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Optimal Investment Strategy for DC Pension Schemes under Partial Information

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Abstract: We consider a defined-contribution (DC)-pension-fund-management problem under partial information. The fund manager is allowed to invest the wealth from the fund account into a financial market consisting of a risk-free account, a stock and a rolling bond. The aim of the fund manager is to maximize the expected utility of the terminal wealth. In contrast to the traditional literature, we assume that the fund manager can only observe the stock-price process and the interest-rate process, but the expected return rate of the stock is unobservable, following a mean-reverting stochastic process. We apply a martingale approach and Clark's formula to solve this problem and the closed-form representations for the optimal terminal wealth and trading strategy are derived. We further present the results for the constant relative risk aversion (CRRA) function as a special case.

Keywords: defined-contribution plan; stochastic interest rate; partial information; full information; trading strategy; Clark's formula

JEL Classification: G22; C61

AMS Classification: Primary: 91G05; 93E20; 93E11; Secondary: 49J40; 60G15

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1. Introduction

With increase in the human lifespan, the proportion of retired people in the total population has increased, leading to greater economic pressures and many social issues. To deal with this problem, researchers have analyzed the structure of the pension fund system and designed new types of pension schemes based on existing pension fund models.

There are two types of pension funds: defined contribution (DC) pension plans and defined benefit (DB) pension plans. In DC pension plans, the contributions are fixed, usually as a constant or a fixed proportion of the plan participants' salary income, and benefits depend on the returns of the fund portfolio, so the participants bear the investment risk. In DB pension plans, the benefits at the moment of retirement are determined in advance, while contributions need to be adjusted at any time to maintain the balance of the pension. DC pension plans are currently becoming increasingly popular with insurance companies as they do not bear the total risk.

Previously, researchers tended to focus on DB pension plans. However, DB pension plans transfer all the risks to the sponsor and might cause bankruptcy. Thus, DB pension plans are gradually switching to DC pension plans. There has been a number of studies dealing with DC pension plans. For example, Boulier et al. (2001) considered a DC plan with a guarantee on the final benefits. The authors of Deelstra et al. (2003) extended the work of Boulier et al. (2001) by considering the effects of the stochastic interest rate. In Dong and Zheng (2019), an S-shaped utility maximization for DC pension funds under short-selling constraints was described. In Guan and Liang (2015), an optimization problem for DC pension plans under stochastic interest rate models was investigated. See also Baltas et al. (2022), Tang et al. (2018), Zhang and Ewald (2010) as illustrative examples.

In practice, investors might not be certain about the dynamics of the prices of risky assets in the financial market. Even though the volatility term might be estimated very well, estimation of the expected return is particularly difficult (see, for example, Section 4.2 in Rogers (2013)). There has been considerable research into portfolio optimization problems under unobservable or partial information. For example, Björk et al. (2010) provided optimal solutions for a terminal wealth and portfolio strategy under partial information. Similar problems were investigated using a martingale dual approach by Lakner (1995, 1998). The authors of Wang et al. (2021) explored the optimal investment strategy under the mean-variance criterion when the drift term in the stock price is unobservable. See, also, Liang and Song (2015), Mania and Santacroce (2010), Pham and Quenez (2001) and Xiong et al. (2021).

In our model, the fund manager invests the wealth into a financial market with a stock asset, a risk-free asset and a rolling bond, where the drift process $\mu(t)$ of the stock price is stochastic and unobservable. We assume the interest rate in our model is stochastic, satisfying the Vasiček model. The fund manager aims to maximize the expected utility of the terminal wealth in the fund account. To avoid large losses arising from financial risks to the participants, we consider a constraint on the terminal wealth, which is called the "minimum guarantee". We apply a martingale dual approach and filtering technique to solve this problem and closed-form representations for the optimal terminal wealth and trading strategy are derived. We further present the results for the CRRA function as a special case.

The paper is organized as follows: In Section 2, we describe the portfolio model in a financial market with three assets: a cash, a stock and a rolling bond. In Section 3, we formulate the optimal investment problem for DC pension schemes with a minimum guarantee. In Section 4, we first transform the original constrained, non-self-financing optimal investment problem to an equivalent unconstrained, self-financing optimal investment problem, then use a filtering technique to describe the partial observable model within a complete observable framework. By calculating an explicit representation of the optional projection ζ_t, and using Clark's formula to solve (39) in Theorem 1, we finally obtain the closed-form expressions for the optimal trading strategies. In Section 5, we summarize the paper.

2. The Financial Market

We consider the financial market in which there is no arbitrage and which is frictionless and continuously open. We also assume that the transaction amounts are small and have no influence on the prices. The model considered in this paper is defined in a complete probability space $\left(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$, where $\mathcal{F}_t$ denotes all the information in the financial market up to the moment t. We also assume that $\mathbb{F}$ is right continuous and $\mathbb{P}$-complete.

The market is composed of a risk-free asset (a cash), a stock and a rolling bond. We allow the fund manager to invest the wealth in the pension account to the above three financial assets. The price of the riskless asset (also called the cash or the bank account) is given by

$$\frac{\mathrm{d}S_0(t)}{S_0(t)} = r(t)\mathrm{d}t, \qquad S_0(0) = 1, \tag{1}$$

where $r(t)$ is the short-rate process. The evolution of $r(t)$ satisfies the following Vasiček model:

$$\mathrm{d}r(t) = a(b - r(t))\mathrm{d}t - \sigma_r \mathrm{d}W_r(t), \qquad r(0) = r_0. \tag{2}$$

We see that the interest rate in the Vasiček model has mean-reverting properties. We use $a > 0$ to denote the speed of mean reversion, $b > 0$ to denote the interest rate's long-term mean level, and $\sigma_r > 0$ to denote the volatility of the stochastic interest rate. $W_r(t)$ is a standard Brownian motion that denotes the interest rate risk.

The second asset is a stock; we use $S(t)$ to denote the stock price. The dynamic of $S(t)$ is given by

$$\frac{\mathrm{d}S(t)}{S(t)} = r(t)\mathrm{d}t + \sigma_{S_1}(\lambda_r \mathrm{d}t + \mathrm{d}W_r(t)) + \sigma_{S_2}(\mu(t)\mathrm{d}t + \mathrm{d}W_S(t)), \quad S(0) = 1, \tag{3}$$

where $\sigma_{S_1}, \sigma_{S_2} > 0$ denote the volatility of the stock and the constants λ_r and $\mu(t)$ denote the market prices of the interest rate risk and stock price risk, respectively. $W_S(t)$ is a one-dimensional Brownian motion with respect to $\mathcal{F}_t$ and is independent of $W_r(t)$.

We assume that the market price of the stock price risk $\mu(t)$ is stochastic and its dynamic is given by

$$\mathrm{d}\mu(t) = \kappa(\bar{\mu} - \mu(t))\mathrm{d}t + \sigma_\mu \rho \mathrm{d}W_S(t) + \sigma_\mu \sqrt{1-\rho^2}\mathrm{d}W_\mu(t), \qquad \mu(0) = \mu_0, \tag{4}$$

where $\bar{\mu} > 0$ represents the long-term mean value of $\mu(t)$, $\kappa > 0$ denotes the rate of mean reversion, $\sigma_\mu > 0$ is the volatility of the process $\mu(t)$, and ρ denotes the correlation coefficient between the process $\mu(t)$ and the Brownian motion $W_S(t)$. $W_\mu(t)$ is a one-dimensional Brownian motion with respect to $\mathcal{F}_t$ and is independent of $W_r(t)$ and $W_S(t)$.

Generally, the drift term $\mu(t)$ of the stock price process is difficult to estimate accurately in real financial markets. In our model, we assume that the fund manager can only observe the stock price $S(t)$ and the stochastic interest rate $r(t)$, but the drift term $\mu(t)$ in the stock price is unobservable. Let $\mathcal{F}^{S,r} = \{\mathcal{F}_t^{S,r}\}_{t\in[0,T]} (\subset \mathbb{F})$ be the filtration generated by the stock price process $S(t)$ and the interest rate process $r(t)$. In contrast to the full information case, in this paper, we assume that the filtration $\mathcal{F}_t^{S,r}$ is information that can only be observed by the fund manager at time t.

