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Network Formation with Asymmetric Players and Chance Moves

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Abstract: We propose a model of network formation as a two-stage game with chance moves and players of various types. First, the leader suggests a connected communication network for the players to join. Second, nature selects a type vector for players based on the given probability distribution, and each player decides whether or not to join the network keeping in mind only his own type and the leader's type. The game is of incomplete information since each player has only a belief over the payoff functions of others. As a result, the network is formed, and each player gets a payoff related to both the network structure and his type. We prove the existence of the Bayesian equilibrium and propose a new definition of the stable partially Bayesian equilibrium defining the network to be formed and prove its existence. The connection between the stable partially Bayesian equilibrium and the Nash equilibrium in the game is examined. Finally, we investigate the characteristics of the network structures under the stable partially Bayesian equilibrium in a three-player game with the major player as well as in the *n*-player game with a specific characteristic function.

1. Introduction

The network formation process has been widely examined since network games became a hot topic in game theory. There exist many approaches modeling the network formation. Aumann and Myerson [1] were the first to model the network formation as a game, describing it by an extensive-form game using the theory of cooperative games with communication structures. Jackson and Watts [2] propose a model of dynamic network formation process and mainly focus on characterizing the set of stable networks. Goyal and Vega-Redondo [3] develop the dynamic network game to examine the interaction between a partner's choice and his individual behavior in coordination games. The network constructed influences the players' payoffs that is a result of a position in the formed network or the information the players access (see Möhlmeier et al. [4]). In some competition models, the network is a basis for interaction among players. The network is formed at the first stage of the game, and then the players are involved in the game and their payoffs are affected by the network structure formed before. The models of the network formation are described by Mazalov and Chirkova [5]. Multistage network games with perfect information are considered by Petrosyan and Sedakov [6], where players can change the network structure at each stage, and the equilibrium is found in this class of dynamic games. Cooperative two-stage network games are studied by Gao et al. [7], where first players form the network following some rules, and then realize cooperative strategies. Perc and Szolnoki [8] investigate the coevolution of strategy and network, exploring how and why network structures influence the evolution of cooperation. In addition, Wang et al. [9] model friendship relationships by means of complex networks and study how network formation can be affected by various games and rules. Two approaches to the endogenous

graph formation based on sequential link announcement and revision are proposed by Khmelnitskaya et al. [10].

We realize the idea of the dynamic network formation, when the network is formed in two steps. We assume that there is a leader or "the most important" player who initiates the network formation. His strategy is to suggest a network from the set of undirected connected graphs without loops to other players. We keep in mind the idea that in real life the communication is usually encouraged by a "zero" person, and then it can or cannot be supported by other participants. In our model, after the network is proposed, nature or a chance acts by making players heterogeneous in payoffs assigning different types to them. This step allows the modeling of non-symmetry in players' behavior. The result of nature's choice is not known to the players. The player knows only his own type as well as the leader's one. Therefore, we come to the game with a stochastic structure.

Considering stochastic factors involved in the game can refer not only to the randomness in transition from state to state similar to the one in stochastic games (see Shapley [11]), but also to uncertainty about game duration. Parilina and Zaccour [12] consider the class of dynamic games played over event trees representing both stochastic nature of parameters of the game and random termination time. Gromova and Plekhanova [13] examine properties of cooperative solutions in multistage games when the game duration is stochastic, and it affects the players' behavior. Ballard [14] provides game-tree search algorithms for games that include chance moves. Our model is referred to the class of dynamic network formation games with the chance moves.

In most models, the players are assumed to be symmetric that means the individuals are the same except the labels. Models with heterogeneous players are a trend, since in practice, heterogeneous people are fairly common, e.g., females and males, individuals with various education backgrounds, etc. In addition, it is also reasonable that various people have different standards and face various cases although being in the same community, such as different levels of salary, getting different information about the community, etc. Heterogeneous players in networks are introduced by Larrosa and Tohmé [15], where payoff to a player is associated not only with the number of links in those paths, the end of which is this player, but also with his own value. Sun and Parilina [16] propose a dynamic network formation game model with heterogenous players. In the model, payoff functions are defined in a different way for two types of players based on the private information that players obtain from the leader.

In our network formation model, after nature's move the players simultaneously and independently choose whether to join the network proposed by the leader or not. They choose their actions based on the information they have about their own types, in particular, on how to calculate their payoffs in the network that can be potentially formed. Once the network is formed as a result of the players' moves, the payoff to a player is defined according to his type taking into account the network structure and the values of the initially given characteristic function. The way to define the payoff is to assign the component of a cooperative solution of a graph-restricted cooperative game (e.g., see Myerson [17], Mazalov and Trukhina [18], Tejada and Álvarez-Mozos [19], and Khmelnitskaya et al. [20]).

Combining the chance moves and non-symmetric players, the game introduced in the paper is of incomplete information. The way to find the solutions in such games is to apply the concept of the Bayesian Nash equilibrium proposed by Harsanyi [21]. We define the stable partially Bayesian equilibrium in network formation game and introduce the relation between this equilibrium and the Nash equilibrium in the game. The equilibrium existence proof is based on the results of Radner and Rosenthal [22] who present sufficient conditions for the existence of the pure strategy Bayesian equilibrium. The stable partially Bayesian equilibrium determines, maybe non-unique, network structure which is finally formed. We should notice that the resulting network structure depends on players' types, probability distribution over players' types and the player who is assigned as a leader.

The rest of the paper is organized as follows. In Section 2 basic definitions and notations are briefly introduced. In Section 3, the model of the two-stage network formation game with non-symmetric players and chance moves is presented. A special form of a two-stage game in an extensive form is described and investigated in Section 4. In Section 5 we introduce the main results, providing the proof of the Bayesian equilibrium existence in any subgame. The definition of the stable partially Bayesian equilibrium and its existence are provided in Section 5, in which we also show the connection between the stable partially Bayesian equilibrium and the Nash equilibrium. A three-player game with a major player is investigated in Section 6. Then an *n*-player game with two projects for the leader is considered in Section 7. We briefly conclude in the last section.

2. Preliminaries

Let the set of players be $N = \{1, \ldots, n\}$, $n \geq 3$. Suppose there is a player called the leader. We give a brief description of a cooperative game with a communication structure. A cooperative game with transferable utility is a pair (N, v), where $v : 2^N \to \mathbb{R}$ is a characteristic function that assigns the worth $v(S)$ to every coalition $S \subseteq N$, with $v(\varnothing) = 0$. For the simplicity of notation and if no ambiguity appears, we write v when we refer to game (N, v). It is natural to allow the formation of not only a grand coalition, but also any coalition $S \subseteq N$. Denote by v_S a cooperative game within coalition $S \subseteq N$ while considering limited cooperation within S, where $v_S : 2^S \to \mathbb{R}$, and $v_S(S') = v(S')$, $S' \subseteq S$.

The communication structure on N is specified by graph (N, Γ), where $\Gamma \subseteq \Gamma_N^c = \{ij \mid i, j \in N, i \neq j\}$ is a collection of unordered pairs of nodes (links). For ease of notation and if no ambiguity appears, we write Γ when we refer to the graph (N, Γ). Moreover, for any graph Γ and any coalition $S \subseteq N$, the subgraph of Γ on S is denoted by Γ_S, where $\Gamma_S = \{ij \in \Gamma \mid i, j \in S\}$. In graph Γ, a sequence of different nodes $(i_1, \ldots, i_k)$, $k \geq 2$ is a path from i_1 to i_k, if for all $h = 1, \ldots, k-1$, $i_h i_{h+1} \in \Gamma$. We say two nodes are connected, if there exists a path from one node to another. The graph is connected, if any two nodes are connected. Given graph Γ, $S \subseteq N$ is connected, if Γ_S is connected. Denote by C^Γ the set of maximally connected coalitions, called components of graph Γ, and let $C^\Gamma(i)$ be the component containing Player i.

