

$$h(\gamma_3) \leq h(9a) + h(g) + h(h) \leq \log(81) + \log(4k+4) + \log(4k+4m+4) < 11.3 \log k,$$

where we used that $4k+4m+4 \leq 4k+40 \leq k^{5.6}$, for all $k \geq 2$.

Thus, we can choose

$$A_1 := 6k^2 \log 10, A_2 := 12k \log 2, A_3 := 67.9k^2 \log k.$$

Note that $\max\{|b_1|, |b_2|, |b_3|\} = \max\{\ell, n-1\}$. However, by (3), we deduce that

$$2^{2n-4} \geq F_n^{(k)} F_n^{(k+m)} = a \left(\frac{10^\ell - 1}{9} \right) > 10^{\ell-1}$$

yielding $n > 1.6\ell + 0.3 > \ell$ and so we can take $B = n$.

Now, we are in a position to use Lemma 1 which provides us (after some manipulations)

$$\left| 1 - a(9gh)^{-1}(\alpha\beta)^{n-1}10^\ell \right| > \exp(-1.4 \cdot 10^{15} k^9 (\log k)^2 \log n). \quad (11)$$

By combining (10) and (11), we obtain

$$\frac{n}{\log n} < 4.7 \cdot 10^{16} k^9 (\log k)^2. \quad (12)$$

Since, for $x > e$, the function $x \mapsto x/\log x$ is increasing, then it is a simple exercise to show that

$$\frac{x}{\log x} < A \text{ implies that } x < 2A \log A. \quad (13)$$

Thus, by using (13) in (12) for $x := n$ and $A := 2.7 \cdot 10^{15} k^9 (\log k)^2$, we get

$$n < 2(4.7 \cdot 10^{16} k^9 (\log k)^2) \log(4.7 \cdot 10^{16} k^9 (\log k)^2) < 2.4 \cdot 10^{18} k^9 (\log k)^3,$$

where

$$\log(4.7) + 16 \log 10 + 9 \log k + 2 \log \log k < 46 \log k,$$

for all $k \geq 2$.

In conclusion, we arrive at

$$\ell < n < 4.4 \cdot 10^{18} k^9 (\log k)^3 \quad (14)$$

Now, the proof splits into two cases, as follows.

3.2. The Case $k \leq 322$

By using (14), we have that, if $k \leq 322$, then

$$\ell < n < 2.3 \cdot 10^{43}.$$

Now, we desire to apply the reduction method (based on Lemma 3) to make the bounds much smaller. The further arguments work for $\Lambda > 0$ and $\Lambda < 0$ in a similar way. So, to avoid unnecessary repetitions, we shall consider only the case when $\Lambda > 0$ (and then $\Lambda < e^\Lambda - 1$). Thus, by (8), we have

$$0 < \Lambda = \ell \log 10 - (n - 1) \log(\alpha\beta) + \log \theta_{a,m} < 96 \cdot \alpha^{-n}.$$

We divide by $\log(\alpha\beta)$ to get

$$0 < \ell \gamma_{k,m} - (n - 1) + \mu_{k,a,m} < 32 \cdot (1.4)^{-n} \quad (15)$$

where $\gamma_{k,m} := \log 10 / \log(\alpha\beta)$ and $\mu_{k,a,m} := \log \theta_a / \log(\alpha\beta)$.

We claim that $\gamma_{k,m}$ is irrational for any integer $k \geq 2$ (In fact, it suffices to notice that $(\alpha\beta)^r$ is irrational for all integer $r \geq 1$). Let $q_{v,k,m}$ be the denominator of the v -th convergent of the continued fraction of $\gamma_{k,m}$. After taking $M := 2.3 \cdot 10^{43}$, we use Mathematica to obtain

$$\min_{(k,m) \in [2,322] \times [1,9]} q_{600,k} > 6M.$$

Moreover

$$q' := \max_{(k,m) \in [2,322] \times [1,9]} q_{600,k} < 3.4 \cdot 10^{423}.$$

Define $\epsilon_{k,a,m} := \|\mu_{k,a,m} q_{600,k,m} - M\| \gamma_{k,m} q_{600,k,m}$, for $k \in [3,322]$ and $a, m \in [1,9]$. Then,

$$\epsilon' := \min_{(k,a,m) \in [3,322] \times [1,9]^2} \epsilon_{k,a} > 2.6 \cdot 10^{-8}.$$

