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Article

A Subclass of q-Starlike Functions Defined by Using a Symmetric q-Derivative Operator and Related with Generalized Symmetric Conic Domains

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Abstract: In this paper, the concepts of symmetric q-calculus and conic regions are used to define a new domain $\widetilde{\Omega}_{k,q,\alpha}$, which generalizes the symmetric conic domains. By using the domain $\widetilde{\Omega}_{k,q,\alpha}$, we define a new subclass of analytic and q-starlike functions in the open unit disk U and establish some new results for functions of this class. We also investigate a number of useful properties and characteristics of this subclass, such as coefficients estimates, structural formulas, distortion inequalities, necessary and sufficient conditions, closure and subordination results. The proposed approach is also compared with some existing methods to show the reliability and effectiveness of the proposed methods.

Keywords: quantum (or q-)calculus; symmetric q-derivative operator; conic domain; generalized symmetric conic domain

MSC: primary 05A30; 30C45; secondary 11B65; 47B38

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1. Introduction

Let $\mathcal{A}$ be the set of all analytic functions in open unit disk $U = \{w \in \mathbb{C} : |w| < 1\}$ and every $g \in \mathcal{A}$ have the series representation of the form:

$$g(w) = w + \sum_{m=2}^{\infty} a_m w^m. \tag{1}$$

Let $S \subset \mathcal{A}$ be the set of functions which are univalent in U (see [1]). Goodman [2] introduced the class of uniformly convex ($\mathcal{UCV}$) and uniformly starlike functions ($\mathcal{UST}$) that are defined as:

$$g \in \mathcal{UCV} \Longleftrightarrow g \in \mathcal{A} \text{ and } Re\left\{1 + \frac{wg''(w)}{g'(w)}\right\} > \left|\frac{wg''(w)}{g'(w)}\right|, \quad w \in U$$

and:

$$g \in \mathcal{UST} \Longleftrightarrow g \in \mathcal{A} \text{ and } Re\left\{\frac{wg'(w)}{g(w)}\right\} > \left|\frac{wg'(w)}{g(w)} - 1\right|, \quad w \in U.$$

Later in [3], for $k \geq 0$, Kanas and Wisniowska introduced the class of k-uniformly convex $(k-\mathcal{UCV})$ and k-uniformly starlike functions $(k-\mathcal{UST})$ that are defined as:

$$g \in k-\mathcal{UCV} \Longleftrightarrow g \in \mathcal{A} \text{ and } Re\left\{1+\frac{wg''(w)}{g'(w)}\right\} > k\left|\frac{wg''(w)}{g'(w)}\right|, \quad w \in U$$

and:

$$g \in k-\mathcal{UST} \Longleftrightarrow g \in \mathcal{A} \text{ and } \left\{\frac{wg'(w)}{g(w)}\right\} > k\left|\frac{wg'(w)}{g(w)}-1\right|, \quad w \in U.$$

A function $g \in \mathcal{A}$ is said to be subordinate to h, written as $(g \prec h)$, if there exists a Schwarz function $u(w)$ analytic in U with:

$$u(0) = 0 \text{ and } |u(w)| < 1,$$

such that $g(w) = h(u(w))$. If $h(w)$ is univalent in U, then (see [1,4]):

$$g(w) \prec h(w) \Longleftrightarrow g(0) - h(0) = 0 \text{ and } h(U) \supset g(U).$$

The convolution (Hadamard product) of $g(w)$ and $h(w)$ is defined as:

$$g(w) * h(w) = \sum_{m=0}^{\infty} a_m b_m w^m,$$

where:

$$g(w) = \sum_{m=0}^{\infty} a_m w^m \text{ and } h(w) = \sum_{m=0}^{\infty} b_m w^m, \qquad (w \in U).$$

Let $\mathcal{P}$ be the class of Carathéodory functions, consisting of all analytic functions p that satisfy the condition $Re(p(w)) > 0$, $w \in U$ and:

$$p(w) = 1 + \sum_{m=1}^{\infty} c_m w^m. \tag{2}$$

In [3,5], Kanas and Wisniowska defined the conic domain $\Omega_{k,\alpha}$, $k \geq 0$, $0 \leq \alpha < 1$, as:

$$\Omega_{k,\alpha} = \left\{u_0 + iv : k\sqrt{(u_0-1)^2 + v^2} + \alpha < u_0\right\}$$

or:

$$\Omega_{k,\alpha} = \{l : Rel > k|l-1| + \alpha\}.$$

Note that $\Omega_{k,\alpha}$ is such that $1 \in \Omega_{k,\alpha}$ and $\partial\Omega_{k,\alpha}$ is curve defined by:

$$\partial\Omega_{k,\alpha} = \left\{l = u_0 + iv : k^2(u-1)^2 + k^2v^2\right\} = (u-\alpha)^2.$$

