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Article

Ruin Time and Severity for a Lévy Subordinator Claim Process: A Simple Approach

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Abstract: This paper is concerned with an insurance risk model whose claim process is described by a Lévy subordinator process. Lévy-type risk models have been the object of much research in recent years. Our purpose is to present, in the case of a subordinator, a simple and direct method for determining the finite time (and ultimate) ruin probabilities, the distribution of the ruin severity, the reserves prior to ruin, and the Laplace transform of the ruin time. Interestingly, the usual net profit condition will be essentially relaxed. Most results generalize those known for the compound Poisson claim process.

Keywords: Lévy subordinator; time reversal; ruin probability; (in)finite time horizon; ruin severity; reserves prior to ruin; ruin time

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1. Introduction

Let us consider a continuous-time risk model, whose reserves R_t at time t are of the form:

$$R_t = u + ct - S_t, \quad t \geq 0 \tag{1}$$

where u represents the initial reserves, c the (constant) premium rate and S_t the total claim amount over the period $(0, t]$. Ruin of the insurance occurs at the first time, $T(u)$, when the reserves become negative.

In the classical Cramér–Lundberg model, $\{S_t\}$ is modeled by a compound Poisson process. A large literature is devoted to the derivation of the ruin probabilities and other related ruin quantities, for this model and various extensions or modifications. Much can be found, e.g., in the comprehensive books [1–3].

The present paper is concerned with the case where $\{S_t\}$ is a Lévy subordinator without the drift, implying that $\{R_t\}$ is a spectrally negative Lévy process. This model and more general Lévy risk processes have been the object of much research work in recent years. The reader is referred, e.g., to the books [4,5] and the papers [6–13].

Our purpose is to present a direct approach for determining the finite time (and ultimate) ruin probabilities (Section 2), the distribution of the ruin severity and the reserves prior to ruin (Section 3) and the Laplace transform of the ruin time (Section 4). This method relies on simple probabilistic arguments. In particular, we will operate a time reversal that allows us to argue with a dual risk model for which ruin problems are more easily studied. The power of duality is well recognized in ruin theory (see, e.g., [14–16]), as well as for stochastic processes with independent stationary increments (see, e.g., the books [17–19]).

In the sequel, it is assumed that $\mu \equiv E(S_1) < \infty$. Special attention is paid to the standard case where $c > \mu$, that is when the net profit condition holds, so that ultimate ruin is not a.s. Nevertheless, we also examine the case where $c < \mu$, which could arise and be temporarily allowed for certain branches in a large insurance company. The case $c = \mu$, quite different, will not be considered in Sections 3 and 4; its practical relevance, however, is minor.

Most results obtained when $c > \mu$ generalize those known for the compound Poisson claim process (see, e.g., [16,20]). This is in agreement with an observation of [21]. A main interest of our work comes from the direct and systematic study of the model made using simple probabilistic methods. Note that a similar approach could be applied to certain queueing and storage models (as, e.g., in [22]). Of course, other methods can be followed to analyze ruin problems with a subordinator. For instance, one can approximate the subordinator by a sequence of compound Poisson processes (a strategy adopted, e.g., in [8,11]). A powerful alternative is by using the fluctuation theory for Lévy processes (see, e.g., [5]).

In forthcoming work, we will show, for the classical compound Poisson risk model, that different results can be obtained by exploiting the analycity in time of the distribution of S_t. For the ruin probabilities, the formulas are those obtained by [23–25]; see, also, [26].

Throughout the paper, we denote $P_t(A) = P(S_t \in A)$ for any Borel set, A. In particular, $P_t(dy) = P_t((y, y + dy]) = P(y < S_t \leq y + dy)$ and $P_t(a + dy) = P_t((a + y, a + y + dy])$, $a, y \geq 0$. For clarity, the main results will be stated at the beginning of each Section, their proofs being presented afterwards. Some more technical results are also given in an Appendix.

2. Non-Ruin Probabilities

The claim process is a Lévy subordinator, $\{S_t\}$, without the drift and with $S_0 = 0$ (see, e.g., the books [19,27,28]). By the Lévy–Khintchine formula, the Laplace transform of its probability distribution, P_t, is given by:

$$E(e^{-\theta S_t}) = \exp\left(t \int_{]0,\infty)} (e^{-\theta x} - 1)\, \Pi(dx)\right), \quad \theta > 0 \tag{2}$$

where $t\Pi$ is a measure on $]0,\infty)$, called the Lévy measure of P_t. The tail of Π is the function $\bar{\Pi}(x) = \Pi(]x,\infty))$. Therefore, $\bar{\Pi}(\infty) = 0$ and an integration by parts shows that:

$$\int_{]0,\infty)} (e^{-\theta x} - 1)\,\Pi(dx) = -\theta \int_{]0,\infty)} e^{-\theta x}\,\bar{\Pi}(x)dx \tag{3}$$

A basic subordinator is the classical compound Poisson process with parameter λ and positive i.i.d. claim amounts (distributed as X, say); here, $\Pi(x) = \lambda P(X \leq x)$ and $\bar{\Pi}(0) = \lambda$. For the other subordinators, $\bar{\Pi}(0) = \infty$, i.e., the process has infinitely many small jumps. This is the case with the gamma process, the α-stable subordinator and the (generalized) inverse Gaussian process (see the Appendix).

2.1. Results

As a preliminary, consider the case where there are no initial reserves ($u = 0$). Non-ruin probabilities then have a (known) explicit expression.

Proposition 1.

$$P(T(0) > t) = \int_{[0,ct[} \left(1 - \frac{y}{ct}\right) P_t(dy) \tag{4}$$

Formula (4) was first derived by [29] in a special case. Later, it was obtained by [30] (Theorem 2) for a process with non-negative interchangeable increments. It was shown to hold, too, for a general spectrally negative Lévy process (see, e.g., [19] (Corollary 7.3), [31]). For this reason, a proof of Equation (4) will not be included here.

Let us now consider $u \geq 0$. Non-ruin probabilities can then be evaluated by the (known) Formula (5) for a finite horizon and Equation (6) for an infinite horizon.

Proposition 2.

$$P(T(u) > t) = P_t([0, u+ct[) - \int_{]u,u+ct[} P_{\frac{y-u}{c}}(dy) \int_{[0,u+ct-y[} (1 - \frac{z}{u+ct-y})\, P_{\frac{u+ct-y}{c}}(dz) \tag{5}$$

Let $\mu = E(S_1)$ be the expected claim amount per time unit, i.e.:

$$\mu = \int_{]0,\infty]} x\Pi(dx) = \int_{]0,\infty]} \bar{\Pi}(dx)$$

Ultimate ruin is known to arise almost surely when $c \leq \mu$. Suppose now $c > \mu$, and denote $\psi(u) = P(T(u) < \infty)$.

Corollary 3. *If $c > \mu$:*

$$\psi(u) = \left(1 - \frac{\mu}{c}\right) \int_{]u,\infty)} P_{\frac{y-u}{c}}(dy) \tag{6}$$

In particular:

$$\psi(0) = \mu/c. \tag{7}$$

For the compound Poisson model, these formulas can be found in most books on ruin theory (e.g., [2]). As shown by [30] (Theorems 3 and 4), Equation (5) holds, too, for a process with non-negative interchangeable increments and Equation (6) for a Lévy subordinator process. We will rederive Formulas (5) and (6) by arguing through the dual risk model.

It is worth mentioning that an alternative expression for $\psi(u)$ is provided by a Pollaczek–Khintchine-type formula. This result is omitted here for brevity reasons. We refer, e.g., to [30] (Theorem 5) for the case of a Lévy subordinator and to [6,32] for a perturbed subordinator model.