Observe that the conditions of Lemma 3 are fulfilled for $A = 32$ and $B = 1.4$ and hence there is no solution to inequality (15) (and then no solution to the Diophantine Equation (3)) for n and l satisfying

$$\ell < M \text{ and } n \geq \frac{\log(Aq'/\epsilon')}{\log B}.$$

Since $\ell < n < M$, then

$$n \geq \frac{\log(Aq'/\epsilon')}{\log B} < 3155.$$

Therefore, the possible solutions of (3) are in the range $\ell, n \in [2,3155]$ (since $\ell > 1$ was already studied) and $k \in [3,322]$. By using Mathematica, we can explicit $F_n^{(k)} F_n^{(k+m)} \pmod{10^{10}}$, for $n \in [3,3155]$, $k \in [3,322]$ and $m \in [1,9]$. Now, we search for 10 digits repdigits among these remainders $\pmod{10^{10}}$ and none a solution was found. This completes the proof.

3.3. The Case $k \geq 323$

3.3.1. Bound for k in Terms of n

In this case, we have

$$n < 4.4 \cdot 10^{18} k^9 (\log k)^3 < 2^{k/2} < 2^{(k+9)/2}.$$

This allows us to apply the Bravo and Luca method (see [39]). The main idea of their approach is to approximate α and β to 2 (in an explicit form). We shall omit the details since it is very well explained in [39]. Therefore, we can write

$$g\alpha^{n-1} = 2^{n-2} + \delta_1/2 + 2^{n-1}\eta_1 + \eta_1\delta_1$$

and

$$h\beta^{n-1} = 2^{n-2} + \delta_2/2 + 2^{n-1}\eta_2 + \eta_2\delta_2,$$

where $\eta_1 := g - 1/2$, $\eta_2 := h - 1/2$, $\delta_1 := 2^{n-1} - \alpha$ and $\delta_2 := 2^{n-1} - \beta$. Moreover, they proved that

$$|\eta_i| < 2(k+i-1)/2^{k+i-1} \text{ and } |\delta_i| < 2^{n+2}/2^{(k+i-1)/2}, \text{ for } i \in \{1, 2\}.$$

Now, we can use all these information to derive (after many straightforward manipulations) that

$$2^{2n-4} = (g\alpha^{n-1} - \delta_1/2 - 2^{n-1}\eta_1 - \eta_1\delta_1)(h\beta^{n-1} - \delta_2/2 - 2^{n-1}\eta_2 - \eta_2\delta_2) = gh(\alpha\beta)^{n-1} + t_{n,k},$$

where $|t_{n,k}| < 15 \cdot 2^{2n-1}/2^{k/2}$, for $k \geq 323$. Thus

$$\left| 2^{2n-4} - a \cdot \frac{10^\ell}{9} \right| \leq \left| gh(\alpha\beta)^{n-1} - a \frac{10^\ell}{9} \right| + |t_{n,k}| < 3\beta^{n-1} + 15 \cdot \frac{2^{2n-1}}{2^{k/2}} < 18 \cdot \frac{2^{2n-1}}{2^{k/2}},$$

where we used the inequality (7). We divide the previous inequality by 2^{2n-4} to have

$$\left| 1 - \frac{16a}{9} 10^\ell \cdot 4^{-n} \right| < \frac{144}{2^{k/2}}. \quad (16)$$