Given the characteristic function $v(S)$, $S \subseteq N$, and graph Γ, determine the new characteristic function using the following approach [17]:

$$v^\Gamma(S) = \sum_{T \in S/\Gamma} v(T), \tag{1}$$

where $S/\Gamma = \big\{ \{i \mid i, j \text{ are connected in } S \text{ by } \Gamma\} \mid j \in S \big\}$.

Given $v(S)$, $S \subseteq N$, and Γ, vector $\xi(\Gamma) = (\xi_1(\Gamma), \ldots, \xi_n(\Gamma))$ is defined as a payoff vector determined for cooperative game v with graph Γ. Moreover, $\xi_i(\Gamma_S)$ denotes the payoff to Player $i \in S$ in game v_S with graph Γ_S. For instance, if the Myerson value $Y(\Gamma)$ is chosen as a cooperative solution [17], then

$$\xi_i(\Gamma) = Y_i(\Gamma) = Sh_i(v^\Gamma) \tag{2}$$

for all $i \in N$, where $Sh_i(v^\Gamma)$ is the *i*-th component of the Shapley value of Player *i* in game (N, v^Γ) (see [23]).

Example 1. *Consider a three-player game. The values of characteristic function are* $v(\{1\}) = 1$, $v(\{2\}) = v(\{1,2\}) = 2$, $v(\{3\}) = 1/2$, $v(\{2,3\}) = 5/2$ *and* $v(\{1,3\}) = v(\{1,2,3\}) = 3$. *Let the communication structure be given by* $\Gamma = \{12, 13\}$. *Using Formula* (1), *we calculate the values of the new characteristic function and obtain*

$$v^\Gamma(\{2,3\}) = v(\{2\}) + v(\{3\}) = \frac{5}{2},$$

and for any coalition $S \subseteq \{1, 2, 3\}$ such that $S \neq \{2, 3\}$, $v^{\Gamma}(\{S\}) = v(S)$ since such a coalition S is connected in Γ. Using the values of characteristic function v^{Γ} we calculate the Myerson value by Formula (2), i.e.,

$$Y(\Gamma) = \left(\frac{11}{12}, \frac{7}{6}, \frac{11}{12} \right).$$

3. Model

3.1. Players of Various Types

We assume that there can be a finite number of players' types. In addition, with all other similar situations the payoffs to the players of various types are defined in different ways. Let $T = \{I_1, \ldots, I_l\}$ denote a finite set of types for any player i, $i \in N$. The set of type vectors over N is denoted by $\bar{T} = \times_{i \in N} T$. Let $p \in \triangle(\bar{T})$ be a probability distribution over the set of type vectors that satisfies $p(t_i) = \sum_{t_{-i} \in \bar{T}_{-i}} p(t_i, t_{-i}) > 0$ for each player $i \in N$ and every type $t_i \in T$, where $\bar{T}_{-i} = \times_{j \in N \setminus \{i\}} T$, $t_{-i} = (t_j)_{j \neq i} \in \bar{T}_{-i}$. The probability distribution p is common knowledge for all players.

With characteristic function $v(S)$, $S \subseteq N$, and graph Γ formed, the payoff to Player i of type I_k, $1 \leq k \leq l$, denoted as $K_i(\Gamma, I_k)$, is assumed to be defined.

Remark 1. *Payoff $K_i(\Gamma, I_k)$, a function of both the network structure and the type of player $i \in N$, can be defined in different ways. For instance, it can be defined as*

$$K_i(\Gamma, I_k) = \xi_i(\Gamma, I_k) + \theta_i(\Gamma, I_k), \tag{3}$$

where $\xi_i(\Gamma, I_k)$ is the payoff value of Player i determined by some cooperative solution relative to type I_k in game $v_{C^{\Gamma}(i)}$, and $\theta_i(\Gamma, I_k)$ is a kind of extra encouragement or punishment to Player i of type I_k in network Γ.

Example 2. *Consider the three-player game presented in Example 1. Now we demonstrate how we define the payoff function assuming that $\xi_i(\Gamma, I_1)$ is the ith component of the Myerson value with the given network Γ, and $\theta_i(\Gamma, I_1)$ is the number of direct links Player i possesses in network Γ. As a result, the payoff to Player 2 of type I_1 is*

$$K_2(\Gamma, I_1) = \frac{7}{6} + 1 = \frac{13}{6}.$$

3.2. Games with Chance Moves and Players of Various Types

The transition from one stage to another in a dynamic game is accomplished by the players' actions. It is possible to produce the situations in which the dynamic game proceeds by some chance factors. These sorts of transitions are chance moves. Player 0, being nature, selects a type vector $t = (t_1, t_2, \ldots, t_n) \in \bar{T}$ according to some given probability distribution p, then the game process proceeds to the next stage. We should notice that the whole game does not start from the nature move in our model. With vector t, Player i knows the type t_i which has been selected for him, also he knows the type of the leader, but does not know the types of other players. Hence, Player i has a belief over the types of other players, which is defined by a conditional probability distribution. Thus, the network formation game with the chance moves and unknown players' types is of incomplete information under the given assumptions.

3.3. Two-Stage Network Formation Game

The two-stage network formation game with the chance moves, players of various types and the leader takes places as follows:

Stage 1. The leader, say Player 1, chooses network (graph) Γ from set U_0 (e.g., he proposes a joint project), where U_0 is the subset of undirected connected graphs without loops on N. The cooperative game v representing the power of any coalition S is given and known for all players.

Remark 2. *We assume that Player 1 is determined as the leader before the game starts, and in fact, the case, where Player $i \in N$, $i \neq 1$, is set to be the leader can be described in the same way. Moreover, any player who is set as the leader possesses a unique type, $I' \in T$ in the game, which is known for all players.*

Stage 2. Based on the proposal from Player 1 of the network structure, players $N \setminus \{1\}$ and nature choose actions at the second stage.

Nature selects a type vector $t = (t_1, \ldots, t_n)$ according to $p(\cdot | t_1 = I')$, which is the conditional probability distribution based on the fact that Player 1 is determined as the leader. The set of all possible type vectors nature is able to select is $\{t \mid t_1 = I'\}$, which is denoted by $\bar{T}(1)$. Indeed, Player $i, i \in N \setminus \{1\}$, knows the type t_i that has been selected for him and the type of the leader, I', but does not know types t_j, $j \in N \setminus \{1, i\}$ of other players.

The players from $N \setminus \{1\}$, knowing the proposed network and their respective types selected by nature, simultaneously and independently choose the actions from their common action set A, which is {accept, reject}. By choosing the action 'accept', player $i \in N \setminus \{1\}$ joins the network. Otherwise, if he chooses the action 'reject', he starts playing as an individual player. Please note that in the following part, we denote the action 'accept' by 'a', and the action 'reject' by 'r'.

We denote by Γ^u the network which is finally formed, where $u = (u_2, \ldots, u_n)$ is the vector of actions that have been selected by players $N \setminus \{1\}$ at the second stage, and it is specified as $\Gamma^u = \{\{ij \mid ij \in \Gamma, u_i = u_j = a\} \cup \{1i \mid 1i \in \Gamma, u_i = a\}\}$. Finally, Player $i \in N \setminus \{1\}$ gets payoff $K_i(\Gamma^u, t_i)$, and the leader, Player 1, gets payoff $K_1(\Gamma^u, I')$.