Note that for $k = 0$, $0 < \kappa < 1$, $\kappa = 1$, and $\kappa > 1$, the domain $\partial\Omega_{k,\alpha}$ represents a right half plane $\{l \in \mathbb{C} : Rel > \alpha\}$, the right branch of a hyperbola, a parabola and an ellipse, respectively. The functions $\widetilde{p_{k,\alpha}}(w)$ are the extremal functions for conic domain $\Omega_{k,\alpha}$, defined by:

$$\widetilde{p_{k,\alpha}}(w) = \begin{cases} \frac{1+(1-2\alpha)w}{1-w} & \text{for } k = 0, \\ 1 + \frac{2(1-\alpha)}{\pi^2}\left(\log \frac{1+\sqrt{w}}{1-\sqrt{w}}\right)^2 & \text{for } k = 1, \\ \frac{1-\alpha}{1-k^2}\cos\left(A(k)\log\frac{1+\sqrt{w}}{1-\sqrt{w}}\right) - \frac{k^2-\alpha}{1-k^2} & \text{for } 0 < k < 1, \\ \frac{1-\alpha}{k^2-1}\sin\left(\frac{\pi}{2K(t)}\int_0^{\frac{u(w)}{\sqrt{t}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right) + \frac{k^2-\alpha}{1-k^2} & \text{for } k > 1, \end{cases} \tag{3}$$

where $A(k) = \frac{2}{\pi}\arccos k$, $t \in (0,1)$, $k = \cosh\left(\frac{\pi K'(t)}{4K(t)}\right)$, and $K(t)$ is the first kind of Legendre's complete elliptic integral; for details see [3,5,6]. Indeed, from (3), we have:

$$\widetilde{p_{k,\alpha}}(w) = 1 + P_1 w + P_2 w^2 + \ldots, \tag{4}$$

$$P_1 = \begin{cases} \frac{8(1-\alpha)(\arccos k)^2}{\pi^2(1-k^2)} & \text{for } 0 \leq k < 1, \\ \frac{8(1-\alpha)}{\pi^2} & \text{for } k = 1, \\ \frac{\pi^2(1-\alpha)}{4(1+t)\sqrt{t}K^2(t)(k^2-1)} & \text{for } k > 1, \end{cases} \tag{5}$$

and:

$$P_2 = \begin{cases} \frac{8(1-\alpha)(\arccos k)^2+2}{3\pi^2}P_1 & \text{for } 0 \leq k < 1, \\ \frac{2}{3}P_1 & \text{for } k = 1, \\ \frac{4\left(t^2+6t+1\right)K^2(t)-\pi^2(1-\alpha)}{24(1+t)K^2(t)\sqrt{t}}P_1 & \text{for } k > 1. \end{cases} \tag{6}$$

Quantum theory is an important tool to deal with complicated and difficult information. The notion of quantum theory has wide applications in many fields such as special functions and quantum physics. The study of the theory of q-calculus and its numerous applications in various fields has gained the great interest of researchers. In 1909, Jackson [7] was among the first few researchers who defined the q-analogous of the derivative and integral operator as well as providing some of their applications. Later on, several researchers suggested many applications of q-analysis in mathematics and related fields; see [8–16]. Recently, in [17], Srivastava discussed operators of basic (or q-)calculus and fractional q-calculus and their applications in the geometric function theory of complex analysis. Keeping in view the significance of q-operators rather than of ordinary operators and because of the wide range of applications of q-calculus, many researchers exhaustively studied q-calculus in different fields; see [8,18–20].

The symmetric q-calculus finds its applications in different fields, specially in quantum mechanics; see [21,22].

The objective of this paper is to further develop the theory of symmetric q-calculus operators in geometric function theory. Here we will give a few essential definitions and the concepts of q-calculus and symmetric q-calculus, which will be useful in subsequent work.

Definition 1. *The q-number $[\delta]_q$ is defined by:*

$$[\delta]_q = -\frac{q^\delta - 1}{1-q}, \quad (\delta \in \mathbb{C}), \tag{7}$$

and:

$$[\delta]_q =_{t=0}^{m-1} q^t = 1 + q + q^2 + \cdots + q^{m-1} \quad (\delta = m \in \mathbb{N}).$$

In addition, the q-factorial $[m]_q!$ is defined by:

$$[m]_q! =_{t=1}^{m} [t]_q, \;(m \in \mathbb{N}) \tag{8}$$

and:

$$[m]_q! = 1, \ (m = 0).$$

Furthermore, the q-gamma function is defined as:

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t) \quad \text{and} \quad \Gamma_q(1) = 1.$$

Definition 2. *For $m \in \mathbb{N}$, the symmetric q-number is defined as:*

$$\widetilde{[m]}_q = \frac{q^{-m} - q^m}{q^{-1} - q}, \quad \widetilde{[0]}_q = 0.$$

Note that the symmetric q-numbers cannot be reduced to q-numbers.