Clearly, the left-hand side above is nonzero (otherwise 5 would divide $9 \cdot 4^n$). We shall apply Lemma 1 again for $t := 3$,

$$\gamma_1 := 10, \gamma_2 := 4, \gamma_3 := 16a/9$$

and

$$b_1 := \ell, b_2 := -n, b_3 := 1.$$

Therefore, $D = 1$ and $h(\gamma_i) = \log \gamma_i$ ($i \in \{1, 2\}$), and by Lemma 2, $h(\gamma_3) < 7.2$. Thus, we can choose

$$A_1 := \log 10, A_2 := \log 4, A_3 := 7.2$$

and $B = n$ and Lemma 1 gives

$$\left| 1 - \frac{16a}{9} 10^\ell \cdot 4^{-n} \right| > \exp(-6.6 \cdot 10^{12} \log n). \quad (17)$$

By combining (16) and (17), we get

$$k < 1.8 \cdot 10^{13} \log n. \quad (18)$$

3.3.2. Explicit Bounds for n, ℓ and k and the Reduction Method

By (14) and (18), we have

$$k < 1.8 \cdot 10^{13} \log(4.4 \cdot 10^{18} k^9 (\log k)^3).$$

Since the left-hand side is $O(k)$, while the right-hand one is $O(\log k)$, then the previous inequality is valid only for finitely many values of k (since $k / \log k \rightarrow 0$ as $k \rightarrow \infty$). In order to make this explicitly, after some calculations, the previous inequality becomes

$$k < 5.2 \cdot 10^{19} \log k.$$

Now, by using Mathematica software, we infer that $k < 2.7 \cdot 10^{21}$ and $n < 5.4 \cdot 10^{214}$. Therefore, we need to make these bounds smaller. For that, we shall use Lemma 3. Take $\Gamma := (\ell - 1) \log 10 - n \log 4 + \log(160a/9)$. By (16),

$$|e^\Gamma - 1| < \frac{144}{2^{k/2}}.$$

Again, we can suppose that $\Gamma > 0$ (otherwise, use that $1 - e^{-x} = |e^x - 1|$, if $x < 0$) to write

$$0 < (\ell - 1) \log 10 - n \log 4 + \log(160a/9) < 144 \cdot 2^{k/2}.$$

Dividing through by $\log 4$, we obtain

$$0 < (\ell - 1)\gamma - n + \mu_a < 104 \cdot (1.4)^{-k}, \quad (19)$$

where $\gamma := \log 10 / \log 4$ and $\mu_a := \log(160a/9) / \log 4$, for $a \in [1, 9]$.

Clearly, γ is irrational and Let q_ν be the denominator of the ν -th convergent of its continued fraction. Taking $M := 3.4 \cdot 10^{214}$, we use Mathematica again to obtain that $q_{454} > 6M$.

Define $\epsilon_a := \|\mu_a q_{454}\| - M \|\gamma q_{454}\|$, for $a \in [1, 9]$. Then, we obtain

$$\epsilon' := \min_{a \in [1, 9]} \epsilon_a > 0.009.$$

Thus, all conditions of Lemma 3 are satisfied for $A = 104$ and $B = 1.4$ and hence there is no solution to inequality (19) for ℓ and k satisfying

$$\ell - 1 < M \text{ and } k \geq \frac{\log(A q_{454} / \epsilon')}{\log B}.$$

Since $\ell - 1 < n < M$, we have that $k < \frac{\log(A q_{454} / \epsilon')}{\log B} < 1506$. Thus $n < 3.8 \cdot 10^{49}$. By repeating this process again for the new $M = 3.8 \cdot 10^{49}$ (we use q_{120}), we obtain $k \leq 365$ and $n < 5.7 \cdot 10^{493}$. To conclude, we apply one more time Lemma 3 for the new choice of $M := 3.8 \cdot 10^{49}$ (for q_{107}) and hence $k \leq 322$.

This contradicts our assumption that $k \geq 323$. In conclusion, there is no solution to the Diophantine Equation (3) for $k \geq 323$. \square

4. Other Similar Equations: The Elementary Method

It is important to notice that an elementary method does not provide a reasonable approach to deal with the Equation (1). The possible reasons can be because the product (in the left-hand side) possesses only two terms, n can be much larger than $k + m$ as well as the very limited knowledge about arithmetic properties of repdigits (from that equation we infer only that ℓ must be a composite number).

However, we shall provide here some similar Diophantine equations which can be solved by using basic tools.

