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On Hermite-Hadamard Type Inequalities for Coordinated Convex Functions via (p,q)-Calculus

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Abstract: In this paper, we define (p,q)-integrals for continuous functions of two variables. Then, we prove the Hermite-Hadamard type inequalities for coordinated convex functions by using (p,q)-integrals. Many results obtained in this paper provide significant extensions of other related results given in the literature. Finally, we give some examples of our results.

Keywords: Hermite-Hadamard inequality; (p,q)-derivative; (p,q)-integral; (p,q)-calculus; coordinated convex function

MSC: 05A30; 26D10; 26D15; 81P68

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1. Introduction

Quantum calculus or q-calculus is the modern name of the study of calculus without limits. It has been studied since the early eighteenth century. The famous mathematician, Euler, established q-calculus and, in 1910, F. H. Jackson [1] determined the definite q-integral known as the q-Jackson integral. Quantum calculus has many applications in mathematics and physics such as combinatorics, orthogonal polynomials, number theory, basic hypergeometric functions, quantum theory, mechanics, and theory of relativity, see for instance [2–23] and the references therein. The book by V. Kac and P. Cheung [24] covers the fundamental knowledge and also the basic theoretical concepts of quantum calculus.

In 2013, J. Tariboon and S. K. Ntouyas [25] defined the q-derivative and q-integral of a continuous function on finite intervals and proved some of its significant properties. In addition, they firstly extended Hölder, Hermite-Hadamard, trapezoid, Ostrowski, Cauchy-Bunyakovsky-Schwarz, Grüss and Grüss-Čebyšev inequalities to q-calculus by using such the definitions, see [26] for more details. Based on these results, there are many publications about q-calculus, see [27–37] and the references cited therein. The further generalization of quantum calculus is post-quantum calculus, denoted by (p,q)-calculus, which was first considered by R. Chakrabati and R. Jagannathan [38].

In 2016, M. Tunç and E. Göv [39,40] introduced the (p,q)-derivative and (p,q)-integral on finite intervals, proved some of its properties and gave many integral inequalities via (p,q)-calculus. Recently, according to works of M. Tunç and E. Göv, many researchers started working in this direction, some more results on the study of (p,q)-calculus can be found in [41–55].

The Hermite-Hadamard inequality is a classical inequality that has fascinated many researchers, stated as: If $f : [a,b] \to \mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Inequality (1) was introduced by C. Hermite [56] in 1883 and was investigated by J. Hadamard [57] in 1893. If $f : [a,b] \times [c,d] \to \mathbb{R}$ is a convex function for coordinates, then S. Dragomir [58] stated the Hermite-Hadamard type inequalities in 2001 as follows:

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) dy\right] \\ &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx \\ &\leq \frac{1}{2}\left[\frac{1}{b-a}\left(\int_a^b f(x,c)\,dx + \int_a^b f(x,d)\,dx\right) + \frac{1}{d-c}\left(\int_c^d f(a,y)\,dy + \int_c^d f(b,y)\,dy\right)\right] \\ &\leq \frac{1}{4}[f(a,c)+f(b,c)+f(a,d)+f(b,d)]. \end{aligned} \tag{2}$$

In 2019, the Hermite-Hadamard type inequalities for coordinates via q-calculus was presented by M. Kunt et al. [27]:

$$\begin{aligned} f\left(\frac{q_1a+b}{1+q_1}, \frac{q_2c+d}{1+q_2}\right) &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{q_2c+d}{1+q_2}\right) {}_ad_{q_1}x + \frac{1}{d-c}\int_c^d f\left(\frac{q_1a+b}{1+q_1}, y\right) {}_cd_{q_2}y\right] \\ &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,{}_cd_{q_2}y\,{}_ad_{q_1}x \\ &\leq \frac{q_1}{2(1+q_1)(d-c)}\int_c^d f(a,y)\,{}_cd_{q_2}y + \frac{q_1}{2(1+q_1)(d-c)}\int_c^d f(b,y)\,{}_cd_{q_2}y \\ &\quad + \frac{q_2}{2(1+q_2)(b-a)}\int_a^b f(x,c)\,{}_ad_{q_1}x + \frac{q_2}{2(1+q_2)(b-a)}\int_a^b f(x,d)\,{}_ad_{q_1}x \\ &\leq \frac{q_1q_2f(a,c)+q_1f(b,c)+q_2f(a,d)+f(b,d)}{(1+q_1)(1+q_2)}, \end{aligned}$$

for all $q_1, q_2 \in (0,1)$.

Recently, S. Bermudo, P. Korus and J. E. N. Valdes [28] defined new ${}^bq$-derivative, ${}^bq$-integral and also gave the Hermite-Hadamard inequality via q-calculus by using such the definitions. Consequently, H. Budak, M. A. Ali and M. Tarhanachi [29] defined some new ${}^bq$-integrals for coordinates and gave the following inequalities

$$\begin{aligned} f\left(\frac{q_1a+b}{1+q_1}, \frac{c+q_2d}{1+q_2}\right) &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+q_2d}{1+q_2}\right) {}_ad_{q_1}x + \frac{1}{d-c}\int_c^d f\left(\frac{q_1a+b}{1+q_1}, y\right) {}^dd_{q_2}y\right] \\ &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,{}^dd_{q_2}y\,{}_ad_{q_1}x \\ &\leq \frac{q_1}{2(1+q_1)(d-c)}\int_c^d f(a,y)\,{}^dd_{q_2}y + \frac{1}{2(1+q_1)(d-c)}\int_c^d f(b,y)\,{}^dd_{q_2}y \\ &\quad + \frac{1}{2(1+q_2)(b-a)}\int_a^b f(x,c)\,{}_ad_{q_1}x + \frac{q_2}{2(1+q_2)(b-a)}\int_a^b f(x,d)\,{}_ad_{q_1}x \\ &\leq \frac{q_1f(a,c)+q_1q_2f(a,d)+f(b,c)+q_2f(b,d)}{(1+q_1)(1+q_2)}, \end{aligned} \tag{3}$$

$$
\begin{aligned}
f\left(\frac{a+q_1b}{1+q_1},\frac{q_2c+d}{1+q_2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{q_2c+d}{1+q_2}\right)\,{}^bd_{q_1}x+\frac{1}{d-c}\int_c^d f\left(\frac{a+q_1b}{1+q_1},y\right)\,{}_cd_{q_2}y\right]\\
&\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,{}_cd_{q_2}y\,{}^bd_{q_1}x\\
&\le \frac{1}{2(1+q_1)(d-c)}\int_c^d f(a,y)\,{}_cd_{q_2}y+\frac{q_1}{2(1+q_1)(d-c)}\int_c^d f(b,y)\,{}_cd_{q_2}y\\
&\quad+\frac{q_2}{2(1+q_2)(b-a)}\int_a^b f(x,c)\,{}^bd_{q_1}x+\frac{1}{2(1+q_2)(b-a)}\int_a^b f(x,d)\,{}^bd_{q_1}x\\
&\le \frac{q_2f(a,c)+f(a,d)+q_1q_2f(b,c)+q_1f(b,d)}{(1+q_1)(1+q_2)}
\end{aligned}
\tag{4}
$$

and

$$
\begin{aligned}
f\left(\frac{a+q_1b}{1+q_1},\frac{c+q_2d}{1+q_2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{q_2c+d}{1+q_2}\right)\,{}^bd_{q_1}x+\frac{1}{d-c}\int_c^d f\left(\frac{a+q_1b}{1+q_1},y\right)\,{}^dd_{q_2}y\right]\\
&\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,{}^dd_{q_2}y\,{}^bd_{q_1}x\\
&\le \frac{1}{2(1+q_1)(d-c)}\int_c^d f(a,y)\,{}^dd_{q_2}y+\frac{q_1}{2(1+q_1)(d-c)}\int_c^d f(b,y)\,{}^dd_{q_2}y\\
&\quad+\frac{1}{2(1+q_2)(b-a)}\int_a^b f(x,c)\,{}^bd_{q_1}x+\frac{q_2}{2(1+q_2)(b-a)}\int_a^b f(x,d)\,{}^bd_{q_1}x\\
&\le \frac{f(a,c)+q_2f(a,d)+q_1f(b,c)+q_1q_2f(b,d)}{(1+q_1)(1+q_2)},
\end{aligned}
\tag{5}
$$

for all $q_1,q_2\in(0,1)$. Moreover, Yu-Ming Chu et al. [51] presented the definitions for new ${}^b(p,q)$-derivatives, ${}^b(p,q)$-integrals and gave the Hermite-Hadamard type inequality for convex functions by using (p,q)-calculus. Our present work was motivated by the above mentioned literatures, we propose to define new ${}^b(p,q)$-integrals for coordinates and then extend the Hermite-Hadamard type inequality in q-calculus for coordinated convex functions to (p,q)-calculus for coordinated convex functions.

2. Preliminaries

Throughout this paper, we let $[a,b],[c,d]\subseteq\mathbb{R}$, $0<q<p\le 1, 0<q_i<p_i\le 1$ for $i=1,2$. The definitions of coordinated functions, (p,q)-calculus, q-calculus and (p,q)-calculus for coordinates are given in [27,29,39,42,51,53,54].

Definition 1. [39] *Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Then the ${}_a(p,q)$-derivative ${}_aD_{p,q}f(x)$ of f at $x\in(a,b]$ is defined by*

$$ {}_aD_{p,q}f(x)=\frac{f(px+(1-p)a)-f(qx+(1-q)a)}{(p-q)(x-a)},\quad x\neq a. $$

The ${}_a(p,q)$-integral $\int_a^x f(t)\,{}_ad_{p,q}t$ is defined by

$$ \int_a^x f(t)\,{}_ad_{p,q}t=(p-q)(x-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right). $$

Definition 2. [51] *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then the $^{b}(p,q)$-derivative $^{b}D_{p,q}f(x)$ of f at $x \in [a,b)$ is defined by*

$$^{b}D_{p,q}f(x) = \frac{f(px+(1-p)b) - f(qx+(1-q)b)}{(p-q)(b-x)}, \quad x \neq b.$$

The $^{b}(p,q)$-integral $\int_x^b f(t)\,^{b}d_{p,q}t$ is defined by

$$\int_x^b f(t)\,^{b}d_{p,q}t = (p-q)(b-x)\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}x + \left(1 - \frac{q^n}{p^{n+1}}\right)b\right).$$

Definition 3. [58] *A function $f : [a,b] \times [c,d] \to \mathbb{R}$ is said to be convex on coordinates, if the partial mappings*

$$f_x : [c,d] \ni v \mapsto f(x,v) \in \mathbb{R} \quad \text{and}$$
$$f_y : [a,b] \ni u \mapsto f(u,y) \in \mathbb{R}$$

are convex for all $x \in (a,b)$ and $y \in (c,d)$.

