

Article

Philos-Type Oscillation Results for Third-Order Differential Equation with Mixed Neutral Terms

Marappan Sathish Kumar ^{1,†}, Omar Bazighifan ^{2,†}, Alanoud Almutairi ^{3,†} and Dimplekumar N. Chalishajar ^{4,*,†}¹ Department of Mathematics, Paavai Engineering College (Autonomous), Namakkal 637 018, Tamil Nadu, India; msksjv@gmail.com² Department of Mathematics, Faculty of Science, Hadhramout University, Hadhramout 50512, Yemen; o.bazighifan@gmail.com³ Department of Mathematics, Faculty of Science, University of Hafr Al Batin, P.O. Box 1803, Hafr Al Batin 31991, Saudi Arabia; amalmutairi@uhb.edu.sa⁴ Department of Applied Mathematics, Virginia Military Institute, 435 Mallory Hall, Letcher Av., Lexington, VA 24450, USA

* Correspondence: chalishajardn@vmi.edu

† These authors contributed equally to this work.

Abstract: The motivation for this paper is to create new Philos-type oscillation criteria that are established for third-order mixed neutral differential equations with distributed deviating arguments. The key idea of our approach is to use the triple of the Riccati transformation techniques and the integral averaging technique. The established criteria improve, simplify and complement results that have been published recently in the literature. An example is also given to demonstrate the applicability of the obtained conditions.

Keywords: oscillation; third-order; mixed neutral differential equation; distributed deviating arguments



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1. introduction

It is prudent to say that neutral differential equations have drawn obvious regard because of their wide uses and applications in science and technology, including physical sciences, gas and fluid mechanics, signal processing, robotics and traffic systems, engineering, population dynamics, medicine and the like. Of late, the theory of oscillation of differential equations of the third order has become an important topic, and therefore the oscillatory properties of this type of equation have already been obtained [1–6]. In particular, it is a necessary and invaluable issue, either theoretically or practically, to probe into neutral differential equations with distributed deviating arguments. Hence, a scientific study of the qualitative properties of solutions of these equations is proposed for applications, see for example the book [7,8] and the papers [9–15].

Tongxing Li et al. [16,17], Yunsong Qi et al. [18], Chenghui Zhang et al. [19], Zhenlai Han et al. [20], Ethiraj Thandapani et al. [21,22], Jianga et al. [23], considered nonlinear second/third-order mixed neutral differential equations.

Cuimei et al. [2] established an important extension of the Kamenev oscillation criterion for a third-order equation with a middle term. Ganesan et al. [3] studied the oscillatory properties of a third-order equation with a neutral type. Kumar et al. [6] extended the oscillation results of a third-order equation with distributed deviating arguments.

Based on these background details, this paper is concerned with the oscillation of third-order mixed neutral differential equations with distributed arguments:

$$(r(\mu)u''(\mu))' + \int_a^b q(\mu, \sigma)y(\mu - \sigma) d\sigma + \int_a^b p(\mu, \sigma)y(\mu + \sigma) d\sigma = 0, \quad (1)$$

where $u(\mu) = y(\mu) + p_1(\mu)y(\mu - \eta_1) + p_2(\mu)y(\mu + \eta_2)$, $\mu \geq \mu_0 > 0$ and $a < b$. Throughout this work, we formulate the following assumptions:

- (H₁) $r(\mu), p_i(\mu) \in C([\mu_0, +\infty)$ and $r(\mu), 0 \leq p_i(\mu) \leq \xi_i$ for $i = 1, 2$;
 (H₂) $p(\mu, \sigma) \in C([\mu_0, +\infty) \times [a, b], [0, +\infty))$,
 $q(\mu, \sigma) \in C([\mu_0, +\infty) \times [a, b], [0, +\infty))$ and $p(\mu, \sigma), q(\mu, \sigma)$ is not identically zero on
 $[\mu_*, +\infty) \times [a, b], \mu_* \geq \mu$;
 (H₃) $\eta_i \geq 0$ are constants, for $i = 1, 2$, and the integral of (1) is taken in the sense of Riemann–Stieltjes.

We recall that the function y is said to be a solution of (1) if the functions $y(\mu) + p_1(\mu)y(\mu - \eta_1) + p_2(\mu)y(\mu + \eta_2)$, $(y(\mu) + p_1(\mu)y(\mu - \eta_1) + p_2(\mu)y(\mu + \eta_2))'$ and $r(\mu)(y(\mu) + p_1(\mu)y(\mu - \eta_1) + p_2(\mu)y(\mu + \eta_2))''$ are continuous differentiable functions and y satisfies Equation (1). We study this equation under the condition

$$R[\mu, \mu_0] = \int_{\mu_0}^{\mu} r^{-1}(s)ds, \quad R[\mu, \mu_0] = \infty \quad \text{as} \quad \mu \rightarrow \infty. \quad (2)$$

Our main goal in this manuscript is to provide new oscillatory conditions for Equation (1) using the Riccati method. We have also discussed one example that confirms the main results.