Throughout this paper, we assume $0 < q < 1$, and U is a unit disk unless otherwise stated.

Definition 3. *For any $m \in \mathbb{Z}^+ \cup \{0\}$, the symmetric q-number shift factorial is defined as:*

$$\widetilde{[m]}_q! = \begin{cases} \widetilde{[m]}_q \widetilde{[m-1]}_q \widetilde{[m-2]}_q \ldots \widetilde{[2]}_q \widetilde{[1]}_q, & m \geq 1 \\ 1 & m = 0. \end{cases}$$

Note that:

$$\lim_{q \to 1-} \widetilde{[m]}_q! = \widetilde{m} \left(\widetilde{m-1}\right) \left(\widetilde{m-2}\right) \ldots \widetilde{2}.\widetilde{1}.$$

Definition 4 ([7])**.** *Let $g \in \mathcal{A}$. Then the q-derivative operator or q-difference operator are defined by:*

$$(\partial D)_q g(w) = \frac{g(qw) - g(w)}{(q-1)w}, \quad w \in U. \tag{9}$$

Using the series representation of g, (9) can be written as:

$$(\partial D)_q g(w) = 1 + \sum_{m=2}^{\infty} [m]_q a_m w^{m-1}.$$

Note that:

$$(\partial D)_q w^m = [m]_q w^{m-1}, \quad (\partial D)_q \left\{ \sum_{m=1}^{\infty} a_m w^m \right\} = \sum_{m=1}^{\infty} [m]_q a_m w^{m-1},$$

and:

$$\lim_{q \to 1-} (\partial D)_q g(w) = g^{'}(w).$$

Definition 5. *([23]). Let $g \in \mathcal{A}$. Then the symmetric q-derivative operator is defined as:*

$$\widetilde{(\partial D)}_q g(w) = \frac{1}{w} \left[\frac{g(qw) - g(q^{-1}w)}{q - q^{-1}} \right], \quad w \in U.$$

Note that:

$$\widetilde{(\partial D)}_q w^m = \widetilde{[m]}_q w^{m-1}, \quad \widetilde{(\partial D)}_q \{_{m=1}^{\infty} a_m w^m\} =_{m=1}^{\infty} \widetilde{[m]}_q a_m w^{m-1},$$

and:

$$\lim_{q \to 1-} \widetilde{(\partial D)}_q g(w) = g^{'}(w).$$

Definition 6 ([24]). *Let $g \in \mathcal{A}$ and $0 < q < 1$, then $g \in S_q^*$ if $g(0) = g^{'}(0) = 1$ and:*

$$\left| \frac{w(\partial D)_q g(w)}{g(w)} - \frac{1}{1-q} \right| \le \frac{1}{1-q}. \tag{10}$$

Identically by utilizing the principle of subordination, the condition (10) can be written as (see [16]):

$$\frac{w(\partial D)_q g(w)}{g(w)} \prec \frac{1+w}{1-qw}.$$

By taking inspiration from the above cited work [24] we define the following symmetric q-starlike $\widetilde{S_q^*}$ function as:

Definition 7. *Let $g \in \mathcal{A}$ and $0 < q < 1$, then $g \in \widetilde{S_q^*}$ if $g(0) = g^{'}(0) = 1$ and:*

$$\left| \frac{w\widetilde{(\partial D)}_q g(w)}{g(w)} - \frac{1}{1-\frac{q}{q^{-1}}} \right| \le \frac{1}{1-\frac{q}{q^{-1}}}. \tag{11}$$

Using the principle of subordination, the condition (11) can be written as:

$$\frac{w\widetilde{(\partial D)}_q g(w)}{g(w)} \prec \frac{1+w}{1-\frac{q}{q^{-1}}w}.$$