4. Two-Stage Game as Game in Extensive Form

The described two-stage game with the chance moves and players of various types can be considered to be the extensive-form game Φ with the set of players $N \cup \{0\}$ on a game tree describing the game process and denoted by Z.

4.1. Construction of a Tree Graph

Let $X = X_0 \cup X_1 \cup \ldots \cup X_n \cup X_{n+1}$ be the finite set of vertices, where X_0 denotes the set of vertices at which nature makes the chance moves, $X_1 = \{x_0\}$ is a unique vertex of the leader, X_i is the set of vertices at which player $i \in N \setminus \{1\}$ makes moves and $X_{n+1} = \{x : F_x = \varnothing\}$ is the set of terminal vertices. For any vertex $x \in X \setminus X_{n+1}$, F_x is the set of those vertices which can be realized immediately after the vertex x has been realized. For any set $V \subseteq X$, $F(V) = \bigcup_{x \in V} F_x$, and $F_x^2 = F(F_x)$, $F_x^k = F(F_x^{k-1})$, $\hat{F}_x = \{x\} \cup F_x \cup F_x^2 \cup \ldots \cup F_x^k \cup \ldots$. By construction, $F_{x_0} = X_0$, $F(X_0) = X_2$, $F(X_i) = X_{i+1}$ for $i = 2, \ldots, n$.

We denote the vertex to which the game process moves after Player 1 suggests network Γ by $x_\Gamma \in X_0$. The subgame which begins at vertex x_Γ is denoted by $\Phi(x_\Gamma)$. Let $x_2^{(\Gamma, t)}$ denote the personal vertex of Player 2 at which the game process arrives after nature selects the type vector $t \in \bar{T}(1)$ at x_Γ. In terms of the incomplete information players get, the set of personal vertices of player $i \in N \setminus \{1\}$ is partitioned into finite number of subsets $X_i^{(\Gamma, j)}$, $\Gamma \in U_0, j \in \{1, \ldots, l\}$, which are referred to as the information sets of player i, where $X_i^{(\Gamma, j)} = \{x \in X_i \mid x \in \hat{F}_{x_2^{(\Gamma, t)}}, t_i = I_j\}$. In addition, for any $x \in X_i, i \in N \setminus \{1\}$, Player i does not know the vertex itself, but knows that such a vertex is in a certain information set $X_i^{(\Gamma, j)} \subset X_i$. For Player 1, there is a unique information set, X_1.

4.2. Two-Stage Game on a Tree Graph

Before we describe how a two-stage game proceeds on the finite tree graph constructed above, we first define each player's behavior strategy in extensive-form game. Th behavior strategy of the leader (Player 1) in game Φ is function b_1 which is a mapping from X_1 to a probability distribution over set U_0, $b_1 : X_1 \longrightarrow \triangle(U_0)$. The behavior strategy of

Player $i \in N \setminus \{1\}$ in game Φ is function b_i which assigns probability distribution over set A to each of his information set, $X_i^{(\Gamma, j)} \subset X_i$. In addition, let $b_i^{\Gamma}, i \in N \setminus \{1\}$ denote the truncation of b_i to subgame $\Phi(x_{\Gamma})$, where $b_i^{\Gamma} : \{X_i^{(\Gamma, j)} \mid X_i^{(\Gamma, j)} \subseteq \hat{F}_{x_{\Gamma}}\} \longrightarrow \prod_{X_i^{(\Gamma, j)} \subseteq \hat{F}_{x_{\Gamma}}} \triangle(A)$.

We denote the terminal vertex which can be reached with probability 1 from vertex $x_2^{(\Gamma, t)}$ if each player $k \in N \setminus \{1\}$ chooses action $u_k \in A$ with probability 1 in his information set $X_k^{(\Gamma, j)}$, where $t_k = I_j$, by $x_{(\Gamma, t)}^u$. And at each terminal vertex $x \in X_{n+1}$, payoff vector $H(x) = (H_1(x), \ldots, H_n(x))$ is given. The payoff for player $i \in N$ at terminal vertex $x_{(\Gamma, t)}^u$ is defined as

$$H_i(x_{(\Gamma, t)}^u) = K_i(\Gamma^u, t_i). \tag{4}$$

The two-stage network formation game on the finite tree graph Z proceeds as follows. At vertex x_0, the leader (Player 1) chooses the network $\Gamma \in U_0$, then the game process moves to vertex x_{Γ}. At x_{Γ}, nature (Player 0) selects the type vector $t \in \bar{T}(1)$ according to the conditional probability distribution $p(\cdot | t_1 = I')$ which is a common belief of all players. Because of the incomplete information over the type vector chosen by nature, the game process moves to the information sets $X_2^{(\Gamma, j(2))}, \ldots, X_n^{(\Gamma, j(n))}$ simultaneously, where $I_{j(i)} = t_i$ for $i = 2, \ldots, n$. Then each player $i \in N \setminus \{1\}$ chooses an action $u_i \in A$ for the corresponding information set independently. Finally, the game process terminates at vertex $x_{(\Gamma, t)}^u$, and Player $i \in N$ gets the payoff $H_i(x_{(\Gamma, t)}^u)$.

4.3. The Expected Payoff in the Extensive-Form Game Φ

Combining the probability distribution over type vectors selected by nature and the players' behavior strategies, below we investigate the expected payoff of each player in the extensive-form game Φ.

The behavior strategy b_i of player $i \in N \setminus \{1\}$ satisfies the following:

$$b_i(X_i^{(\Gamma, j)}) = \left(b_i(X_i^{(\Gamma, j)}; u_i)\right)_{u_i \in A} \in \triangle(A)$$

for $j = 1, \ldots, l, \Gamma \in U_0$, where $b_i(X_i^{(\Gamma, j)}; u_i)$ is the probability that player i chooses the action $u_i \in A$ at information set $X_i^{(\Gamma, j)}$. Then we consider subgames $\Phi(x_{\Gamma}), \Gamma \in U_0$. If the behavior strategy vector of players $N \setminus \{1\}$ is $b_{-1} = (b_2, \ldots, b_n)$, and the type vector selected by nature is $t \in \bar{T}(1)$, then the action vector $u = (u_2, \ldots, u_n)$ is chosen with the probability

$$b_2^{\Gamma}(X_2^{(\Gamma, j(2))}; u_2) \times \ldots \times b_n^{\Gamma}(X_n^{(\Gamma, j(n))}; u_n) \tag{5}$$

in subgame $\Phi(x_{\Gamma})$, where $I_{j(i)} = t_i$ for $i \in N \setminus \{1\}$. Therefore, given the type vector $t \in \bar{T}(1)$ selected by nature and vector b_{-1}, the expected payoff to Player $i \in N$ in subgame $\Phi(x_{\Gamma})$, denoted by $E_i(t; b_{-1}^{\Gamma})$, is

$$E_i(t; b_{-1}^{\Gamma}) = \sum_{u \in \prod_{j \in N \setminus \{1\}} A} b_2^{\Gamma}(X_2^{(\Gamma, j(2))}; u_2) \cdot \ldots \cdot b_n^{\Gamma}(X_n^{(\Gamma, j(n))}; u_n) H_i(x_{(\Gamma, t)}^u), \tag{6}$$

where $I_{j(k)} = t_k$ for $k \in N \setminus \{1\}$.

Now consider the conditional probability distribution over the type vectors selected by nature. The expected payoff to Player $i \in N$ in subgame $\Phi(x_{\Gamma})$ with the chosen vector b_{-1} can be defined as follows:

$$\bar{E}_i(b_{-1}^{\Gamma}) = \sum_{t_{-1} \in \bar{T}_{-1}} p(t_{-1} | t_1 = I') \cdot E_i((I', t_{-1}); b_{-1}^{\Gamma}), \tag{7}$$

where $p(t_{-1} | t_1 = I') = p(I', t_{-1}) / \sum_{t' \in \bar{T}(1)} p(t')$.