The third asset is the zero-coupon bond with maturity T; we use $B(t,T)$ to denote its price process at moment t. In the Vasiček model, it satisfies

$$\frac{\mathrm{d}B(t,T)}{B(t,T)} = r(t)\mathrm{d}t + \sigma_B(T-t)(\lambda_r \mathrm{d}t + \mathrm{d}W_r(t)),$$

i.e.,

$$B(t,T) = \exp\left\{-\int_t^T r(u)\mathrm{d}u + \sigma_B(T-u)(\lambda_r \mathrm{d}u + \mathrm{d}W_r(u)) + \frac{1}{2}\int_t^T \sigma_B^2(T-u)\mathrm{d}u\right\}, \tag{5}$$

where $\sigma_B(t) = \frac{1-e^{-at}}{a}\sigma_r(>0)$ is the volatility of the price of the zero-coupon bond and $B(T,T) = 1$.

Since $B(t,T)$ has a residual maturity of $T-t > 0$, the fund manager needs to adjust the zero-coupon bond at any time according to the residual maturity but there is no zero-coupon bond with any residual maturity in the financial market. So, we consider a rolling bond with a residual maturity of K as a substitute for a zero-coupon bond with any residual maturity (See also Boulier et al. (2001) and Wang et al. (2021) for more details). The price of the rolling bond process $B_K(t)$ is described by

$$\frac{\mathrm{d}B_K(t)}{B_K(t)} = r(t)\mathrm{d}t + \sigma_K(\lambda_r \mathrm{d}t + \mathrm{d}W_r(t)), \tag{6}$$

where $\sigma_K = \frac{1-e^{-aK}}{a}\sigma_r(>0)$ is the volatility of the rolling bond $B_K(t)$. By rearranging terms, one can easily derive the following relationship between the zero-coupon bond $B(t,T)$ and the rolling bond $B_K(t)$ through the cash asset $S_0(t)$:

$$\frac{\mathrm{d}B(t,T)}{B(t,T)} = \left(1 - \frac{\sigma_B(T-t)}{\sigma_K}\right)\frac{\mathrm{d}S_0(t)}{S_0(t)} + \frac{\sigma_B(T-t)}{\sigma_K}\frac{\mathrm{d}B_K(t)}{B_K(t)}. \tag{7}$$

3. Statement of the Pension Fund Management

In this section, we describe the pension fund management model. Assume that there is only one cohort of contributors in the fund; they start to subscript the fund from time $t = 0$ until the retirement time T.

3.1. The Random Contribution Rate $C(t)$

We introduce the stochastic contribution rate $C(t),\ t \in [0, T]$, which represents the total contributions made instantaneously by the pension members. Since the members' salaries are influenced by many factors, we assume that $C(t)$ is stochastic and satisfies the following stochastic differential equation(SDE):

$$\frac{\mathrm{d}C(t)}{C(t)} = \eta \mathrm{d}t + \sigma_{C_1}(\lambda_r \mathrm{d}t + \mathrm{d}W_r(t)) + \sigma_{C_2}(\mu(t)\mathrm{d}t + \mathrm{d}W_S(t)), \quad C(0) = C_0, \tag{8}$$

where $\eta, \sigma_{C_1}, \sigma_{C_2}$ are positive constants. We can also see that the contribution $C(t)$ shares the same stochastic sources as those in the stock price $S(t)$.

3.2. The Guarantee $G(T)$

In this paper, we consider a guarantee constraint on the terminal wealth. Let $f(t),\ t \in [T, T']$ be a minimal annuity, where T' is the random time of death; then, the actuarial present value of the target at time T is given by

$$G(T) = \int_T^w f(s) B(T, s)_{s-T} p_T \mathrm{d}s, \tag{9}$$

where w is the maximum age of survival, the minimal annuity function $f(s) = f(T)e^{g(s-T)}$, where $g \geq 0$ represents the coefficient of increase in the cost of living as time increases. The pension guarantees an annuity of at least $f(t)$ at time t. Here $B(T, s)$ is given by (5) and ${}_{s-T}p_T$ represents the probability that the members will survive to s given that they are alive at T, which can be calculated from the mortality rate $\lambda(\nu)$ as ${}_{s-T}p_T = e^{-\int_T^s \lambda(\nu)\mathrm{d}\nu}$. In our model, we assume that $\lambda(\nu) = w_1 e^{w_2 \nu}$, where $w_1, w_2 > 0$. Then, we obtain

$${}_{s-T}p_T = \exp\left(-\int_T^s \lambda(\nu)\mathrm{d}\nu\right) = \exp\left(-\int_T^s w_1 e^{w_2 \nu} d\nu\right) = \exp\left[\frac{w_1}{w_2}(e^{w_2 T} - e^{w_2 s})\right].$$

3.3. The Wealth Process $X^\pi(T),\ t \in [0, T]$

We use $\pi_0(t)$, $\pi_S(t)$ and $\pi_K(t)$ to represent the amount of wealth invested in the risk-free asset $S_0(t)$, the stock index $S(t)$ and the rolling bond $B_K(t)$, respectively, which satisfy

$$\pi_0(t) + \pi_S(t) + \pi_K(t) = X^\pi(t),$$

where $X^\pi(t)$ denotes the total wealth at time t under the investment strategy π. So, we have

$$\begin{aligned} dX^\pi(t) &= \pi_0(t)\frac{\mathrm{d}S_0(t)}{S_0(t)} + \pi_S(t)\frac{\mathrm{d}S(t)}{S(t)} + \pi_K(t)\frac{\mathrm{d}B_K(t)}{B_K(t)} + C(t)\mathrm{d}t \\ &= X^\pi(t)r(t)\mathrm{d}t + \pi_S(t)[\sigma_{S_1}(\lambda_r \mathrm{d}t + \mathrm{d}W_r(t)) + \sigma_{S_2}(\mu(t)\mathrm{d}t + \mathrm{d}W_S(t))] \\ &\quad + \pi_K(t)\sigma_K(\lambda_r \mathrm{d}t + \mathrm{d}W_r(t)) + C(t)\mathrm{d}t, \end{aligned} \tag{10}$$

with $X^\pi(0) = x$. In our paper, we require that the fund wealth should be greater than the guarantee $G(T)$ almost certainly at terminal time; that is,

$$X^\pi(T) \geq G(T),\ a.s.$$

Now, we define the set of admissible strategies as follows.

Definition 1. *(Admissible strategy) An investment strategy* $\pi = \{(\pi_S(t), \pi_K(t))^{\mathrm{T}}\}_{t\in[0,T]}$ *is called admissible if*

(i) $\{(\pi_S(t), \pi_K(t))^{\mathrm{T}}\}_{t\in[0,T]} \in \mathcal{F}_t^{S,r}$;

(ii) $\mathbb{E}\Big\{\int_0^T (X^\pi(t))^2[(\sigma_K\pi_K(t) + \sigma_{S_1}\pi_S(t))^2 + (\sigma_{S_2}\pi_S(t))^2]\mathrm{d}t\Big\} < \infty$;

(iii) $X^\pi(T) \geq G(T)$, *a.s.;*

(iv) *The SDE (10) has a pathwise unique solution* $\{X^\pi(t)\}_{t\in[0,T]}$ *associated with* π *satisfying (i)–(iii).*