2. Main Results

Here we present some notations:

$$\begin{aligned} S^{[1]}u(\mu) &= u'(\mu), \quad S_{\alpha}^{[1]}u(\mu) = u'(\mu + \alpha), \\ S^{[2]}u(\mu) &= r(\mu)u''(\mu), \quad S_{\alpha}^{[2]}u(\mu) = r(\mu + \alpha)u''(\mu + \alpha), \\ Q(\mu, \sigma) &= \min\{q(\mu, \sigma), q(\mu - \eta_1, \sigma), q(\mu + \eta_2, \sigma)\}, \\ P(\mu, \sigma) &= \min\{p(\mu, \sigma), p(\mu - \eta_1, \sigma), p(\mu + \eta_2, \sigma)\} \\ \text{and} \\ \Psi(\mu) &= \int_a^b [Q(\mu, \sigma) + P(\mu, \sigma)] d\sigma. \end{aligned}$$

Now, we discuss Philos-type oscillation criteria [24] of Equation (1) under the conditions of (2).

Consider $\mathbb{S}_0 = \{(\mu, s) : a \leq s < \mu < +\infty\}$ and $\mathbb{S} = \{(\mu, s) : a \leq s \leq \mu < +\infty\}$, the continuous function $K(\mu, s) : \mathbb{S} \rightarrow \mathbb{R}$ belongs to the class $\mathcal{R}$, if it satisfies

- (i) $K(\mu, \mu) = 0$ for $\mu \geq \mu_0$ and $K(\mu, s) > 0$ for $(\mu, s) \in \mathbb{S}_0$,
- (ii) $\frac{\partial K(\mu, s)}{\partial s} \leq 0$, $(\mu, s) \in \mathbb{S}_0$ and $e(\mu, s)$ is a locally integrable function.

Theorem 1. Let (2) hold, $a + b \geq 0$ and $b \geq \eta_1$. If there exists $\phi \in C^1([\mu_0, \infty), (0, \infty))$, for all sufficiently large $\mu_i \geq \mu_0$ ($i = 1, 2, 3, 4$), such that

$$\limsup_{\mu \rightarrow \infty} \frac{1}{K(\mu, \mu_3)} \int_{\mu_3}^{\mu} \left[K(\mu, s)\phi'(s)\Psi(s) - \frac{(1 + \xi_1 + \xi_2)}{4} \frac{\phi(s)(e_+(\mu, s))^2}{R[s - b, \mu_1]} \right] ds = \infty \quad (3)$$

and

$$\int_{\mu_4}^{\infty} \int_v^{\infty} \left[r^{-1}(\mu) \int_u^{\infty} \left(\int_a^b (q(\mu, \sigma) + p(\mu, \sigma)) d\sigma \right) ds \right] du dv = \infty, \quad (4)$$

where $e_+(\mu, s) := \max\{0, e(\mu, s)\}$ and

$$-\frac{\partial}{\partial s} K(\mu, s) - \frac{\phi'(s)}{\phi(s)} K(\mu, s) = e(\mu, s)(K(\mu, s))^{\frac{1}{2}} \quad \text{for all } (\mu, s) \in \mathbb{S}_0 \quad (5)$$

then all $y(\mu)$ of (1) either oscillates or satisfies $y(\mu) \rightarrow 0$ as $\mu \rightarrow \infty$.

Proof. Assume, for sake of contradiction, that Equation (1) has an eventually positive solution $y(\mu)$. That is $y(\mu) > 0, y(\mu - \eta_1) > 0, y(\mu + \eta_2) > 0, y(\mu - \sigma) > 0$ and $y(\mu + \sigma) > 0$ for all $\mu \geq \mu_1$, some $\mu_1 \geq \mu_0$ and $\sigma \in [a, b]$. Then we have $u(\mu) > 0$ for all $\mu \geq \mu_1$, in view of (1), we have

$$(S^{[2]}u(\mu))' = - \int_a^b q(\mu, \sigma)y(\mu - \sigma) d\sigma - \int_a^b p(\mu, \sigma)y(\mu + \sigma) d\sigma \leq 0. \quad (6)$$

Thus, $S^{[2]}u(\mu)$ is nonincreasing.