Definition 8 ([25]). *Let $k \in [0, \infty)$, $q \in (0,1)$ and $\gamma \in \mathbb{C}/\{0\}$. For a function we have $p(w) \in k - \mathcal{P}_{q,\gamma}$ if and only if:*

$$p(w) \prec p_{k,q,\gamma}$$

where:

$$p_{k,q,\gamma} = \frac{2p_{k,\gamma}}{(1+q) + (1-q)p_{k,\gamma}}.$$

Geometrically, the function $p(w) \in k - \mathcal{P}_{q,\gamma}$ takes on all values from the domain $\Omega_{k,q,\alpha}$, given by:

$$\Omega_{k,q,\gamma} = \gamma\Omega_{k,q} + (1-\gamma),$$

where:

$$\Omega_{k,q} = \left\{ l : Re\left(\frac{(1+q)l}{(q-1)l+2} \right) > k\left| \frac{(1+q)l}{(q-1)l+2} - 1 \right| \right\}.$$

Recently many researchers [3,6,19,20,26–29] investigated several classes of analytic and univalent functions in different types of domains. For example, let $p(w)$ be analytic in U and $p(0) = 1$, then:

(i) If $p(w) \prec \frac{1+w}{1-w}$, then the image domain of U under $p(w)$ lies in right half plane, see [2].
(ii) For $-1 \le B < A \le 1$, if $p(w) \prec \frac{1+Aw}{1+Bw}$, then the image of U under $p(w)$ lies inside a circle centered on real axis, see [30].
(iii) In [3,5] Kanas showed that if $p(w) \prec p_{k,\alpha}(w)$, then the image of U under $p(w)$ lies inside the conic domains Ω_k and $\Omega_{k,\alpha}$.

By taking inspiration from the above cited work, we introduce the following classes:

Definition 9. *Let $k \in [0, \infty)$, $q \in (0,1)$ and $0 \le \alpha < 1$. A function $p(w)$ is said to be in the class $k - \widetilde{\mathcal{P}}_q(\alpha)$ if and only if:*

$$p(w) \prec \widetilde{p_{k,q,\alpha}}, \tag{12}$$

where:

$$\widetilde{p_{k,q,\alpha}} = \frac{2q^{-1}\widetilde{p_{k,\alpha}}}{(q^{-1}+q)+(q^{-1}-q)\widetilde{p_{k,\alpha}}}. \tag{13}$$

Geometrically, the function $p(w) \in k-\widetilde{\mathcal{P}}_q(\alpha)$ takes all values in the domain $\widetilde{\Omega_{k,q,\alpha}}$, which is defined as:

$$\widetilde{\Omega_{k,q,\alpha}} = \left\{ l : Re\left(\frac{(q^{-1}+q)l}{(q-q^{-1})l+2q^{-1}}\right) > k\left|\frac{(q^{-1}+q)l}{(q-q^{-1})l+2q^{-1}}-1\right|\right\} + \alpha. \tag{14}$$

It can be seen that:

- *(i)* *For $q \to 1-$, we have $\widetilde{\Omega_{k,q,\alpha}} = \Omega_{k,\alpha}$, see [5].*
- *(ii)* *For $\alpha = 0$ and $q \to 1-$, we have $\widetilde{\Omega_{k,q}} = \Omega_k$, see [3].*
- *(iii)* *If $\alpha = 0$ and $q \to 1-$, then $k-\widetilde{\mathcal{P}}_q(\alpha) = \mathcal{P}(p_k)$, where $\mathcal{P}(p_k)$ is the well-known class of function initiated by Kanas [3].*
- *(iv)* *Let $\alpha = 0$, $k = 0$, and $q \to 1-$; then $k-\widetilde{\mathcal{P}}_q = \mathcal{P}$.*

Definition 10. *A function $g \in \mathcal{A}$ is said to be in class $k-\widetilde{\mathcal{UST}}(q,\alpha)$, if it satisfies the condition:*

$$Re\left(\frac{(q^{-1}+q)\frac{w\widetilde{(\partial D)}_q g(w)}{g(w)}}{(q-q^{-1})\frac{w\widetilde{(\partial D)}_q g(w)}{g(w)}+2q^{-1}}\right) > k\left|\frac{(q^{-1}+q)\frac{w\widetilde{(\partial D)}_q g(w)}{g(w)}}{(q-q^{-1})\frac{w\widetilde{(\partial D)}_q g(w)}{g(w)}+2q^{-1}}-1\right| + \alpha. \tag{15}$$

We set $k-\widetilde{\mathcal{UST}}^{-}(q,\alpha) = k-\widetilde{\mathcal{UST}}(q,\alpha) \cap T$. T is the subclass $\mathcal{A}$ of consisting of functions of the form:

$$g(w) = w - \sum_{m=2}^{\infty} a_m w^m, \quad a_m \geq 0, \text{ for all } m \geq 2. \tag{16}$$

2. Main Results

Theorem 1. *Let $g \in \mathcal{A}$ of the form (1) and:*

$$\begin{aligned} &\sum_{m=2}^{\infty}\left\{2q^{-1}(k+1)\left(\widetilde{[m]}_q-1\right)+(1-\alpha)\left\{\left|\left(q-q^{-1}\right)\widetilde{[m]}_q+2q^{-1}\right|\right\}\right\}|a_m| \\ \leq\ &\left(q+q^{-1}\right)(1-\alpha). \end{aligned} \tag{17}$$

Then $g \in k-\widetilde{\mathcal{UST}}(q,\alpha)$.