Denote by Π *the set of all admissible strategies.*

3.4. The Optimization Criterion

We call function $u : [0,\infty) \to \mathbb{R}\bigcup\{-\infty\}$ a utility function if it is strictly increasing, strictly concave, continuous on its domain of definition and its derivative function $u'(\cdot)$ is continuously differentiable on $(0,\infty)$ satisfying the following conditions:

$$\lim_{x\to\infty} u'(x) = 0, \quad \lim_{x\to 0} u'(x) = \infty.$$

In our model, the fund manager aims to maximize the expected utility of the terminal wealth in the fund account. We assume that the initial wealth is x, and require that the terminal wealth should be greater than the guarantee. Therefore, we consider the following optimization problem:

$$(\mathrm{P}) = \begin{cases} \max\limits_{\pi\in\Pi} & \mathbb{E}[u(X^\pi(T) - G(T))], \\ & \text{diffusion Equation (10)}, \\ & X^\pi(0) = x, \\ \text{s.t.} & X^\pi(T) \geq G(T),\ \textit{a.s.} \end{cases}$$

4. Solution to the Optimization Problem

In this section, we do three main things: (1) transform the original optimization problem (P) into an auxiliary problem; (2) solve the optimal investment strategy for the auxiliary problem; and (3) solve the optimal investment strategy for the original optimization problem (P). The original problem (P) is different from the traditional optimal portfolio problem: On the one hand, due to the continuous cash inflows, the problem is non-self-financing; On the other hand, we consider a minimum guarantee constraint for the terminal fund account. In Section 4.1, we transform the original problem (P) into a simple investment optimization problem by introducing an auxiliary process. In Section 4.2, we give closed-form solutions of the terminal wealth and trading strategy for the auxiliary problem. Since the drift term $\mu(t)$ of the stock price is unobservable, in Section 4.3, we use the filtering technique to give an estimation of $\mu(t)$, and then use the estimation to solve the auxiliary problem. In Section 4.4, solutions to the original optimization problem (P) are derived.

Define the process L_t as follows:

$$\mathrm{d}L_t = -L_t[\lambda_r \mathrm{d}W_r(t) + \mu(t)\mathrm{d}W_S(t)], \quad L_0 = 1.$$

Then, L_t is a positive local martingale and has the following expression:

$$L_t = \exp\bigg\{-\int_0^t [\lambda_r \mathrm{d}W_r(s) + \mu(s)\mathrm{d}W_S(s)] - \frac{1}{2}\int_0^t \Big[\lambda_r^2 + \mu^2(s)\Big]\mathrm{d}s\bigg\}. \tag{11}$$

Assumption 1. *We assume that* $\{L_t\}_{t\geq 0}$ *is a martingale on* $(\mathcal{F}, \mathbb{P})$.

From the Novikov condition, we know that if λ_r and $\mu(s)$ satisfy $\mathbb{E}\Big[e^{\int_0^T(\lambda_r^2+\mu^2(s))\mathrm{d}s}\Big] < \infty$, then L_t is a martingale.

Define

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_T} = L_T,$$

and we define $\widetilde{\mathbb{E}}$ as the expectation operator under the risk-neutral measure $\widetilde{\mathbb{P}}$. By the above assumption and applying the Girsanov Theorem, we know that

$$\mathrm{d}\widetilde{W}(t) = (\mathrm{d}\widetilde{W}_r(t), \mathrm{d}\widetilde{W}_S(t))^{\mathrm{T}} = (\lambda_r \mathrm{d}t + \mathrm{d}W_r(t), \mu(t)\mathrm{d}t + \mathrm{d}W_S(t))^{\mathrm{T}}$$

is a two-dimensional Brownian motion under the probability measure $\widetilde{\mathbb{P}}$.

With the Brownian motion $(\widetilde{W}_r(t), \widetilde{W}_S(t))$, we can rewrite the dynamics of the instantaneous interest rate and the price of the stock as

$$\begin{aligned} \mathrm{d}r(t) &= a(b - r(t))\mathrm{d}t + \sigma_r\lambda_r\mathrm{d}t - \sigma_r\mathrm{d}\widetilde{W}_r(t), \\ \frac{\mathrm{d}S(t)}{S(t)} &= r(t)\mathrm{d}t + \sigma_{S_1}\mathrm{d}\widetilde{W}_r(t) + \sigma_{S_2}\mathrm{d}\widetilde{W}_S(t). \end{aligned} \tag{12}$$

By solving SDE in (12), we get

$$r(t) = r_0 e^{-at} + \int_0^t e^{a(s-t)}(ab\ \mathrm{d}s + \sigma_r\lambda_r\mathrm{d}s - \sigma_r\mathrm{d}\widetilde{W}_r(s)). \tag{13}$$

4.1. Transformation of the Problem

Since the wealth process (10) is not a self-financing process, there is no direct approach to solve the optimization problem (P). As in Boulier et al. (2001), Deelstra et al. (2003) and Wang et al. (2021), we first introduce some auxiliary processes to transfer the original problem to an equivalent problem.

We denote $D(t, s, r(t), C(t))$ as a loan corresponding to all contributions that the members will inject into the fund in the future. It is important to note that t, s, $r(t)$, $C(t)$ are the different variables in $D(t, s, r(t), C(t))$, respectively. This loan will be paid back with the contribution; thus, its value can be written as:

$$D(t, s, r(t), C(t)) = \widetilde{\mathbb{E}}\left[e^{-\int_t^s r(u)\mathrm{d}u} C(s) \middle| \mathcal{F}_t\right].$$

We will replicate it with the rolling bond, the cash asset and the stock. Under the risk-neutral measure $\widetilde{\mathbb{P}}$, the dynamic of the stochastic interest rate process is given by (12), and the stochastic contribution rate process is given by

$$\frac{\mathrm{d}C(t)}{C(t)} = \eta\mathrm{d}t + \sigma_{C_1}\mathrm{d}\widetilde{W}_r(t) + \sigma_{C_2}\mathrm{d}\widetilde{W}_S(t).$$

So, from the risk-neutral pricing formula, we find $D(t, s, r(t), C(t))$ satisfies the following partial differential equation:

$$\begin{gathered} D_t + D_r[a(b - r) + \lambda_r\sigma_r] + D_C C\eta + \tfrac{1}{2}D_{rr}\sigma_r^2 + \tfrac{1}{2}D_{CC}C^2(\sigma_{C_1}^2 + \sigma_{C_2}^2) \\ -D_{rC}C\sigma_{C_1}\sigma_r - rD = 0, \end{gathered} \tag{14}$$

with $D(s, s, r(s), C(s)) = C(s)$, where D_t, D_r, D_C are the first-order partial derivatives of $D(t, s, r(t), C(t))$ with respect to the variables t, $r(t)$, $C(t)$, respectively, and D_{rr}, D_{CC}, D_{rC} are the second-order partial derivatives of $D(t, s, r(t), C(t))$ with respect to the variables $r(t)$ and $C(t)$. For simplicity, we sometimes use D to mean $D(t, s, r(t), C(t))$.

To solve (14), we assume that $D(t, s, r(t), C(t))$ has the following form

$$D(t, s, r(t), C(t)) = C(t)\exp[f_1(s - t) + f_2(s - t)r(t)]. \tag{15}$$

Let $s - t = \tau$. Then, by differentiating $D(t, s, r(t), C(t))$ with respect to t, r, C, respectively, we get

$$D_t = -D_\tau = -D\big[f_2'(\tau) + f_2'(\tau)r\big], \quad D_C = \exp\{f_1(\tau) + f_2(\tau)r\},$$
$$D_r = Df_2(\tau), \quad D_{CC} = 0, \quad D_{rr} = Df_2^2(\tau), \quad D_{Cr} = \exp\{f_1(\tau) + f_2(\tau)r\}f_2(\tau),$$

where the sign $'$ represents the first derivative of the function. By substituting the above expressions into (14), we obtain

$$\begin{aligned} Df_1'(\tau) + D\eta + Df_2(\tau)(ab + \sigma_r\lambda_r) + \tfrac{1}{2}Df_2^2(\tau)\sigma_r^2 - Df_2(\tau)\sigma_r\sigma_{C_1} \\ -(Df_2'(\tau) + Df_2(\tau)a + D)r = 0, \end{aligned} \tag{16}$$

with $D(s,s,r(s),C(s)) = C(s)$. Note that (16) is equivalent to the following two equations with boundary conditions

$$\begin{cases} f_1'(\tau) + \eta + f_2(\tau)(ab + \sigma_r\lambda_r) + \frac{1}{2}f_2^2(\tau)\sigma_r^2 - f_2(\tau)\sigma_r\sigma_{C_1} = 0, \\ f_1(0) = 0, \end{cases}$$
$$\begin{cases} f_2'(\tau) + f_2(\tau)a + 1 = 0, \\ f_2(0) = 0. \end{cases}$$