$$\begin{aligned} (S^{[2]}u(\mu))' + \int_a^b q(\mu, \sigma)y(\mu - \sigma) d\sigma + \int_a^b p(\mu, \sigma)y(\mu + \sigma) d\sigma + \xi_1(S_{-\eta_1}^{[2]}u(\mu))' \\ + \xi_1 \int_a^b q(\mu - \eta_1, \sigma)y(\mu - \eta_1 - \sigma) d\sigma + \xi_1 \int_a^b p(\mu - \eta_1, \sigma)y(\mu - \eta_1 + \sigma) d\sigma + \xi_2(S_{+\eta_1}^{[2]}u(\mu))' \\ + \xi_2 \int_a^b q(\mu + \eta_2, \sigma)y(\mu + \eta_2 - \sigma) d\sigma + \xi_2 \int_a^b p(\mu + \eta_2, \sigma)y(\mu + \eta_2 + \sigma) d\sigma = 0. \end{aligned} \quad (7)$$

On the other hand,

$$q(\mu, \sigma)y(\mu - \sigma) d\sigma + \xi_1 q(\mu - \eta_1, \sigma)y(\mu - \eta_1 - \sigma) + \xi_2 q(\mu + \eta_2, \sigma)y(\mu + \eta_2 - \sigma) \geq Q(\mu, \sigma)u(\mu - \sigma). \quad (8)$$

Similarly, we obtain

$$p(\mu, \sigma)y(\mu + \sigma) d\sigma + \xi_1 p(\mu - \eta_1, \sigma)y(\mu - \eta_1 + \sigma) + \xi_2 p(\mu + \eta_2, \sigma)y(\mu + \eta_2 + \sigma) \geq P(\mu, \sigma)u(\mu + \sigma). \quad (9)$$

It follows from (7)–(9) that we obtain

$$(S^{[2]}u(\mu))' + \xi_1(S_{-\eta_1}^{[2]}u(\mu))' + \xi_2(S_{+\eta_1}^{[2]}u(\mu))' + \int_a^b Q(\mu, \sigma)u(\mu - \sigma) d\sigma + \int_a^b P(\mu, \sigma)u(\mu + \sigma) d\sigma \leq 0. \quad (10)$$

By assumption (2), there exist the following cases

$$(C_1) : u(\mu) > 0, \quad u'(\mu) > 0, \quad u''(\mu) > 0, \quad \text{and} \quad (S^{[2]}u(\mu))' \leq 0,$$

or

$$(C_2) : u(\mu) > 0, \quad u'(\mu) < 0, \quad u''(\mu) > 0, \quad \text{and} \quad (S^{[2]}u(\mu))' \leq 0,$$

Assume (C_1) . By virtue of $u(\mu) > 0, a + b \geq 0$, it follows that

$$(S^{[2]}u(\mu))' + \xi_1(S_{-\eta_1}^{[2]}u(\mu))' + \xi_2(S_{+\eta_1}^{[2]}u(\mu))' + \Psi(\mu)u(\mu - b) \leq 0. \quad (11)$$

Since $S^{[2]}u(\mu) > 0$ is decreasing, then

$$u'(\mu) \geq \int_{\mu_1}^{\mu} \frac{1}{r(s)} [S^{[2]}u(s)] ds \geq S^{[2]}u(\mu) \int_{\mu_1}^{\mu} \frac{ds}{r(s)} = (S^{[2]}u(\mu))R[\mu, \mu_1]. \quad (12)$$

Now, we define the function

$$\chi_1(\mu) = \phi(\mu) \frac{S^{[2]}u(\mu)}{u(\mu - b)}. \quad (13)$$

Then $\chi_1(\mu)$ is positive for $\mu \geq \mu_1$. We obtain

$$\chi_1'(\mu) = \phi'(\mu) \frac{S^{[2]}u(\mu)}{u(\mu - b)} + \phi(\mu) \frac{(S^{[2]}u(\mu))'}{u(\mu - b)} - \phi(\mu) \frac{S^{[2]}u(\mu)}{u^2(\mu - b)} S_{-b} u(\mu). \quad (14)$$

By (6) and (12), one obtains $S_{-b}^{[1]}u(\mu) \geq (S_{-b}^{[2]}u(\mu))R[\mu - b, \mu_1] \geq (S^{[2]}u(\mu))R[\mu - b, \mu_1]$. Therefore

$$\chi'_1(\mu) \leq \phi'(\mu) \frac{S^{[2]}u(\mu)}{u(\mu-b)} + \phi(\mu) \frac{(S^{[2]}u(\mu))'}{u(\mu-b)} - \phi(\mu) \frac{S^{[2]}u(\mu)}{u^2(\mu-b)} (S^{[2]}u(\mu)) R[\mu-b, \mu_1]. \quad (15)$$

