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On the Ninth Coefficient of the Inverse of a Convex Function

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Abstract: We consider the inverse function $z = g(w) = w + b_2 w^2 + \cdots$ of a normalized convex univalent function $w = f(z) = z + a_2 z^2 + \cdots$ on the unit disk in the complex plane. So far, it is known that $|b_n| \le 1$ for $n = 2, 3, \ldots, 8$. On the other hand, the inequality $|b_n| \le 1$ is not valid for $n = 10$. It is conjectured that $|b_9| \le 1$. The present paper offers the estimate $|b_9| < 1.617$.

Keywords: convex functions; inverse function; coefficient estimates

MSC: Primary 30C45; Secondary 30C50

1. Introduction

An analytic function f on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$ is called convex if f maps $\mathbb{D}$ univalently onto a convex domain in $\mathbb{C}$. Let $\mathcal{K}$ denote the class of convex functions f normalized so that $w = f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$. For a function $f \in \mathcal{K}$, we expand the inverse function $g = f^{-1} : f(\mathbb{D}) \to \mathbb{D}$ as a power series of the form

$$g(w) = w + b_2 w^2 + b_3 w^3 + \cdots .$$

It is known that $|a_n| \le 1$ for every $n \ge 2$. This is sharp for every n and, indeed, the function $f_0(z) = z/(1-z) = z + z^2 + z^3 + \cdots$ satisfies the equality for all n. Note that $g_0(w) = f_0^{-1}(w) = w/(1+w) = w - w^2 + w^3 - \cdots$ satisfies $|b_n| = 1$ for all n. Since $f(\mathbb{D})$ contains the disk $|w| < 1/2$ for every $f \in \mathcal{K}$, the radius of convergence of the above $g(w)$ is at least $1/2$; namely, $\limsup |b_n|^{1/n} \le 2$, and the number 2 is sharp. Thus we cannot expect small bounds for b_n. Nevertheless, it has been proved so far that $|b_n| \le 1$ for $n = 2, 3, \ldots, 8$ (Libera-Złotkiewicz [1] for $n \le 7$ and Campschroer [2] for $n = 8$). For clarity, we define the quantity

$$M_n = \sup_{f \in \mathcal{K}} |b_n| = \sup_{f \in \mathcal{K}} \frac{\left|(f^{-1})^{(n)}(0)\right|}{n!}$$

for $n \ge 2$. Then $M_n = 1$ for $n = 2, 3, \ldots, 8$. On the other hand, Kirwan and Schober [3] showed that $M_{10} > 1$. In the same paper, Kirwan and Schober also gave the estimate

$$M_n < \frac{2^n \Gamma(\frac{n+1}{2})}{\sqrt{\pi} n \Gamma(\frac{n}{2} + 1)} \sim \sqrt{\frac{2}{\pi}} \frac{2^n}{n^{3/2}}. \tag{1}$$

Moreover, for each $0 < \varepsilon < 2$, there is a number n_ε such that

$$M_n > \frac{2 - \varepsilon}{e^2} \cdot \frac{2^n}{n^3}, \quad n \ge n_\varepsilon.$$

Clunie [4] showed that $M_n = O(2^n n^{-3} \log n)$ as $n \to \infty$ and conjectured that $M_n = O(2^n n^{-3})$. The conjecture was confirmed by Campschroer [5].

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It is believed that $M_9 = 1$ but this has neither been proved nor disproved so far. The estimate in Equation (1) gives in this case

$$M_9 < \frac{2^9 \cdot 4!}{9\sqrt{\pi}\,\Gamma(11/2)} = \frac{131072}{2835\pi} \approx 14.7166.$$

The main purpose of this short note is to show the following.

Theorem 1.

$$M_9 \le \frac{12223}{7560} \approx 1.6168.$$

The estimate is not optimal. We may find a better partition of the expression of b_9 for the proof in Section 3. However, it seems difficult to prove $|b_9| \le 1$.

2. Some Results on Carathéodory Functions

An analytic function P on the unit disk $\mathbb{D}$ is called Carathéodory if $\operatorname{Re} P(z) > 0$ for $z \in \mathbb{D}$ and $P(0) = 1$. We denote by $\mathcal{P}$ the class of Carathéodory functions. We expand $P \in \mathcal{P}$ in the forms

$$P(z) = 1 + \sum_{n=1}^{\infty} d_n z^n = 1 + 2\sum_{n=1}^{\infty} p_n z^n, \quad |z| < 1.$$

The following general estimates are useful.