Let $f_{T(u)}$ be the density function of $T(u)$, if it exists.

Proposition 4.

$$\begin{aligned} f_{T(u)}(t) &= \int_{[0,u+ct[} \bar{\Pi}(u+ct-x)P_t(dx) - \int_{[u,u+ct[} P_{\frac{y-u}{c}}(dy) \\ &\quad \int_{[0,u+ct-y[} \bar{\Pi}(u+ct-y-z)\,(1-\frac{z}{u+ct-y})\,P_{\frac{u+ct-y}{c}}(dz). \end{aligned} \tag{8}$$

2.1.1. Special Cases

(i) Consider the Cramér–Lundberg model, i.e., S_t is a compound Poisson process with parameter λ and i.i.d. claim amounts X_i. The case where the X_i's are positive arithmetic random variables was studied in [23] when the cumulated premium income is a linear function, $u+ct$, as here, but also for any non-decreasing deterministic function. Let us rather examine the case where the X_i's are valued in $]0,\infty)$ with density $f(x)$, $x>0$. Therefore, S_t has an atom at state 0, and otherwise, it is continuous with density:

$$f_t(x) = e^{-\lambda t}\sum_{i=1}^{\infty}\frac{(\lambda t)^i}{i!}f^{*i}(x), \quad x>0$$

where f^{*i} is the i-th convolution of f.

Then, Equation (5) yields Seal's relation:

$$\begin{aligned} P(T(u)>t) &= e^{-\lambda t} + \int_0^{u+ct} f_t(y)dy - \int_u^{u+ct} e^{-\lambda(u+ct-y)/c} f_{\frac{y-u}{c}}(y)dy \\ &\quad - \int_u^{u+ct} f_{\frac{y-u}{c}}(y)\,[\int_0^{u+ct-y}\left(1-\frac{z}{u+ct-y}\right) f_{\frac{u+ct-y}{c}}(z)dz]\,dy \end{aligned}$$

and Equation (6) becomes, when $c>\lambda E(X)$:

$$\psi(u) = \left(1-\frac{\lambda E(X)}{c}\right)\int_u^{\infty} f_{\frac{y-u}{c}}(dy).$$

These results are given, e.g., in [33]. For the density, $f_{T(u)}$, an explicit formula is easily written from Equation (8) using $\bar{\Pi}(x)=\lambda P(X>x)$.

(ii) Suppose that the Lévy subordinator is a gamma process with parameters $a,b>0$ (see the Appendix). Then, Equation (6) gives, when $c>a/b$:

$$\begin{aligned} \psi(u) &= \left(1-\frac{a}{bc}\right)\int_u^{\infty}\frac{b^{a(y-u)/c}}{\Gamma(a(y-u)/c)}x^{a(y-u)/c-1}e^{-by}dy \\ &= \left(\frac{c}{a}-\frac{1}{b}\right)e^{-bu}\int_0^{\infty}\frac{(be^{-bc/a})^z(u+cz/a)^{z-1}}{\Gamma(z)}dz. \end{aligned}$$

For a study of this model, we refer, e.g., to [34–36].

(iii) If the Lévy subordinator is an inverse Gaussian process with parameter $b > 0$, then Equation (6) gives, when $c > 1/b$:

$$\psi(u) = \left(1 - \frac{1}{bc}\right) \int_u^\infty \frac{y-u}{c\sqrt{2\pi y^3}}\, e^{-(by-(y-u)/c)^2/2y} dy.$$

Such a model was investigated, e.g., in [11,37].

2.2. Proofs

Non-ruin until time t means, of course, that over the period $[0, t]$, the process, $\{S_\tau\}$, remains below the upper boundary, F, of equation $y = u + c\tau$. It is easily seen that non-crossing through a lower boundary is an easier problem, because crossing here means necessarily meeting. Therefore, we choose to tackle the non-ruin problem by first studying the case of a lower boundary.

2.2.1. Step 1: First-Meeting in a Lower Boundary.

Consider a point, (t, x), with $x < u + ct$ and a trajectory $\tau \to S_\tau$ joining the origin $(0, 0)$ to (t, x). To construct the dual model, we make a rotation of $180°$ with center (t, x) and then reverse the two axes. In these new axes, the corresponding trajectory, $\tau \to \tilde{S}_\tau$, joins $(0, 0)$ to (t, x), and the straight line becomes a lower boundary, $\tilde{F}$, for this trajectory. Clearly, $\tilde{S}_\tau = -(S_{t-\tau} - S_t)$, $0 \le \tau \le t$, and $\tilde{F}$ is of equation $y = x - u - ct + c\tau$. Moreover, the process, $\{\tilde{S}_\tau\}$, over $[0, t]$ is defined exactly as $\{S_\tau\}$ (e.g., [19], p. 43).

Let us now pass to the whole positive quadrant in which $\{\tilde{S}_\tau, \tau \ge 0\}$ is again a Lévy subordinator. Two lower boundaries are considered: $\tilde{F}_t$ defined by $\tilde{F}$ for $\tau \le t$ followed by a vertical line of abscissa t and $\tilde{F}$ defined as before, but on $[0, \infty)$. The first-meeting times of $\{\tilde{S}_\tau\}$ with these boundaries are denoted $\tilde{T}_t$ and $\tilde{T}$, respectively.

We want to determine $\nu_t(dx)$, the probability that $\{\tilde{S}_\tau\}$ meets $\tilde{F}_t$ for the first time at the point (t, x), i.e., $\nu_t(dx) = P(x < \tilde{S}_{\tilde{T}_t} \le x + dx)$. For that, we will have to compute $\nu(dz)$, the probability that $\{\tilde{S}_\tau\}$ meets $\tilde{F}$ for the first time at the level, z, i.e., $\nu(dz) = P(z < \tilde{S}_{\tilde{T}} \le z + dz)$.

The lemma below shows that the probability, ν, can be easily calculated.

Lemma 5. *For $z \ge 0$,*

$$\nu(dz) = (1 - \frac{z}{z - x + u + ct})\, P_{\frac{z-x+u+ct}{c}}(dz) \tag{9}$$

Proof. Consider a first-meeting of $\{\tilde{S}_\tau\}$ with $\tilde{F}$ at some point M of height z and, thus, at time $\tau_z = (z - x + u + ct)/c$; so, $M = (\tau_z, z)$. Returning to the original axes, but with origin M, we observe that $\nu(dz)$ corresponds to the probability $P(T(0) > \tau_z, S_{\tau_z} = z)$. Formula (9) then follows directly from Equation (4). □

Thanks to ν, the probability ν_t can be determined from the result (10) below. Subsequently, a product measure between two measures $\tilde{\nu}$ and ν will be denoted by $\tilde{\nu} \otimes \nu$ (this should not be confused with a convolution product, which has already been used and denoted by *).

Lemma 6. *For* $x \geq 0$*,*

$$\nu_t(dx) = P_t(dx) - \int_{[0,x-u[} P_{\frac{x-u-z}{c}}(dx-z) \otimes \nu(dz) \tag{10}$$

the integral vanishing, of course, for $x \leq u$*.*

Proof. The argument is standard (see e.g., [1]). Any trajectory of $\tau \to \tilde{S}_\tau$ can reach the point (t,x) either without meeting the boundary $\tilde{F}$, or after a first meeting with $\tilde{F}$ at some point of height z, thus at time $(z-x+u+ct)/c$, followed by an increment of $x-z$ during the period $((z-x+u+ct)/c, t)$. Therefore,

$$P_t(dx) = \nu_t(dx) + \int_{[0,x-u[} P_{\frac{x-u-z}{c}}(dx-z) \otimes \nu(dz)$$

hence the Formula (10). □

Note that Equation (10) can be extended to any increasing lower boundary, $\tilde{F}$ (i.e., linearity has no simplifying role at this step). The next lemma is straightforward.