A formal definition for coordinated convex functions may be stated as follows:

Definition 4. *A function $f : [a,b] \times [c,d] \to \mathbb{R}$ is said to be convex on coordinates, if*

$$f(tx+(1-t)z, \lambda y + (1-\lambda)w) \leq t\lambda f(x,y) + t(1-\lambda)f(x,w) + (1-t)\lambda f(z,y) + (1-t)\lambda(1-])f(z,w)$$

holds for all $t, \lambda \in [0,1]$, $(x,y),(z,w) \in [a,b] \times [c,d]$.

Definition 5. [30] *Suppose that $f : [a,b] \times [c,d] \to \mathbb{R}$ is a continuous function of two variables. Then the derivatives are given by*

$$\frac{{}_{a}\partial_{q_1} f(x,y)}{{}_{a}\partial_{q_1} x} = \frac{f(q_1x+(1-q_1)a, y) - f(x,y)}{(1-q_1)(x-a)}, \quad x \neq a,$$
$$\frac{{}_{c}\partial_{q_2} f(x,y)}{{}_{c}\partial_{q_2} x} = \frac{f(x, q_2y+(1-q_2)c, y) - f(x,y)}{(1-q_2)(y-c)}, \quad y \neq c,$$

and

$$\frac{{}_{a,c}\partial^2{}_{q_1,q_2} f(x,y)}{{}_{a}\partial_{q_1} x\,{}_{c}\partial_{q_2} y} = \frac{f(q_1x+(1-q_1)a, q_2y+(1-q_2)c) - f(q_1x+(1-q_1)a, y) - f(x, q_2y+(1-q_2)c) + f(x,y)}{(1-q_1)(1-q_2)(x-a)(y-c)},$$

for $x \neq a$, $y \neq c$.

Definition 6. [30] *Suppose that $f : [a,b] \times [c,d] \to \mathbb{R}$ is a continuous function of two variables. Then the definite integral is given by*

$$\int_a^x \int_c^y f(t,s)\, {}_{c}d_{q_2}s\, {}_{a}d_{q_1}t = (1-q_1)(1-q_2)(x-a)(y-c)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m f(q_1^n x + (1-q_1^n)a, q_2^m y + (1-q_2^m)c).$$

Definition 7. [29] *Suppose that $f : [a,b] \times [c,d] \to \mathbb{R}$ is a continuous function of two variables. Then the definite integrals are given by*

$$\int_a^x \int_y^d f(t,s)\ {}^d d_{q_2} s\ {}_a d_{q_1} t$$
$$= (1-q_1)(1-q_2)(x-a)(d-y) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m f(q_1^n x + (1-q_1^n)a, q_2^m y + (1-q_2^m)d),$$

$$\int_x^b \int_c^y f(t,s)\ {}_c d_{q_2} s\ {}^b d_{q_1} t$$
$$= (1-q_1)(1-q_2)(b-x)(y-c) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m f(q_1^n x + (1-q_1^n)b, q_2^m y + (1-q_2^m)c)$$

and

$$\int_x^b \int_y^d f(t,s)\ {}^d d_{q_2} s\ {}^b d_{q_1} t$$
$$= (1-q_1)(1-q_2)(b-x)(d-y) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m f(q_1^n x + (1-q_1^n)b, q_2^m y + (1-q_2^m)d).$$

Definition 8. [52] *Suppose that $f : [a,b] \times [c,d] \to \mathbb{R}$ is a continuous function of two variables. Then the derivatives are given by*

$$\frac{{}_a\partial_{p_1,q_1} f(x,y)}{{}_a\partial_{p_1,q_1} x} = \frac{f(p_1 x + (1-p_1)a, y) - f(q_1 x + (1-q_1)a, y)}{(p_1 - q_1)(x-a)}, \quad x \neq a,$$
$$\frac{{}_c\partial_{p_2,q_2} f(x,y)}{{}_c\partial_{p_2,q_2} x} = \frac{f(x, p_2 y + (1-p_2)c, y) - f(x, q_2 y + (1-q_2)c, y)}{(p_2 - q_2)(y-c)}, \quad y \neq c,$$

and

$$\frac{{}_{a,c}\partial^2{}_{p_1,p_2,q_1,q_2} f(x,y)}{{}_a\partial_{p_1,q_1} x\, {}_c\partial_{p_2,q_2} y} = \frac{f(q_1 x + (1-q_1)a, q_2 y + (1-q_2)c) - f(q_1 x + (1-q_1)a, p_2 y + (1-p_2)c)}{(p_1-q_1)(p_2-q_2)(x-a)(y-c)}$$
$$- \frac{f(p_1 x + (1-p_1)a, q_2 y + (1-q_2)c) + f(p_1 x + (1-p_1)a, p_2 y + (1-p_2)c)}{(p_1-q_1)(p_2-q_2)(x-a)(y-c)},$$

for $x \neq a$, $y \neq c$.

Definition 9. [52] *Suppose that $f : [a,b] \times [c,d] \to \mathbb{R}$ is a continuous function of two variables. Then the definite integral is given by*

$$\int_a^x \int_c^y f(t,s)\ {}_c d_{p_2,q_2} s\ {}_a d_{p_1,q_1} t = (p_1-q_1)(p_2-q_2)(x-a)(y-c)$$
$$\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}} \frac{q_2^m}{p_2^{m+1}} f\left(\frac{q_1^n}{p_1^{n+1}} x + \left(1 - \frac{q_1^n}{p_1^{n+1}}\right) a, \frac{q_2^m}{p_2^{m+1}} y + \left(1 - \frac{q_2^m}{p_2^{m+1}}\right) c \right).$$

For convenience, we call the integral defined in Definition 10 as L-L (Left-Left) integral. Next, we define another integrals for continuous functions of two variables.

Definition 10. *Suppose that $f : [a,b] \times [c,d] \to \mathbb{R}$ is a continuous function of two variables. Then the L-R integral, R-L integral and R-R integral are given by*

$$\int_a^x \int_y^d f(t,s)\ {}^d d_{p_2,q_2}s\ {}_a d_{p_1,q_1}t = (p_1 - q_1)(p_2 - q_2)(x-a)(d-y) \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}} \frac{q_2^m}{p_2^{m+1}} f\left(\frac{q_1^n}{p_1^{n+1}}x + \left(1 - \frac{q_1^n}{p_1^{n+1}}\right)a, \frac{q_2^m}{p_2^{m+1}}y + \left(1 - \frac{q_2^m}{p_2^{m+1}}\right)d\right),$$

$$\int_x^b \int_c^y f(t,s)\ {}_c d_{p_2,q_2}s\ {}^b d_{p_1,q_1}t = (p_1 - q_1)(p_2 - q_2)(b-x)(y-c) \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}} \frac{q_2^m}{p_2^{m+1}} f\left(\frac{q_1^n}{p_1^{n+1}}x + \left(1 - \frac{q_1^n}{p_1^{n+1}}\right)b, \frac{q_2^m}{p_2^{m+1}}y + \left(1 - \frac{q_2^m}{p_2^{m+1}}\right)c\right)$$

and

$$\int_x^b \int_y^d f(t,s)\ {}^d d_{p_2,q_2}s\ {}^b d_{p_1,q_1}t = (p_1 - q_1)(p_2 - q_2)(b-x)(d-y) \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}} \frac{q_2^m}{p_2^{m+1}} f\left(\frac{q_1^n}{p_1^{n+1}}x + \left(1 - \frac{q_1^n}{p_1^{n+1}}\right)b, \frac{q_2^m}{p_2^{m+1}}y + \left(1 - \frac{q_2^m}{p_2^{m+1}}\right)d\right).$$

Obviously, if $p_1 = p_2 = 1$, then Definition 10 reduces to Definition 7.

Example 1. *Define a function $f : [0,1] \times [0,1] \to \mathbb{R}$ by $f(xy) = xy$, which is a continuous function of two variables. Then by Definitions 9 and 10, for $p_1 = p_2 = \frac{3}{4}$ and $q_1 = q_2 = \frac{2}{4}$, we obtain*

$$\int_0^1 \int_0^1 xy\ {}_0 d_{\frac{3}{4},\frac{2}{4}}y\ {}_0 d_{\frac{3}{4},\frac{2}{4}}x = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)(1)(1) \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\left(\frac{2}{4}\right)^n}{\left(\frac{3}{4}\right)^{n+1}} \frac{\left(\frac{2}{4}\right)^m}{\left(\frac{3}{4}\right)^{m+1}} \frac{\left(\frac{2}{4}\right)^n}{\left(\frac{3}{4}\right)^{n+1}} \frac{\left(\frac{2}{4}\right)^m}{\left(\frac{3}{4}\right)^{m+1}} = \frac{16}{25},$$

$$\int_0^1 \int_0^1 xy\ {}^1 d_{\frac{3}{4},\frac{2}{4}}y\ {}_0 d_{\frac{3}{4},\frac{2}{4}}x = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)(1)(1) \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\left(\frac{2}{4}\right)^n}{\left(\frac{3}{4}\right)^{n+1}} \frac{\left(\frac{2}{4}\right)^m}{\left(\frac{3}{4}\right)^{m+1}} \frac{\left(\frac{2}{4}\right)^n}{\left(\frac{3}{4}\right)^{n+1}} \left(1 - \frac{\left(\frac{2}{4}\right)^m}{\left(\frac{3}{4}\right)^{m+1}}\right) = \frac{4}{25},$$

$$\int_0^1 \int_0^1 xy\ {}_0 d_{\frac{3}{4},\frac{2}{4}}y\ {}^1 d_{\frac{3}{4},\frac{2}{4}}x = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)(1)(1) \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\left(\frac{1}{4}\right)^n}{\left(\frac{3}{4}\right)^{n+1}} \frac{\left(\frac{2}{4}\right)^m}{\left(\frac{3}{4}\right)^{m+1}} \left(1 - \frac{\left(\frac{2}{4}\right)^n}{\left(\frac{3}{4}\right)^{n+1}}\right) \frac{\left(\frac{2}{4}\right)^m}{\left(\frac{3}{4}\right)^{m+1}} = \frac{4}{25}$$

and

$$\int_0^1 \int_0^1 xy\ {}^1 d_{\frac{3}{4},\frac{2}{4}}y\ {}^1 d_{\frac{3}{4},\frac{2}{4}}x = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)(1)(1) \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\left(\frac{2}{4}\right)^n}{\left(\frac{3}{4}\right)^{n+1}} \frac{\left(\frac{2}{4}\right)^m}{\left(\frac{3}{4}\right)^{m+1}} \left(1 - \frac{\left(\frac{2}{4}\right)^n}{\left(\frac{3}{4}\right)^{n+1}}\right)\left(1 - \frac{\left(\frac{2}{4}\right)^m}{\left(\frac{3}{4}\right)^{m+1}}\right) = \frac{1}{25}.$$

At the end of this section, we give some known theorems needed to prove our main results.