By solving the above two equations, we deduce the solutions of $f_1(\tau)$ and $f_2(\tau)$ as

$$f_1(\tau) = \int_0^\tau [abf_2(s) + \frac{1}{2}f_2^2(s)\sigma_r^2 + (\lambda_r - \sigma_{C_1})\sigma_r f_2(s) + \eta]\mathrm{d}s, \quad f_2(\tau) = \frac{e^{-a\tau} - 1}{a}.$$

By applying the Itô formula, we find that $D(t,s,r(t),C(t))$ satisfies the following SDE:

$$\begin{aligned} \mathrm{d}D =& D_t\mathrm{d}t + D_r(\mathrm{d}r(t)) + D_C(\mathrm{d}C(t)) + \frac{1}{2}D_{rr}(\mathrm{d}r(t))^2 + \frac{1}{2}D_{CC}(\mathrm{d}C(t))^2 + D_{rC}(\mathrm{d}r(t))(\mathrm{d}C(t)) \\ =& D\big[r(t)\mathrm{d}t + (\sigma_{C_1} - f_2(s-t)\sigma_r)(\lambda_r\mathrm{d}t + \mathrm{d}W_r(t)) + \sigma_{C_2}(\mu(t)\mathrm{d}t + \mathrm{d}W_S(t))\big], \end{aligned} \tag{17}$$

with boundary condition $D(s,s,r(s),C(s)) = C(s)$.

Let $F(t,T) = \int_t^T D(t,s,r(t),C(t))\mathrm{d}s$, according to (1), (3), (6) and (17), we have

$$\mathrm{d}F(t,T) = -C(t)\mathrm{d}t + \pi_0^F(t)\frac{\mathrm{d}S_0(t)}{S_0(t)} + \pi_K^F(t)\frac{\mathrm{d}B_K(t)}{B_K(t)} + \pi_S^F(t)\frac{\mathrm{d}S(t)}{S(t)}, \tag{18}$$

where

$$\begin{aligned} \pi_K^F(t) &= \frac{\int_t^T \sigma_{S_2}D(t,s,r(t),C(t))(\sigma_{C_1} - f_2(s-t)\sigma_r)\mathrm{d}s - \sigma_{S_1}\sigma_{C_2}F(t,T)}{\sigma_{S_2}\sigma_K}, \\ \pi_S^F(t) &= \frac{\sigma_{C_2}}{\sigma_{S_2}}F(t,T), \\ \pi_0^F(t) &= F(t,T) - \pi_K^F(t) - \pi_S^F(t). \end{aligned}$$

Given $F(t,T)$ in (18), we are able to transfer the original optimal problem (P) into a self-financing investment problem. However, we also require that the terminal wealth $X^\pi(T)$ should be greater than a minimum guarantee. So, in the remaining part of this section, we continue to transfer this problem into an unconstrained problem.

Define $G(t)$:

$$G(t) = \mathbb{E}[H_T^t G(T)|\mathcal{F}_t], \quad 0 < t < T, \tag{19}$$

where

$$H_T^t = \frac{L_T}{L_t}e^{-\int_t^T r(u)\mathrm{d}u}. \tag{20}$$

By substituting the expression of $G(T)$ defined in (9) into (19), we obtain

$$G(t) = \mathbb{E}\Big[H_T^t\int_T^w f(s)B(T,s)_{s-T}p_T\mathrm{d}s\big|\mathcal{F}_t\Big]. \tag{21}$$

In addition, because

$$B(t,T) = \widetilde{\mathbb{E}}\left[e^{-\int_t^T r(s)\mathrm{d}s}\middle|\mathcal{F}_t\right],$$

where $\widetilde{\mathbb{E}}$ is the expectation under the measure $\widetilde{\mathbb{P}}$. By using the expressions of L_t in (11), H_T^t in (20), and from the Bayes formula, one can rewrite $G(t)$ in (21) as

$$\begin{aligned}
G(t) &= \int_T^w f(s)\mathbb{E}\left[H_T^t\widetilde{\mathbb{E}}\left[e^{-\int_T^s r(u)\mathrm{d}u}\middle|\mathcal{F}_T\right]\middle|\mathcal{F}_t\right]_{s-T}p_T\mathrm{d}s \\
&= \int_T^w f(s)\widetilde{\mathbb{E}}\left[e^{-\int_t^T r(u)\mathrm{d}u}\widetilde{\mathbb{E}}\left[e^{-\int_T^s r(u)\mathrm{d}u}\middle|\mathcal{F}_T\right]\middle|\mathcal{F}_t\right]_{s-T}p_T\mathrm{d}s \\
&= \int_T^w f(s)\widetilde{\mathbb{E}}\left[e^{-\int_t^s r(u)\mathrm{d}u}\middle|\mathcal{F}_t\right]_{s-T}p_T\mathrm{d}s = \int_T^w f(s)B(t,s)_{s-T}p_T\mathrm{d}s.
\end{aligned}$$

Moreover, from (7), we know that

$$\begin{aligned}
\mathrm{d}G(t) &= \int_T^w f(s)(\mathrm{d}B(t,s))_{s-T}p_T\mathrm{d}s \\
&= \pi_0^G(t)\frac{\mathrm{d}S_0(t)}{S_0(t)} + \pi_K^G(t)\frac{\mathrm{d}B_K(t)}{B_K(t)},
\end{aligned} \tag{22}$$

where $\pi_K^G(t)$ and $\pi_0^G(t)$ are compositions of $G(t)$ as follows:

$$\begin{aligned}
\pi_K^G(t) &= \int_T^w f(s)B(t,s)_{s-T}p_T\frac{\sigma_B(s-t)}{\sigma_K}\mathrm{d}s, \\
\pi_0^G(t) &= G(t) - \pi_K^G(t).
\end{aligned}$$

Define a new portfolio

$$Z^\pi(t) = X^\pi(t) + F(t,T) - G(t), \tag{23}$$

then, by (10), (18) and (22), we obtain the differential expression of the new transformed wealth process $Z^\pi(t)$ as

$$\begin{aligned}
\mathrm{d}Z^\pi(t) =&\pi_0^Z(t)\frac{\mathrm{d}S_0(t)}{S_0(t)} + \pi_S^Z(t)\frac{\mathrm{d}S(t)}{S(t)} + \pi_K^Z(t)\frac{\mathrm{d}B_K(t)}{B_K(t)} \\
=&Z^\pi(t)r(t)\mathrm{d}t + \pi_S^Z(t)[\sigma_{S_1}(\lambda_r\mathrm{d}t + \mathrm{d}W_r(t)) + \sigma_{S_2}(\mu(t)\mathrm{d}t + \mathrm{d}W_S(t))] \\
&+ \pi_K^Z(t)\sigma_K(\lambda_r\mathrm{d}t + \mathrm{d}W_r(t)),
\end{aligned} \tag{24}$$

where $\pi_0^Z(t)$, $\pi_S^Z(t)$ and $\pi_K^Z(t)$ denote the proportion of the new wealth process $Z^\pi(t)$ invested in the risk-free asset $S_0(t)$, $S(t)$ and $B_K(t)$ at moment t, respectively. Let $\pi^Z = \{(\pi_S^Z(t), \pi_K^Z(t))^T\}_{t\in[0,T]}$. From (18), the differential form of $F(t,T)$ can be expressed by $S_0(t), S(t), B_K(t)$. Similarly, by (22), the differential form of $D(t)$ can be expressed by $S_0(t), B_K(t)$. So, from the relationship between $X^\pi(t)$ and $Z^\pi(t)$ in (23), we can obtain the relationship between π and π^Z as

$$\begin{aligned}
\pi_S^Z(t) &= \pi_S(t) + \pi_S^F(t), \\
\pi_K^Z(t) &= \pi_K(t) + \pi_K^F(t) - \pi_K^G(t), \\
\pi_0^Z(t) &= \pi_0(t) + \pi_0^F(t) - \pi_0^G(t),
\end{aligned} \tag{25}$$

we also know that $Z^\pi(T) = X^\pi(T) - G(T)$.