Lemma 7. *For* $x \geq 0$*,*

$$P(\tilde{T}_t = t | \tilde{S}_t = x) = \frac{\nu_t(dx)}{P_t(dx)} \tag{11}$$

Proof. Evidently, $\nu_t(A) \leq P_t(A)$ for all $A \subset]u, \infty)$, which implies that $\nu_t << P_t$. Given any probability measure ρ satisfying $\nu_t << P_t << \rho$ (for instance, $\rho = P_t$), the l.h.s. of Equation (11) corresponds to the quotient $\nu_t'(x)/P_t'(x)$ of the Radon–Nikodym derivatives with respect to ρ, which is equivalent to $\nu_t'(x)\rho(dx)/P_t'(x)\rho(dx) = \nu_t(dx)/P_t(dx)$. □

2.2.2. Step 2: Back to the Ruin Problem

We are now ready to derive the announced formulas for the ruin time distribution.

Lemma 8.

$$P(T(u) > t) = \int_{[0,u+ct[} \nu_t(dx) \tag{12}$$

Proof. Operating the rotation of axes described above, we get:

$$\begin{aligned} P(T(u) > t) &= \int_{[0,u+ct[} P(\tau \to S_\tau \text{ does not cross } F | S_t = x)\, P_t(dx) \\ &= \int_{[0,u+ct[} P(\tau \to \tilde{S}_\tau \text{ does not cross } \tilde{F} | \tilde{S}_t = x)\, P_t(dx) \\ &= \int_{[0,u+ct[} P(\tilde{T}_t = t | \tilde{S}_t = x)\, P_t(dx) \\ &= \int_{[0,u+ct[} \frac{\nu_t(dx)}{P_t(dx)}\, P_t(dx) \end{aligned}$$

using Equation (11), which gives the desired result. □

Proof of Proposition 3. Inserting in Equation (12) Formulas (10) and (9) yields:

$$\begin{aligned} P(T(u)>t) &= P_t([0,u+ct[) - \int_{x\in[0,u+ct[}\int_{z\in[0,x-u[} P_{\frac{x-u-z}{c}}(dx-z)\otimes\nu(dz) \\ &= P_t([0,u+ct[) - \int_{x\in]u,u+ct[}\int_{z\in[0,x-u[}\left(1-\frac{z}{z-x+u+ct}\right) \\ &\qquad P_{\frac{x-u-z}{c}}(dx-z)\otimes P_{\frac{z-x+u+ct}{c}}(dz) \end{aligned} \tag{13}$$

In Equation (13), let us substitute for x a new variable $y = x - z$. The integration is now over the domain $\{u < y < u+ct, 0 \le z < u+ct-y\}$, and applying Fubini's theorem leads to Equation (5). □

Proof of Corollary 4. Formula (5) can be rewritten as:

$$P(T(u)>t) = P_t([0,u+ct[) - \int_{]u,u+ct[} E\left[\left(1-\frac{S_{(u+ct-y)/c}}{u+ct-y}\right) 1_{\{0\le S_{(u+ct-y)/c}<u+ct-y\}}\right] P_{\frac{y-u}{c}}(dy). \tag{14}$$

When $c > \mu$, we have by the strong law of large numbers (SLLN) theorem:

$$\lim_{t\to\infty} P_t([0,u+ct[) = \lim_{t\to\infty} E[1_{\{0\le S_t/t<(u+ct)/t\}}] = 1.$$

Thus, we can write that:

$$\psi(u) = 1 - P(T(u)=\infty) = \lim_{t\to\infty}[P_t([0,u+ct[) - P(T(u)>t)] \tag{15}$$

Moreover, it is proved in Property 18 of the Appendix that, as $c > \mu$:

$$\int_{]u,\infty)} P_{\frac{y-u}{c}}(dy) < \infty \text{ a.e.}$$

From Equations (14) and (15), we then obtain, using the dominated convergence theorem:

$$\begin{aligned} \psi(u) &= \lim_{t\to\infty}\int_{]u,u+ct[} E\left[\left(1-\frac{S_{(u+ct-y)/c}}{u+ct-y}\right)1_{\{0\le S_{(u+ct-y)/c}<u+ct-y\}}\right] P_{\frac{y-u}{c}}(dy) \\ &= \int_{]u,\infty)} E\left[\left(1-\frac{\mu}{c}\right)1_{\{0\le\mu<c\}}\right] P_{\frac{y-u}{c}}(dy) \end{aligned}$$

which is Formula (6).

When $u = 0$, Equation (6) gives:

$$\psi(0) = \left(1-\frac{\mu}{c}\right)\int_{]0,\infty)} P_{\frac{y}{c}}(dy) = \left(1-\frac{\mu}{c}\right)\left[\int_{[0,\infty)} P_{\frac{y}{c}}(dy) - 1\right].$$

From Equation (A9) below, where $\theta_0 = 0$ by Equation (20) and, thus, $l_0 = \mu$ by Equation (45), we get:

$$\int_{[0,\infty)} P_{\frac{y}{c}}(dy) = 1/\left(1-\frac{\mu}{c}\right)$$

hence, Formula (7). □

Lemma 9.

$$f_{T(u)}(t) = \int_{]u,u+ct[} \bar{\Pi}(u+ct-x)\,\nu_t(dx) \tag{16}$$

Proof. By the independence and stationarity of the increments in S_t, we have:

$$\begin{aligned} &P(x < S_t < x + dx, t < T(u) < t + dt) \\ &\quad = P(x < S_t < x + dx, T(u) > t, S_{t+dt} - S_t > u + ct - x) \\ &\quad = P(S_t \in (x, x + dx), T(u) > t) P(S_{dt} > u + ct - x) \\ &\quad = \nu_t(dx) P(S_{dt} > u + ct - x) \end{aligned} \tag{17}$$

as seen in proving Lemma 8. Now, by Property 17 of the Appendix:

$$P(S_{dt} > u + ct - x) = \bar{\Pi}(u + ct - x)dt \tag{18}$$

so that inserting Equation (18) in Equation (17) yields Equation (16). □

Proof of Proposition 4. It suffices to adapt the proof of Proposition 2, starting from Equation (16). □

3. Reserves at and Prior to Ruin

In this Section, we focus on the joint distribution of the reserves at and just prior to ruin (when it occurs) and some of its implications.