Moreover, the expected payoff to Player $i \in N \setminus \{1\}$ of type $t_i \in T$ in subgame $\Phi(x_\Gamma)$ can be defined since Player i does not know the types of players from the set $N \setminus \{1, i\}$, and it is defined as

$$\hat{E}_i(b_{-1}^\Gamma | t_i) = \sum_{(t_{-i})_{-1} \in (\bar{T}_{-i})_{-1}} p((t_{-i})_{-1} | t_1 = I', t_i) \cdot E_i((I', t_i, (t_{-i})_{-1}); b_{-1}^\Gamma), \tag{8}$$

where $p((t_{-i})_{-1} | t_1 = I', t_i) = p(I', t_i, (t_{-i})_{-1}) / \sum_{t' \in \bar{T}(1)} p(t_i, t'_{-i})$.

The following proposition shows the correlation between the expected payoffs $\bar{E}_i(b_{-1}^\Gamma)$ and $\hat{E}_i(b_{-1}^\Gamma | t_i)$.

Proposition 1. *The expected payoff $\bar{E}_i(b_{-1}^\Gamma)$ can be expressed by $\hat{E}_i(b_{-1}^\Gamma | t_i)$ as*

$$\bar{E}_i(b_{-1}^\Gamma) = \sum_{t_i \in T} p(t_i | t_1 = I') \cdot \hat{E}_i(b_{-1}^\Gamma | t_i), \tag{9}$$

where $p(t_i | t_1 = I') = \sum_{t' \in \bar{T}(1)} p(t_i, t'_{-i}) / \sum_{t' \in \bar{T}(1)} p(t')$.

Proof. From (8), we can get

$$\sum_{t_i \in T} p(t_i | t_1 = I') \cdot \hat{E}_i(b_{-1}^\Gamma | t_i)$$

$$= \sum_{t_i \in T} p(t_i | t_1 = I') \cdot \sum_{(t_{-i})_{-1} \in (\bar{T}_{-i})_{-1}} p((t_{-i})_{-1} | t_1 = I', t_i) \cdot E_i((I', t_i, (t_{-i})_{-1}); b_{-1}^\Gamma)$$

$$= \sum_{t_i \in T} \sum_{(t_{-i})_{-1} \in (\bar{T}_{-i})_{-1}} p(t_i | t_1 = I') \cdot p((t_{-i})_{-1} | t_1 = I', t_i) E_i((I', t_i, (t_{-i})_{-1}); b_{-1}^\Gamma).$$

Then we may simplify the following equation as

$$p(t_i | t_1 = I') p((t_{-i})_{-1} | t_1 = I', t_i) = \frac{\sum_{t' \in \bar{T}(1)} p(t_i, t'_{-i})}{\sum_{t' \in \bar{T}(1)} p(t')} \cdot \frac{p(I', t_i, (t_{-i})_{-1})}{\sum_{t' \in \bar{T}(1)} p(t_i, t'_{-i})}$$

$$= \frac{p(I', t_i, (t_{-i})_{-1})}{\sum_{t' \in \bar{T}(1)} p(t')} = p(t_{-1} | t_1 = I').$$

Consequently, the following relationship is obtained:

$$\sum_{t_i \in T} p(t_i | t_1 = I') \cdot \hat{E}_i(b_{-1}^\Gamma | t_i) = \sum_{t_{-1} \in \bar{T}_{-1}} p(t_{-1} | t_1 = I') \cdot E_i((I', t_{-1}); b_{-1}^\Gamma) = \bar{E}_i(b_{-1}^\Gamma).$$

$\square$

Finally, we define the expected payoff to any player in the extensive-form game Φ. Below we consider only Player 1's behavior strategies under which a certain element from the set U_0 is chosen with probability 1 that is

$$b_1(X_1) = (0, \ldots, 1, \ldots, 0).$$

Given behavior strategy profile $b = (b_1, \ldots, b_n)$, the expected payoff to Player $i \in N$ in game Φ is defined as

$$G_i(b) = \bar{E}_i(b_{-1}^\Gamma),$$

where network $\Gamma \in U_0$ is chosen with probability 1 under b_1.

5. Main Results

As the two-stage game is represented in an extensive form, in every subgame $\Phi(x_\Gamma)$, $\Gamma \in U_0$, each type $t_i \in T$ corresponds to the information set $X_i^{(\Gamma,j)}$, $I_j = t_i$. Since there are various types for each player, below we consider the Bayesian equilibrium which provides the actions for players of different types.

Definition 1. *Behavior strategy vector $\tilde{b}_{-1}^\Gamma$ of players from $N \setminus \{1\}$, is the Bayesian equilibrium in subgame $\Phi(x_\Gamma)$, $\Gamma \in U_0$, if for any player $i \in N \setminus \{1\}$, any type $t_i \in T$, and any possible action $u_i \in A$ the following inequality holds:*

$$\hat{E}_i(\tilde{b}_{-1}^\Gamma | t_i) \geq \hat{E}_i\big((u_i, (\tilde{b}_{-1})_{-i}^\Gamma) | t_i\big). \tag{10}$$

Proposition 2. *Every subgame $\Phi(x_\Gamma)$, $\Gamma \in U_0$ of the extensive-form game Φ on tree graph Z admits the Bayesian equilibrium.*

Proof. It follows from [24], in which it is stated stating that every game with the incomplete information in which the set of types is finite and the set of actions of each type is finite has the Bayesian equilibrium (in behavior strategies). Indeed, in each subgame $\Phi(x_\Gamma)$ of game Φ, every player does not know the payoffs of other players except the leader's payoff since he knows only the leader's type and his own type chosen by nature. Thus, the game is of incomplete information. Moreover, the leader has a unique type, other players possess a finite number of types, and for any player of any type the set of actions, A, is finite. Above all, at least one Bayesian equilibrium $\tilde{b}_{-1}^\Gamma$ can be found in each subgame $\Phi(x_\Gamma)$, $\Gamma \in U_0$. $\square$

As the existence of the Bayesian equilibrium in each subgame is proved in Proposition 2, we investigate the connection between the Nash equilibrium and the Bayesian equilibrium in any subgame of the extensive-form game Φ.

Definition 2. *Behavior strategy vector $\bar{b}_{-1}^\Gamma$ of players from $N \setminus \{1\}$ is the Nash equilibrium in subgame $\Phi(x_\Gamma)$, $\Gamma \in U_0$, if for any player $i \in N \setminus \{1\}$, each strategy b_i^Γ of Player i the following inequality holds:*

$$\bar{E}_i(\bar{b}_{-1}^\Gamma) \geq \bar{E}_i\big(b_i^\Gamma, (\bar{b}_{-1})_{-i}^\Gamma\big). \tag{11}$$

Proposition 3. *If $\tilde{b}_{-1}^\Gamma$ is the Bayesian equilibrium in subgame $\Phi(x_\Gamma)$, $\Gamma \in U_0$, then $\tilde{b}_{-1}^\Gamma$ is also the Nash equilibrium, and vice versa.*

Proof. We prove it based on [25] that in a game with the incomplete information in which the number of each player's types is finite, each Bayesian equilibrium is also the Nash equilibrium, and conversely every Nash equilibrium is also a Bayesian equilibrium. Each subgame $\Phi(x_\Gamma)$ is of incomplete information since players $N \setminus \{1\}$ do not know the exact payoffs of others, and T, the common set of types for each player is also finite. Thus, we can directly get the result. $\square$

Combining the results from Proposition 2 and Proposition 3, we can get Theorem 1 below.