Using (13) in (15), we have

$$\chi'_1(\mu) \leq \frac{\phi'(\mu)}{\phi(\mu)} \chi_1(\mu) + \phi(\mu) \frac{(S^{[2]}u(\mu))'}{u(\mu-b)} - R[\mu-b, \mu_1] \frac{\chi_1^2(\mu)}{\phi(\mu)}. \quad (16)$$

Next, we define the function

$$\chi_2(\mu) = \phi(\mu) \frac{S^{[2]}_{-\eta_1} u(\mu)}{u(\mu-b)}. \quad (17)$$

Then $\chi_2(\mu)$ is positive for $\mu \geq \mu_1$. We obtain

$$\chi'_2(\mu) = \phi'(\mu) \frac{S^{[2]}_{-\eta_1} u(\mu)}{u(\mu-b)} + \phi(\mu) \frac{(S^{[2]}_{-\eta_1} u(\mu))'}{u(\mu-b)} - \phi(\mu) \frac{S^{[2]}_{-\eta_1} u(\mu)}{u^2(\mu-b)} S_{-b} u(\mu). \quad (18)$$

By (6) and (12), one obtains $S^{[1]}_{-b} u(\mu) \geq (S^{[2]}_{-b} u(\mu)) R[\mu-b, \mu_1] \geq (S^{[2]}_{-\eta_1} u(\mu)) R[\mu-b, \mu_1]$. Therefore

$$\begin{aligned} \chi'_2(\mu) &\leq \phi'(\mu) \frac{S^{[2]}_{-\eta_1} u(\mu)}{u(\mu-b)} + \phi(\mu) \frac{(S^{[2]}_{-\eta_1} u(\mu))'}{u(\mu-b)} \\ &\quad - \phi(\mu) \frac{S^{[2]}_{-\eta_1} u(\mu)}{u^2(\mu-b)} (S^{[2]}_{-\eta_1} u(\mu)) R[\mu-b, \mu_1]. \end{aligned} \quad (19)$$

Using (17) in (19), we have

$$\chi'_2(\mu) \leq \frac{\phi'(\mu)}{\phi(\mu)} \chi_2(\mu) + \phi(\mu) \frac{(S^{[2]}_{-\eta_1} u(\mu))'}{u(\mu-b)} - R[\mu-b, \mu_1] \frac{\chi_2^2(\mu)}{\phi(\mu)}. \quad (20)$$

Finally, define

$$\chi_3(\mu) = \phi(\mu) \frac{S^{[2]}_{+\eta_1} u(\mu)}{u(\mu-b)}. \quad (21)$$

Then $\chi_3(\mu)$ is positive for $\mu \geq \mu_1$. We obtain

$$\chi'_3(\mu) = \phi'(\mu) \frac{S^{[2]}_{+\eta_1} u(\mu)}{u(\mu-b)} + \phi(\mu) \frac{(S^{[2]}_{+\eta_1} u(\mu))'}{u(\mu-b)} - \phi(\mu) \frac{S^{[2]}_{+\eta_1} u(\mu)}{u^2(\mu-b)} S_{-b} u(\mu). \quad (22)$$

By (6) and (12), one obtains $S^{[1]}_{-b} u(\mu) \geq (S^{[2]}_{-b} u(\mu)) R[\mu-b, \mu_1] \geq (S^{[2]}_{-\eta_2} u(\mu)) R[\mu-b, \mu_1]$. Hence by (21), we have

$$\chi'_3(\mu) \leq \frac{\phi'(\mu)}{\phi(\mu)} \chi_3(\mu) + \phi(\mu) \frac{(S^{[2]}_{+\eta_1} u(\mu))'}{u(\mu-b)} - R[\mu-b, \mu_1] \frac{\chi_3^2(\mu)}{\phi(\mu)}. \quad (23)$$