Before this, we introduce a useful parameter inside the subordinator model. For any $s > 0$, consider the equation in θ:

$$E(e^{-\theta S_t}) = e^{(s-\theta c)t}.$$

Note that with such a θ, the process $\{\exp(\theta R_t - st)\}$ is a martingale. By Equation (2), the previous equation becomes:

$$\int_{]0,\infty)} (e^{-\theta x} - 1)\, \Pi(dx) = s - c\theta \tag{19}$$

which is referred to as Lundberg's equation (see, e.g., [16]). It is easily checked that Equation (19) admits a unique non-negative root, $s \to \theta(s)$, say. In this section, we will only need the limiting value $\theta_0 = \lim_{s \to 0} \theta(s)$ (but $\theta(s)$ for $s > 0$ will be used in Section 4). From (19), we observe that:

$$\theta_0 = 0 \text{ if } c \geq \mu, \text{ and } \theta_0 > 0 \text{ if } c < \mu \tag{20}$$

3.1. Results

Let R_{T-0} be the reserves just prior to ruin and $|R_T|$ the severity of ruin. Note that $Z = R_{T-0} + |R_T|$ is the claim amount that causes the ruin. Our main goal is to determine the probability:

$$G(u; x, y) = P(T(u) < \infty, R_{T-0} < x, |R_T| < y), \quad x, y > 0.$$

The following function will have a key role. For any real v:

$$\chi(v) = \int_{[0,\infty) \cap]v,\infty)} P_{\frac{y-v}{c}}(dy). \tag{21}$$

Note that the domain of integration in Equation (21) is $]v, \infty)$ if $v \geq 0$, while it is $[0, \infty)$ if $v < 0$. In Property 18, we will show that if $c \neq \mu$, the function, $\chi(v)$, is finite a.e. On the contrary, when $c = \mu$, $\chi(v)$ may be infinite (see the remark in the Appendix).

Proposition 10. *If $c \neq \mu$:*

$$G(u; x, y) = \frac{1}{c} \int_0^x [\bar{\Pi}(\tau) - \bar{\Pi}(\tau + y)] \, [\chi(u - \tau) - e^{-\theta_0 \tau} \chi(u)] d\tau. \tag{22}$$

In particular:

$$G(0; x, y) = \frac{1}{c} \int_0^x [\bar{\Pi}(\tau) - \bar{\Pi}(\tau + y)] \, e^{-\theta_0 \tau} d\tau \tag{23}$$

Corollary 11. *If $c > \mu$:*

$$G(u; x, y) = \frac{1}{c - \mu} \int_0^x [\bar{\Pi}(\tau) - \bar{\Pi}(\tau + y)] \, [\psi(u - \tau) - \psi(u)] d\tau. \tag{24}$$

In particular:

$$G(0; x, \infty) = G(0; \infty, x) = \frac{1}{c} \int_0^x \bar{\Pi}(\tau) d\tau \tag{25}$$

Corollary 12. *If $c \neq \mu$, for $z > 0$:*

$$P(Z < z | T(u) < \infty) = \frac{1}{c\, \psi(u)} \int_0^z [\bar{\Pi}(\tau) - \bar{\Pi}(z)] \, [\chi(u - \tau) - e^{-\theta_0 \tau} \chi(u)] d\tau \tag{26}$$

and if $c > \mu$:

$$P(Z < z | T(u) < \infty) = \frac{1}{(c - \mu) \psi(u)} \int_0^z [\bar{\Pi}(\tau) - \bar{\Pi}(z)] \, [\psi(u - \tau) - \psi(u)] d\tau \tag{27}$$

Corollary 13. *If $c \neq \mu$, for $\alpha \geq 0$ and $\beta > 0$, and provided these moments exist:*

$$E(R_{T-0}^{\alpha} 1_{\{T(u) < \infty\}}) = \frac{1}{c} \int_0^{\infty} x^{\alpha} \bar{\Pi}(x) \, [\chi(u - x) - e^{-\theta_0 x} \chi(u)] dx \tag{28}$$

$$E(R_{T-0}^{\alpha} |R_T|^{\beta} 1_{\{T(u) < \infty\}}) = \frac{\beta}{c} \int_0^{\infty} \int_0^{\infty} x^{\alpha} y^{\beta - 1} \bar{\Pi}(x + y) \, [\chi(u - x) - e^{-\theta_0 x} \chi(u)] dx dy \tag{29}$$

$$E(Z 1_{\{T(u) < \infty\}}) = \frac{1}{c} \int_0^{\infty} [\int_x^{\infty} \tau \Pi(d\tau)] \, [\chi(u - x) - e^{-\theta_0 x} \chi(u)] dx \tag{30}$$

Formula (22) and its corollaries (especially Equation (25)) cover several results obtained in the compound Poisson model, mainly for the case where $c > \mu$ (see, e.g., [38–40]).

3.2. Proofs

The result below is proved by following the same argument as for Lemma 9.

Lemma 14.

$$G(u; x, y) = \int \int_{L(x)} [\bar{\Pi}(u + ct - \xi) - \bar{\Pi}(u + ct - \xi + y)] \, \nu_t(d\xi) \otimes dt \tag{31}$$

where:

$$L(x) = \{(t, \xi) : t > 0, \xi \geq 0, u + ct - x < \xi < u + ct\}$$

Proof. We have, for $y > 0$:

$$\begin{aligned} &P(\xi < S_t < \xi + d\xi, t < T(u) < t + dt, |R_T| < y) \\ &\quad = P(\xi < S_t < \xi + d\xi, T(u) > t)P(u + ct - \xi < S_{t+dt} - S_t < u + ct - \xi + y) \\ &\quad = \nu_t(d\xi) \otimes dt[\bar{\Pi}(u + ct - \xi) - \bar{\Pi}(u + ct - \xi + y)] \end{aligned}$$

by virtue of Equation (A1). Formula (31) then follows. Since $R_{T-0} < x$, $x > 0$, requires $u + cT - x < \xi < u + cT$, the domain of integration is $(t, \xi) \in L(x)$, as stated. □

Proof of Proposition 10. Inserting Equations (9) and (10) inside Equation (31), we can write that:

$$G(u; x, y) = I_1(u; x, y) - I_2(u; x, y) \tag{32}$$

where:

$$I_1(u; x, y) = \int\int_{L(x)} [\bar{\Pi}(u + ct - \xi) - \bar{\Pi}(u + ct - \xi + y)]P_t(d\xi) \otimes dt \tag{33}$$

$$\begin{aligned} I_2(u; x, y) = \int\int\int_{L'(x)} &[\bar{\Pi}(u + ct - \xi) - \bar{\Pi}(u + ct - \xi + y)]\, P_{\frac{\xi-u-z}{c}}(d\xi - z) \\ &\otimes (1 - \frac{z}{z - \xi + u + ct})\, P_{\frac{z-\xi+u+ct}{c}}(dz) \otimes dt \end{aligned} \tag{34}$$

with:

$$L'(x) = \{(t, \xi, z) : t > 0, \xi \geq 0, 0 \leq z < \xi - u, u + ct - x < \xi < u + ct\}.$$

Let us make the change of variable $\tau = u + ct - \xi$ in both integrals.

For Equation (33), we get:

$$I_1(u; x, y) = \int\int_{M(x)} [\bar{\Pi}(\tau) - \bar{\Pi}(\tau + y)]\, P_{\frac{\tau-u+\xi}{c}}(d\xi) \otimes \frac{d\tau}{c}$$

where $M(x) = \{(\tau, \xi) : 0 < \tau < x, \xi \geq 0, u - \tau < \xi\}$ (since $t > 0$ means $\xi + \tau > u$). Thus:

$$\begin{aligned} I_1(u; x, y) &= \frac{1}{c}\int_0^x [\bar{\Pi}(\tau) - \bar{\Pi}(\tau + y)]d\tau \int_{[0,\infty)\cap]u-\tau,\infty)} P_{\frac{\xi-(u-\tau)}{c}}(d\xi) \\ &= \frac{1}{c}\int_0^x [\bar{\Pi}(\tau) - \bar{\Pi}(\tau + y)]\, \chi(u - \tau)d\tau \end{aligned} \tag{35}$$

using the definition (21) of $\chi(v)$ ($<\infty$ a.e. by Equation (A4)).