Theorem 1. *Every subgame $\Phi(x_\Gamma)$, $\Gamma \in U_0$, of the extensive-form game Φ on tree graph Z admits the Nash equilibrium.*

Obtaining the Bayesian equilibrium (if there are many Bayesian equilibria, then we can choose one randomly) in each subgame, we can get the strategy profile of players $N \setminus \{1\}$ in the extensive-form game. Below, we propose a new concept of the stable partially Bayesian equilibrium for game Φ taking into account the leader's strategy.

Definition 3. *Behavior strategy profile $b^* = (b_1^*, \ldots, b_n^*)$, where network $\Gamma^* \in U_0$ is chosen with probability 1 under b_1^*, is called the stable partially Bayesian equilibrium in game Φ if the following two conditions are satisfied:*

1. *For any $\Gamma \in U_0$, $(b^*)_{-1}^{\Gamma}$ is the Bayesian equilibrium in subgame $\Phi(x_{\Gamma})$;*
2. *$G_1(b^*) \geq G_1(b_1, (b^*)_{-1})$ for any strategy b_1 of the leader (Player 1).*

Theorem 2. *There exists the stable partially Bayesian equilibrium in the extensive-form game Φ.*

Proof. By Proposition 2, at least one vector $(b^*)_{-1}$ can be found such that $(b^*)_{-1}^{\Gamma}$ is the Bayesian equilibrium in subgame $\Phi(x_{\Gamma})$ for any $\Gamma \in U_0$. Let b_1^* be the behavior strategy of Player 1 such that

$$b_1^* = \arg\max_{b_1} G_1(b_1, (b^*)_{-1}). \tag{12}$$

Then the strategy profile b^* constructed above satisfies both conditions in Definition 3, thus it is the stable partially Bayesian equilibrium in the extensive-form game Φ. $\square$

Remark 3. *We call the equilibrium concept as the stable partially Bayesian equilibrium to highlight that in any subgame starting from stage 2, the equilibrium concept is the Bayesian equilibrium and the expected payoff of the leader is larger when he chooses strategy b_1^* rather than any other strategy b_1. In addition, the set of all stable partially Bayesian equilibria in game Φ is denoted by SPBE. The theorem below shows the connection between the stable partially Bayesian equilibrium and the Nash equilibrium in the extensive-form game Φ.*

Theorem 3. *Each stable partially Bayesian equilibrium is also the Nash equilibrium in the extensive-form game Φ, and the converse is not necessarily true.*

Proof. Let $\bar{b}$ be the stable partially Bayesian equilibrium in game Φ, and Γ^* is chosen with probability 1 under $\bar{b}_1$. Then by condition 2 of Definition 3, we know that for Player 1, and any strategy b_1,

$$G_1(\bar{b}) \geq G_1(b_1, \bar{b}_{-1}).$$

Then for any player $i \in N \setminus \{1\}$, first consider any strategy b_i such that $b_i^{\Gamma^*} = \bar{b}_i^{\Gamma^*}$, and we obtain that

$$G_i(\bar{b}) = \bar{E}_i(\bar{b}_{-1}^{\Gamma^*}) = G_i(b_i, \bar{b}_{-i}).$$

Then consider any strategy b_i such that $b_i^{\Gamma^*} \neq \bar{b}_i^{\Gamma^*}$, and from Proposition 3 we have

$$G_i(\bar{b}) = \bar{E}_i(\bar{b}_{-1}^{\Gamma^*}) \geq \bar{E}_i\left(b_i^{\Gamma^*}, (\bar{b}_{-1})_{-i}^{\Gamma^*}\right) = G_i(b_i, \bar{b}_{-i}).$$

Above all, for any player $i \in N$ and any strategy b_i, the inequality

$$G_i(\bar{b}) \geq G_i(b_i, \bar{b}_{-i})$$

holds. Therefore, $\bar{b}$ is also the Nash equilibrium in game Φ. Conversely, let $\bar{b}$ be the Nash equilibrium in game Φ, while it cannot be guaranteed that $\bar{b}_{-1}^{\Gamma}$ is the Bayesian equilibrium in subgame $\Phi(x_{\Gamma})$ for any $\Gamma \in U_0$. Thus, the converse is not necessarily true. $\square$

Remark 4. *We continue to consider the three-player game described in Example 1 by defining more essential parameters such as the probability distribution of the chance move, the payoff functions for players of various types, etc. We provide the link: http://hdl.handle.net/11701/27022, accessed on 8 March 2021 (and see http://hdl.handle.net/11701/27021 for the program code, accessed on 8 March 2021) to Supplementary Materials where the tree graph describing the game and detailed calculations are provided.*

6. Three-Player Game with Major Player

Below we investigate a three-player game with a major player where the major player (Player 1) differs from other players, and other players are supposed to be identical, that indicates that conditions:

$$v(\{1,3\}) = v(\{1,2\}), \ v(\{2\}) = v(\{3\})$$

are satisfied. Following [26], the characteristic function in the game is defined in the following form:

$$v(S) = \begin{cases} \alpha(s-1) + \beta/s, 1 \in S, \\ 0, \qquad\qquad 1 \notin S, \end{cases} \tag{13}$$

where α and β are positive constants.

The possible interpretation of the characteristic function (13) is shown further. Player 1 can be the leader and he/she gets funding for the project. Without Player 1, coalition S can obtain nothing. As other players are identical, their contributions into any coalition are equal. In addition, if Player 1 belongs to coalition S, the productivity or payoff of any Player 2 or 3 is measured by $\alpha > 0$, and the productivity $\beta > 0$ of the leader, Player 1, is divided by the number of coalition members. Thus, Player 1's productivity decreases over the size of coalition S. If we assume the productivity of the leader is larger than the one of other players, we may set $\alpha < \beta$ that has been supposed in [26], but we will not use this condition in the calculations.

The set of network structures which can be proposed by the leader is $U_0 = \{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\}$, where $\Gamma_1 = \{12, 13\}$, $\Gamma_2 = \{13, 23\}$, $\Gamma_3 = \{12, 23\}$, $\Gamma_4 = \{12, 13, 23\}$ shown in Figure 1. We assume the player can be of two types. The payoff function for Player i of type I_1 is defined as

$$K_i(\Gamma, I_1) = Y_i(\Gamma), \tag{14}$$

where $Y_i(\Gamma)$ is a component of the Myerson value [17], and for Player i of type I_2, it is defined as

$$K_i(\Gamma, I_2) = ES_i(\Gamma), \tag{15}$$

where $ES_i(\Gamma)$ is the component of the ES value [27] for Player i in game (N, v^Γ). We assume type I_1 for the leader.
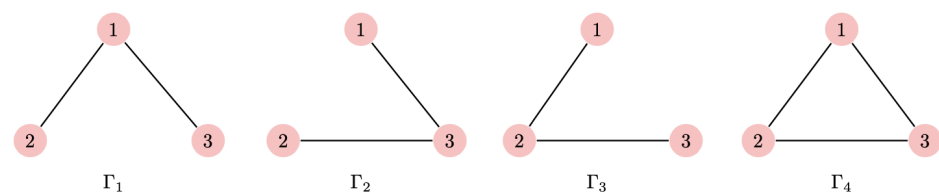

Figure 1. Network structures $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4.

The following two propositions characterize the features of the network structures formed under the stable partially Bayesian equilibria in the game above for the case where the probability distribution of the chance move is uniform.