Theorem 1. [53] *Suppose that $f : [a,b] \to \mathbb{R}$ is a convex differentiable function on $[a,b]$. Then we have*

$$f\left(\frac{qa + pb}{p+q}\right) \leq \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(x)\ {}_a d_{p,q}x \leq \frac{qf(a) + pf(b)}{p+q}. \tag{6}$$

Theorem 2. [42] *Suppose that $f : [a,b] \times [c,d] \to \mathbb{R}$ is a continuous function of two variables. Then the following inequalities hold:*

$$
\begin{aligned}
& f\left(\frac{q_1 a + p_1 b}{p_1+q_1}, \frac{q_2 c + p_2 d}{p_2+q_2}\right) \\
& \leq \frac{1}{2p_1(b-a)} \int_a^{p_1 b + (1-p_1)a} f\left(x, \frac{q_2 c + p_2 d}{p_2+q_2}\right) \, {}_a d_{p_1,q_1} x + \frac{1}{2p_2(b-a)} \int_c^{p_2 d + (1-p_2)c} f\left(\frac{q_1 a + p_1 b}{p_1+q_1}, y\right) \, {}_c d_{p_2,q_2} y \\
& \leq \frac{1}{p_1 p_2 (b-a)(d-c)} \int_a^{p_1 b + (1-p_1)a} \int_c^{p_2 d + (1-p_2)c} f(x,y) \, {}_c d_{p_2,q_2} y \, {}_a d_{p_1,q_1} x \\
& \leq \frac{q_2}{2p_1(p_2+q_2)(b-a)} \int_a^{p_1 b + (1-p_1)a} f(x,c) \, {}_a d_{p_1,q_1} x + \frac{p_2}{2p_1(p_2+q_2)(b-a)} \int_a^{p_1 b + (1-p_1)a} f(x,d) \, {}_a d_{p_1,q_1} x \\
& \quad + \frac{q_1}{2p_2(p_1+q_1)(d-c)} \int_c^{p_2 d + (1-p_2)c} f(a,y) \, {}_c d_{p_2,q_2} y + \frac{p_1}{2p_2(p_1+q_1)(d-c)} \int_c^{p_2 d + (1-p_2)c} f(b,y) \, {}_c d_{p_2,q_2} y \\
& \leq \frac{q_1 q_2 f(a,c) + q_1 p_2 f(a,d) + p_1 q_2 f(b,c) + p_1 p_2 f(b,d)}{(p_1+q_1)(p_2+q_2)}. \qquad (7)
\end{aligned}
$$

Theorem 3. [54] *Suppose that $f : [a,b] \to \mathbb{R}$ is a convex differentiable function on $[a,b]$. Then we have*

$$
f\left(\frac{pa+qb}{p+q}\right) \leq \frac{1}{p(b-a)} \int_{pa+(1-p)b}^{b} f(x) \, {}^b d_{p,q} x \leq \frac{pf(a)+qf(b)}{p+q}. \qquad (8)
$$

3. Main Results

In this section, we give new (p,q)-Hermite-Hadamard type inequalities for coordinated convex functions and verify them.

Theorem 4. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a convex differentiable function of two variables. Then the following inequalities hold:*

$$
\begin{aligned}
& f\left(\frac{q_1 a + p_1 b}{p_1+q_1}, \frac{p_2 c + q_2 d}{p_2+q_2}\right) \\
& \leq \frac{1}{2p_1(b-a)} \int_a^{p_1 b + (1-p_1)a} f\left(x, \frac{p_2 c + q_2 d}{p_2+q_2}\right) \, {}_a d_{p_1,q_1} x + \frac{1}{2p_2(b-a)} \int_{p_2 c + (1-p_2)d}^{d} f\left(\frac{q_1 a + p_1 b}{p_1+q_1}, y\right) \, {}^d d_{p_2,q_2} y \\
& \leq \frac{1}{p_1 p_2 (b-a)(d-c)} \int_a^{p_1 b + (1-p_1)a} \int_{p_2 c + (1-p_2)d}^{d} f(x,y) \, {}^d d_{p_2,q_2} y \, {}_a d_{p_1,q_1} x \\
& \leq \frac{q_1}{2p_2(p_1+q_1)(d-c)} \int_{p_2 c + (1-p_2)d}^{d} f(a,y) \, {}^d d_{p_2,q_2} y + \frac{p_1}{2p_2(p_1+q_1)(d-c)} \int_{p_2 c + (1-p_2)d}^{d} f(b,y) \, {}^d d_{p_2,q_2} y \\
& \quad + \frac{p_2}{2p_1(p_2+q_2)(b-a)} \int_a^{p_1 b + (1-p_1)a} f(x,c) \, {}_a d_{p_1,q_1} x + \frac{q_2}{2p_1(p_2+q_2)(b-a)} \int_a^{p_1 b + (1-p_1)a} f(x,d) \, {}_a d_{p_1,q_1} x \\
& \leq \frac{q_1 p_2 f(a,c) + q_1 q_2 f(a,d) + p_1 p_2 f(b,c) + p_1 q_2 f(b,d)}{(p_1+q_1)(p_2+q_2)}. \qquad (9)
\end{aligned}
$$

Proof. Let $h_x : [c,d] \to \mathbb{R}$ defined by $h_x(y) = f(x,y)$ be a convex differentiable function on $[c,d]$. Using the inequality (8) on $[c,d]$, we have

$$
h_x\left(\frac{p_2 c + q_2 d}{p_2+q_2}\right) \leq \frac{1}{p_2(d-c)} \int_{p_2 c + (1-p_2)d}^{d} h_x(y) \, {}^d d_{p_2,q_2} y \leq \frac{p_2 h_x(c) + q_2 h_x(d)}{p_2+q_2},
$$

i.e.,

$$
f\left(x, \frac{p_2 c + q_2 d}{p_2+q_2}\right) \leq \frac{1}{p_2(d-c)} \int_{p_2 c + (1-p_2)d}^{d} f(x,y) \, {}^d d_{p_2,q_2} y \leq \frac{p_2 f(x,c) + q_2 f(x,d)}{p_2+q_2}, \quad (10)
$$

for all $x \in [a,b]$.

By $(p_1, q_1)_a$-integrating both sides of (10) on $[a, p_1a + (1-p_1)b]$, we have

$$\begin{aligned}
&\frac{1}{p_1(b-a)} \int_a^{p_1b+(1-p_1)a} f\left(x, \frac{p_2c+q_2d}{p_2+q_2}\right) \, {}_ad_{p_1,q_1}x \\
&\quad \le \frac{1}{p_1p_2(b-a)(d-c)} \int_a^{p_1b+(1-p_1)a} \int_{p_2c+(1-p_2)d}^{d} f(x,y) \, {}^dd_{p_2,q_2}y \, {}_ad_{p_1,q_1}x \\
&\quad \le \frac{1}{p_1(b-a)} \int_a^{p_1b+(1-p_1)a} \frac{p_2f(x,c)+q_2f(x,d)}{p_2+q_2} \, {}_ad_{p_1,q_1}x. \qquad (11)
\end{aligned}$$

Similarly, let $h_y : [a,b] \to \mathbb{R}$ defined by $h_y(x) = f(x,y)$ be a convex function on $[a,b]$. Using the inequality (6) on $[a,b]$, we have

$$h_y\left(\frac{q_1a+p_1b}{p_1+q_1}\right) \le \frac{1}{p_1(b-a)} \int_a^{p_1b+(1-p_1)a} h_y(x) \, {}_ad_{p_1,q_1}x \le \frac{q_1h_y(a)+p_1h_y(b)}{p_1+q_1},$$

i.e.,

$$f\left(\frac{q_1a+p_1b}{p_1+q_1}, y\right) \le \frac{1}{p_1(b-a)} \int_a^{p_1b+(1-p_1)a} f(x,y) \, {}_ad_{p_1,q_1}x \le \frac{q_1f(a,y)+p_1f(b,y)}{p_1+q_1}, \qquad (12)$$

for all $y \in [a,b]$.

By $(p_2, q_2)^d$-integrating both sides of (12) on $[p_2c + (1-p_2)d, d]$, we have

$$\begin{aligned}
&\frac{1}{p_2(d-c)} \int_{p_2c+(1-p_2)d}^{d} f\left(\frac{q_1a+p_1b}{p_1+q_1}, y\right) {}^dd_{p_2,q_2}y \\
&\quad \le \frac{1}{p_1p_2(b-a)(d-c)} \int_{p_2c+(1-p_2)d}^{d} \int_a^{p_1b+(1-p_1)a} f(x,y) \, {}_ad_{p_1,q_1}x \, {}^dd_{p_2,q_2}y \\
&\quad \le \frac{1}{p_2(d-c)} \int_{p_2c+(1-p_2)d}^{d} \frac{q_1f(a,y)+p_1f(b,y)}{p_1+q_1} \, {}^dd_{p_2,q_2}y. \qquad (13)
\end{aligned}$$

Adding (11) and (13), we obtain

$$\begin{aligned}
&\frac{1}{p_1(b-a)} \int_a^{p_1b+(1-p_1)a} f\left(x, \frac{p_2c+q_2d}{p_2+q_2}\right) {}_ad_{p_1,q_1}x + \frac{1}{p_2(d-c)} \int_{p_2c+(1-p_2)d}^{d} f\left(\frac{q_1a+p_1b}{p_1+q_1}, y\right) {}^dd_{p_2,q_2}y \\
&\le \frac{2}{p_1p_2(b-a)(d-c)} \int_a^{p_1b+(1-p_1)a} \int_{p_2c+(1-p_2)d}^{d} f(x,y) \, {}^dd_{p_2,q_2}y \, {}_ad_{p_1,q_1}x \\
&\le \frac{1}{p_1(b-a)} \int_a^{p_1b+(1-p_1)a} \frac{p_2f(x,c)+q_2f(x,d)}{p_2+q_2} \, {}_ad_{p_1,q_1}x \\
&\quad + \frac{1}{p_2(d-c)} \int_{p_2c+(1-p_2)d}^{d} \frac{q_1f(a,y)+p_1f(b,y)}{p_1+q_1} \, {}^dd_{p_2,q_2}y \\
&= \frac{p_2}{p_1(b-a)(p_2+q_2)} \int_a^{p_1b+(1-p_1)a} f(x,c) \, {}_ad_{p_1,q_1}x + \frac{q_2}{p_1(b-a)(p_2+q_2)} \int_a^{p_1b+(1-p_1)a} f(x,d) \, {}_ad_{p_1,q_1}x \\
&\quad + \frac{q_1}{p_2(d-c)(p_1+q_1)} \int_{p_2c+(1-p_2)d}^{d} f(a,y) \, {}^dd_{p_2,q_2}y + \frac{p_1}{p_2(d-c)(p_1+q_1)} \int_{p_2c+(1-p_2)d}^{d} f(b,y) \, {}^dd_{p_2,q_2}y,
\end{aligned}$$

which proves the second and the third inequalities of the theorem.