For Equation (34), we have:

$$I_2(u; x, y) = \int\int\int_{M'(x)} [\bar{\Pi}(\tau) - \bar{\Pi}(\tau + y)]\, P_{\frac{\xi-u-z}{c}}(d\xi - z) \otimes \frac{\tau}{\tau + z}\, P_{\frac{\tau+z}{c}}(dz) \otimes \frac{d\tau}{c}$$

where $M'(x) = \{(\tau, \xi, z) : 0 < \tau < x, \xi \geq 0, 0 \leq z < \xi - u\}$ (since $t > 0$ means $\xi + \tau > u$, which is less restrictive than $\xi > u + z$). Putting $\zeta = \xi - z$, then $M'(x)$ becomes $M''(x) = \{(\tau, \zeta, z) : 0 < \tau < x, \zeta > u, z \geq 0\}$, so that:

$$I_2(u; x, y) = \frac{1}{c}\int_0^x [\bar{\Pi}(\tau) - \bar{\Pi}(\tau + y)]d\tau \int_{[0,\infty)} \frac{\tau}{\tau + z}\, P_{\frac{\tau+z}{c}}(dz) \int_{]u,\infty)} P_{\frac{\zeta-u}{c}}(d\zeta).$$

By Equation (21), the third integral is $\chi(u)$. From Equation (A7) below where $s = 0$, we know that the second integral is equal to $\exp(-\theta_0 \tau)$. Thus:

$$I_2(u; x, y) = \frac{1}{c}\,\chi(u) \int_0^x [\bar{\Pi}(\tau) - \bar{\Pi}(\tau + y)]\, e^{-\theta_0 \tau} d\tau. \tag{36}$$

Inserting Equations (35) and (36) in Equation (32) then yields Equation (22).

Finally, suppose that $u = 0$. In the r.h.s. of Equation (22), the second factor, $[\ldots]$, becomes:

$$\chi(-\tau) - e^{-\theta_0 \tau}\chi(0_+) = \chi(-\tau) + e^{-\theta_0 \tau}[1 - \chi(0_-)]. \tag{37}$$

It will be proven in Equation (A9) (with Equation (45)) that:

$$\chi(v) = e^{\theta_0 v} / [1 - \frac{1}{c}\int_0^\infty x e^{-\theta_0 x} \Pi(dx)] \;\text{ when }\; v < 0. \tag{38}$$

Thus, Equation (37) reduces to $\exp(-\theta_0 \tau)$, which leads to Equation (23). □

Proof of Corollary 11. Since $c > \mu$, we know by Equation (20) that $\theta_0 = 0$. From Equations (6) and (21), we have, as $u \geq 0$:

$$\psi(u) = (1 - \frac{\mu}{c})\,\chi(u). \tag{39}$$

Obviously, an analogous identity Equation (39) holds for $\psi(u - \tau)$ when $u - \tau \geq 0$. This is also true when $u - \tau < 0$, since in that case, $\psi(u - \tau) = 1$ by definition, and from Equation (38) with $\theta_0 = 0$, we see that:

$$\chi(u - \tau) = 1/(1 - \frac{\mu}{c})$$

as well. Formula (24) then follows, with Equation (25) as a direct consequence. □

Proof of Corollary 12. By Equation (22), we can write:

$$\begin{aligned} P(R_{T-0} < x, |R_T| < y | T(u) < \infty) &= \frac{G(u; x, y)}{P(T(u) < \infty)} \equiv \int_0^x g(\tau, y) d\tau \\ &= \int_0^x \frac{g(\tau, y)}{g(\tau, \infty)}\, g(\tau, \infty) d\tau \end{aligned} \tag{40}$$

where a.e.:

$$g(\tau, y) = \frac{1}{c\,\psi(u)}[\bar{\Pi}(\tau) - \bar{\Pi}(\tau + y)]\,[\chi(u - \tau) - e^{-\theta_0 \tau}\chi(u)] \tag{41}$$

In particular:

$$P(R_{T-0} < x | T(u) < \infty) = \int_0^x g(\tau, \infty) d\tau \tag{42}$$

which shows that the function, $\tau \to g(\tau, \infty)$, is a conditional density for R_{T-0}. Thus, we see from Equation (40) that the function, $\tau \to g(\tau, y)/g(\tau, \infty)$, is a version of the conditional probability $P(|R_T| < y | T(u) < \infty, R_{T-0} = \tau)$. As a consequence:

$$\begin{aligned} P(Z < z | T(u) < \infty, R_{T-0} = \tau) &= P(|R_T| < z - \tau | T(u) < \infty, R_{T-0} = \tau) \\ &= \frac{g(\tau, z - \tau)}{g(\tau, \infty)}. \end{aligned} \tag{43}$$

From Equations (42) and (43), we then obtain:

$$\begin{aligned} P(Z<z|T(u)<\infty) &= \int_0^z P(Z<z|T(u)<\infty, R_{T-0}=\tau)g(\tau,\infty)d\tau \\ &= \int_0^z g(\tau, z-\tau)d\tau \end{aligned}$$

where $g(\tau, z-\tau)$ is given by Equation (41) for $y = z-\tau$. This yields Formulas (26) and (27) (using Equation (6)). □

Proof of Corollary 13. By Equation (22):

$$\begin{aligned} E(R_{T-0}^{\alpha}|R_T|^{\beta}1_{\{T(u)<\infty\}}) &= \int_0^\infty\int_0^\infty x^\alpha y^\beta d_{x,y}G(u;x,y) \\ &= \frac{1}{c}\int_0^\infty x^\alpha[\chi(u-x)-e^{-\theta_0 x}\chi(u)][\int_0^\infty y^\beta(-1)d_y\bar{\Pi}(x+y)]\,dx. \end{aligned}$$

An integration by parts of the inner integral gives:

$$[-y^\beta\bar{\Pi}(x+y)]_0^\infty + \int_0^\infty \bar{\Pi}(x+y)d(y^\beta)$$

which becomes:

$$\bar{\Pi}(x) \text{ if } \beta=0, \text{ and } \int_0^\infty \beta y^{\beta-1}\bar{\Pi}(x+y)dy \text{ if } \beta>0.$$

This yields the desired Formulas (28) and (29).

For Equation (30), a similar argument leads to:

$$E(Z1_{\{T(u)<\infty\}}) = \frac{1}{c}\int_0^\infty [x\bar{\Pi}(x)+\int_0^\infty \bar{\Pi}(x+y)dy][\chi(u-x)-e^{-\theta_0 x}\chi(u)]dx.$$

Noting that the first factor, $[\ldots]$, is equal to:

$$x\bar{\Pi}(x)+\int_x^\infty \bar{\Pi}(\tau)d\tau = \int_x^\infty \tau\Pi(d\tau)$$

we deduce the announced formula. □

4. Ruin Time

In this Section, our main goal is to determine the expectation:

$$H(u;s,x,y) = E[e^{-sT(u)}\,1_{\{R_{T-0}<x, |R_T|<y\}}], \quad s,x,y>0.$$

For that, we will use the non-negative root, $\theta(s)$, of Equation (19). Note that by Equation (3), $\theta(s)$ for $s>0$ satisfies the relation:

$$\int_{]0,\infty)} e^{-\theta(s)x}\,\bar{\Pi}(x)dx = c - \frac{s}{\theta(s)}. \tag{44}$$

Moreover, differentiating Equation (3) with respect to s gives:

$$-\theta'(s)\int_{]0,\infty)} xe^{-\theta(s)x}\,\Pi(dx) = 1 - c\theta'(s)$$

so that:

$$\theta'(0) = \frac{1}{c - l_0} \quad \text{and} \quad \theta''(0) = \frac{-k_0}{(c - l_0)^3} \tag{45}$$

where:

$$l_0 = \int_0^\infty x e^{-\theta_0 x} \Pi(dx) \quad \text{and} \quad k_0 = \int_0^\infty x^2 e^{-\theta_0 x} \Pi(dx).$$

In particular, $l_0 = \mu$ and $k_0 = E(S_1^2)$ when $\theta_0 = 0$.