Finally, after the above series of transformations, we simplify the original problem (P) into the following optimization problem on $Z^{\pi}(t)$ without constraint,

$$(\mathrm{Q}) = \begin{cases} \max\limits_{\pi\in\Pi} & \mathbb{E}[u(Z^{\pi}(T))], \\ & \text{diffusion Equation (24)}, \\ & Z^{\pi}(0) = z, \\ \text{s.t.} & Z^{\pi}(T) \geq 0,\ a.s. \end{cases}$$

where

$$z = x + F(0,T) - G(0). \tag{26}$$

4.2. Explicit Representation of the Optimal Terminal Wealth

In the financial market, the only observable information to the fund manager is the stock price process $S(t)$ and the stochastic interest rate $r(t)$, while the drift term $\mu(t)$ of the stock price process is unobservable; that is, we consider the optimal investment problem under incomplete information. We use $\mathcal{F}^{S,r}$ to denote the filtering of observable information. In this subsection, we first compute an explicit solution of the optional projection of $\mathbb{P}$-martingale L to $\mathcal{F}^{S,r}$ and then explore expressions for the optimal terminal wealth and the optimal trading strategy under incomplete information.

Define $\zeta = \{\zeta_t, t \in [0,T]\}$ as

$$\zeta_t = \mathbb{E}[L_t|\mathcal{F}_t^{S,r}]. \tag{27}$$

In geometrical terms, this means that $\mathbb{E}[L_t|\mathcal{F}_t^{S,r}]$ is the orthogonal projection (in $(\mathcal{F},\mathbb{P})$ of L onto the subspace $(\mathcal{F}^{S,r},\mathbb{P})$). Note that ζ is a martingale on $(\mathcal{F}^{S,r},\mathbb{P})$. For every $\mathcal{F}_t^{S,r}$-measurable random variable V, $\mathcal{F}_u$-measurable random variable Y and $\mathcal{F}_u^{S,r}$-measurable random variable W, $0 \leq t \leq u \leq T$, we have

$$\begin{aligned} \widetilde{\mathbb{E}}[V] &= \mathbb{E}[\zeta_t V], \\ \widetilde{\mathbb{E}}[Y|\mathcal{F}_t^{S,r}] &= \frac{1}{\zeta_t}\mathbb{E}[L_u Y|\mathcal{F}_t^{S,r}], \\ \widetilde{\mathbb{E}}[W|\mathcal{F}_t^{S,r}] &= \frac{1}{\zeta_t}\mathbb{E}[\zeta_u W|\mathcal{F}_t^{S,r}]. \end{aligned} \tag{28}$$

The only information that can be observed is the stock price $S(t)$ and the stochastic interest rate $r(t)$ but the drift process $\mu(t)$ of the stock price is unobservable. Thus, to solve this problem, we consider the conditional mean and covariance of $\mu(t)$ as:

$$m(t) = \mathbb{E}[\mu(t)|\mathcal{F}_t^{S,r}], \qquad m(0) = m_0,$$

$$\gamma(t) = \mathbb{E}[(\mu(t) - m(t))^2|\mathcal{F}_t^{S,r}], \qquad \gamma(0) = \gamma_0.$$

By applying a similar approach to the use of Theorem 3.1 in Lakner (1998), in the following proposition, we show that ζ_t can be explicitly expressed with the conditional expectation $m(t)$, which implies that ζ_t is observable under $\mathcal{F}_t^{S,r}$.

Proposition 1. *The process ζ_t has the following explicit representation:*

$$\zeta_t = \exp\left\{-\int_0^t \left(\lambda_r \mathrm{d}\widetilde{W}_r(s) + m(s)\mathrm{d}\widetilde{W}_S(s)\right) + \frac{1}{2}\int_0^t \left(\lambda_r^2 + m^2(s)\right)\mathrm{d}s\right\}. \tag{29}$$

Proof. We know that $(\mathrm{d}\widetilde{W}_r(t), \mathrm{d}\widetilde{W}_S(t))^{\mathrm{T}} = (\lambda_r \mathrm{d}t + \mathrm{d}W_r(t), \mu(t)\mathrm{d}t + \mathrm{d}W_S(t))^{\mathrm{T}}$; then L_t given in (11) can be rewritten as

$$L_t = \exp\left\{-\int_0^t (\lambda_r \mathrm{d}\widetilde{W}_r(s) + \mu(s)\mathrm{d}\widetilde{W}_S(s)) + \frac{1}{2}\int_0^t (\lambda_r^2 + \mu^2(s))\mathrm{d}s\right\},$$

and $1/L_t$ satisfies

$$\mathrm{d}\left(\frac{1}{L_t}\right) = \frac{1}{L_t}\Big(\lambda_r \mathrm{d}\widetilde{W}_r(t) + \mu(t)\mathrm{d}\widetilde{W}_S(t)\Big).$$

So, we have

$$\widetilde{\mathbb{E}}\left[\int_0^t \mathrm{d}\left(\frac{1}{L_u}\right)\middle|\mathcal{F}_t^{S,r}\right] = \widetilde{\mathbb{E}}\left[\int_0^t \frac{1}{L_u}\Big(\lambda_r \mathrm{d}\widetilde{W}_r(u) + \mu(u)\mathrm{d}\widetilde{W}_S(u)\Big)\mathrm{d}u\middle|\mathcal{F}_t^{S,r}\right]. \tag{30}$$

To apply Theorem 5.14 in Liptser and Shiryayev (1977), we need to check the following two conditions:

$$\widetilde{\mathbb{E}}\left[\frac{|\lambda_r|}{L_u}\right] + \widetilde{\mathbb{E}}\left[\frac{1}{L_u}|\mu(u)|\right] < \infty, \qquad u \in [0, T] \tag{31}$$

and

$$\int_0^T \left\{\widetilde{\mathbb{E}}\left[\frac{|\lambda_r|}{L_u}\middle|\mathcal{F}_u^{S,r}\right]\right\}^2 + \left\{\widetilde{\mathbb{E}}\left[\frac{1}{L_u}|\mu(u)|\middle|\mathcal{F}_u^{S,r}\right]\right\}^2 \mathrm{d}u < \infty, \quad a.s. \tag{32}$$

The inequality in (31) is true because

$$\widetilde{\mathbb{E}}\left[\frac{1}{L_u}|\lambda_r|\right] + \widetilde{\mathbb{E}}\left[\frac{1}{L_u}|\mu(u)|\right] = \lambda_r + \mathbb{E}[|\mu(u)|] < \infty.$$

According to (28), the left-hand side of (32) can be rewritten as

$$\int_0^T \frac{1}{\zeta_u^2}\left\{\mathbb{E}\left[|\lambda_r||\mathcal{F}_u^{S,r}\right]\right\}^2 + \frac{1}{\zeta_u^2}\left\{\mathbb{E}\left[|\mu(u)||\mathcal{F}_u^{S,r}\right]\right\}^2 \mathrm{d}u = \int_0^T \frac{1}{\zeta_u^2}\Big(\lambda_r^2 + m^2(u)\Big)\mathrm{d}u. \tag{33}$$

Due to the continuity of m and ζ, we know that the above equation is also finite; thus, (32) is true.