Lemma 1. *Let $P \in \mathcal{P}$ be expanded as above. Then*

(i) $|d_n| \le 2 \ (n = 1, 2, \dots)$,
(ii) $|d_{n+k} - d_n d_k| \le 2 \ (k, n = 1, 2, \dots)$,
(iii) $|d_{n+k} - d_n d_k/2| \le 2 - |d_n d_k|/2 \ (k, n = 1, 2, \dots)$.

The inequalities in (i), (ii) and (iii) are due to Carathéodory [6], Livingston [7] and Campschroer [2], respectively. See also [8]. Note that (ii) follows from (iii); in other words, (iii) is a refinement of (ii). Let A and B be square matrices of order n. We will say that A is majorized by B and write $A \ll B$ if the inequality $|Ax| \le |Bx|$ holds for each vector $x \in \mathbb{C}^n$. Here, the norm of a vector $x = (x_1, \dots, x_n)^{\mathrm{T}} \in \mathbb{C}^n$ is defined by $|x| = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$ as usual. For $P(z) = 1 + d_1 z + d_2 z^2 + \cdots$, we define two kinds of Toeplitz matrices of order n by

$$A_n = \begin{pmatrix} d_1 & d_2 & d_3 & \cdots & d_n \\ 0 & d_1 & d_2 & \cdots & d_{n-1} \\ 0 & 0 & d_1 & \cdots & d_{n-2} \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & d_1 \end{pmatrix} \quad \text{and} \quad B_n = \begin{pmatrix} 2 & d_1 & d_2 & \cdots & d_{n-1} \\ 0 & 2 & d_1 & \cdots & d_{n-2} \\ 0 & 0 & 2 & \cdots & d_{n-3} \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}.$$

Then Campschroer [2] (Example 1.XIII) showed the following.

Lemma 2. *$A_n \ll B_n$ for a Carathédory function $P(z) = 1 + d_1 z + d_2 z^2 + \cdots$.*

For $n = 6$, we take $x = (0, 0, d_1 d_2 - d_3, -d_2, 0, 1)^{\mathrm{T}}$ and apply the above lemma to get $|A_n x| \le |B_n x|$ for $P \in \mathcal{P}$; that is,

$$|d_1 d_2 d_3 - d_3^2 - d_2 d_4 + d_6| \le 2.$$

Similarly, for $n = 8$, $x = (0, 0, d_1 d_4 - d_5, -d_4, 0, 0, 0, 1)^{\mathrm{T}}$, we obtain

$$|d_1 d_3 d_4 - d_3 d_5 - d_4^2 + d_8| \le 2.$$

The coefficient inequalities for $P \in \mathcal{P}$ are more convenient for the later use if we express them in terms of $p_n = d_n/2$. Thus we summarize the above inequalities in the following form.

Lemma 3. *Let $P(z) = 1 + 2p_1z + 2p_2z^2 + \cdots$ be a Carathédory function. Then the following inequalities hold.*

(i) $|p_n| \le 1$ $(n = 1, 2, \ldots)$,
(ii) $|p_{n+k} - 2p_np_k| \le 1$ $(k, n = 1, 2, \ldots)$,
(iii) $|p_{n+k} - p_np_k| \le 1 - |p_np_k|$ $(k, n = 1, 2, \ldots)$,
(iv) $|R| \le 1$, *where* $R = 4p_1p_2p_3 - 2p_3^2 - 2p_2p_4 + p_6$,
(v) $|S| \le 1$, *where* $S = 4p_1p_3p_4 - 2p_3p_5 - 2p_4^2 + p_8$.

Note that the above inequalities are all sharp because the function $P_0(z) = (1 + z)/(1 - z) = 1 + 2z + 2z^2 + \cdots$ satisfies equalities.

3. Proof of the Theorem

We will show Theorem 1 in this section. Since computations are often involved, we need symbolic computations by computers. Suppose that a function $w = f(z) = z + a_2z^2 + \cdots$ belongs to $\mathcal{K}$ and let $z = g(w) = f^{-1}(w) = w + b_2w^2 + \cdots$. Then, by a formal calculation, we have the formula

$$\begin{aligned} b_9 = {} & 1430a_2^8 - 5005a_2^6a_3 + 2002a_2^5a_4 + 5005a_2^4a_3^2 - 715a_2^4a_5 - 2860a_2^3a_3a_4 + 220a_2^3a_6 \\ & - 1430a_2^2a_3^3 + 330a_2^2a_4^2 + 660a_2^2a_3a_5 - 55a_2^2a_7 + 660a_2a_3^2a_4 - 110a_2a_4a_5 - 110a_2a_3a_6 \\ & + 10a_2a_8 + 55a_3^4 - 55a_3a_4^2 + 5a_5^2 - 55a_3^2a_5 + 10a_4a_6 + 10a_3a_7 + 10a_2a_8 - a_9. \end{aligned}$$