Since $\dfrac{q_1a+p_1b}{p_1+q_1} \in [a,b]$, it follows from the first inequality of (10) that

$$f\left(\frac{q_1a+p_1b}{p_1+q_1}, \frac{p_2c+q_2d}{p_2+q_2}\right) \le \frac{1}{p_2(d-c)} \int_{p_2c+(1-p_2)d}^{d} f\left(\frac{q_1a+p_1b}{p_1+q_1}, y\right) {}^dd_{p_2,q_2}y.$$

Since $\dfrac{p_2c+q_2d}{p_2+q_2} \in [c,d]$, it follows from the first inequality of (12) that

$$f\left(\frac{q_1a+p_1b}{p_1+q_1}, \frac{p_2c+q_2d}{p_2+q_2}\right) \le \frac{1}{p_1(b-a)} \int_a^{p_1b+(1-p_1)a} f\left(x, \frac{p_2c+q_2d}{p_2+q_2}\right) {}_a d_{p_1,q_1}x.$$

Adding two inequalities above, we obtain the first inequality in the theorem. Finally, using the second inequality of (10) and (12), we have

$$\begin{aligned}
\frac{q_1}{p_2(d-c)(p_1+q_1)} \int_{p_2c+(1-p_2)d}^{d} f(a,y)\ {}^d d_{p_2,q_2}y &\le \frac{q_1p_2f(a,c)+q_1q_2f(a,d)}{(p_1+q_1)(p_2+q_2)},\\
\frac{p_1}{p_2(d-c)(p_1+q_1)} \int_{p_2c+(1-p_2)d}^{d} f(b,y)\ {}^d d_{p_2,q_2}y &\le \frac{p_1p_2f(b,c)+p_1q_2f(b,d)}{(p_1+q_1)(p_2+q_2)},\\
\frac{q_2}{p_1(b-a)(p_2+q_2)} \int_a^{p_1b+(1-p_1)a} f(x,d)\ {}_a d_{p_1,q_1}x &\le \frac{q_1q_2f(a,d)+p_1q_2f(b,d)}{(p_1+q_1)(p_2+q_2)},
\end{aligned}$$

$$\frac{p_2}{p_1(b-a)(p_2+q_2)} \int_a^{p_1b+(1-p_1)a} f(x,c)\ {}_a d_{p_1,q_1}x \le \frac{q_1p_2f(a,c)+p_1p_2f(b,c)}{(p_1+q_1)(p_2+q_2)}.$$

Combining the inequalities above, we get the last inequality in the theorem. This completes the proof. □

Remark 1. *If $p_1 = p_2 = 1$, then* (9) *reduces to* (3)*, which was appeared in* [29].

Remark 2. *If $p_1 = p_2 = 1$, q_1 and q_2 tend to* 1*, then* (9) *reduces to* (2)*, which was appeared in* [58].

Theorem 5. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a convex differentiable function of two variables. Then the following inequalities hold:*

$$\begin{aligned}
& f\left(\frac{p_1a+q_1b}{p_1+q_1}, \frac{q_2c+p_2d}{p_2+q_2}\right)\\
& \le \frac{1}{2p_1(b-a)} \int_{p_1a+(1-p_1)b}^{b} f\left(x, \frac{q_2c+p_2d}{p_2+q_2}\right) {}^b d_{p_1,q_1}x + \frac{1}{2p_2(b-a)} \int_c^{p_2d+(1-p_2)c} f\left(\frac{p_1a+q_1b}{p_1+q_1}, y\right) {}_c d_{p_2,q_2}y\\
& \le \frac{1}{p_1p_2(b-a)(d-c)} \int_{p_1a+(1-p_1)b}^{b} \int_c^{p_2d+(1-p_2)c} f(x,y)\ {}_c d_{p_2,q_2}y\ {}^b d_{p_1,q_1}x\\
& \le \frac{p_1}{2p_2(p_1+q_1)(d-c)} \int_c^{p_2d+(1-p_2)c} f(a,y)\ {}_c d_{p_2,q_2}y + \frac{q_1}{2p_2(p_1+q_1)(d-c)} \int_c^{p_2d+(1-p_2)c} f(b,y)\ {}_c d_{p_2,q_2}y\\
& \quad + \frac{q_2}{2p_1(p_2+q_2)(b-a)} \int_{p_1a+(1-p_1)b}^{b} f(x,c)\ {}^b d_{p_1,q_1}x + \frac{p_2}{2p_1(p_2+q_2)(b-a)} \int_{p_1a+(1-p_1)b}^{b} f(x,d)\ {}^b d_{p_1,q_1}x\\
& \le \frac{p_1q_2f(a,c)+p_1p_2f(a,d)+q_1q_2f(b,c)+q_1p_2f(b,d)}{(p_1+q_1)(p_2+q_2)}.
\end{aligned} \tag{14}$$

Proof. Let $h_x : [c,d] \to \mathbb{R}$ defined by $h_x(y) = f(x,y)$ be a convex differentiable function on $[c,d]$. Using the inequality (6) on $[c,d]$, we have

$$h_x\left(\frac{q_2c+p_2d}{p_2+q_2}\right) \le \frac{1}{p_2(d-c)} \int_c^{p_2d+(1-p_2)c} h_x(y)\ {}_c d_{p_2,q_2}y \le \frac{q_2h_x(c)+p_2h_x(d)}{p_2+q_2},$$

i.e.,

$$f\left(x, \frac{q_2c+p_2d}{p_2+q_2}\right) \le \frac{1}{p_2(d-c)} \int_c^{p_2d+(1-p_2)c} f(x,y)\ {}_c d_{p_2,q_2}y \le \frac{q_2f(x,c)+p_2f(x,d)}{p_2+q_2}, \tag{15}$$

for all $x \in [a,b]$.

By $(p_1,q_1)^b$-integrating both sides of (15) on $[p_1a+(1-p_1)b, b]$, we have

$$\begin{aligned}
&\frac{1}{p_1(b-a)}\int_{p_1a+(1-p_1)b}^{b} f\left(x, \frac{q_2c+p_2d}{p_2+q_2}\right)\, {}^{b}d_{p_1,q_1}x \\
&\quad\leq \frac{1}{p_1p_2(b-a)(d-c)}\int_{p_1a+(1-p_1)b}^{b}\int_{c}^{p_2d+(1-p_2)c} f(x,y)\, {}_{c}d_{p_2,q_2}y\, {}^{b}d_{p_1,q_1}x \\
&\quad\leq \frac{1}{p_1(b-a)}\int_{p_1a+(1-p_1)b}^{b} \frac{q_2f(x,c)+p_2f(x,d)}{p_2+q_2}\, {}^{b}d_{p_1,q_1}x. \qquad (16)
\end{aligned}$$

Similarly, let $h_y : [a,b] \to \mathbb{R}$ defined by $h_y(x) = f(x,y)$ be a convex function on $[a,b]$. Using the inequality (8) on $[a,b]$, we have

$$h_y\left(\frac{p_1a+q_1b}{p_1+q_1}\right) \leq \frac{1}{p_1(b-a)}\int_{p_1a+(1-p_1)b}^{b} h_y(x)\, {}^{b}d_{p_1,q_1}x \leq \frac{p_1h_y(a)+q_1h_y(b)}{p_1+q_1},$$

i.e.,

$$f\left(\frac{p_1a+q_1b}{p_1+q_1}, y\right) \leq \frac{1}{p_1(b-a)}\int_{p_1a+(1-p_1)b}^{b} f(x,y)\, {}^{b}d_{p_1,q_1}x \leq \frac{p_1f(a,y)+q_1f(b,y)}{p_1+q_1}, \qquad (17)$$

for all $y \in [a,b]$.

By $(p_2,q_2)_c$-integrating both sides of (17) on $[c, p_2d+(1-p_2)c]$, we have

$$\begin{aligned}
&\frac{1}{p_2(d-c)}\int_{c}^{p_2d+(1-p_2)c} f\left(\frac{p_1a+q_1b}{p_1+q_1}, y\right)\, {}_{c}d_{p_2,q_2}y \\
&\quad\leq \frac{1}{p_1p_2(b-a)(d-c)}\int_{c}^{p_2d+(1-p_2)c}\int_{p_1a+(1-p_1)b}^{b} f(x,y)\, {}^{b}d_{p_1,q_1}x\, {}_{c}d_{p_2,q_2}y \\
&\quad\leq \frac{1}{p_2(d-c)}\int_{c}^{p_2d+(1-p_2)c} \frac{p_1f(a,y)+q_1f(b,y)}{p_1+q_1}\, {}_{c}d_{p_2,q_2}y. \qquad (18)
\end{aligned}$$

Adding (16) and (18), we obtain

$$\begin{aligned}
&\frac{1}{p_1(b-a)}\int_{p_1a+(1-p_1)b}^{b} f\left(x, \frac{q_2c+p_2d}{p_2+q_2}\right)\, {}^{b}d_{p_1,q_1}x + \frac{1}{p_2(d-c)}\int_{c}^{p_2d+(1-p_2)c} f\left(\frac{p_1a+q_1b}{p_1+q_1}, y\right)\, {}_{c}d_{p_2,q_2}y \\
&\leq \frac{2}{p_1p_2(b-a)(d-c)}\int_{p_1a+(1-p_1)b}^{b}\int_{c}^{p_2d+(1-p_2)c} f(x,y)\, {}_{c}d_{p_2,q_2}y\, {}^{b}d_{p_1,q_1}x \\
&\leq \frac{1}{p_1(b-a)}\int_{p_1a+(1-p_1)b}^{b} \frac{q_2f(x,c)+p_2f(x,d)}{p_2+q_2}\, {}^{b}d_{p_1,q_1}x \\
&\quad + \frac{1}{p_2(d-c)}\int_{c}^{p_2d+(1-p_2)c} \frac{p_1f(a,y)+q_1f(b,y)}{p_1+q_1}\, {}_{c}d_{p_2,q_2}y \\
&= \frac{q_2}{p_1(b-a)(p_2+q_2)}\int_{p_1a+(1-p_1)b}^{b} f(x,c)\, {}^{b}d_{p_1,q_1}x + \frac{p_2}{p_1(b-a)(p_2+q_2)}\int_{p_1a+(1-p_1)b}^{b} f(x,d)\, {}^{b}d_{p_1,q_1}x \\
&\quad + \frac{p_1}{p_2(d-c)(p_1+q_1)}\int_{c}^{p_2d+(1-p_2)c} f(a,y)\, {}_{c}d_{p_2,q_2}y + \frac{q_1}{p_2(d-c)(p_1+q_1)}\int_{c}^{p_2d+(1-p_2)c} f(b,y)\, {}_{c}d_{p_2,q_2}y,
\end{aligned}$$

which proves the second and the third inequalities of the theorem.