Proposition 4. *When $\alpha > \frac{\beta}{2}$ and the probability distribution over the set of type vectors is uniform, then the set of networks formed in SPBE is $\{\Gamma_1, \Gamma_4\}$.*

Proof. For each subgame x_Γ, $\Gamma \in U_0$ in the extensive-form game, the corresponding strategic form is shown in Tables 1–4, in which the payoff vector is the expected payoffs to Players 2 and 3 for the corresponding situation in the subgame when the distribution of the chance move is uniform. Please note that under strategy (a, r), Player i of type I_1 chooses the action 'a', and Player i of type I_2 chooses the action 'r'. In Tables 1–4, Player 2 chooses the rows, and Player 3 chooses the columns.

Table 1. Strategic form of subgame x_{Γ_1}.

Strategy	(a,a)	(a,r)	(r,a)	(r,r)
(a,a)	$(\frac{7\alpha}{3} - \frac{13\beta}{18}, \frac{7\alpha}{3} - \frac{13\beta}{18})$	$(2\alpha - \frac{7\beta}{9}, \frac{5\alpha}{3} - \frac{11\beta}{18})$	$(2\alpha - \frac{7\beta}{9}, \frac{4\alpha}{3} - \frac{4\beta}{9})$	$(\frac{5\alpha}{3} - \frac{5\beta}{6}, \frac{2\alpha}{3} - \frac{\beta}{3})$
(a,r)	$(\frac{5\alpha}{3} - \frac{11\beta}{18}, 2\alpha - \frac{7\beta}{9})$	$(\frac{4\alpha}{3} - \frac{5\beta}{9}, \frac{4\alpha}{3} - \frac{5\beta}{9})$	$(\frac{4\alpha}{3} - \frac{5\beta}{9}, \alpha - \frac{7\beta}{18})$	$(\alpha - \frac{\beta}{2}, \frac{\alpha}{3} - \frac{\beta}{6})$
(r,a)	$(\frac{4\alpha}{3} - \frac{4\beta}{9}, 2\alpha - \frac{7\beta}{9})$	$(\alpha - \frac{7\beta}{18}, \frac{4\alpha}{3} - \frac{5\beta}{9})$	$(\alpha - \frac{7\beta}{18}, \alpha - \frac{7\beta}{18})$	$(\frac{2\alpha}{3} - \frac{\beta}{3}, \frac{\alpha}{3} - \frac{\beta}{6})$
(r,r)	$(\frac{2\alpha}{3} - \frac{\beta}{3}, \frac{5\alpha}{3} - \frac{5\beta}{6})$	$(\frac{\alpha}{3} - \frac{\beta}{6}, \alpha - \frac{\beta}{2})$	$(\frac{\alpha}{3} - \frac{\beta}{6}, \frac{2\alpha}{3} - \frac{\beta}{3})$	$(0,0)$

Table 2. Strategic form of subgame x_{Γ_2}.

Strategy	(a,a)	(a,r)	(r,a)	(r,r)
(a,a)	$(2\alpha - \frac{5\beta}{9}, 3\alpha - \frac{19\beta}{18})$	$(\alpha - \frac{5\beta}{18}, \frac{5\alpha}{3} - \frac{11\beta}{18})$	$(\alpha - \frac{5\beta}{18}, \frac{4\alpha}{3} - \frac{4\beta}{9})$	$(0,0)$
(a,r)	$(\frac{4\alpha}{3} - \frac{4\beta}{9}, \frac{7\alpha}{3} - \frac{17\beta}{18})$	$(\frac{2\alpha}{3} - \frac{2\beta}{9}, \frac{4\alpha}{3} - \frac{5\beta}{9})$	$(\frac{2\alpha}{3} - \frac{2\beta}{9}, \alpha - \frac{7\beta}{18})$	$(0,0)$
(r,a)	$(\frac{4\alpha}{3} - \frac{4\beta}{9}, \frac{7\alpha}{3} - \frac{17\beta}{18})$	$(\frac{2\alpha}{3} - \frac{2\beta}{9}, \frac{4\alpha}{3} - \frac{5\beta}{9})$	$(\frac{2\alpha}{3} - \frac{2\beta}{9}, \alpha - \frac{7\beta}{18})$	$(0,0)$
(r,r)	$(\frac{2\alpha}{3} - \frac{\beta}{3}, \frac{5\alpha}{3} - \frac{5\beta}{6})$	$(\frac{\alpha}{3} - \frac{\beta}{6}, \alpha - \frac{\beta}{2})$	$(\frac{\alpha}{3} - \frac{\beta}{6}, \frac{2\alpha}{3} - \frac{\beta}{3})$	$(0,0)$

Table 3. Strategic form of subgame x_{Γ_3}.

Strategy	(a,a)	(a,r)	(r,a)	(r,r)
(a,a)	$(3\alpha - \frac{19\beta}{18}, 2\alpha - \frac{5\beta}{9})$	$(\frac{7\alpha}{3} - \frac{17\beta}{18}, \frac{4\alpha}{3} - \frac{4\beta}{9})$	$(\frac{7\alpha}{3} - \frac{17\beta}{18}, \frac{4\alpha}{3} - \frac{4\beta}{9})$	$(\frac{5\alpha}{3} - \frac{5\beta}{6}, \frac{2\alpha}{3} - \frac{\beta}{3})$
(a,r)	$(\frac{5\alpha}{3} - \frac{11\beta}{18}, \alpha - \frac{5\beta}{18})$	$(\frac{4\alpha}{3} - \frac{5\beta}{9}, \frac{2\alpha}{3} - \frac{2\beta}{9})$	$(\frac{4\alpha}{3} - \frac{5\beta}{9}, \frac{2\alpha}{3} - \frac{2\beta}{9})$	$(\alpha - \frac{\beta}{2}, \frac{\alpha}{3} - \frac{\beta}{6})$
(r,a)	$(\frac{4\alpha}{3} - \frac{4\beta}{9}, \alpha - \frac{5\beta}{18})$	$(\alpha - \frac{7\beta}{18}, \frac{2\alpha}{3} - \frac{2\beta}{9})$	$(\alpha - \frac{7\beta}{18}, \frac{2\alpha}{3} - \frac{2\beta}{9})$	$(\frac{2\alpha}{3} - \frac{\beta}{3}, \frac{\alpha}{3} - \frac{\beta}{6})$
(r,r)	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$

Table 4. Strategic form of subgame x_{Γ_4}.

Strategy	(a,a)	(a,r)	(r,a)	(r,r)
(a,a)	$(\frac{7\alpha}{3} - \frac{13\beta}{18}, \frac{7\alpha}{3} - \frac{13\beta}{18})$	$(2\alpha - \frac{7\beta}{9}, \frac{5\alpha}{3} - \frac{11\beta}{18})$	$(2\alpha - \frac{7\beta}{9}, \frac{4\alpha}{3} - \frac{4\beta}{9})$	$(\frac{5\alpha}{3} - \frac{5\beta}{6}, \frac{2\alpha}{3} - \frac{\beta}{3})$
(a,r)	$(\frac{5\alpha}{3} - \frac{11\beta}{18}, 2\alpha - \frac{7\beta}{9})$	$(\frac{4\alpha}{3} - \frac{5\beta}{9}, \frac{4\alpha}{3} - \frac{5\beta}{9})$	$(\frac{4\alpha}{3} - \frac{5\beta}{9}, \alpha - \frac{7\beta}{18})$	$(\alpha - \frac{\beta}{2}, \frac{\alpha}{3} - \frac{\beta}{6})$
(r,a)	$(\frac{4\alpha}{3} - \frac{4\beta}{9}, 2\alpha - \frac{7\beta}{9})$	$(\alpha - \frac{7\beta}{18}, \frac{4\alpha}{3} - \frac{5\beta}{9})$	$(\alpha - \frac{7\beta}{18}, \alpha - \frac{7\beta}{18})$	$(\frac{2\alpha}{3} - \frac{\beta}{3}, \frac{\alpha}{3} - \frac{\beta}{6})$
(r,r)	$(\frac{2\alpha}{3} - \frac{\beta}{3}, \frac{5\alpha}{3} - \frac{5\beta}{6})$	$(\frac{\alpha}{3} - \frac{\beta}{6}, \alpha - \frac{\beta}{2})$	$(\frac{\alpha}{3} - \frac{\beta}{6}, \frac{2\alpha}{3} - \frac{\beta}{3})$	$(0,0)$