The first one is when the order is larger than the index. More precisely

Proposition 1. *The Diophantine equation*

$$F_n^{(k+n)} = a \left(\frac{10^\ell - 1}{9} \right), \quad (20)$$

does not have a solution in positive integers n, a, k, ℓ , with $k \geq 2$, $\ell > 1$ and $a \in [1, 9]$.

The proof follows because $F_n^{(k+n)} = 2^{n-2}$, for all $n \in [2, k+n+1]$ and so $F_n^{(k+n)} \equiv 0 \pmod{16}$, for all $n \geq 6$. Since none repdigit is a multiple of 16, then $n \in [1, 5]$. However, $F_5^{(k+5)} < 10$ which contradicts $\ell > 1$.

Another possible problem is a symmetric equation (between k and n) which forces the previous case. For example:

Proposition 2. *The Diophantine equation*

$$F_n^{(k)} F_k^{(n)} = a \left(\frac{10^\ell - 1}{9} \right), \quad (21)$$

does not have a solution in positive integers n, a, k, ℓ , with $k \geq 2$, $\ell > 1$ and $a \in [1, 9]$.

The proof is similar to the previous one by using that either $n \leq k+1$ or $k \leq n+1$.

In the previous propositions, we used the fact that $F_n^{(k)}$ is a power of 2, when $n \leq k+1$. However, we can use another approach (since $F_n^{(k)}$ is never a power of two, for $n > k+1$, see [39]) for equations related to the product of “many” consecutive k -bonacci numbers. More precisely:

Proposition 3. *The Diophantine equation*

$$F_n^{(k)} F_{n+1}^{(k)} \cdots F_{n+4k}^{(k)} = a \left(\frac{10^\ell - 1}{9} \right), \quad (22)$$

does not have a solution in positive integers n, a, k, ℓ , with $k \geq 2$, $\ell > 1$ and $a \in [1, 9]$.

For proving this, we notice that the recurrence of $(F_n^{(k)})_n$ yields

$$F_{n+1}^{(k)} = 2F_n^{(k)} - F_{n-k}^{(k)}.$$

Thus, $F_{n+(k+1)}^{(k)} \equiv F_n^{(k)} \pmod{2}$, i.e., $(F_n^{(k)})_n$ is a $(k+1)$ -periodic sequence $\pmod{2}$. Thus, the sequence $(F_n^{(k)})_n$ contains infinitely many even numbers (for example, $F_{3+t(k+1)}^{(k)}$, for all $t \geq 0$). For this reason, at least one among the numbers $F_n^{(k)}, F_{n+1}^{(k)}, \dots, F_{n+k}^{(k)}$ is even. The same happens for the lists $\{F_{n+k+1}^{(k)}, \dots, F_{n+2k}^{(k)}\}$, $\{F_{n+2k+1}^{(k)}, \dots, F_{n+3k}^{(k)}\}$ and $\{F_{n+3k+1}^{(k)}, \dots, F_{n+4k}^{(k)}\}$. Hence, the product

$$F_n^{(k)} F_{n+1}^{(k)} \cdots F_{n+4k}^{(k)}$$

is a multiple of 16 and so it can not be a repdigit.

5. Conclusions

For any integer $k \geq 2$, the sequence of the k -generalized Fibonacci numbers (or k -bonacci numbers) is defined by the k initial values $F_{-(k-2)}^{(k)} = \dots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$ and such that each term afterwards is the sum of the k preceding ones. In this paper, we search for solutions of the Diophantine equation $F_n^{(k)} F_n^{(k+m)} = a(10^\ell - 1)/9$ for positive

integers k, n, ℓ, m and a , with $k \geq 2$, $\ell \geq 2$ and $a, m \in [1, 9]$. In particular, the only repdigits, which can be written as a product of n th terms of two generalized Fibonacci sequences with consecutive orders, has only one digit. Our approach to proving this fact is to combine the Baker's theory (on lower bounds for linear forms in the logarithms) with a reduction method from the theory of continued fractions (due to Dujella and Pethő). In the concluding section, we present some similar problems which can be solved by using only elementary tools.

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