Since $\dfrac{p_1a+q_1b}{p_1+q_1} \in [a,b]$, it follows from the first inequality of (15) that

$$f\left(\frac{p_1a+q_1b}{p_1+q_1}, \frac{q_2c+p_2d}{p_2+q_2}\right) \le \frac{1}{p_2(d-c)} \int_c^{p_2d+(1-p_2)c} f\left(\frac{p_1a+q_1b}{p_1+q_1}, y\right) \, {}_c d_{p_2,q_2} y.$$

Since $\dfrac{q_2c+p_2d}{p_2+q_2} \in [c,d]$, it follows from the first inequality of (17) that

$$f\left(\frac{p_1a+q_1b}{p_1+q_1}, \frac{q_2c+p_2d}{p_2+q_2}\right) \le \frac{1}{p_1(b-a)} \int_{p_1a+(1-p_1)b}^{b} f\left(x, \frac{q_2c+p_2d}{p_2+q_2}\right) \, {}^b d_{p_1,q_1} x.$$

Adding the two inequalities above, we obtain the first inequality in the theorem. Finally, using the second inequality of (15) and (17), we have

$$\frac{p_1}{p_2(d-c)(p_1+q_1)} \int_c^{p_2d+(1-p_2)c} f(a,y) \, {}_c d_{p_2,q_2} y \le \frac{p_1q_2f(a,c)+p_1p_2f(a,d)}{(p_1+q_1)(p_2+q_2)},$$
$$\frac{q_1}{p_2(d-c)(p_1+q_1)} \int_c^{p_2d+(1-p_2)c} f(b,y) \, {}_c d_{p_2,q_2} y \le \frac{q_1q_2f(b,c)+q_1p_2f(b,d)}{(p_1+q_1)(p_2+q_2)},$$
$$\frac{p_2}{p_1(b-a)(p_2+q_2)} \int_{p_1a+(1-p_1)b}^{b} f(x,d) \, {}^b d_{p_1,q_1} x \le \frac{p_1p_2f(a,d)+q_1p_2f(b,d)}{(p_1+q_1)(p_2+q_2)},$$

$$\frac{q_2}{p_1(b-a)(p_2+q_2)} \int_{p_1a+(1-p_1)b}^{b} f(x,c) \, {}^b d_{p_1,q_1} x \le \frac{p_1q_2f(a,c)+q_1q_2f(b,c)}{(p_1+q_1)(p_2+q_2)}.$$

Combining inequalities above, we get the last inequality in the theorem. This completes the proof. □

Remark 3. *If $p_1 = p_2 = 1$, then (14) reduces to (4), which was appeared in [29].*

Remark 4. *If $p_1 = p_2 = 1$, q_1 and q_2 tend to 1, then (14) reduces to (2), which was appeared in [58].*

Theorem 6. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a convex differentiable function of two variables. Then the following inequalities hold:*

$$\begin{aligned}
& f\left(\frac{p_1a+q_1b}{p_1+q_1}, \frac{p_2c+q_2d}{p_2+q_2}\right) \\
& \le \frac{1}{2p_1(b-a)} \int_{p_1a+(1-p_1)b}^{b} f\left(x, \frac{p_2c+q_2d}{p_2+q_2}\right) \, {}^b d_{p_1,q_1} x + \frac{1}{2p_2(b-a)} \int_{p_2c+(1-p_2)d}^{d} f\left(\frac{p_1a+q_1b}{p_1+q_1}, y\right) \, {}^d d_{p_2,q_2} y \\
& \le \frac{1}{p_1p_2(b-a)(d-c)} \int_{p_1a+(1-p_1)b}^{b} \int_{p_2c+(1-p_2)d}^{d} f(x,y) \, {}^d d_{p_2,q_2} y \, {}^b d_{p_1,q_1} x \\
& \le \frac{p_1}{2p_2(p_1+q_1)(d-c)} \int_{p_2c+(1-p_2)d}^{d} f(a,y) \, {}^d d_{p_2,q_2} y + \frac{q_1}{2p_2(p_1+q_1)(d-c)} \int_{p_2c+(1-p_2)d}^{d} f(b,y) \, {}^d d_{p_2,q_2} y \\
& \quad + \frac{p_2}{2p_1(p_2+q_2)(b-a)} \int_{p_1a+(1-p_1)b}^{b} f(x,c) \, {}^b d_{p_1,q_1} x + \frac{q_2}{2p_1(p_2+q_2)(b-a)} \int_{p_1a+(1-p_1)b}^{b} f(x,d) \, {}^b d_{p_1,q_1} x \\
& \le \frac{p_1p_2f(a,c)+p_1q_2f(a,d)+q_1p_2f(b,c)+q_1q_2f(b,d)}{(p_1+q_1)(p_2+q_2)}.
\end{aligned} \tag{19}$$

Proof. Let $h_x : [c,d] \to \mathbb{R}$ defined by $h_x(y) = f(x,y)$ be a convex differentiable function on $[c,d]$. Using the inequality (8) on $[c,d]$, we have

$$h_x\left(\frac{p_2c+q_2d}{p_2+q_2}\right) \leq \frac{1}{p_2(d-c)} \int_{p_2c+(1-p_2)d}^{d} h_x(y)\ {}^{d}d_{p_2,q_2}y \leq \frac{p_2h_x(c)+q_2h_x(d)}{p_2+q_2},$$

i.e.,

$$f\left(x,\frac{p_2c+q_2d}{p_2+q_2}\right) \leq \frac{1}{p_2(d-c)} \int_{p_2c+(1-p_2)d}^{d} f(x,y)\ {}^{d}d_{p_2,q_2}y \leq \frac{p_2f(x,c)+q_2f(x,d)}{p_2+q_2}, \quad (20)$$

for all $x \in [a,b]$.

By $(p_1,q_1)^b$-integrating both sides of (20) on $[p_1a+(1-p_1)b, b]$, we have

$$\begin{aligned}
&\frac{1}{p_1(b-a)} \int_{p_1a+(1-p_1)b}^{b} f\left(x,\frac{p_2c+q_2d}{p_2+q_2}\right)\ {}^{b}d_{p_1,q_1}x \\
&\quad \leq \frac{1}{p_1p_2(b-a)(d-c)} \int_{p_1a+(1-p_1)b}^{b} \int_{p_2c+(1-p_2)d}^{d} f(x,y)\ {}^{d}d_{p_2,q_2}y\ {}^{b}d_{p_1,q_1}x \\
&\quad \leq \frac{1}{p_1(b-a)} \int_{p_1a+(1-p_1)b}^{b} \frac{p_2f(x,c)+q_2f(x,d)}{p_2+q_2}\ {}^{b}d_{p_1,q_1}x. \qquad (21)
\end{aligned}$$

Similarly, let $h_y : [a,b] \to \mathbb{R}$ defined by $h_y(x) = f(x,y)$ be a convex function on $[a,b]$. Using the inequality (8) on $[a,b]$, we have

$$h_y\left(\frac{p_1a+q_1b}{p_1+q_1}\right) \leq \frac{1}{p_1(b-a)} \int_{p_1a+(1-p_1)b}^{b} h_y(x)\ {}^{b}d_{p_1,q_1}x \leq \frac{p_1h_y(a)+q_1h_y(b)}{p_1+q_1},$$

i.e.,

$$f\left(\frac{p_1a+q_1b}{p_1+q_1},y\right) \leq \frac{1}{p_1(b-a)} \int_{p_1a+(1-p_1)b}^{b} f(x,y)\ {}^{b}d_{p_1,q_1}x \leq \frac{p_1f(a,y)+q_1f(b,y)}{p_1+q_1}, \quad (22)$$

for all $y \in [a,b]$.