The calculations show that when $\alpha > \frac{\beta}{2}$, strategy (a,a) of Player i, $i = 2,3$ strictly dominates strategies $(a,r), (r,a)$ and (r,r) in any game represented in Tables 1–4. Thus, strategy profile $\big((a,a),(a,a)\big)$ is the unique Nash equilibrium in any game. Then from Proposition 3, we know that profile $(\bar{b}_2^\Gamma, \bar{b}_3^\Gamma)$, where $\bar{b}_i^\Gamma(X_i^{(\Gamma,j)}; a) = 1$, $i = 2,3$, $j = 1,2$, is the unique Bayesian equilibrium in subgame x_Γ for any $\Gamma \in U_0$. The inequality

$$G_1(b_1^1, \bar{b}_2, \bar{b}_3) = G_1(b_1^4, \bar{b}_2, \bar{b}_3) = \alpha + \frac{11\beta}{18} > \frac{25\beta}{36} + \frac{5\alpha}{6} = G_1(b_1^2, \bar{b}_2, \bar{b}_3) = G_1(b_1^3, \bar{b}_2, \bar{b}_3)$$

holds if $\alpha > \frac{\beta}{2}$, where network Γ_k is chosen with probability 1 under the leader's strategy b_1^k.

Summarizing we obtain that strategy profiles $(b_1^1, \bar{b}_2, \bar{b}_3)$ and $(b_1^4, \bar{b}_2, \bar{b}_3)$ form the set of all stable partially Bayesian equilibria in the extensive-form game, under which only networks Γ_1 and Γ_4 can be formed. □

Proposition 5. *When $\alpha < \frac{\beta}{6}$ and the probability distribution over the set of type vectors is uniform, the empty network is a unique network that can be formed under the stable partially Bayesian equilibria.*

Proof. When $\alpha < \frac{\beta}{6}$, strategy (r,r) Player i, $i = 2,3$ strictly dominates strategies (a,a), (r,a) and (a,r) in each game defined in Tables 1–4. Thus, strategy profile $((r,r),(r,r))$ is the unique Nash equilibrium in each game. Then by Proposition 3, we can obtain that profile $(\hat{b}_2^\Gamma, \hat{b}_3^\Gamma)$, where $\hat{b}_i^\Gamma(X_i^{(\Gamma,j)}; r) = 1$, $i = 2,3$, $j = 1,2$, is the unique Bayesian equilibrium in subgame x_Γ for any $\Gamma \in U_0$. Therefore,

$$G_1(b_1^k, \hat{b}_2, \hat{b}_3) = \beta$$

for $k = 1, \ldots, 4$, where network Γ_k is chosen with probability 1 under strategy b_1^k.

Summarizing we obtain that when $\alpha < \frac{\beta}{6}$, any strategy profile $(b_1^k, \hat{b}_2, \hat{b}_3)$, $k = 1, \ldots, 4$ is the stable partially Bayesian equilibrium in the extensive-form game, under which only the empty network can be formed. □

Remark 5. *For a value of $\alpha \in [\frac{\beta}{6}, \frac{\beta}{2}]$, there may exist several Bayesian equilibria, or the Bayesian equilibrium in mixed strategies, resulting in various networks that can be formed with positive possibilities under the stable partially Bayesian equilibria. Thus, here we cannot give the specific network structure features under the stable partially Bayesian equilibria when $\frac{\beta}{6} \leq \alpha \leq \frac{\beta}{2}$.*

7. Special n-Player Game

We introduce an n-player game with the characteristic function defined in the following way:

$$v(S) = \frac{k}{n - s + 1}, \quad S \subseteq N, \tag{16}$$

where k is a positive constant, $s = |S|$, $n = |N|$. The set of network structures which can be proposed by the leader as the joint projects is $U_0 = \{\Gamma_1, \Gamma_2\}$, where $\Gamma_1 = \{12, 13, \ldots, 1n\}$, the star network with the leader, say Player 1, being in the central position, and $\Gamma_2 = \{ij \mid i, j \in N, i \neq j\}$, the complete network of set N. In addition, there are two types for each player: types I_1 and I_2. The payoff function for Player i of type I_1 is defined as

$$K_i(\Gamma, I_1) = Y_i(\Gamma) + \theta_i(\Gamma, I_1), \tag{17}$$

where $Y_i(\Gamma)$ is the component of the Myerson value, and for Player i of type I_2, it is defined as

$$K_i(\Gamma, I_2) = ES_i(\Gamma), \tag{18}$$

where $ES_i(\Gamma)$ is the component of the ES value for Player i in game (N, v^Γ).

The following theorems characterize the features of the network structures formed under the stable partially Bayesian equilibria.

Proposition 6. *When for any player $i \in N$ and any network Γ, $\theta_i(\Gamma, I_1) = 0$, then independently of the leader's type, there exist at least three stable partially Bayesian equilibria in the extensive-form game regardless of the probability distribution of the chance move.*

Proof. First note that the proof below is conducted assuming the type for the leader is I_1 skipping the case of type I_2 to save the space for which the proof is similar. For any player $i \in N \setminus \{1\}$, consider strategy $\bar{b}_i$ such that $\bar{b}_i(X_i^{(\Gamma,j)}; r) = 1$ for $\Gamma \in \{\Gamma_1, \Gamma_2\}$ and $j = 1, 2$. Consider the truncation of strategy $\bar{b}_i$ to subgame x_{Γ_1}, and the calculations show that for player $i \in N \setminus \{1\}$ and network $\bar{\Gamma} = \{1i\}$ which is formed when only player i accepts network Γ_1 or Γ_2, we obtain expressions:

$$Y_i(\bar{\Gamma}) = \frac{k}{n}\left[1 + \frac{2 - n}{2n(n-1)}\right] < \frac{k}{n}$$

and

$$ES_i(\bar{\Gamma}) = \frac{k}{n}\left[1 + \frac{2-n}{n(n-1)}\right] < \frac{k}{n}.$$

While

$$Y_i(\varnothing) = ES_i(\varnothing) = \frac{k}{n},$$

then for any $\Gamma \in \{\Gamma_1, \Gamma_2\}$ and any strategy b_i^Γ of Player i, we have

$$\bar{E}_i(\bar{b}_{-1}^\Gamma) = p(t_i = I_1 \mid t_1 = I_1) \cdot Y_i(\varnothing) + p(t_i = I_2 \mid t_1 = I_1) \cdot ES_i(\varnothing) = \frac{k}{n} > \bar{E}_i(b_i^\Gamma, (\bar{b}_{-1})_{-i}^\Gamma),$$

which indicates that strategy profile $(\bar{b}_2^\Gamma, \ldots, \bar{b}_n^\Gamma)$ is the Bayesian equilibrium in subgame x_Γ, $\Gamma \in \{\Gamma_1, \Gamma_2\}$. Therefore, regardless of the leader's type, we get

$$G_1(b_1^1, \bar{b}_2, \ldots, \bar{b}_n) = G_1(b_1^2, \bar{b}_2, \ldots, \bar{b}_n) = \frac{k}{n},$$

where networks Γ_1 and Γ_2 are chosen with probability 1 respectively under strategies b_1^1 and b_1^2. By Definition 3, we conclude that both strategy profiles $(b_1^1, \bar{b}_2, \ldots, \bar{b}_n)$ and $(b_1^2, \bar{b}_2, \ldots, \bar{b}_n)$ are the stable partially Bayesian equilibria in the extensive-form game.