By (16), (20) and (23), we arrive at

$$\begin{aligned}
 & \chi_1'(\mu) + \xi_1 \chi_2'(\mu) + \xi_2 \chi_3'(\mu) \\
 & \leq \phi(\mu) \left[\frac{(S^{[2]}u(\mu))' + \xi_1 (S_{-\eta_1}^{[2]}u(\mu))' + \xi_2 (S_{+\eta_1}^{[2]}u(\mu))'}{u(\mu - b)} \right] \\
 & + \frac{\phi'(\mu)}{\phi(\mu)} \chi_1(\mu) - R[\mu - b, \mu_1] \frac{\chi_1^2(\mu)}{\phi(\mu)} + \xi_1 \frac{\phi'(\mu)}{\phi(\mu)} \chi_2(\mu) \\
 & - \xi_1 R[\mu - b, \mu_1] \frac{\chi_2^2(\mu)}{\phi(\mu)} + \xi_2 \frac{\phi'(\mu)}{\phi(\mu)} \chi_3(\mu) - \xi_2 R[\mu - b, \mu_1] \frac{\chi_3^2(\mu)}{\phi(\mu)}. \quad (24)
 \end{aligned}$$

Using (11), we have

$$\begin{aligned}
 \chi_1'(\mu) + \xi_1 \chi_2'(\mu) + \xi_2 \chi_3'(\mu) & \leq -\phi'(\mu)\Psi(\mu) + \frac{\phi'(\mu)}{\phi(\mu)} \chi_1(\mu) - R[\mu - b, \mu_1] \frac{\chi_1^2(\mu)}{\phi(\mu)} \\
 & + \xi_1 \frac{\phi'(\mu)}{\phi(\mu)} \chi_2(\mu) - \xi_1 R[\mu - b, \mu_1] \frac{\chi_2^2(\mu)}{\phi(\mu)} \\
 & + \xi_2 \frac{\phi'(\mu)}{\phi(\mu)} \chi_3(\mu) - \xi_2 R[\mu - b, \mu_1] \frac{\chi_3^2(\mu)}{\phi(\mu)}. \quad (25)
 \end{aligned}$$

Hence by (25), we obtain

$$\begin{aligned}
 & \int_{\mu_3}^{\mu} K(\mu, s) \phi'(s) \Psi(s) ds \leq K(\mu, \mu_3) \chi_1(\mu_3) + \int_{\mu_3}^{\mu} \frac{\partial K(\mu, s)}{\partial s} \chi_1(s) ds \\
 & + \xi_1 K(\mu, \mu_3) \chi_2(\mu_3) + \xi_1 \int_{\mu_3}^{\mu} \frac{\partial K(\mu, s)}{\partial s} \chi_2(s) ds + \xi_2 K(\mu, \mu_3) \chi_3(\mu_3) \\
 & + \xi_2 \int_{\mu_3}^{\mu} \frac{\partial K(\mu, s)}{\partial s} \chi_3(s) ds + \int_{\mu_3}^{\mu} K(\mu, s) \frac{\phi'(s)}{\phi(s)} \chi_1(s) ds \\
 & + \xi_1 \int_{\mu_3}^{\mu} K(\mu, s) \frac{\phi'(s)}{\phi(s)} \chi_2(s) ds + \xi_2 \int_{\mu_3}^{\mu} K(\mu, s) \frac{\phi'(s)}{\phi(s)} \chi_3(s) ds \\
 & - \int_{\mu_3}^{\mu} \frac{K(\mu, s) R[s - b, \mu_1]}{\phi(s)} \chi_1^2(s) ds - \xi_1 \int_{\mu_3}^{\mu} \frac{K(\mu, s) R[s - b, \mu_1]}{\phi(s)} \chi_2^2(s) ds \\
 & - \xi_2 \int_{\mu_3}^{\mu} \frac{K(\mu, s) R[s - b, \mu_1]}{\phi(s)} \chi_3^2(s) ds. \quad (26)
 \end{aligned}$$

Inequality (24) can be also written as

$$\begin{aligned}
 & \int_{\mu_3}^{\mu} K(\mu, s) \phi'(s) \Psi(s) ds \leq - \int_{\mu_3}^{\mu} \left\{ -\frac{\partial K(\mu, s)}{\partial s} - \frac{\phi'(s)}{\phi(s)} K(\mu, s) \right\} \chi_1(s) ds \\
 & - \xi_1 \int_{\mu_3}^{\mu} \left\{ -\frac{\partial K(\mu, s)}{\partial s} - \frac{\phi'(s)}{\phi(s)} K(\mu, s) \right\} \chi_2(s) ds - \xi_2 \int_{\mu_3}^{\mu} \left\{ -\frac{\partial K(\mu, s)}{\partial s} - \frac{\phi'(s)}{\phi(s)} K(\mu, s) \right\} \chi_3(s) ds \\
 & - \int_{\mu_3}^{\mu} \frac{K(\mu, s) R[s - b, \mu_1]}{\phi(s)} \chi_1^2(s) ds - \xi_1 \int_{\mu_3}^{\mu} \frac{K(\mu, s) R[s - b, \mu_1]}{\phi(s)} \chi_2^2(s) ds \\
 & - \xi_2 \int_{\mu_3}^{\mu} \frac{K(\mu, s) R[s - b, \mu_1]}{\phi(s)} \chi_3^2(s) ds + K(\mu, \mu_3) \chi_1(\mu_3) + \xi_1 K(\mu, \mu_3) \chi_2(\mu_3) + \xi_2 K(\mu, \mu_3) \chi_3(\mu_3). \quad (27)
 \end{aligned}$$