By $(p_2,q_2)^d$-integrating both sides of (22) on $[p_2c+(1-p_2)d, d]$, we have

$$\begin{aligned}
&\frac{1}{p_2(d-c)} \int_{p_2c+(1-p_2)d}^{d} f\left(\frac{p_1a+q_1b}{p_1+q_1},y\right)\ {}^{d}d_{p_2,q_2}y \\
&\quad \leq \frac{1}{p_1p_2(b-a)(d-c)} \int_{p_2c+(1-p_2)d}^{d} \int_{p_1a+(1-p_1)b}^{b} f(x,y)\ {}^{b}d_{p_1,q_1}x\ {}^{d}d_{p_2,q_2}y \\
&\quad \leq \frac{1}{p_2(d-c)} \int_{p_2c+(1-p_2)d}^{d} \frac{p_1f(a,y)+q_1f(b,y)}{p_1+q_1}\ {}^{d}d_{p_2,q_2}y. \qquad (23)
\end{aligned}$$

Adding (21) and (23), we obtain

$$
\begin{aligned}
&\frac{1}{p_1(b-a)}\int_{p_1a+(1-p_1)b}^{b} f\left(x,\frac{p_2c+q_2d}{p_2+q_2}\right)\,{}^{b}d_{p_1,q_1}x+\frac{1}{p_2(d-c)}\int_{p_2c+(1-p_2)d}^{d} f\left(\frac{p_1a+q_1b}{p_1+q_1},y\right)\,{}^{d}d_{p_2,q_2}y\\
&\leq \frac{2}{p_1p_2(b-a)(d-c)}\int_{p_1a+(1-p_1)b}^{b}\int_{p_2c+(1-p_2)d}^{d} f(x,y)\,{}^{d}d_{p_2,q_2}y\,{}^{b}d_{p_1,q_1}x\\
&\leq \frac{1}{p_1(b-a)}\int_{p_1a+(1-p_1)b}^{b}\frac{p_2f(x,c)+q_2f(x,d)}{p_2+q_2}\,{}^{b}d_{p_1,q_1}x\\
&\quad+\frac{1}{p_2(d-c)}\int_{p_2c+(1-p_2)d}^{d}\frac{p_1f(a,y)+q_1f(b,y)}{p_1+q_1}\,{}^{d}d_{p_2,q_2}y\\
&=\frac{p_2}{p_1(b-a)(p_2+q_2)}\int_{p_1a+(1-p_1)b}^{b} f(x,c)\,{}^{b}d_{p_1,q_1}x+\frac{q_2}{p_1(b-a)(p_2+q_2)}\int_{p_1a+(1-p_1)b}^{b} f(x,d)\,{}^{b}d_{p_1,q_1}x\\
&\quad+\frac{p_1}{p_2(d-c)(p_1+q_1)}\int_{p_2c+(1-p_2)d}^{d} f(a,y)\,{}^{d}d_{p_2,q_2}y+\frac{q_1}{p_2(d-c)(p_1+q_1)}\int_{p_2c+(1-p_2)d}^{d} f(b,y)\,{}^{d}d_{p_2,q_2}y,
\end{aligned}
$$

which proves the second and the third inequalities of the theorem.

Since $\frac{p_1a+q_1b}{p_1+q_1}\in[a,b]$, it follows from the first inequality of (20) that

$$f\left(\frac{p_1a+q_1b}{p_1+q_1},\frac{p_2c+q_2d}{p_2+q_2}\right)\leq\frac{1}{p_2(d-c)}\int_{p_2c+(1-p_2)d}^{d} f\left(\frac{p_1a+q_1b}{p_1+q_1},y\right)\,{}^{d}d_{p_2,q_2}y.$$

Since $\frac{p_2c+q_2d}{p_2+q_2}\in[c,d]$, it follows from the first inequality of (22) that

$$f\left(\frac{p_1a+q_1b}{p_1+q_1},\frac{p_2c+q_2d}{p_2+q_2}\right)\leq\frac{1}{p_1(b-a)}\int_{p_1a+(1-p_1)b}^{b} f\left(x,\frac{p_2c+q_2d}{p_2+q_2}\right)\,{}^{b}d_{p_1,q_1}x.$$

Adding two inequalities above, we obtain the first inequality in the theorem. Finally, using the second inequality of (20) and (22), we have

$$
\begin{aligned}
\frac{p_1}{p_2(d-c)(p_1+q_1)}\int_{p_2c+(1-p_2)d}^{d} f(a,y)\,{}^{d}d_{p_2,q_2}y&\leq\frac{p_1p_2f(a,c)+p_1q_2f(a,d)}{(p_1+q_1)(p_2+q_2)},\\
\frac{q_1}{p_2(d-c)(p_1+q_1)}\int_{p_2c+(1-p_2)d}^{d} f(b,y)\,{}^{d}d_{p_2,q_2}y&\leq\frac{q_1p_2f(b,c)+q_1q_2f(b,d)}{(p_1+q_1)(p_2+q_2)},\\
\frac{q_2}{p_1(b-a)(p_2+q_2)}\int_{p_1a+(1-p_1)b}^{b} f(x,d)\,{}^{b}d_{p_1,q_1}x&\leq\frac{p_1q_2f(a,d)+q_1q_2f(b,d)}{(p_1+q_1)(p_2+q_2)},\\
\frac{p_2}{p_1(b-a)(p_2+q_2)}\int_{p_1a+(1-p_1)b}^{b} f(x,c)\,{}^{b}d_{p_1,q_1}x&\leq\frac{p_1p_2f(a,c)+q_1p_2f(b,c)}{(p_1+q_1)(p_2+q_2)}.
\end{aligned}
$$

Combining inequalities above, we get the last inequality in the theorem. This completes the proof. □

Remark 5. *If $p_1=p_2=1$, then* (19) *reduces to* (5)*, which was appeared in* [29].

Remark 6. *If $p_1=p_2=1$, q_1 and q_2 tend to 1, then* (19) *reduces to* (2)*, which was appeared in* [58].

Corollary 1. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a convex differentiable function of two variables. Then we have the inequalities*

$$\begin{aligned}
&\frac{1}{4}\left[f\left(\frac{q_1a+p_1b}{p_1+q_1},\frac{q_2c+p_2d}{p_2+q_2}\right)+f\left(\frac{q_1a+p_1b}{p_1+q_1},\frac{p_2c+q_2d}{p_2+q_2}\right)\right.\\
&\quad\left.+f\left(\frac{p_1a+q_1b}{p_1+q_1},\frac{q_2c+p_2d}{p_2+q_2}\right)+f\left(\frac{p_1a+q_1b}{p_1+q_1},\frac{p_2c+q_2d}{p_2+q_2}\right)\right]\\
&\quad\le \frac{1}{8p_1(b-a)}\int_a^{p_1b+(1-p_1)a}\left[f\left(x,\frac{q_2c+p_2d}{p_2+q_2}\right)+f\left(x,\frac{p_2c+q_2d}{p_2+q_2}\right)\right]\,{}_a d_{p_1,q_1}x\\
&\quad\quad+\frac{1}{8p_1(b-a)}\int_{p_1a+(1-p_1)b}^{b}\left[f\left(x,\frac{q_2c+p_2d}{p_2+q_2}\right)+f\left(x,\frac{p_2c+q_2d}{p_2+q_2}\right)\right]\,{}^b d_{p_1,q_1}x\\
&\quad\quad+\frac{1}{8p_2(b-a)}\int_c^{p_2d+(1-p_2)c}\left[f\left(\frac{q_1a+p_1b}{p_1+q_1},y\right)+f\left(\frac{p_1a+q_1b}{p_1+q_1},y\right)\right]\,{}_c d_{p_2,q_2}y\\
&\quad\quad+\frac{1}{8p_2(b-a)}\int_{p_2c+(1-p_2)d}^{d}\left[f\left(\frac{q_1a+p_1b}{p_1+q_1},y\right)+f\left(\frac{p_1a+q_1b}{p_1+q_1},y\right)\right]\,{}^d d_{p_2,q_2}y\\
&\quad\le\frac{1}{4p_1p_2(b-a)(d-c)}\left[\int_a^{p_1b+(1-p_1)a}\int_c^{p_2d+(1-p_2)c}f(x,y)\,{}_c d_{p_2,q_2}y\,{}_a d_{p_1,q_1}x\right.\\
&\quad\quad+\int_a^{p_1b+(1-p_1)a}\int_{p_2c+(1-p_2)d}^{d}f(x,y)\,{}^d d_{p_2,q_2}y\,{}_a d_{p_1,q_1}x\\
&\quad\quad+\int_{p_1a+(1-p_1)b}^{b}\int_c^{p_2d+(1-p_2)c}f(x,y)\,{}_c d_{p_2,q_2}y\,{}^b d_{p_1,q_1}x\\
&\quad\quad\left.+\int_{p_1a+(1-p_1)b}^{b}\int_{p_2c+(1-p_2)d}^{d}f(x,y)\,{}^d d_{p_2,q_2}y\,{}^b d_{p_1,q_1}x\right]\\
&\quad\le\frac{q_2}{8p_1(p_2+q_2)(b-a)}\left[\int_a^{p_1b+(1-p_1)a}f(x,c)\,{}_a d_{p_1,q_1}x+\int_a^{p_1b+(1-p_1)a}f(x,d)\,{}_a d_{p_1,q_1}x\right.\\
&\quad\quad\left.+\int_{p_1a+(1-p_1)b}^{b}f(x,c)\,{}^b d_{p_1,q_1}x+\int_{p_1a+(1-p_1)b}^{b}f(x,d)\,{}^b d_{p_1,q_1}x\right]\\
&\quad\quad+\frac{p_2}{8p_1(p_2+q_2)(b-a)}\left[\int_a^{p_1b+(1-p_1)a}f(x,d)\,{}_a d_{p_1,q_1}x+\int_a^{p_1b+(1-p_1)a}f(x,c)\,{}_a d_{p_1,q_1}x\right.\\
&\quad\quad\left.+\int_{p_1a+(1-p_1)b}^{b}f(x,d)\,{}^b d_{p_1,q_1}x+\int_{p_1a+(1-p_1)b}^{b}f(x,c)\,{}^b d_{p_1,q_1}x\right]\\
&\quad\quad+\frac{q_1}{8p_2(p_1+q_1)(d-c)}\left[\int_c^{p_2d+(1-p_2)c}f(a,y)\,{}_c d_{p_2,q_2}y+\int_{p_2c+(1-p_2)d}^{d}f(a,y)\,{}^d d_{p_2,q_2}y\right.\\
&\quad\quad\left.+\int_c^{p_2d+(1-p_2)c}f(b,y)\,{}_c d_{p_2,q_2}y+\int_{p_2c+(1-p_2)d}^{d}f(b,y)\,{}^d d_{p_2,q_2}y\right]\\
&\quad\quad+\frac{p_1}{8p_2(p_1+q_1)(d-c)}\left[\int_c^{p_2d+(1-p_2)c}f(b,y)\,{}_c d_{p_2,q_2}y+\int_{p_2c+(1-p_2)d}^{d}f(b,y)\,{}^d d_{p_2,q_2}y\right.\\
&\quad\quad\left.+\int_c^{p_2d+(1-p_2)c}f(a,y)\,{}_c d_{p_2,q_2}y+\int_{p_2c+(1-p_2)d}^{d}f(a,y)\,{}^d d_{p_2,q_2}y\right]\\
&\quad\le\frac{1}{4(p_1+q_1)(p_2+q_2)}[(q_1q_2+q_1p_2+p_1q_2+p_1p_2)f(a,c)+(q_1p_2+q_1q_2+p_1p_2+p_1q_2)f(a,d)\\
&\quad\quad+(p_1q_2+p_1p_2+q_1q_2+q_1p_2)f(b,c)+(p_1p_2+p_1q_2+q_1p_2+q_1q_2)f(b,d)]. \qquad (24)
\end{aligned}$$