Next for Player $i \in N \setminus \{1\}$, we consider another strategy $\hat{b}_i$ such that $\hat{b}_i(X_i^{(\Gamma_1,j)}; a) = 1$, $\hat{b}_i(X_i^{(\Gamma_2,j)}; r) = 1$ for $j = 1, 2$. From the proof above, it follows that strategy profile $(\hat{b}_2^{\Gamma_2}, \ldots, \hat{b}_n^{\Gamma_2})$ is the Bayesian equilibrium in subgame x_{Γ_2}. Below we prove that $(\hat{b}_2^{\Gamma_1}, \ldots, \hat{b}_n^{\Gamma_1})$ is also the Bayesian equilibrium in subgame x_{Γ_1}. The calculations show that for each player $i \in N \setminus \{1\}$, we have

$$Y_i(\Gamma_1) = \frac{k}{n}\left[1 + \frac{\frac{n-1}{2} - \sum_{t=2}^{n}\frac{1}{t}}{n-1}\right] > \frac{k}{n}$$

and

$$ES_i(\Gamma_1) = \frac{k}{n}.$$

As a result,

$$\bar{E}_i(\hat{b}_{-1}^{\Gamma_1}) = p(t_i = I_1 \mid t_1 = I_1) \cdot Y_i(\Gamma_1) + p(t_i = I_2 \mid t_1 = I_1) \cdot ES_i(\Gamma_1) > \frac{k}{n}.$$

While in network $\hat{\Gamma}_1 = \Gamma_1 \setminus \{1i\}$ which is formed when only player i rejects joining network Γ_1, we have

$$Y_i(\hat{\Gamma}_1) = \frac{k}{n} < Y_i(\Gamma_1),$$

and

$$ES_i(\hat{\Gamma}_1) = \frac{\frac{k}{n} + \frac{k}{2}}{n} < \frac{k}{n} = ES_i(\Gamma_1).$$

Therefore, for any strategy $b_i^{\Gamma_1}$ of Player i in subgame x_{Γ_1}, it is true that

$$\bar{E}_i(\hat{b}_{-1}^{\Gamma_1}) > \frac{k}{n} > \bar{E}_i(b_i^{\Gamma_1}, (\hat{b}_{-1})_{-i}^{\Gamma_1}),$$

which shows that $\hat{b}_{-1}^{\Gamma_1}$ is the Bayesian equilibrium in subgame x_{Γ_1}, no matter which type is defined for the leader,

$$G_1(b_1^2, \hat{b}_2, \ldots, \hat{b}_n) = \frac{k}{n} \geq G_1(b_1^1, \hat{b}_2, \ldots, \hat{b}_n).$$

By Definition 3, we obtain that strategy profile $(b_1^2, \hat{b}_2, \ldots, \hat{b}_n)$ is also the stable partially Bayesian equilibrium in the extensive-form game.

In conclusion, given any probability distribution of the chance move, strategy profiles $(b_1^1, \bar{b}_2, \ldots, \bar{b}_n)$, $(b_1^2, \bar{b}_2, \ldots, \bar{b}_n)$ and $(b_1^2, \hat{b}_2, \ldots, \hat{b}_n)$ are all stable partially Bayesian equilibria in the extensive-form game no matter which type is defined for the leader. $\square$

Proposition 7. *When $\theta_i(\Gamma, I_1) = 0$ is satisfied for any player $i \in N$ and any network Γ, there are two cases:*

1. *If the leader's type is I_1, then, regardless of the probability distribution of the chance move, the empty network $\varnothing$ can be realized under SPBE;*
2. *If the leader's type is I_2, then, whatever the probability distribution over the set of type vectors is given, networks Γ_1 and $\varnothing$ can be realized under SPBE.*

Proof. 1. From Proposition 6, we know that if the leader's type is I_1 (or I_2), then strategy profiles $(b_1^1, \bar{b}_2, \ldots, \bar{b}_n)$, $(b_1^2, \bar{b}_2, \ldots, \bar{b}_n)$ and $(b_1^2, \hat{b}_2, \ldots, \hat{b}_n)$ are all stable partially Bayesian equilibria in the extensive-form game regardless of the probability distribution by the chance move. In addition, under any of the three equilibria, the empty network can be formed.

2. For the empty network, the result can be obtained similarly to Item 1. Then for network Γ_1 consider strategy profile $(b_1^1, \hat{b}_2, \ldots, \hat{b}_n)$, when the type for the leader is I_2, so we have

$$G_1(b_1^1, \hat{b}_2, \ldots, \hat{b}_n) = G_1(b_1^2, \hat{b}_2, \ldots, \hat{b}_n) = \frac{k}{n}.$$

Thus, profile $(b_1^1, \hat{b}_2, \ldots, \hat{b}_n)$ is the stable partially Bayesian equilibrium in the game, under which network Γ_1 is formed. $\square$

Then we turn to set $\theta_i(\Gamma, I_1) = n_i(\Gamma)/(n-1)$, where $n_i(\Gamma)$ is the number of neighbors of Player i in network Γ. Table 5 below summaries the resulting networks (the last column in Table 5) which can be formed under the stable partially Bayesian equilibria by different conditions for any given probability distribution of the chance move.

Table 5. Summary of the results with $\theta_i(\Gamma, I_1) = n_i(\Gamma)/(n-1)$.

Leader's Type	Conditions	Networks
I_1/I_2	$2n^2 - nk + 2k \leq 0$	$\varnothing$
I_1/I_2	$\begin{cases} 2n^2 - nk + 2k \leq 0 \\ n(n+1) \geq k \end{cases}$	$\{\varnothing, \Gamma_2\}$
I_1	$\begin{cases} 2n^2 - nk + 2k \leq 0 \\ \frac{k}{n}\left[\sum\limits_{t=2}^{n} \frac{1}{t} - \frac{n-1}{2}\right] + 1 \geq 0 \end{cases}$	$\{\varnothing, \Gamma_1\}$
I_1	$\begin{cases} \sum\limits_{t=2}^{n+1} \frac{1}{t} - \frac{n-1}{2} > 0 \\ n(n+1) \geq k \end{cases}$	Γ_1
I_1	$\begin{cases} \sum\limits_{t=2}^{n+1} \frac{1}{t} - \frac{n-1}{2} < 0 \\ n(n+1) \geq k \end{cases}$	Γ_2
I_2	$2n^2 - nk + 2k \leq 0$	$\{\varnothing, \Gamma_1\}$
I_2	$n(n+1) \geq k$	$\{\Gamma_1, \Gamma_2\}$

8. Conclusions

We propose a model of the network formation when players are non-symmetric in their payoff functions or types. The players' types are the private information. First, one player is assigned to be the leader and he proposes a network to be formed to other players. Second, other players make decisions to accept or reject the network simultaneously. After that,

the network is formed by the given rule. We prove the existence of an equilibrium called the stable partially Bayesian equilibrium in the network formation game with the incomplete information about players' types. We examine the relation between the equilibrium and the Nash equilibrium. The three-player game with a major player with the specific characteristic function is considered to be an example of our model. We also examine an n-player game with the particular characteristic function and two projects, the star network and the complete network are investigated if they can be formed under the stable partially Bayesian equilibria. As a direction for future research, we mention an interesting problem of defining the steady network which can be formed with the positive probability as a result of the stable partially Bayesian equilibrium regardless of the appointed leader.

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