Using (5), we obtain

$$\begin{aligned}
 \int_{\mu_3}^{\mu} K(\mu, s) \phi'(s) \Psi(s) ds &\leq - \int_{\mu_3}^{\mu} e(\mu, s) (K(\mu, s))^{1/2} \chi_1(s) ds \\
 &\quad - \xi_1 \int_{\mu_3}^{\mu} e(\mu, s) (K(\mu, s))^{1/2} \chi_2(s) ds - \xi_2 \int_{\mu_3}^{\mu} e(\mu, s) (K(\mu, s))^{1/2} \chi_3(s) ds \\
 &\quad - \int_{\mu_3}^{\mu} \frac{K(\mu, s) R[s-b, \mu_1]}{\phi(s)} \chi_1^2(s) ds - \xi_1 \int_{\mu_3}^{\mu} \frac{K(\mu, s) R[s-b, \mu_1]}{\phi(s)} \chi_2^2(s) ds \\
 &\quad - \xi_2 \int_{\mu_3}^{\mu} \frac{K(\mu, s) R[s-b, \mu_1]}{\phi(s)} \chi_3^2(s) ds + K(\mu, \mu_3) \chi_1(\mu_3) \\
 &\quad + \xi_1 K(\mu, \mu_3) \chi_2(\mu_3) + \xi_2 K(\mu, \mu_3) \chi_3(\mu_3), \tag{28}
 \end{aligned}$$

and

$$\begin{aligned}
 \int_{\mu_3}^{\mu} K(\mu, s) \phi'(s) \Psi(s) ds &\leq - \int_{\mu_3}^{\mu} \left\{ e(\mu, s) (K(\mu, s))^{1/2} \chi_1(s) + \frac{K(\mu, s) R[s-b, \mu_1]}{\phi(s)} \chi_1^2(s) \right\} ds \\
 &\quad - \xi_1 \int_{\mu_3}^{\mu} \left\{ e(\mu, s) (K(\mu, s))^{1/2} \chi_2(s) + \frac{K(\mu, s) R[s-b, \mu_1]}{\phi(s)} \chi_2^2(s) \right\} ds \\
 &\quad - \xi_2 \int_{\mu_3}^{\mu} \left\{ e(\mu, s) (K(\mu, s))^{1/2} \chi_3(s) + \frac{K(\mu, s) R[s-b, \mu_1]}{\phi(s)} \chi_3^2(s) \right\} ds \\
 &\quad + K(\mu, \mu_3) \chi_1(\mu_3) + \xi_1 K(\mu, \mu_3) \chi_2(\mu_3) + \xi_2 K(\mu, \mu_3) \chi_3(\mu_3) \\
 &\leq - \int_{\mu_3}^{\mu} \left[\frac{(R[s-b, \mu_1] K(\mu, s))^{1/2}}{(\phi(s))^{1/2}} \chi_1(s) + \frac{(\phi(s))^{1/2}}{2(R[s-b, \mu_1])^{1/2}} \right]^2 ds \\
 &\quad + \int_{\mu_3}^{\mu} \frac{\phi(s) (e_+(\mu, s))^2}{4R[s-b, \mu_1]} ds \\
 &\quad - \xi_1 \int_{\mu_3}^{\mu} \left[\frac{(R[s-b, \mu_1] K(\mu, s))^{1/2}}{(\phi(s))^{1/2}} \chi_2(s) + \frac{(\phi(s))^{1/2}}{2(R[s-b, \mu_1])^{1/2}} \right]^2 ds \\
 &\quad + \xi_1 \int_{\mu_3}^{\mu} \frac{\phi(s) (e_+(\mu, s))^2}{4R[s-b, \mu_1]} ds \\
 &\quad - \xi_2 \int_{\mu_3}^{\mu} \left[\frac{(R[s-b, \mu_1] K(\mu, s))^{1/2}}{(\phi(s))^{1/2}} \chi_3(s) + \frac{(\phi(s))^{1/2}}{2(R[s-b, \mu_1])^{1/2}} \right]^2 ds \\
 &\quad + \xi_2 \int_{\mu_3}^{\mu} \frac{\phi(s) (e_+(\mu, s))^2}{4R[s-b, \mu_1]} ds \\
 &\quad + K(\mu, \mu_3) \chi_1(\mu_3) + \xi_1 K(\mu, \mu_3) \chi_2(\mu_3) + \xi_2 K(\mu, \mu_3) \chi_3(\mu_3). \tag{29}
 \end{aligned}$$