Remark 7. *If $p_1=p_2=1$, then (24) reduces to*

$$\frac{1}{4}\left[f\left(\frac{q_1a+b}{1+q_1},\frac{q_2c+d}{1+q_2}\right)+f\left(\frac{q_1a+b}{1+q_1},\frac{c+q_2d}{1+q_2}\right)+f\left(\frac{a+q_1b}{1+q_1},\frac{q_2c+d}{1+q_2}\right)+f\left(\frac{a+q_1b}{1+q_1},\frac{c+q_2d}{1+q_2}\right)\right]$$
$$\begin{aligned}
&\leq \frac{1}{8(b-a)}\int_a^b\left[f\left(x,\frac{q_2c+d}{1+q_2}\right)+f\left(x,\frac{c+q_2d}{1+q_2}\right)\right]\,{}_ad_{q_1}x \\
&\quad+\frac{1}{8(b-a)}\int_a^b\left[f\left(x,\frac{q_2c+d}{1+q_2}\right)+f\left(x,\frac{c+q_2d}{1+q_2}\right)\right]\,{}^bd_{q_1}x \\
&\quad+\frac{1}{8(b-a)}\int_c^d\left[f\left(\frac{q_1a+b}{1+q_1},y\right)+f\left(\frac{a+q_1b}{1+q_1},y\right)\right]\,{}_cd_{q_2}y \\
&\quad+\frac{1}{8(b-a)}\int_c^d\left[f\left(\frac{q_1a+b}{1+q_1},y\right)+f\left(\frac{a+q_1b}{1+q_1},y\right)\right]\,{}^dd_{q_2}y \\
&\leq \frac{1}{4(b-a)(d-c)}\left[\int_a^b\int_c^d f(x,y)\,{}_cd_{q_2}y\,{}_ad_{q_1}x+\int_a^b\int_c^d f(x,y)\,{}^dd_{q_2}y\,{}_ad_{q_1}x\right. \\
&\quad\left.+\int_a^b\int_c^d f(x,y)\,{}_cd_{q_2}y\,{}^bd_{q_1}x+\int_a^b\int_c^d f(x,y)\,{}^dd_{q_2}y\,{}^bd_{q_1}x\right] \\
&\leq \frac{1}{8(b-a)}\left[\int_a^b f(x,c)\,{}_ad_{q_1}x+\int_a^b f(x,d)\,{}_ad_{q_1}x+\int_a^b f(x,c)\,{}^bd_{q_1}x+\int_a^b f(x,d)\,{}^bd_{q_1}x\right] \\
&\quad+\frac{1}{8(d-c)}\left[\int_c^d f(a,y)\,{}_cd_{q_2}y+\int_c^d f(a,y)\,{}^dd_{q_2}y+\int_c^d f(b,y)\,{}_cd_{q_2}y+\int_c^d f(b,y)\,{}^dd_{q_2}y\right] \\
&\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4},
\end{aligned}$$

which was appeared in [29].

4. Examples

In this section, we give some examples of our main theorems.

Example 2. *Define a function $f:[0,1]\times[0,1]\to\mathbb{R}$ by $f(x,y)=x^2y^2$. Then $f(x,y)$ is a convex differentiable function of two variables on $[0,1]\times[0,1]$. By applying Theorem 4 with $p_1=p_2=\frac{3}{4}$ and $q_1=q_2=\frac{1}{4}$, the first inequality of (9) becomes*

$$\begin{aligned}
\frac{9}{256} &= f\left(\frac{3}{4},\frac{1}{4}\right)=f\left(\frac{\frac{1}{4}\cdot 0+\frac{3}{4}\cdot 1}{\frac{3}{4}+\frac{1}{4}},\frac{\frac{3}{4}\cdot 0+\frac{1}{4}\cdot 1}{\frac{3}{4}+\frac{1}{4}}\right) \\
&\leq \frac{1}{2\cdot\frac{3}{4}\cdot 1}\int_0^{\frac{3}{4}\cdot 1+(1-\frac{3}{4})\cdot 0} f\left(x,\frac{\frac{3}{4}\cdot 0+\frac{1}{4}\cdot 1}{\frac{3}{4}+\frac{1}{4}}\right)\,{}_0d_{\frac{3}{4},\frac{1}{4}}x+\frac{1}{2\cdot\frac{3}{4}\cdot 1}\int_{\frac{3}{4}\cdot 0+(1-\frac{3}{4})\cdot 1}^1 f\left(\frac{\frac{1}{4}\cdot 0+\frac{3}{4}\cdot 1}{\frac{3}{4}+\frac{1}{4}},y\right)\,{}^1d_{\frac{3}{4},\frac{1}{4}}y \\
&= \frac{2}{3}\int_0^{\frac{3}{4}} f\left(x,\frac{1}{4}\right)\,{}_0d_{\frac{3}{4},\frac{1}{4}}x+\frac{2}{3}\int_{\frac{1}{4}}^1 f\left(\frac{3}{4},y\right)\,{}^1d_{\frac{3}{4},\frac{1}{4}}y=\frac{93}{832}.
\end{aligned}$$

The third inequality of (9) becomes

$$\begin{aligned}
\frac{45}{338} &= \frac{16}{9}\int_0^{\frac{3}{4}}\int_{\frac{1}{4}}^1 f(x,y)\,{}^1d_{\frac{3}{4},\frac{1}{4}}y\,{}_0d_{\frac{3}{4},\frac{1}{4}}x=\frac{1}{\frac{3}{4}\cdot\frac{3}{4}\cdot 1\cdot 1}\int_0^{\frac{3}{4}\cdot 1+(1-\frac{3}{4})\cdot 0}\int_{\frac{3}{4}\cdot 0+(1-\frac{3}{4})\cdot 1}^1 f(x,y)\,{}^1d_{\frac{3}{4},\frac{1}{4}}y\,{}_0d_{\frac{3}{4},\frac{1}{4}}x \\
&\leq \frac{\frac{1}{4}}{2\cdot\frac{3}{4}\cdot(\frac{3}{4}+\frac{1}{4})\cdot 1}\int_{\frac{3}{4}\cdot 0+(1-\frac{3}{4})\cdot 1}^1 f(0,y)\,{}^1d_{\frac{3}{4},\frac{1}{4}}y+\frac{\frac{3}{4}}{2\cdot\frac{3}{4}\cdot(\frac{3}{4}+\frac{1}{4})\cdot 1}\int_{\frac{3}{4}\cdot 0+(1-\frac{3}{4})\cdot 1}^1 f(1,y)\,{}^1d_{\frac{3}{4},\frac{1}{4}}y \\
&\quad+\frac{\frac{3}{4}}{2\cdot\frac{3}{4}\cdot(\frac{3}{4}+\frac{1}{4})\cdot 1}\int_0^{\frac{3}{4}\cdot 1+(1-\frac{3}{4})\cdot 0} f(x,0)\,{}_0d_{\frac{3}{4},\frac{1}{4}}x+\frac{\frac{1}{4}}{2\cdot\frac{3}{4}\cdot(\frac{3}{4}+\frac{1}{4})\cdot 1}\int_0^{\frac{3}{4}\cdot 1+(1-\frac{3}{4})\cdot 0} f(x,1)\,{}_0d_{\frac{3}{4},\frac{1}{4}}x
\end{aligned}$$

$$= 0 + \frac{1}{2}\int_{\frac{1}{4}}^{1} f(1,y)\ {}^{1}d_{\frac{3}{4},\frac{1}{4}}y + 0 + \frac{1}{6}\int_{0}^{\frac{3}{4}} f(x,1)\ {}_{0}d_{\frac{3}{4},\frac{1}{4}}x = \frac{156}{832}.$$

We also have

$$\begin{aligned}\frac{q_1p_2f(a,c)+q_1q_2f(a,d)+p_1p_2f(b,c)+p_1q_2f(b,d)}{(p_1+q_1)(p_2+q_2)} &= \frac{1}{(\frac{3}{4}+\frac{1}{4})(\frac{3}{4}+\frac{1}{4})}\left[\frac{1}{4}\cdot\frac{3}{4}\cdot f(0,0)+\frac{1}{4}\cdot\frac{1}{4}\cdot f(0,1)\right.\\ &\left.+\frac{3}{4}\cdot\frac{3}{4}\cdot f(1,0)+\frac{3}{4}\cdot\frac{1}{4}\cdot f(1,1)\right]\\ &= 0+0+0+\frac{3}{4}\cdot\frac{1}{4} = \frac{3}{16}.\end{aligned}$$

It is clear that

$$\frac{9}{256} \le \frac{93}{832} \le \frac{45}{338} \le \frac{156}{832} \le \frac{3}{16},$$

which demonstrates the result described in Theorem 4.

Example 3. *Define a function $f : [0,1]\times[0,1] \to \mathbb{R}$ by $f(x,y) = x^2y^2$. Then $f(x,y)$ is a convex differentiable function of two variables on $[0,1]\times[0,1]$. By applying Theorem 5 with $p_1 = p_2 = \frac{3}{4}$ and $q_1 = q_2 = \frac{1}{4}$, the first inequality of (14) becomes*

$$\begin{aligned}\frac{9}{256} =& f\left(\frac{1}{4},\frac{3}{4}\right) = f\left(\frac{\frac{3}{4}\cdot 0+\frac{1}{4}\cdot 1}{\frac{3}{4}+\frac{1}{4}}, \frac{\frac{1}{4}\cdot 0+\frac{3}{4}\cdot 1}{\frac{3}{4}+\frac{1}{4}}\right)\\ &\le \frac{1}{2\cdot\frac{3}{4}\cdot 1}\int_{\frac{3}{4}\cdot 0+(1-\frac{3}{4})\cdot 1}^{1} f\left(x, \frac{\frac{1}{4}\cdot 0+\frac{3}{4}\cdot 1}{\frac{3}{4}+\frac{1}{4}}\right)\ {}^{1}d_{\frac{3}{4},\frac{1}{4}}x + \frac{1}{2\cdot\frac{3}{4}\cdot 1}\int_{0}^{\frac{3}{4}\cdot 1+(1-\frac{3}{4})\cdot 0} f\left(\frac{\frac{3}{4}\cdot 0+\frac{1}{4}\cdot 1}{\frac{3}{4}+\frac{1}{4}}, y\right)\ {}_{0}d_{\frac{3}{4},\frac{1}{4}}y\\ &= \frac{2}{3}\int_{\frac{1}{4}}^{1} f\left(x,\frac{3}{4}\right)\ {}^{1}d_{\frac{3}{4},\frac{1}{4}}x + \frac{2}{3}\int_{0}^{\frac{3}{4}} f\left(\frac{1}{4},y\right)\ {}_{0}d_{\frac{3}{4},\frac{1}{4}}y = \frac{93}{832}.\end{aligned}$$