By Theorem 5.14 in Liptser and Shiryayev (1977), and relations in (31) and (32), we know that the right-hand side of (30) is equal to

$$\int_0^t \widetilde{\mathbb{E}}\left[\frac{\lambda_r}{L_u}\middle|\mathcal{F}_u^{S,r}\right]\mathrm{d}\widetilde{W}_r(u) + \int_0^t \widetilde{\mathbb{E}}\left[\frac{1}{L_u}\mu(u)\middle|\mathcal{F}_u^{S,r}\right]\mathrm{d}\widetilde{W}_S(u). \tag{34}$$

According to the definitions of m and ζ in (27), the left-hand side of (30) is equal to

$$\widetilde{\mathbb{E}}\left[\frac{1}{L_t}\middle|\mathcal{F}_t^{S,r}\right] - 1 = \frac{1}{\zeta_t} - 1, \tag{35}$$

and according to (34), the right-hand side of (30) becomes

$$\begin{aligned} &\int_0^t \frac{1}{\zeta_u}\mathbb{E}[\lambda_r|\mathcal{F}_u^{S,r}]\mathrm{d}\widetilde{W}_r(u) + \int_0^t \frac{1}{\zeta_u}\mathbb{E}[\mu(u)|\mathcal{F}_u^{S,r}]\mathrm{d}\widetilde{W}_S(u) \\ =& \int_0^t \frac{1}{\zeta_u}\lambda_r \mathrm{d}\widetilde{W}_r(u) + \int_0^t \frac{1}{\zeta_u}m(u)\mathrm{d}\widetilde{W}_S(u). \end{aligned} \tag{36}$$

Thus, from the equivalency of (35) and (36), we get

$$\mathrm{d}\left(\frac{1}{\zeta_t}\right) = \frac{1}{\zeta_t}\lambda_r \mathrm{d}\widetilde{W}_r(t) + \frac{1}{\zeta_t}m(t)\mathrm{d}\widetilde{W}_S(t),$$

that is,

$$\frac{1}{\zeta_t} = \exp\left\{\int_0^t \left(\lambda_r \mathrm{d}\widetilde{W}_r(s) + m(s)\mathrm{d}\widetilde{W}_S(s)\right) - \frac{1}{2}\int_0^t (\lambda_r^2 + m^2(s))\mathrm{d}s\right\}, \tag{37}$$

and (29) is an obvious consequence of (37). Hence, our proof is complete. □

Assuming that u is a utility function, define the (continuous, strictly decreasing) function $I : (0, \infty) \mapsto [0, \infty)$ as the inverse function of u', to satisfy

$$\lim_{y\to\infty} I(y) = 0, \quad \lim_{y\to 0} I(y) = \infty.$$

Through a similar derivation of Theorem 2.5 as in Lakner (1998), we obtain closed solutions for the optimal terminal wealth and the optimal trading strategy for the auxiliary problem.

Theorem 1. *Suppose that, for any constant $x \in (0, \infty)$, there is*

$$\widetilde{\mathbb{E}}[I(x\zeta_T)] < \infty,$$

in which $I(\cdot)$ is the inverse of $u'(\cdot)$.

Then, we can obtain the optimal terminal wealth

$$\hat{Z}^\pi(T) = I(\beta e^{-\int_0^T r(s)\mathrm{d}s}\zeta_T),$$

where the constant β is uniquely determined by the following relationship

$$\widetilde{\mathbb{E}}[e^{-\int_0^T r(s)\mathrm{d}s} I(\beta e^{-\int_0^T r(s)\mathrm{d}s}\zeta_T)] = z. \tag{38}$$

The optimal wealth process $\hat{Z}^\pi(t)$ and the optimal trading strategy $\{\hat{\pi}(t)\}_{t\geq 0}$ can be implicitly determined by

$$\begin{aligned} e^{-\int_0^t r(s)\mathrm{d}s}\hat{Z}^\pi(t) &= \widetilde{\mathbb{E}}[e^{-\int_0^T r(s)\mathrm{d}s} I(\beta e^{-\int_0^T r(s)\mathrm{d}s}\zeta_T) | \mathcal{F}_t^{S,r}] \\ &= z + \int_0^t e^{-\int_0^u r(s)\mathrm{d}s}\Big[\Big(\sigma_{S_1}\hat{\pi}_S^Z(u) + \sigma_K \hat{\pi}_K^Z(u)\Big)\mathrm{d}\widetilde{W}_r(u) + \sigma_{S_2}\hat{\pi}_S^Z(u)\mathrm{d}\widetilde{W}_S(u)\Big]. \end{aligned} \tag{39}$$

4.3. Explicit Formula for the Optimal Trading Strategy

In the optimization problem (Q), the drift term $\mu(t)$ in the stock price is unobservable; thus, the manager would first need to estimate the value of $\mu(t)$ and use the estimated value of $\mu(t)$ to determine the optimal current investment strategy. In this sub-section, we compute some optimal solutions to the auxiliary problem (Q). Similar to Wang et al. (2021), we first estimate the value of $\mu(t)$ by the filtering technique. Then, the estimated value of $\mu(t)$ is substituted into the optimal problem (Q). Finally, we deduce the closed-form expression for the optimal investment strategy.

We use filtering theory to estimate the value of $\mu(t)$ using information from $\mathcal{F}_t^{S,r}$. For the original theorem and the corresponding proof of the filtering theory, we refer to Theorem 8.1 and subsections 8.2 in Liptser and Shiryayev (1977). First, we rearrange Equations (2)–(4) into matrix form as[1]:

$$\begin{aligned} \begin{pmatrix} \mathrm{d}r(t) \\ \frac{\mathrm{d}S(t)}{S(t)} \end{pmatrix} = &\left[\underbrace{\begin{pmatrix} a(b - r(t)) \\ r(t) + \sigma_{S_1}\lambda_r \end{pmatrix}}_{A_0} + \underbrace{\begin{pmatrix} 0 \\ \sigma_{S_2} \end{pmatrix}}_{A_1}\mu(t)\right]\mathrm{d}t + \underbrace{\mathbf{0}}_{B_1}\mathrm{d}W_\mu(t) \\ &+ \underbrace{\begin{pmatrix} -\sigma_r & 0 \\ \sigma_{S_1} & \sigma_{S_2} \end{pmatrix}}_{B_2}\begin{pmatrix} \mathrm{d}W_r(t) \\ \mathrm{d}W_S(t) \end{pmatrix} \end{aligned} \tag{40}$$

and

$$\mathrm{d}\mu(t) = (\underbrace{\kappa\bar{\mu}}_{a_0} + \underbrace{(-\kappa)}_{a_1}\mu(t))\mathrm{d}t + \underbrace{\sigma_\mu\sqrt{1-\rho^2}}_{b_1}\mathrm{d}W_\mu(t) + \underbrace{\begin{pmatrix} 0 & \sigma_\mu\rho \end{pmatrix}}_{b_2}\begin{pmatrix} \mathrm{d}W_r(t) \\ \mathrm{d}W_S(t) \end{pmatrix}.$$

Denote $B = (B_1, B_2), b = (b_1, b_2)$ and introduce the following notations:

$$b \circ b = b_1 \circ b_1^{\mathrm{T}} + b_2 \circ b_2^{\mathrm{T}},$$

$$b \circ B = b_1 \circ B_1^{\mathrm{T}} + b_2 \circ B_2^{\mathrm{T}},$$

$$B \circ B = B_1 \circ B_1^{\mathrm{T}} + B_2 \circ B_2^{\mathrm{T}},$$

where the symbol "$\circ$" is the operator for matrix multiplication. Since $B_1 = \mathbf{0}$, we have $B \circ B = B_2 \circ B_2^{\mathrm{T}}$ and $(B \circ B)^{-1} = (B_2 \circ B_2^{\mathrm{T}})^{-1} = (B_2^{-1})^{\mathrm{T}} \circ B_2^{-1}, b \circ B = b_2 \circ B_2^{\mathrm{T}}$.