Proof. Assume that (17) holds. It is sufficient to show that:

$$\begin{aligned} &\left|k\left(\frac{(q^{-1}+q)\frac{w\widetilde{\partial D}_q g(w)}{g(w)}}{(q-q^{-1})\frac{w\widetilde{\partial D}_q g(w)}{g(w)}+2q^{-1}}-1\right)\right| - Re\left(\frac{(q^{-1}+q)\frac{w\widetilde{\partial D}_q g(w)}{g(w)}}{(q-q^{-1})\frac{w\widetilde{\partial D}_q g(w)}{g(w)}+2q^{-1}}-1\right) \\ <\ &1-\alpha. \end{aligned}$$

Consider:

$$
\begin{aligned}
& \left| k\left(\frac{(q^{-1}+q)\frac{w\partial\widetilde{D}_q g(w)}{g(w)}}{(q-q^{-1})\frac{w\partial\widetilde{D}_q g(w)}{g(w)}+2q^{-1}} - 1 \right) \right| - Re\left(\frac{(q^{-1}+q)\frac{w\partial\widetilde{D}_q g(w)}{g(w)}}{(q-q^{-1})\frac{w\partial\widetilde{D}_q g(w)}{g(w)}+2q^{-1}} - 1 \right) \\
\leq \quad & k\left| \frac{(q^{-1}+q)\frac{w\partial\widetilde{D}_q g(w)}{g(w)}}{(q-q^{-1})\frac{w\partial\widetilde{D}_q g(w)}{g(w)}+2q^{-1}} - 1 \right| + \left| \frac{(q^{-1}+q)\frac{w\partial\widetilde{D}_q g(w)}{g(w)}}{(q-q^{-1})\frac{w\partial\widetilde{D}_q g(w)}{g(w)}+2q^{-1}} - 1 \right| \\
\leq \quad & (k+1)\left| \frac{(q^{-1}+q)\frac{w\partial\widetilde{D}_q g(w)}{g(w)}}{(q-q^{-1})\frac{w\partial\widetilde{D}_q g(w)}{g(w)}+2q^{-1}} - 1 \right| \\
= \quad & 2q^{-1}(k+1)\left| \frac{\sum_{m=2}^{\infty}\left(\widetilde{[m]}_q - 1\right)a_m w^m}{(q^{-1}+q)+\sum_{m=2}^{\infty}\left\{(q-q^{-1})\widetilde{[m]}_q + 2q^{-1}\right\}a_m w^m} \right| \\
\leq \quad & 2q^{-1}(k+1)\left(\frac{\sum_{m=2}^{\infty}\left(\widetilde{[m]}_q - 1\right)|a_m|}{(q^{-1}+q)-\sum_{m=2}^{\infty}\left|(q-q^{-1})\widetilde{[m]}_q + 2q^{-1}\right||a_m|} \right).
\end{aligned}
$$

The last expression is bounded above by $1-\alpha$, as the inequality (17) is true. Hence the proof of Theorem 1 is completed $\square$

If $q \to 1-$ with $0 \leq \alpha < 1$, then we have the following known result proven by Shams et al., see [31].

Corollary 1. *Let $g \in \mathcal{A}$ be of the form (1) and g be in the class $k - \mathcal{UST}(\alpha)$, if it satisfies the condition:*

$$\sum_{m=2}^{\infty}\{m(k+1)-(k+\alpha)\}|a_m| \leq 1-\alpha,$$

where $0 \leq \alpha < 1$ and $k \geq 0$.