It is well known (see [9] for example) that a normalized analytic function f on $\mathbb{D}$ is convex if and only if $1 + zf''(z)/f'(z)$ has a positive real part for each $z \in \mathbb{D}$. Therefore, there is a function $P \in \mathcal{P}$ such that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1}{P(z)}, \quad z \in \mathbb{D}.$$

If we expand P in the form $P(z) = 1 + 2p_1z + 2p_2z^2 + \cdots$, we have the following relations

$$\begin{aligned} a_2 &= -p_1, \quad 3a_3 = 4p_1^2 - p_2, \quad 6a_4 = -12p_1^3 + 7p_1p_2 - p_3, \\ 30a_5 &= 96p_1^4 - 92p_1^2p_2 + 9p_2^2 + 20p_1p_3 - 3p_4, \\ 90a_6 &= -480p_1^5 + 652p_1^3p_2 - 172p_1^2p_3 - 157p_1p_2^2 + 39p_1p_4 + 34p_2p_3 - 6p_5, \\ 630a_7 &= 5760p_1^6 - 10224p_1^4p_2 + 3024p_1^3p_3 + 4184p_1^2p_2^2 - 828p_1^2p_4 - 1468p_1p_2p_3 \\ &\quad + 192p_1p_5 - 225p_2^3 + 80p_3^2 + 165p_2p_4 - 30p_6, \\ 2520a_8 &= -40320p_1^7 + 88848p_1^5p_2 - 28368p_1^4p_3 - 52760p_1^3p_2^2 + 8676p_1^3p_4 + 23368p_1^2p_2p_3 \\ &\quad - 2424p_1^2p_5 + 7227p_1p_2^3 - 2060p_1p_3^2 - 4239p_1p_2p_4 + 570p_1p_6 - 1899p_2^2p_3 \\ &\quad + 465p_3p_4 + 486p_2p_5 - 90p_7, \\ 22680a_9 &= 645120p_1^8 - 1703808p_1^6p_2 + 574848p_1^5p_3 + 1345136p_1^4p_2^2 - 189216p_1^4p_4 \\ &\quad - 686368p_1^3p_2p_3 + 58944p_1^3p_5 - 320648p_1^2p_2^3 + 76304p_1^2p_3^2 + 156696p_1^2p_2p_4 \\ &\quad - 16680p_1^2p_6 + 141880p_1p_2^2p_3 - 27768p_1p_3p_4 - 28944p_1p_2p_5 + 3960p_1p_7 \\ &\quad + 11025p_2^4 - 12488p_2p_3^2 + 1575p_4^2 - 12810p_2^2p_4 + 3192p_3p_5 + 3360p_2p_6 - 630p_8. \end{aligned}$$

We substitute these relations into the above expression of b_9 to obtain

$$\begin{aligned} B := 22680b_9 = {} & 16p_1^8 + 224p_1^6p_2 + 1752p_1^5p_3 - 464p_1^4p_2^2 + 7020p_1^4p_4 \\ & - 3512p_1^3p_2p_3 + 12336p_1^3p_5 + 412p_1^2p_2^3 - 3764p_1^2p_3^2 - 2454p_1^2p_2p_4 \\ & + 10380p_1^2p_6 + 650p_1p_2^2p_3 - 4002p_1p_3p_4 - 36p_1p_2p_5 + 4140p_1p_7 + p_2^4 \\ & + 158p_2p_3^2 - 441p_4^2 + 66p_2^2p_4 - 672p_3p_5 + 240p_2p_6 + 630p_8. \end{aligned}$$

We now write B as follows:

$$\begin{aligned} B = {} & 336S + 1882p_1^2R + 232p_1^2p_2(p_1^2 - p_2)^2 \\ & + (8p_1^6 + 3386p_1^2p_4 + 36p_1p_5)(2p_1^2 - p_2) + 4696p_1p_4(p_1p_2 - p_3) \\ & + 650p_1p_3(p_2^2 - p_4) + 1752p_1^3p_3(p_1^2 - p_2) + 9288p_1^3(p_5 - p_2p_3) \\ & + 180p_1^2p_2^3 + p_2^4 + 158p_2p_3^2 + 248p_1^4p_4 + 66p_2^2p_4 + 231p_4^2 \\ & + 2976p_1^3p_5 + 8498p_1^2p_6 + 240p_2p_6 + 4140p_1p_7 + 294p_8 \end{aligned}$$