From Theorem 10.3 in Liptser and Shiryayev (1977), we know that $m(\cdot)$ is the unique solution of the following linear system of SDE:

$$\begin{aligned} dm(t) = (a_0 + a_1 m(t))\mathrm{d}t + [b_2 \circ B_2^{\mathrm{T}} + \gamma(t)A_1^{\mathrm{T}}](B_2 \circ B_2^{\mathrm{T}})^{-1} \times \\ \left\{\begin{pmatrix} \mathrm{d}r(t) \\ \frac{\mathrm{d}S(t)}{S(t)} \end{pmatrix} - (A_0 + A_1 m(t))\mathrm{d}t\right\}, \end{aligned} \tag{41}$$

where $\gamma(\cdot)$ is the unique solution of the following deterministic Riccati equation:

$$\begin{aligned} \gamma'(t) =& a_1\gamma(t) + \gamma(t)a_1^{\mathrm{T}} + b \circ b - [b_2 \circ B_2^{\mathrm{T}} + \gamma(t)A_1^{\mathrm{T}}](B_2 \circ B_2^{\mathrm{T}})^{-1}[b_2 \circ B_2^{\mathrm{T}} + \gamma(t)A_1^{\mathrm{T}}]^{\mathrm{T}} \\ =& -\gamma^2(t) - 2(\kappa + \sigma_\mu\rho)\gamma(t) + \left(\sigma_\mu\sqrt{1-\rho^2}\right)^2, \end{aligned} \tag{42}$$

with $\gamma(0) = \gamma_0$.

By calculation, we know that

$$(B_2)^{-1} = \begin{pmatrix} -\sigma_r & 0 \\ \sigma_{S_1} & \sigma_{S_2} \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{1}{\sigma_r} & 0 \\ \frac{\sigma_{S_1}}{\sigma_r\sigma_{S_2}} & \frac{1}{\sigma_{S_2}} \end{pmatrix}, \tag{43}$$

and from (40), we have

$$\begin{aligned} &\begin{pmatrix} \mathrm{d}r(t) \\ \frac{\mathrm{d}S(t)}{S(t)} \end{pmatrix} - (A_0 + A_1 m(t))\mathrm{d}t \\ =& \begin{pmatrix} -\sigma_r \mathrm{d}W_r(t) \\ \sigma_{S_1}\mathrm{d}W_r(t) + \sigma_{S_2}(\mu(t)\mathrm{d}t + \mathrm{d}W_S(t)) - \sigma_{S_2}m(t)\mathrm{d}t \end{pmatrix} \\ =& \begin{pmatrix} -\sigma_r & 0 \\ \sigma_{S_1} & \sigma_{S_2} \end{pmatrix}\begin{pmatrix} \mathrm{d}\widetilde{W}_r(t) - \lambda_r\mathrm{d}t \\ \mathrm{d}\widetilde{W}_S(t) - m(t)\mathrm{d}t \end{pmatrix}. \end{aligned} \tag{44}$$

By substituting (43) and (44) into (41), we obtain

$$dm(t) = -(\kappa + \sigma_\mu\rho + \gamma(t))m(t)\mathrm{d}t + (\sigma_\mu\rho + \gamma(t))\mathrm{d}\widetilde{W}_S(t) + \kappa\bar{\mu}\mathrm{d}t, \quad m(0) = m_0. \tag{45}$$

It is known that (42) has an explicit solution, the solution is

$$\gamma(t) = \sqrt{C}\frac{C_1 e^{2\sqrt{C}t} + C_2}{C_1 e^{2\sqrt{C}t} - C_2} - (\kappa + \sigma_\mu\rho), \tag{46}$$

where

$$C = \left(\kappa + \sigma_\mu \rho\right)^2 + \sigma_\mu^2(1-\rho^2),$$
$$C_1 = \sqrt{C} + \gamma_0 + \kappa + \sigma_\mu \rho,$$
$$C_2 = -\sqrt{C} + \gamma_0 + \kappa + \sigma_\mu \rho.$$

Define

$$\phi(t) = \exp\left\{-(\kappa + \sigma_\mu \rho)t - \int_0^t \gamma(s)\mathrm{d}s\right\}, \tag{47}$$

then, $m(t)$ in (45) has an explicit expression:

$$m(t) = \phi(t)\left\{m_0 + \int_0^t \phi^{-1}(s)\left[(\sigma_\mu \rho + \gamma(s))\mathrm{d}\widetilde{W}_S(s) + \kappa\overline{\mu}\mathrm{d}s\right]\right\}. \tag{48}$$

Moreover, from the theory of filtering (formula (12.65) in Liptser and Shiryayev (1978)), we know the process

$$\begin{aligned}(\mathrm{d}\overline{W}_r(t), \mathrm{d}\overline{W}_S(t))^{\mathrm{T}} =& B_2^{-1}\left(\mathrm{d}r(t), \frac{\mathrm{d}S(t)}{S(t)}\right)^{\mathrm{T}} - B_2^{-1}(A_0 + A_1 m(t))\mathrm{d}t \\ =& (\mathrm{d}\widetilde{W}_r(t) - \lambda_r \mathrm{d}t, \mathrm{d}\widetilde{W}_S(t) - m(t)\mathrm{d}t)^{\mathrm{T}}\end{aligned} \tag{49}$$

is a two-dimensional Brownian motion with respect to $(\mathbb{P}, \mathcal{F}_t^{S,r})$. Denote $\mathrm{d}\overline{W}(t) = (\mathrm{d}\overline{W}_r(t), \mathrm{d}\overline{W}_S(t))^{\mathrm{T}}$. Then, with the new defined Brownian motion $(\overline{W}_r(t), \overline{W}_S(t))$, the dynamics of the stochastic interest rate and the price of the stock can be rewritten as

$$\mathrm{d}r(t) = a(b - r(t))\mathrm{d}t - \sigma_r \mathrm{d}\overline{W}_r(t),$$
$$\frac{\mathrm{d}S(t)}{S(t)} = r(t)\mathrm{d}t + \sigma_{S_1}(\lambda_r + \mathrm{d}\overline{W}_r(t)) + \sigma_{S_2}(m(t)\mathrm{d}t + \mathrm{d}\overline{W}_S(t)).$$

Based on the above analysis and calculations, we provide the expressions of the optimal strategies for our transformed portfolio problem (Q).

Theorem 2. *Suppose that for some* $K^* > 0$,[2]

$$I(x) < K^*(1 + x^{-5/2}), \tag{50}$$

$$-I'(x) < K^*(1 + x^{-2}). \tag{51}$$

Then the optimal trading strategies for the auxiliary problem (Q) *satisfy the following equation*

$$\begin{aligned}\hat{\pi}_S^Z(t) =& -\frac{\beta}{\sigma_{S_2}} e^{\int_0^t r(s)\mathrm{d}s}\widetilde{\mathbb{E}}\Big[e^{-2\int_0^T r(s)\mathrm{d}s}\zeta_T I'(\beta e^{-\int_0^T r(s)\mathrm{d}s}\zeta_T) \times \\ & \left(\int_t^T \phi(s)\phi^{-1}(t)(\sigma_\mu\rho + \gamma(t))\mathrm{d}\overline{W}_S(s) + m(t)\right)\Big|\mathcal{F}_t^{S,r}\Big],\end{aligned}$$

$$\begin{aligned}\hat{\pi}_K^Z(t) =& -\frac{\sigma_r(e^{-a(T-t)} - 1)}{a\sigma_K}\hat{Z}^\pi(t) - e^{\int_0^t r(s)\mathrm{d}s}\widetilde{\mathbb{E}}\Bigg\{\beta e^{-2\int_0^T r(s)\mathrm{d}s}\zeta_T I'(\beta e^{-\int_0^T r(s)\mathrm{d}s}\zeta_T)\Bigg[\frac{\sigma_r(e^{-a(T-t)} - 1)}{a\sigma_K} \\ & + \frac{\lambda_r}{\sigma_K} - \frac{\sigma_{S_1}}{\sigma_K\sigma_{S_2}}\left(\int_t^T \phi(s)\phi^{-1}(t)(\sigma_\mu\rho + \gamma(t))\mathrm{d}\overline{W}_S(s) + m(t)\right)\Bigg]\Bigg|\mathcal{F}_t^{S,r}\Bigg\},\end{aligned}$$

where

$$\hat{Z}^\pi(t) = e^{\int_0^t r(s)\mathrm{d}s}\widetilde{\mathbb{E}}\big[e^{-\int_0^T r(s)\mathrm{d}s}I(\beta e^{-\int_0^T r(s)\mathrm{d}s}\zeta_T)\big|\mathcal{F}_t^{S,r}\big],$$

ζ_t, β, $\gamma(t)$, $\phi(t)$ and $m(t)$ are given in (29), (38), (46), (47) and (48), respectively.