The third inequality of (14) becomes

$$\begin{aligned}\frac{45}{338} &= \frac{16}{9}\int_{\frac{1}{4}}^{1}\int_{0}^{\frac{3}{4}} f(x,y)\ {}_{0}d_{\frac{3}{4},\frac{1}{4}}y\ {}^{1}d_{\frac{3}{4},\frac{1}{4}}x = \frac{1}{\frac{3}{4}\cdot\frac{3}{4}\cdot 1\cdot 1}\int_{\frac{3}{4}\cdot 0+(1-\frac{3}{4})\cdot 1}^{1}\int_{0}^{\frac{3}{4}\cdot 1+(1-\frac{3}{4})\cdot 0} f(x,y)\ {}_{0}d_{\frac{3}{4},\frac{1}{4}}y\,{}^{1}d_{\frac{3}{4},\frac{1}{4}}x\\ &\le \frac{\frac{3}{4}}{2\cdot\frac{3}{4}\cdot(\frac{3}{4}+\frac{1}{4})\cdot 1}\int_{0}^{\frac{3}{4}\cdot 1+(1-\frac{3}{4})\cdot 0} f(0,y)\ {}_{0}d_{\frac{3}{4},\frac{1}{4}}y + \frac{\frac{1}{4}}{2\cdot\frac{3}{4}\cdot(\frac{3}{4}+\frac{1}{4})\cdot 1}\int_{0}^{\frac{3}{4}\cdot 1+(1-\frac{3}{4})\cdot 0} f(1,y)\ {}_{0}d_{\frac{3}{4},\frac{1}{4}}y\\ &\quad + \frac{\frac{1}{4}}{2\cdot\frac{3}{4}\cdot(\frac{3}{4}+\frac{1}{4})\cdot 1}\int_{\frac{3}{4}\cdot 0+(1-\frac{3}{4})\cdot 1}^{1} f(x,0)\ {}^{1}d_{\frac{3}{4},\frac{1}{4}}x + \frac{\frac{3}{4}}{2\cdot\frac{3}{4}\cdot(\frac{3}{4}+\frac{1}{4})\cdot 1}\int_{\frac{3}{4}\cdot 0+(1-\frac{3}{4})\cdot 1}^{1} f(x,1)\ {}^{1}d_{\frac{3}{4},\frac{1}{4}}x\\ &= 0 + \frac{1}{6}\int_{0}^{\frac{3}{4}} f(1,y)\ {}_{0}d_{\frac{3}{4},\frac{1}{4}}y + 0 + \frac{1}{2}\int_{\frac{1}{4}}^{1} f(x,1)\ {}^{1}d_{\frac{3}{4},\frac{1}{4}}x = \frac{156}{832}.\end{aligned}$$

We also have

$$\begin{aligned}\frac{q_1p_2f(a,c)+q_1q_2f(a,d)+p_1p_2f(b,c)+p_1q_2f(b,d)}{(p_1+q_1)(p_2+q_2)} &= \frac{1}{(\frac{3}{4}+\frac{1}{4})(\frac{3}{4}+\frac{1}{4})}\left[\frac{3}{4}\cdot\frac{1}{4}\cdot f(0,0)+\frac{3}{4}\cdot\frac{3}{4}\cdot f(0,1)\right.\\ &\left.+\frac{1}{4}\cdot\frac{1}{4}\cdot f(1,0)+\frac{1}{4}\cdot\frac{3}{4}\cdot f(1,1)\right]\\ &= 0+0+0+\frac{1}{4}\cdot\frac{3}{4} = \frac{3}{16}.\end{aligned}$$

It is clear that

$$\frac{9}{256} \le \frac{93}{832} \le \frac{45}{338} \le \frac{156}{832} \le \frac{3}{16},$$

which demonstrates the result described in Theorem 5.

Example 4. *Define a function $f : [0,1] \times [0,1] \to \mathbb{R}$ by $f(x,y) = x^2y^2$. Then $f(x,y)$ is a convex differentiable function of two variables on $[0,1] \times [0,1]$. By applying Theorem 6 with $p_1 = p_2 = \frac{3}{4}$ and $q_1 = q_2 = \frac{1}{4}$, the first inequality of (19) becomes*

$$\begin{aligned}
\frac{1}{256} =& f\left(\frac{1}{4}, \frac{1}{4}\right) = f\left(\frac{\frac{3}{4}\cdot 0 + \frac{1}{4}\cdot 1}{\frac{3}{4}+\frac{1}{4}}, \frac{\frac{3}{4}\cdot 0 + \frac{1}{4}\cdot 1}{\frac{3}{4}+\frac{1}{4}}\right) \\
&\le \frac{1}{2\cdot\frac{3}{4}\cdot 1}\int_{\frac{3}{4}\cdot 0+(1-\frac{3}{4})\cdot 1}^{1} f\left(x, \frac{\frac{3}{4}\cdot 0 + \frac{1}{4}\cdot 1}{\frac{3}{4}+\frac{1}{4}}\right)\, {}^{1}d_{\frac{3}{4},\frac{1}{4}}x + \frac{1}{2\cdot\frac{3}{4}\cdot 1}\int_{\frac{3}{4}\cdot 0+(1-\frac{3}{4})\cdot 1}^{1} f\left(\frac{\frac{3}{4}\cdot 0 + \frac{1}{4}\cdot 1}{\frac{3}{4}+\frac{1}{4}}, y\right)\, {}^{1}d_{\frac{3}{4},\frac{1}{4}}y \\
&= \frac{2}{3}\int_{\frac{1}{4}}^{1} f\left(x, \frac{1}{4}\right)\, {}^{1}d_{\frac{3}{4},\frac{1}{4}}x + \frac{2}{3}\int_{\frac{1}{4}}^{1} f\left(\frac{1}{4}, y\right)\, {}^{1}d_{\frac{3}{4},\frac{1}{4}}y = \frac{10}{832}.
\end{aligned}$$

The third inequality of (19) becomes

$$\begin{aligned}
\frac{25}{676} &= \frac{16}{9}\int_{\frac{1}{4}}^{1}\int_{\frac{1}{4}}^{1} f(x,y)\, {}^{1}d_{\frac{3}{4},\frac{1}{4}}y\, {}^{1}d_{\frac{3}{4},\frac{1}{4}}x = \frac{1}{\frac{3}{4}\cdot\frac{3}{4}\cdot 1\cdot 1}\int_{\frac{3}{4}\cdot 0+(1-\frac{3}{4})\cdot 1}^{1}\int_{\frac{3}{4}\cdot 0+(1-\frac{3}{4})\cdot 1}^{1} f(x,y)\, {}^{1}d_{\frac{3}{4},\frac{1}{4}}y\, {}^{1}d_{\frac{3}{4},\frac{1}{4}}x \\
&\le \frac{\frac{1}{4}}{2\cdot\frac{3}{4}\cdot(\frac{3}{4}+\frac{1}{4})\cdot 1}\int_{\frac{3}{4}\cdot 0+(1-\frac{3}{4})\cdot 1}^{1} f(0,y)\, {}^{1}d_{\frac{3}{4},\frac{1}{4}}y + \frac{\frac{3}{4}}{2\cdot\frac{3}{4}\cdot(\frac{3}{4}+\frac{1}{4})\cdot 1}\int_{\frac{3}{4}\cdot 0+(1-\frac{3}{4})\cdot 1}^{1} f(1,y)\, {}^{1}d_{\frac{3}{4},\frac{1}{4}}y \\
&\quad + \frac{\frac{1}{4}}{2\cdot\frac{3}{4}\cdot(\frac{3}{4}+\frac{1}{4})\cdot 1}\int_{\frac{3}{4}\cdot 0+(1-\frac{3}{4})\cdot 1}^{1} f(x,0)\, {}^{1}d_{\frac{3}{4},\frac{1}{4}}x + \frac{\frac{3}{4}}{2\cdot\frac{3}{4}\cdot(\frac{3}{4}+\frac{1}{4})\cdot 1}\int_{\frac{3}{4}\cdot 0+(1-\frac{3}{4})\cdot 1}^{1} f(x,1)\, {}^{1}d_{\frac{3}{4},\frac{1}{4}}x \\
&= 0 + \frac{1}{6}\int_{\frac{1}{4}}^{1} f(1,y)\, {}^{1}d_{\frac{3}{4},\frac{1}{4}}y + 0 + \frac{1}{6}\int_{\frac{1}{4}}^{1} f(x,1)\, {}^{1}d_{\frac{3}{4},\frac{1}{4}}x = \frac{40}{832}.
\end{aligned}$$

We also have

$$\begin{aligned}
\frac{p_1p_2f(a,c)+p_1q_2f(a,d)+q_1p_2f(b,c)+q_1q_2f(b,d)}{(p_1+q_1)(p_2+q_2)} &= \frac{1}{(\frac{3}{4}+\frac{1}{4})(\frac{3}{4}+\frac{1}{4})}\left[\frac{3}{4}\cdot\frac{3}{4}\cdot f(0,0) + \frac{3}{4}\cdot\frac{1}{4}\cdot f(0,1)\right. \\
&\quad \left. + \frac{1}{4}\cdot\frac{3}{4}\cdot f(1,0) + \frac{1}{4}\cdot\frac{1}{4}\cdot f(1,1)\right] \\
&= 0 + 0 + 0 + \frac{1}{4}\cdot\frac{1}{4} = \frac{1}{16}.
\end{aligned}$$

It is clear that

$$\frac{1}{256} \le \frac{10}{832} \le \frac{25}{676} \le \frac{40}{832} \le \frac{1}{16},$$

which demonstrates the result described in Theorem 6.

5. Conclusions

We define (p,q)-integrals for continuous functions of two variables. Moreover, we prove the Hermite-Hadamard type inequalities for coordinated convex functions by using (p,q)-integrals. Some previously published results of other researchers are deduced as special cases of our results for $p = 1$ and $q \to 1$. Finally, some examples are given to illustrate the result obtained in this paper. For further research, we will study some more refinements of the Hermite-Hadamard inequality and study other famous mathematical inequalities by using (p,q)-integrals.

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