where R and S are given in Lemma 3. We now apply Lemma 3 to obtain

$$\begin{aligned} |B| \le {} & 336 + 1882t_1^2 + 232t_1^2t_2(1 - t_1^2)^2 \\ & + (8t_1^6 + 3386t_1^2t_4 + 36t_1t_5) + 4696t_1t_4(1 - t_1t_2) \\ & + 650t_1t_3(1 - t_2^2) + 1752t_1^3t_3(1 - t_1^2) + 9288t_1^3(1 - t_2t_3) \\ & + 180t_1^2t_2^3 + t_2^4 + 158t_2t_3^2 + 248t_1^4t_4 + 66t_2^2t_4 + 231t_4^2 \\ & + 2976t_1^3t_5 + 8498t_1^2t_6 + 240t_2t_6 + 4140t_1t_7 + 294t_8, \end{aligned}$$

where $t_j = |p_j|$ $(j = 1, 2, \ldots, 8)$. Note that $0 \le t_j \le 1$ by Lemma 3 (i). Hence,

$$\begin{aligned} |B| \le {} & 336 + 1882t_1^2 + 232t_1^2t_2(1 - t_1^2)^2 \\ & + (8t_1^6 + 3386t_1^2 + 36t_1) + 4696t_1(1 - t_1t_2) \\ & + 650t_1(1 - t_2^2) + 1752t_1^3(1 - t_1^2) + 9288t_1^3 \\ & + 180t_1^2t_2^3 + t_2^4 + 158t_2 + 248t_1^4 + 66t_2^2 + 231 \\ & + 2976t_1^3 + 8498t_1^2 + 240t_2 + 4140t_1 + 294 \\ = {} & h(t_1, t_2), \end{aligned}$$

where

$$\begin{aligned} h(x, y) = {} & 232x^6y + 8x^6 - 1752x^5 - 464x^4y + 248x^4 + 14016x^3 + 180x^2y^3 \\ & - 4464x^2y + 13766x^2 - 650xy^2 + 9522x + y^4 + 66y^2 + 398y + 861. \end{aligned}$$

Since

$$\begin{aligned} \frac{\partial h}{\partial x} = {} & x^5(1392y + 48) - 8760x^4 + x^3(992 - 1856y) + 42048x^2 \\ & + x\Big(360y^3 - 8928y + 27532\Big) - 650y^2 + 9522 \\ \ge {} & 48x^5 - 8760x^4 - 864x^3 + 42048x^2 + 18964x + 8872 > 0 \end{aligned}$$

for $0 \le x, y \le 1$, we observe that $h(x, y)$ is increasing in $0 \le x \le 1$ for a fixed $y \in [0, 1]$. Therefore, $h(x, y) \le h(1, y) = y^4 + 180y^3 - 584y^2 - 4298y + 36669 =: H(y)$. We compute

$H'(y) = 2(2y^3 + 270y^2 - 584y - 2149) \leq 2(2 + 270 - 2149) < 0$ and thus conclude that $H(y) \leq H(0) = 36669$ for $0 \leq y \leq 1$. In summary, we have obtained

$$|b_9| = \frac{|B|}{22680} \leq \frac{36669}{22680} = \frac{12223}{7560} \approx 1.6167989.$$

The proof is now complete.

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References

1. Libera, R.J.; Złotkiewicz, E.J. Early coefficients of the inverse of a regular convex function. *Proc. Am. Math. Soc.* **1982**, *85*, 225–230. [CrossRef]
2. Campschroer, J.T.P. Inverse Coefficients and Symmetrization of Univalent Functions. Ph.D. Thesis, University of Nijmegen, Nijmegen, The Netherlands, 1984.
3. Kirwan, W.E.; Schober, G. Inverse coefficients for functions of bounded boundary rotation. *J. Anal. Math.* **1979**, *36*, 167–178. [CrossRef]
4. Clunie, J.G. Inverse coefficients for convex univalent functions. *J. Anal. Math.* **1979**, *36*, 31–35. [CrossRef]
5. Campschroer, J.T.P. Growth of the coefficients of the inverse of a convex function. *Indag. Math.* **1984**, *46*, 1–6. [CrossRef]
6. Carathéodory, C. Über den Variabilitätsbereich der Fourier'schen Konstanten von positiven harmonischen Funktionen. *Rend. Circ. Mat. Palermo* **1911**, *32*, 193–217. [CrossRef]
7. Livingston, A.E. The coefficients of multivalent close-to-convex functions. *Proc. Am. Math. Soc.* **1969**, *21*, 545–552. [CrossRef]
8. Efraimidis, I.; Vukotić, D. Applications of Livingston-type inequalities to the generalized Zalcman functional. *Math. Narchr.* **2018**, *291*, 1502–1513. [CrossRef]
9. Duren, P.L. *Univalent Functions*; Springer: Berlin/Heidelberg, Germany, 1983.