4.1. Results

To begin with, we introduce a function that generalizes $\chi(v)$ defined in Equation (21): for any $s \geq 0$ and real v, let:

$$\chi(s, v) = \int_{[0,\infty)\cap]v,\infty)} e^{-s\frac{y-v}{c}} P_{\frac{y-v}{c}}(dy). \tag{46}$$

Therefore, $\chi(0, v) = \chi(v)$.

Proposition 15. *If $c \neq \mu$:*

$$H(u; s, x, y) = \frac{1}{c} \int_0^x [\bar{\Pi}(\tau) - \bar{\Pi}(\tau + y)] \left[\chi(s, u - \tau) - e^{-\theta(s)\tau} \chi(s, u)\right] d\tau. \tag{47}$$

In particular:

$$H(0; s, x, y) = \frac{1}{c} \int_0^x [\bar{\Pi}(\tau) - \bar{\Pi}(\tau + y)]\, e^{-\theta(s)\tau} d\tau \tag{48}$$

$$E(e^{-sT(u)}) = \frac{1}{c} \int_0^\infty \bar{\Pi}(\tau) \chi(s, u - \tau) d\tau + \left(\frac{s}{c\theta(s)} - 1\right) \chi(s, u). \tag{49}$$

Alternatively:

$$E(e^{-sT(u)}) = 1 - \frac{s}{c} \int_0^\infty \chi(s, u - \tau) d\tau + \frac{s}{c\theta(s)} \chi(s, u) \tag{50}$$

Corollary 16. *If $c \neq \mu$:*

$$E(T(u) 1_{\{T(u)<\infty, X<x, Y<y\}}) = -\frac{1}{c} \int_0^x [\bar{\Pi}(\tau) - \bar{\Pi}(\tau + y)]\, \chi'_s(s, u - \tau) d\tau$$
$$+ \frac{\chi'_s(0, u)}{c} \int_0^x [\bar{\Pi}(\tau) - \bar{\Pi}(\tau + y)] e^{-\theta_0 \tau} d\tau - \frac{\chi(u)}{c(c - l_0)} \int_0^x \tau [\bar{\Pi}(\tau) - \bar{\Pi}(\tau + y)] e^{-\theta_0 \tau} d\tau. \tag{51}$$

In particular, when $c < \mu$:

$$E(T(u)) = \frac{1}{c} \int_0^u \chi(\tau) d\tau - \frac{1}{c\theta_0} \chi(u) + \frac{1}{\theta_0(c - l_0)} \tag{52}$$

and when $c > \mu$:

$$E(T(u) 1_{\{T(u)<\infty\}}) = \frac{1}{c - \mu} \int_0^u \psi(\tau) d\tau - \left(1 - \frac{\mu}{c}\right) \chi'_s(0, u) - \frac{k_0(1 + \psi(u))}{2(c - \mu)^2} \tag{53}$$

Formula (47) and its corollaries cover several results known for the compound Poisson case when $c > \mu$ (see, e.g., [16,20]). Clearly, a similar approach would allow us to determine the expected discounted penalty function introduced in [16]. For a study of this function in a Lévy framework, see, e.g., [6–8,10].

4.2. Proofs

Proof of Proposition 15. Starting with Equation (31), we have:

$$H(u;s,x,y) = \int\int_{L(x)} e^{-st}\,[\bar{\Pi}(u+ct-\xi) - \bar{\Pi}(u+ct-\xi+y)]\,\nu_t(d\xi)\otimes dt. \tag{54}$$

We then proceed as with Proposition 10. Therefore, instead of Equation (32), we rewrite Equation (54) as:

$$H(u;s,x,y) = J_1(u;s,x,y) - J_2(u;s,x,y)$$

where J_1 and J_2 are defined by Equations (33) and (34) with the additional factor, $\exp(-st)$, in the integrals. After the same changes of variable as before, we get:

$$J_1(u;s,x,y) = \frac{1}{c}\int_0^x [\bar{\Pi}(\tau) - \bar{\Pi}(\tau+y)]d\tau \int_{[0,\infty)\cap]u-\tau,\infty)} e^{-s\frac{\xi-(u-\tau)}{c}}\, P_{\frac{\xi-(u-\tau)}{c}}(d\xi)$$

instead of Equation (35) and:

$$J_2(u;s,x,y) = \frac{1}{c}\int_0^x [\bar{\Pi}(\tau) - \bar{\Pi}(\tau+y)]d\tau \int_{[0,\infty)} e^{-s\frac{\tau+z}{c}}\,\frac{\tau}{\tau+z}\,P_{\frac{\tau+z}{c}}(dz)\int_{]u,\infty)} e^{-s\frac{\zeta-u}{c}}\,P_{\frac{\zeta-u}{c}}(d\zeta)$$

instead of Equation (36). It is proved in Equation (A7) of the Appendix that the second integral inside J_2 is equal to $\exp(-\theta(s)\tau)$. Using the notation Equation (46), we then deduce Formula (47).

When $u = 0$ in Equation (47), the second factor, $[\ldots]$, reduces to:

$$\chi(s,-\tau) - e^{-\theta(s)\tau}\chi(s,0_+) = \chi(s,-\tau) + e^{-\theta(s)\tau}[1-\chi(s,0_-)], \tag{55}$$

a result that generalizes Equation (37). In Equation (A8) below, we will establish that:

$$\chi(s,v) = c\theta'(s)e^{\theta(s)v} \quad \text{when } v<0. \tag{56}$$

Thus, Equation (55) reduces to $\exp(-\theta(s)\tau)$, and Formula (48) follows.

For Equation (49), putting $x = y = \infty$ in Equation (47) gives:

$$E(e^{-sT(u)}) = \frac{1}{c}\int_0^\infty \bar{\Pi}(\tau)\chi(s,u-\tau) - \frac{1}{c}\chi(s,u)\int_0^\infty \bar{\Pi}(\tau)e^{-\theta(s)\tau}d\tau.$$

It then remains to apply the identity Equation (44) to the second integral.

Formula (50) is obtained by a different method. Clearly:

$$E(e^{-sT(u)}\,1_{\{T(u)<\infty\}}) = 1 - s\int_0^\infty e^{-st}P(T(u)>t)dt.$$

Using (12) to express $P(T(u)>t)$, we then get:

$$\frac{1}{s}[1 - E(e^{-sT(u)}\,1_{\{T(u)<\infty\}})] = \int_0^\infty e^{-st}\int_{[0,u+ct[} \nu_t(dx). \tag{57}$$

By comparison with Equation (54), we note that the r.h.s. of Equation (57) corresponds to the r.h.s. of Equation (54) in which the factor, $[\ldots]$, is deleted and $x = y = \infty$. By arguing as from Equation (54) to Equation (47), we rewrite the double integral in (57) as:

$$\frac{1}{c}\int_0^\infty \chi(s,u-\tau)d\tau - \frac{\chi(s,u)}{c}\int_0^\infty e^{-\theta(s)\tau}d\tau$$

hence, the announced formula. □

Proof of Corollary 16. Let us differentiate Equation (47) with respect to s and put $s = 0$. Using the value of $\theta'(0)$ given in Equation (45), we then obtain Formula (51).