Corollary 2. *Let $0 \leq k < \infty$, $q \in (0,1)$ and $0 \leq \alpha < 1$. If the-inequality*

$$|a_m| \leq \frac{(q+q^{-1})(1-\alpha)}{2q^{-1}(k+1)\left(\widetilde{[m]}_q - 1\right)+(1-\alpha)\left|(q-q^{-1})\widetilde{[m]}_q + 2q^{-1}\right|}, \quad m \geq 2,$$

holds for $g(w) = w + a_m w^m$, then $k - \widetilde{\mathcal{UST}}(q,\alpha)$. In particular:

$$
\begin{aligned}
g(w) \quad &= \quad w + \frac{(q+q^{-1})(1-\alpha)}{2q^{-1}(k+1)\left(\widetilde{[2]}_q - 1\right)+(1-\alpha)\left|(q-q^{-1})\widetilde{[2]}_q + 2q^{-1}\right|}w^2 \\
&\in \quad k - \widetilde{\mathcal{UST}}(q,\alpha),
\end{aligned}
$$

and:

$$|a_2| = \frac{(q+q^{-1})(1-\alpha)}{2q^{-1}(k+1)\left(\widetilde{[2]}_q - 1\right)+(1-\alpha)\left|(q-q^{-1})\widetilde{[2]}_q + 2q^{-1}\right|}.$$

Theorem 2. *Let $k \in [0,\infty)$, $q \in (0,1)$ and $0 \le \alpha < 1$. A necessary and sufficient condition for g of the form (16) to be in the class $k - \widetilde{\mathcal{UST}^{-}}(q,\alpha)$ can be formulated as follows:*

$$\begin{aligned}\sum_{m=2}^{\infty}&\left\{2q^{-1}(k+1)\left(\widetilde{[m]}_q - 1\right) + (1-\alpha)\left|\left(q - q^{-1}\right)\widetilde{[m]}_q + 2q^{-1}\right|\right\}|a_m| \\ \le\ & \left(q + q^{-1}\right)(1-\alpha). \end{aligned} \tag{18}$$

The result is sharp for the function:

$$g(w) = w - \frac{(q+q^{-1})(1-\alpha)}{2q^{-1}(k+1)\left(\widetilde{[m]}_q - 1\right) + (1-\alpha)\left|(q-q^{-1})\widetilde{[m]}_q + 2q^{-1}\right|}w^m.$$

Proof. In view of Theorem 1, it remains to prove the necessity only. If $g \in k - \widetilde{\mathcal{UST}^{-}}(q,\alpha)$, then in virtue of the fact that $|Re(w)| \le |w|$, we have:

$$\begin{aligned}&\left|\frac{1 - \sum_{m=2}^{\infty}\left(\widetilde{[m]}_q - 1\right)a_m w^{m-1}}{1 - \sum_{m=2}^{\infty}\left(\frac{q-q^{-1}}{q+q^{-1}}\right)\left(\widetilde{[m]}_q - 1\right)a_m w^{m-1}} - \alpha\right| \\ \ge\ & k\left|\frac{2q^{-1}}{q+q^{-1}}\left(\frac{\sum_{m=2}^{\infty}\left(\widetilde{[m]}_q - 1\right)a_m w^{m-1}}{1 + \sum_{m=2}^{\infty}\left\{\left(\frac{q-q^{-1}}{q+q^{-1}}\right)\widetilde{[m]}_q + 2q^{-1}\right\}a_m w^{m-1}}\right)\right|. \end{aligned} \tag{19}$$

Letting $w \to 1-$, along the real axis, we obtain the desired inequality (18). □

Corollary 3. *Let the function g of the form (16) be in the class $k - \widetilde{\mathcal{UST}^{-}}(q,\alpha)$. Then:*

$$a_m \le \frac{(q+q^{-1})(1-\alpha)}{2q^{-1}(k+1)\left(\widetilde{[m]}_q - 1\right) + (1-\alpha)\left|(q-q^{-1})\widetilde{[m]}_q + 2q^{-1}\right|}, \quad m \ge 2. \tag{20}$$

Corollary 4. *Let the function g of the form (16) be in the class $k - \widetilde{\mathcal{UST}^{-}}(q,\alpha)$. Then:*

$$a_2 = \frac{(q+q^{-1})(1-\alpha)}{2q^{-1}(k+1)\left(\widetilde{[2]}_q - 1\right) + (1-\alpha)\left|(q-q^{-1})\widetilde{[2]}_q + 2q^{-1}\right|}. \tag{21}$$

Theorem 3. *Let $k \in [0,\infty)$, $q \in (0,1)$, $m \ge 3$ and $0 \le \alpha < 1$. Additionally let $g_1(w) = w$ and*

$$g_m(w) = w - \frac{(q+q^{-1})(1-\alpha)}{\varphi_m(q,q^{-1})}w^m. \tag{22}$$

Then, $g \in k - \widetilde{\mathcal{UST}^{-}}(q,\alpha)$ if and only if g can be expressed as:

$$g(w) = \sum_{m=1}^{\infty}\lambda_m g_m(w), \quad \lambda_m > 0, \quad \text{and} \quad \sum_{m=1}^{\infty}\lambda_m = 1, \tag{23}$$

where $\varphi_m(q,q^{-1})$ is given by (24).