Thus

$$\frac{1}{K(\mu, \mu_3)} \int_{\mu_3}^{\mu} \left[K(\mu, s) \phi'(s) \Psi(s) - \frac{(1 + \xi_2 + \xi_2)}{4} \frac{\phi(s) (e_+(\mu, s))^2}{R[s-b, \mu_1]} \right] ds \leq \chi_1(\mu_3) + \xi_1 \chi_2(\mu_3) + \xi_2 \chi_3(\mu_3) \tag{30}$$

which contradicts (3).

Assume (C₂) holds. Since $u(\mu) > 0$ and $u'(\mu) < 0$, then

$$\lim_{\mu \rightarrow \infty} u(\mu) = \lambda \geq 0.$$

We claim that $\lambda = 0$. Suppose $\lambda > 0$, we have $\lambda + \epsilon > y(\mu) > \lambda$, for any $\epsilon > 0$ and $\mu \geq \mu_1$. Set $0 < \epsilon < \frac{\lambda(1-(\xi_1+\xi_2))}{\xi_1+\xi_2}$. By (H₁), we have

$$y(\mu) \geq u(\mu) - (\xi_1 + \xi_2) y(\mu - \eta_1) > \lambda - (\xi_1 + \xi_2)(\lambda + \epsilon) = g(\lambda + \epsilon) > gu(\mu), \tag{31}$$

where $g = \frac{\lambda(1-(\xi_1+\xi_2))}{(\lambda+\epsilon)} > 0$. Using (6) and (31), we obtain

$$(S^{[2]}u(\mu))' \leq - \int_a^b g q(\mu, \sigma) u(\mu - \sigma) d\sigma - \int_a^b g p(\mu, \sigma) u(\mu + \sigma) d\sigma.$$

Integrating the above from μ ($\mu \geq \mu_3$) to ∞ , we have

$$S^{[2]}u(\mu) \geq g \int_{\mu}^{\infty} \left(\int_a^b q(\mu, \sigma) u(\mu - \sigma) d\sigma + \int_a^b p(\mu, \sigma) u(\mu + \sigma) d\sigma \right) ds,$$

and $u(\mu) > \lambda$, we obtain

$$u''(\mu) \geq g \lambda \left[r^{-1}(\mu) \int_{\mu}^{\infty} \left(\int_a^b (q(\mu, \sigma) + p(\mu, \sigma)) d\sigma \right) ds \right],$$

Again integrating the above from μ ($\mu \geq \mu_4$) to ∞ , we have

$$-u'(\mu) \geq g \lambda \int_{\mu}^{\infty} \left[r^{-1}(\mu) \int_u^{\infty} \left(\int_a^b (q(\mu, \sigma) + p(\mu, \sigma)) d\sigma \right) ds \right] du.$$

Finally, integrating the above from μ_4 to ∞ , we have

$$u(\mu_4) \geq g \lambda \int_{\mu_4}^{\infty} \int_v^{\infty} \left[r^{-1}(\mu) \int_u^{\infty} \left(\int_a^b (q(\mu, \sigma) + p(\mu, \sigma)) d\sigma \right) ds \right] du dv,$$

which contradicts (4), since $\lambda = 0$ and $0 \leq y(\mu) \leq u(\mu)$ implies $\lim_{\mu \rightarrow \infty} y(\mu) = 0$. $\square$

Theorem 2. Let (2) hold and $a + b \leq 0$ and $a + \eta_1 \leq 0$. If there exists $\phi \in C^1([\mu_0, \infty), (0, \infty))$, for all sufficiently large $\mu_i \geq \mu_0$ ($i = 1, 2, 3, 4$), such that (4) and

$$\limsup_{\mu \rightarrow \infty} \frac{1}{K(\mu, \mu_3)} \int_{\mu_3}^{\mu} \left[K(\mu, s) \phi'(s) \Psi(s) - \frac{(1 + \xi_1 + \xi_2)}{4} \frac{\phi(s)(e_+(\mu, s))^2}{R[s + a, \mu_1]} \right] ds = \infty \quad (32)$$

then all $y(\mu)$ of (1) either oscillates or satisfies $y(\mu) \rightarrow 0$ as $\mu \rightarrow \infty$.