For Equations (52) and (53), we start with Formula (50). Note that:

$$\int_0^\infty \chi(s, u-\tau)d\tau = \int_0^u \chi(s, u-\tau)d\tau + \int_{-\infty}^0 \chi(s,\tau)d\tau$$

and from Equation (56):

$$\int_{-\infty}^0 \chi(s,\tau)d\tau = \frac{c\theta'(s)}{\theta(s)}$$

so that:

$$E(e^{-sT(u)}) = 1 - \frac{s}{c}\int_0^u \chi(s, u-\tau)d\tau - \frac{s\theta'(s)}{\theta(s)} + \frac{s}{c\theta(s)}\chi(s,u).$$

By differentiation and L'Hospital's rule as $s \to 0$, we then deduce Equation (52) when $c < \mu$, i.e., $\theta_0 > 0$, and Equation (53) when $c > \mu$, i.e., $\theta_0 = 0$, using θ_0' and θ_0'' given by Equation (45). □

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Conflicts of Interest

The authors declare no conflict of interest.

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Appendix

A. Special Cases

It may be worth recalling a few (standard) examples of a Lévy subordinator that could be used to represent the claim process, $\{S_t\}$ (see, e.g., [19]).

A.1. Compound Poisson Process

For such a process with Poisson parameter λ and claim amounts distributed as X with distribution function F, then:

$$\Pi(dx) = \lambda F(dx), \text{ with } \mu = \lambda E(X)$$

A.2. Gamma Process

For such a process with parameters $a, b > 0$, then:

$$\begin{aligned}
&\Pi(dx) = ax^{-1}e^{-bx}dx \\
&P_t(dx) = \frac{b^{at}}{\Gamma(at)}x^{at-1}e^{-bx}dx \\
&E(e^{-\theta S_t}) = (1+\frac{\theta}{b})^{-ta}, \text{ with } \mu = a/b
\end{aligned}$$

A.3. α-Stable Subordinator

For such a process with parameter $\alpha \in (0,1)$, then:

$$\Pi(dx) = \frac{\alpha}{\Gamma(1-\alpha)} x^{-1-\alpha} dx$$
$$E(e^{-\theta S_t}) = e^{-t\theta^{\alpha}}, \text{ with } \mu = \infty.$$

The expected claim being infinite, this model is of less interest in practice.

A.4. Inverse Gaussian Process

For such a process with parameter $b > 0$, then:

$$\Pi(dx) = \frac{t}{\sqrt{2\pi x^3}} e^{-b^2 x/2} dx$$
$$P_t(dx) = \frac{t}{\sqrt{2\pi x^3}} e^{-(bx-t)^2/2x} dx$$
$$E(e^{-\theta S_t}) = e^{-tb(\sqrt{1+2\theta/b^2}-1)}, \text{ with } \mu = 1/b.$$

A possible extension is the generalized inverse Gaussian process.

B. Useful Properties

The following property is used in the proof of Lemmas 9 and 14. It is surely known, but we have no precise reference to give. A proof is presented below for completeness. Note that the result is easy to see in the compound Poisson case.

Property 17.

$$\lim_{t \downarrow 0} \frac{P(S_t > y)}{t} = \bar{\Pi}(y) \ a.e., \quad y > 0 \tag{A1}$$

Proof. Define σ_t as a compound Poisson measure with parameter $1/t$ and jump distribution P_t (so, $P(S_t > y) = P_t(]y, \infty))$). For $\theta > 0$, let $\hat{\sigma}_t(\theta)$ and $\hat{P}_t(\theta)$ denote the associated Laplace transforms. Observing that:

$$\hat{\sigma}_t(\theta) = \exp\left(\frac{\hat{P}_t(\theta) - 1}{t}\right) = \exp\left(\int_{]0,\infty)} (e^{-\theta x} - 1) \frac{P_t(dx)}{t}\right) \tag{A2}$$

we see that the Lévy measure of σ_t is given by P_t/t.

Let $t \downarrow 0$ in Equation (A2). Using $\hat{P}_t = (\hat{P}_1)^t$, we get:

$$\lim_{t \downarrow 0} \hat{\sigma}_t(\theta) = \lim_{t \downarrow 0} \exp\left(\frac{e^{t \ln \hat{P}_1(\theta)} - 1}{t}\right) = \exp(\ln \hat{P}_1(\theta)) = \hat{P}_1(\theta).$$

By Lévy's continuity theorem, this means that $\lim_{t \downarrow 0} \sigma_t = P_1$. Thus, an analogous result holds for the corresponding Lévy measures (see, e.g., Theorem 8.7 in [27]), that is:

$$\lim_{t \downarrow 0} \frac{P_t}{t} = \Pi.$$

Equivalently:

$$\lim_{t\downarrow 0}\int_{]0,\infty)} f(x)\frac{P_t(dx)}{t} = \int_{]0,\infty)} f(x)\Pi(dx) \tag{A3}$$

for every bounded continuous function, f.

Now, let $y > 0$. For any $\epsilon > 0$, we may consider the function:

$$f_{y,\epsilon}(x) = \begin{cases} 0 & \text{if } x < y \\ (x-y)/\epsilon & \text{if } y \le x \le y+\epsilon \\ 1 & \text{if } x > y+\epsilon \end{cases}$$

Evidently:

$$f_{y,\epsilon}(x) \le 1_{]y,\infty)}(x) \le f_{y-\epsilon,\epsilon}(x)$$

so that after multiplication by P_t/t and integration over $]0,\infty)$, Equation (A3) yields:

$$\int_{]0,\infty)} f_{y,\epsilon}(x)\Pi(dx) \le \liminf_{t\downarrow 0}\frac{P_t(]y,\infty))}{t} \le \limsup_{t\downarrow 0}\frac{P_t(]y,\infty))}{t} \le \int_{]0,\infty)} f_{y-\epsilon,\epsilon}(x)\Pi(dx) \text{ a.e.}$$

Moreover:

$$\begin{aligned} \bar{\Pi}(y+\epsilon) &= \int_{]0,\infty)} 1_{]y+\epsilon,\infty)}(x)\Pi(dx) \le \int_{]0,\infty)} f_{y,\epsilon}(x)\Pi(dx) \\ \bar{\Pi}(y-\epsilon) &= \int_{]0,\infty)} 1_{]y-\epsilon,\infty)}(x)\Pi(dx) \ge \int_{]0,\infty)} f_{y-\epsilon,\epsilon}(x)\Pi(dx). \end{aligned}$$

Therefore, as $\epsilon \downarrow 0$:

$$\bar{\Pi}(y+0) \le \liminf_{t\downarrow 0}\frac{P_t(]y,\infty))}{t} \le \limsup_{t\downarrow 0}\frac{P_t(]y,\infty))}{t} \le \bar{\Pi}(y-0) \text{ a.e.}$$

As the non-increasing function, Π, is continuous, except perhaps on a countable set, we deduce that $\lim_{t\downarrow 0} P_t(]y,\infty))/t = \bar{\Pi}(y)$ a.e. □

The next property is used when proving Corollary 3 and Proposition 10.