Proof. Suppose that:

$$\begin{aligned}
g(w) &= \sum_{m=1}^{\infty} \lambda_m g_m(w) = \lambda_1 g_1(w) + \sum_{m=2}^{\infty} \lambda_m g_m(w) \\
&= \lambda_1 g_1(w) + \sum_{m=2}^{\infty} \lambda_m - \left\{ w - \frac{(q+q^{-1})(1-\alpha)}{\varphi_m(q,q^{-1})} w^m \right\} \\
&= \lambda_1 w + \sum_{m=2}^{\infty} \lambda_m w - \sum_{m=2}^{\infty} \lambda_m \frac{(q+q^{-1})(1-\alpha)}{\varphi_m(q,q^{-1})} w^m \\
&= \left(\sum_{m=1}^{\infty} \lambda_m \right) w - \sum_{m=2}^{\infty} \lambda_m \frac{(q+q^{-1})(1-\alpha)}{\varphi_m(q,q^{-1})} w^m \\
&= w - \sum_{m=2}^{\infty} \lambda_m \frac{(q+q^{-1})(1-\alpha)}{\varphi_m(q,q^{-1})} w^m .
\end{aligned}$$

Now:

$$\begin{aligned}
&\sum_{m=2}^{\infty} \lambda_m \left(\frac{\{(q+q^{-1})(1-\alpha)\}\varphi_m(q,q^{-1})}{\varphi_m(q,q^{-1})(q+q^{-1})(1-\alpha)} \right) \\
= &\sum_{m=2}^{\infty} \lambda_m = \sum_{m=1}^{\infty} \lambda_m - \lambda_1 = 1 - \lambda_1 \le 1,
\end{aligned}$$

where:

$$\varphi_m\left(q,q^{-1}\right) = 2q^{-1}(k+1)\left(\widetilde{[m]}_q - 1\right) + (1-\alpha)\left|\left(q-q^{-1}\right)\widetilde{[m]}_q + 2q^{-1}\right| \tag{24}$$

and we find $k - \widetilde{\mathcal{UST}}^{-}(q,\alpha)$.

Conversely, assume that $k - \widetilde{\mathcal{UST}}^{-}(q,\alpha)$. Since:

$$|a_m| \le \frac{(q+q^{-1})(1-\alpha)}{\varphi_m(q,q^{-1})},$$

we can set:

$$\lambda_m = \frac{\varphi_m(q,q^{-1})}{(q+q^{-1})(1-\alpha)} |a_m|$$

and:

$$\lambda_1 = 1 - \sum_{m=2}^{\infty} \lambda_m .$$

Then:

$$\begin{aligned}
g(w) &= w + \sum_{m=2}^{\infty} a_m w^m, \\
&= w + \sum_{m=2}^{\infty} \lambda_m \frac{(q+q^{-1})(1-\alpha)}{\varphi_m(q,q^{-1})} w^m, \\
&= w + \sum_{m=2}^{\infty} \lambda_m (w + g_m(w)) = w + \sum_{m=2}^{\infty} \lambda_m w + \sum_{m=2}^{\infty} \lambda_m g_m(w), \\
&= \left(1 - \sum_{m=2}^{\infty} \lambda_m \right) w + \sum_{m=2}^{\infty} \lambda_m g_m(w) = \lambda_1 w + \sum_{m=2}^{\infty} \lambda_m g_m(w), \\
&= \sum_{m=1}^{\infty} \lambda_m g_m(w).
\end{aligned}$$

This completes the proof of Theorem 3. □

Theorem 4. *Let $k \in [0,\infty)$, $q \in (0,1)$, $0 \le \alpha < 1$, and g given by (16) belongs to the class $k - \widetilde{\mathcal{UST}}^{-}(q,\alpha)$. Thus, for $|w| = r < 1$, the following inequality is true:*

$$r - \frac{(q+q^{-1})(1-\alpha)}{\varphi_2(q,q^{-1})}r^2 \le |g(w)| \le r + \frac{(q+q^{-1})(1-\alpha)}{\varphi_2(q,q^{-1})}r^2. \tag{25}$$

The result is sharp for the function:

$$g(w) = w + \frac{(q+q^{-1})(1-\alpha)}{\varphi_2(q,q^{-1})}w^2, \tag{26}$$

where:

$$\varphi_2\left(q,q^{-1}\right) = 2q^{-1}(k+1)\left(\widetilde{[2]}_q - 1\right) + (1-\alpha)\left|\left(q-q^{-1}\right)\widetilde{[2]}_q + 2q^{-1}\right|.$$