Proof. Let $y(\mu)$ be a non-oscillatory solution of Equation (1). Proceeding from the proof of Theorem 1, by virtue of $u(\mu) > 0$ and $a + b \leq 0$, from (10), we obtain

$$(S^{[2]}u(\mu))' + \xi_1(S_{-\eta_1}^{[2]}u(\mu))' + \xi_2(S_{+\eta_1}^{[2]}u(\mu))' + \Psi(\mu)u(\mu + a) \leq 0. \quad (33)$$

By defining

$$\tilde{\chi}_1(\mu) = \phi(\mu) \frac{S^{[2]}u(\mu)}{u(\mu + a)}, \quad (34)$$

$$\hat{\chi}_2(\mu) = \phi(\mu) \frac{S_{-\eta_1}^{[2]}u(\mu)}{u(\mu + a)}, \quad (35)$$

and

$$\hat{\chi}_3(\mu) = \phi(\mu) \frac{S_{+\eta_1}^{[2]}u(\mu)}{u(\mu + a)}, \quad (36)$$

for all $\mu \geq \mu_1$, respectively, and the proof of Theorem 1, we obtain

$$\frac{1}{K(\mu, \mu_3)} \int_{\mu_3}^{\mu} \left[K(\mu, s) \phi'(s) \Psi(s) - \frac{(1 + \xi_2 + \xi_3)}{4} \frac{\phi(s)(e_+(\mu, s))^2}{R[s + a, \mu_1]} \right] ds \leq \chi_1(\mu_3) + \xi_1 \chi_2(\mu_3) + \xi_2 \chi_3(\mu_3), \quad (37)$$

which contradicts (32). $\square$

Example 1. Consider a third-order differential equation

$$\left(\frac{1}{2} (u(\mu) + u(\mu - \pi/4) + u(\mu + \pi/2))'' \right)' + \int_0^{\pi} u(\mu - \sigma) d\sigma + \frac{3}{2} \int_0^{\pi} u(\mu + \sigma) d\sigma = 0, \quad (38)$$

where $p_1(\mu) = p_2(\mu) = 1$, $r(\mu) = 1/2$, $\eta_1 = \frac{\pi}{4}$, $\eta_2 = \frac{\pi}{2}$, $q(\mu, \sigma) = 1$, $r(\mu, \sigma) = 3/2$, $a = 0$, $b = \pi$, $q(\mu, \sigma) = 1$ and $p(\mu, \sigma) = 3/2$. We obtain $\psi(\mu) = 5\pi/2$, $R[\mu, \mu_1] = 2(\mu - \mu_1)$. Choose, $\phi(\mu) = \mu$ and $K(\mu, s) = (\mu - s)^2$. Then $e(\mu, s) = (3 - \mu s^{-1})$. It is easy to verify that

$$\int_{\mu_4}^{\infty} \int_v^{\infty} \left[2 \int_u^{\infty} \left(\int_0^{\pi} (5/2) d\sigma \right) ds \right] du dv = \infty$$

and

$$\begin{aligned} & \limsup_{\mu \rightarrow \infty} \frac{1}{K(\mu, \mu_3)} \int_{\mu_3}^{\mu} \left[K(\mu, s) \phi'(s) \Psi(s) - \frac{(1 + \xi_1 + \xi_2)}{4} \frac{\phi(s)(e_+(\mu, s))^2}{R[s - b, \mu_1]} \right] ds \\ &= \limsup_{\mu \rightarrow \infty} \frac{1}{(\mu - \mu_3)} \int_{\mu_3}^{\mu} \left[\frac{5\pi(\mu - s)^2}{2} - \frac{3s(3 - \mu s^{-1})^2}{8(s - \pi - \mu_1)} \right] ds = \infty. \end{aligned}$$

Therefore, by Theorem 1 all solutions of (38) either oscillate or tend to 0 and $u(\mu) = \sin \mu$ is such a solution of (38).

Remark 1. From Theorems 1 and 2, one can easily obtain various asymptotic criteria for (1) by choosing different ϕ , $K(\mu, s)$.

Remark 2. In this paper, we suggest a new Philos-type oscillation criterion for a third-order mixed neutral differential Equation (1) by using the triple Riccati transformation technique and inequalities technique.

3. Conclusions

In this paper, we used the triple of the Riccati transformation techniques to establish Philos-type oscillation theorems for (1) in the case of (2). Our result improves and complements results in the cited papers. It would also be of interest to find another method to study (1) in the case where $R[\mu, \mu_0] < \infty$ as $\mu \rightarrow \infty$. These results easily extend to the corresponding dynamic equations on time scales. The details are left to the reader.

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