Property 18. *If $c \neq \mu$, then for any $u \ge 0$:*

$$\chi(u) \equiv \int_{]u,\infty)} P_{\frac{y-u}{c}}(dy) < \infty \text{ a.e.} \tag{A4}$$

Proof. First, suppose $c > \mu$. By Equation (5) rewritten as Equation (14), we have:

$$P(S_t < u+ct) - P(T(u) > t) = \int_{]u,u+ct[} E\left[\left(1-\frac{S_\tau}{c\tau}\right) 1_{\{0\le S_\tau < c\tau\}}\right] P_{\frac{y-u}{c}}(dy)$$

Evidently, this difference of probabilities is in $[0,1]$. Thus, for any positive real n_0, such that $ct > n_0$:

$$0 \le \int_{]u,u+n_0[} E\left[\left(1-\frac{S_\tau}{c\tau}\right) 1_{\{0\le S_\tau < c\tau\}}\right] P_{\frac{y-u}{c}}(dy) \le 1.$$

Letting $t \to \infty$, we get by the SLLN and the dominated convergence theorem:

$$0 \le \int_{]u,u+n_0[} E\left[\left(1-\frac{\mu}{c}\right) 1_{\{0\le \mu < c\}}\right] P_{\frac{y-u}{c}}(dy) \le 1.$$

As this holds for any n_0, we deduce that Equation (A4) is true in that case.

Now, consider $c < \mu$. It is sufficient to establish that the Laplace transform of χ, i.e.:

$$L_\theta(\chi) = \int_{[0,\infty)} e^{-\theta u}\chi(u)du = \int\int_{\{(y,u):0\le u<y\}} e^{-\theta u} P_{\frac{y-u}{c}}(dy)\otimes du$$

is finite for some $\theta > 0$. Putting $z = y - u$, we have:

$$L_\theta(\chi) = \int_{]0,\infty)} e^{\theta z}\left[\int_{[0,\infty)} e^{-\theta y} P_{\frac{z}{c}}(dy)\right] dz$$

and using Equation (2):

$$L_\theta(\chi) = \int_{]0,\infty)} \exp\left\{\theta z + \frac{z}{c}\int_0^\infty (e^{-\theta x} - 1)\Pi(dx)\right\} dz. \tag{A5}$$

To get $L_\theta(\chi) < \infty$, it suffices to choose θ, such that the term, $\{\ldots\}$, in Equation (A5) is negative, i.e.:

$$\int_0^\infty (e^{-\theta x} - 1)\Pi(dx) < -c\theta$$

By Equation (20), $c < \mu$ implies $\theta_0 > 0$ and from Equation (19), we see that any θ in $(0, \theta_0)$ will guarantee the validity of this condition. □

Remark. Property 18 is not true when $c = \mu$. To show this, consider, for instance, the compound Poisson model with parameter λ and i.i.d. claim amounts X_i of the exponential law with parameter $\alpha = 1/E(X)$. Of course, $\mu = \lambda/\alpha$. By definition:

$$\chi(u) = \sum_{n=1}^\infty \int_{]u,\infty)} e^{-\lambda(y-u)/c}\frac{[\lambda(y-u)/c]^n}{n!} e^{-\alpha y}\frac{\alpha^n y^{n-1}}{(n-1)!}dy + \int_{]u,\infty)} e^{-\lambda(y-u)/c}\delta_0(dy)$$

where δ_0 is the Dirac measure at zero. It is easily checked that $\chi(u) < \infty$ iff $M(u) < \infty$, where:

$$M(u) = \sum_{n=1}^\infty \int_{]0,\infty)} e^{-\lambda y/c}\frac{(\lambda y/c)^n}{n!} e^{-\alpha y}\frac{\alpha^n y^{n-1}}{(n-1)!}dy.$$

Now, we have:

$$M(u) = \sum_{n=1}^\infty \frac{(\lambda\alpha/c)^n}{n!(n-1)!}\int_{]0,\infty)} e^{-(\alpha+\lambda/c)y}y^{2n-1}dy = \sum_{n=1}^\infty \frac{(\lambda\alpha/c)^n}{n!(n-1)!}\frac{(2n-1)!}{(\alpha+\lambda/c)^{2n}}.$$

By Stirling's formula:

$$\frac{(2n-1)!}{n!(n-1)!} \approx \frac{\sqrt{2\pi(2n-1)}}{\sqrt{2\pi n}\sqrt{2\pi(n-1)}}\frac{e^{-(2n-1)}(2n-1)^{2n-1}}{e^{-n}n^n e^{-(n-1)}(n-1)^{n-1}} \approx \frac{1}{\sqrt{\pi n}}2^{2n-1}$$

so that:

$$M(u) \approx \sum_{n=1}^\infty \left[\frac{4\lambda\alpha/c}{(\alpha+\lambda/c)^2}\right]^n \frac{1}{2\sqrt{\pi n}}. \tag{A6}$$

Denote by $h(\alpha)$ the factor, $[\ldots]$, in the r.h.s. of Equation (A6). We see that $h(\alpha)$ has a maximum at $\alpha = \lambda/c$, with $h(\lambda/c) = 1$. Consequently, the series, $M(u)$, and, thus, $\chi(u)$, diverges when $c = \lambda/\alpha$. Note that when $c \neq \lambda/\alpha$, then $h(\alpha) < 1$, so that $\chi(u) < \infty$, which is in agreement with Equation (A4).

The following identities involve $\theta(s)$, the non-negative root of Equation (19). They are used several times in Sections 2–4.

Property 19. *For* $s, u \geq 0$,

$$\int_{[0,\infty)} e^{-s\frac{u+z}{c}} \frac{u}{u+z} P_{\frac{u+z}{c}}(dz) = e^{-\theta(s)u}. \tag{A7}$$

As a consequence:

$$\chi(s, -u) = \int_{[0,\infty)} e^{-s\frac{u+z}{c}} P_{\frac{u+z}{c}}(dz) = c\theta'(s)e^{-\theta(s)u} \tag{A8}$$

and in particular:

$$\chi(-u) = e^{-\theta_0 u}/(1 - l_0/c) \tag{A9}$$

Proof. Let us go back to the dual model studied in Step 1 of Section 2. Take $x = ct$ (which is allowed). For the first-meeting level, $\tilde{S}_{\tilde{T}}$, Equation (9) then gives:

$$\nu(dz) = \frac{u}{u+z} P_{\frac{u+z}{c}}(dz).$$

Thus, the l.h.s. of Equation (A7) can be written as:

$$\int_{[0,\infty)} e^{-s\frac{u+z}{c}} \nu(dz) = E(e^{-s\frac{u+\tilde{S}_{\tilde{T}}}{c}} 1_{\{\tilde{S}_{\tilde{T}}<\infty\}}) = E(e^{-s\frac{u+\tilde{S}_{\tilde{T}}}{c}}).$$

Moreover, by construction, $\tilde{S}_{\tilde{T}} = -u + c\tilde{T}$, so that:

$$\int_{[0,\infty)} e^{-s\frac{u+z}{c}} \frac{u}{u+z} P_{\frac{u+z}{c}}(dz) = E(e^{-s\tilde{T}}). \tag{A10}$$

Now, for $\theta(s)$ defined through Equation (19), we know that the process, $\{\exp[\theta(s)(ct - \tilde{S}_t) - st]\}$, is a martingale. Applying the optional stopping theorem at time $\tilde{T}$ then gives:

$$E[e^{\theta(s)(c\tilde{T} - \tilde{S}_{\tilde{T}}) - s\tilde{T}}] = 1$$

which shows that:

$$E(e^{-s\tilde{T}}) = e^{-\theta(s)u}. \tag{A11}$$

Combining Equations (A10) and (A11) then yields Formula (A7).

By differentiating with respect to s, we have:

$$\int_{[0,\infty)} e^{-s\frac{u+z}{c}} \frac{u}{c} P_{\frac{u+z}{c}}(dz) = -\theta'(s)ue^{-\theta(s)u}$$

and Formula (A8) follows. For $s = 0$, Equation (A8) becomes Equation (A9) after using Equation (45). □