Proof. Since $g \in k - \widetilde{\mathcal{UST}}^{-}(q,\alpha)$, in view of Theorem 2, we find:

$$\begin{aligned}\varphi_2\left(q,q^{-1}\right)\sum_{m=2}^{\infty} a_m &\le \sum_{m=2}^{\infty}\varphi_m\left(q,q^{-1}\right)|a_m| \\ &\le \left(q+q^{-1}\right)(1-\alpha).\end{aligned}$$

This gives:

$$\sum_{m=2}^{\infty} a_m \le \frac{(q+q^{-1})(1-\alpha)}{\varphi_2(q,q^{-1})}. \tag{27}$$

Therefore:

$$|g(w)| \le |w| + \sum_{m=2}^{\infty} a_m|w|^m \le r + \frac{(q+q^{-1})(1-\alpha)}{\varphi_2(q,q^{-1})}r^2$$

and:

$$|g(w)| \ge |w| - \sum_{m=2}^{\infty} a_m|w|^m \ge r - \frac{(q+q^{-1})(1-\alpha)}{\varphi_2(q,q^{-1})}r^2.$$

The required results follows by letting $r \to 1-$. □

Theorem 5. *Let $k \in [0,\infty)$, $q \in (0,1)$, and $0 \le \alpha < 1$, and let g given by (16), which belongs to the class $\widetilde{\mathcal{UST}}^{-}(q,\alpha)$. Thus, for $|w| = r < 1$, the following inequality is true:*

$$1 - \frac{2(q+q^{-1})(1-\alpha)}{\varphi_2(q,q^{-1})}r \le \left|g'(w)\right| \le 1 + \frac{2(q+q^{-1})(1-\alpha)}{\varphi_2(q,q^{-1})}r. \tag{28}$$

Proof. By differentiating g and using the triangle inequality, we obtain:

$$\left|g'(w)\right| \le 1 + \sum_{m=2}^{\infty} ma_m|w|^{m-1} \le 1 + r\sum_{m=2}^{\infty} ma_m \tag{29}$$

and:

$$\left|g'(w)\right| \ge 1 - \sum_{m=2}^{\infty} ma_m|w|^{m-1} \ge 1 - r\sum_{m=2}^{\infty} ma_m. \tag{30}$$

Assertion (28) now follows from (29) and (30) in view of the rather simple consequence of (27), given by the inequality:

$$\sum_{m=2}^{\infty} ma_m \le \frac{2(q+q^{-1})(1-\alpha)}{\varphi_2(q,q^{-1})}.$$

Hence we complete the proof of Theorem 5. □

Theorem 6. *The class $k - \widetilde{\mathcal{UST}}^{-}(q,\alpha)$ is closed under convex linear combination.*

Proof. Let the functions $h(w)$ and $g(w)$ be in class $k - \widetilde{\mathcal{UST}}^{-}(q,\alpha)$. Suppose $g(w)$ is given by (16) and:

$$h(w) = w - \sum_{m=2}^{\infty} d_m w^m, \tag{31}$$

where $a_m, d_m \geq 0$.

It is sufficient to prove that, for $0 \leq \lambda \leq 1$, the function:

$$H(w) = \lambda g(w) + (1-\lambda)h(w), \tag{32}$$

is also in the class $k - \widetilde{\mathcal{UST}}^{-}(q,\alpha)$. From (16), (31) and (32), we have:

$$H(w) = w - \sum_{m=2}^{\infty} \{\lambda a_m + (1-\lambda)d_m\} w^m. \tag{33}$$

By using Theorem 2, we obtain:

$$\sum_{m=2}^{\infty} \varphi_m\left(q, q^{-1}\right)\{\lambda a_m + (1-\lambda)d_m\} \leq \left(q + q^{-1}\right)(1-\alpha). \tag{34}$$

By using Theorem 2 and inequality (34), we have $H \in k - \widetilde{\mathcal{UST}}^{-}(q,\alpha)$. This completes the proof of Theorem 6. □

3. Conclusions

In this paper, we used the concept of symmetric quantum calculus and conic regions to define a new domain $\widetilde{\Omega}_{k,q,\alpha}$, which generalizes the symmetric conic domains. Additionally, by using certain generalized symmetric conic domains we defined and investigated a new subclasses $k - \widetilde{\mathcal{UST}}(q,\alpha)$ and $k - \widetilde{\mathcal{UST}}^{-}(q,\alpha)$ of analytic and q-starlike functions in the open unit disk U. We also derived several properties and characteristics of newly defined subclasses of analytic functions such as coefficients estimates, structural formulas, distortion inequalities, necessary and sufficient conditions, closure theorems and subordination results. We have highlighted some consequences of our main results as corollaries.

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