For simplicity, we defer the detailed proof of Theorem 2 to Appendix A.

Remark 1. *For the optimal problem* (Q), *if we already know the investment proportion* π_S^Z *invested in the stock price and the investment proportion* π_K^Z *invested in the rolling bond, then the investment proportion* π_0^Z *invested in the risk-free asset can be obtained from the relation* $\pi_0^Z + \pi_S^Z + \pi_K^Z = Z^\pi$.

Define a CRRA function u as

$$u(x) = \frac{x^\delta}{\delta}, \quad \forall x \geq 0, \tag{52}$$

where $\delta < 0$, and $I(x) = x^{\frac{1}{\delta-1}}$.

The relative risk aversion coefficient of this utility function is $-\frac{xu''(x)}{u'(x)} = \delta - 1$, which is independent of x; therefore, it is denoted a constant relative risk aversion function. In particular, for the CRRA utility function, the optimal investment strategies have much simpler representations.

Corollary 1. *When u is a CRRA utility function. Then, the optimal trading strategies become*

$$\begin{aligned}
\hat{\pi}_S^Z(t) = & \frac{\beta^{-\frac{1}{1-\delta}}}{\sigma_{S_2}(1-\delta)} e^{\int_0^t r(s)\mathrm{d}s} \widetilde{\mathbb{E}}\bigg[e^{\frac{\delta}{1-\delta}\int_0^T r(s)\mathrm{d}s} \zeta_T^{-\frac{1}{1-\delta}} \times \\
& \left(\int_t^T \phi(s)\phi^{-1}(t)(\sigma_\mu \rho + \gamma(t))\mathrm{d}\overline{W}_S(s) + m(t) \right) \bigg| \mathcal{F}_t^{S,r} \bigg], \\
\hat{\pi}_K^Z(t) = & \frac{\beta^{-\frac{1}{1-\delta}} e^{\int_0^t r(s)\mathrm{d}s}}{\sigma_K(1-\delta)} \left\{ \left(\lambda_r + \frac{\delta\sigma_r(e^{-a(T-t)} - 1)}{a} \right) \widetilde{\mathbb{E}}\left[e^{\frac{\delta}{1-\delta}\int_0^T r(s)\mathrm{d}s} \zeta_T^{-\frac{1}{1-\delta}} \bigg| \mathcal{F}_t^{S,r} \right] \right. \\
& \left. - \frac{\sigma_{S_1}}{\sigma_{S_2}} \widetilde{\mathbb{E}}\left[e^{\frac{\delta}{1-\delta}\int_0^T r(s)\mathrm{d}s} \zeta_T^{-\frac{1}{1-\delta}} \left(\int_t^T \phi(s)\phi^{-1}(t)(\sigma_\mu \rho + \gamma(t))\mathrm{d}\overline{W}_S(s) + m(t) \right) \bigg| \mathcal{F}_t^{S,r} \right] \right\},
\end{aligned}$$

where z is given in (26) and

$$\beta^{-\frac{1}{1-\delta}} = z \left\{ \widetilde{\mathbb{E}}\left[e^{\frac{\delta}{1-\delta}\int_0^T r(s)\mathrm{d}s} \zeta_T^{-\frac{1}{1-\delta}} \right] \right\}^{-1}.$$

Remark 2. *(The case when* δ *is positive) Let* $\theta \in (0,1)$ *and we consider* $\delta \in (0, \theta]$*. In this case, the inequalities (50) and (51) might not hold. To overcome the difficulties, we substitute the parameters into Proposition 4.6 in Lakner (1995), so that, in order for Equations (50) and (51) to hold, we propose the following stronger condition:*

$$\gamma_0 + T\sigma_\mu^2 < \frac{1}{8K\|B_2^{-1}\|^2 T} \min\left\{ \frac{1}{45}, \frac{(1-\theta)^2}{(\theta+3)(\theta+7)} \right\},$$

where

$$K = \max_{t \leq T} e^{-2\kappa t}.$$

Then for the CRRA utility function with $\delta \in (0, \theta]$*, the optimal investment strategies are the same as those provided in Corollary 1.*

4.4. Optimal Trading Strategy for the Original Problem

Following the relationship between π and π^Z in (25), in this sub-section, we can obtain the optimal investment strategy for the original optimization problem (P).

Corollary 2. *Based on relationships in (23) and (25), the solution of the optimal investment strategy for the original problem* (P) *can be shown as*

$$\hat{\pi}_S(t) = \hat{\pi}_S^Z(t) - \pi_S^F(t) = -\frac{\beta}{\sigma_{S_2}} e^{\int_0^t r(s)\mathrm{d}s} \widetilde{\mathbb{E}}\Bigg\{ e^{-2\int_0^T r(s)\mathrm{d}s} \zeta_T I'(\beta e^{-\int_0^T r(s)\mathrm{d}s} \zeta_T) \times \\ \left(\int_t^T \phi(s)\phi^{-1}(t)(\sigma_\mu \rho + \gamma(t))\mathrm{d}\overline{W}_S(s) + m(t) \right) \Big| \mathcal{F}_t^{S,r} \Bigg\} - \frac{\sigma_{C_2}}{\sigma_{S_2}} \int_t^T D(t,s)\mathrm{d}s,$$

$$\begin{aligned} \hat{\pi}_K(t) &= \hat{\pi}_K^Z(t) - \pi_K^F(t) + \pi_K^G(t) \\ &= -\frac{\sigma_r(e^{-a(T-t)} - 1)}{a\sigma_K} \left(\hat{X}^\pi(t) + \int_t^T D(t,s)\mathrm{d}s - \int_T^w f(s)B(t,s)_{s-T}p_T \mathrm{d}s \right) \\ &\quad - e^{\int_0^t r(s)\mathrm{d}s} \widetilde{\mathbb{E}}\Bigg[\beta e^{-2\int_0^T r(s)\mathrm{d}s} \zeta_T I'(\beta e^{-\int_0^T r(s)\mathrm{d}s} \zeta_T) \Bigg(\frac{\sigma_r(e^{-a(T-t)} - 1)}{a\sigma_K} + \frac{\lambda_r}{\sigma_K} \\ &\quad - \frac{\sigma_{S_1}}{\sigma_K \sigma_{S_2}} \left(\int_t^T \phi(s)\phi^{-1}(t)(\sigma_\mu \rho + \gamma(t))\mathrm{d}\overline{W}_S(s) + m(t) \right) \Bigg) \Big| \mathcal{F}_t^{S,r} \Bigg] \\ &\quad - \frac{\int_t^T \sigma_{S_2} D(t,s)(\sigma_{C_1} - f_2(s-t)\sigma_r)\mathrm{d}s - \sigma_{S_1}\sigma_{C_2} F(t,T)}{\sigma_{S_2}\sigma_K} \\ &\quad + \int_T^w f(s)B(t,s)_{s-T}p_T \frac{\sigma_B(s-t)}{\sigma_K} \mathrm{d}s. \end{aligned}$$

Similarly, for the CRRA utility, we derive the following results.

Corollary 3. *When u is a CRRA function in (52), the optimal trading strategy is given by*

$$\hat{\pi}_S(t) = \frac{\beta^{-\frac{1}{1-\delta}}}{\sigma_{S_2}(1-\delta)} e^{\int_0^t r(s)\mathrm{d}s} \widetilde{\mathbb{E}}\Bigg[e^{\frac{\delta}{1-\delta}\int_0^T r(s)\mathrm{d}s} \zeta_T^{-\frac{1}{1-\delta}} \times \\ \left(\int_t^T \phi(s)\phi^{-1}(t)(\sigma_\mu \rho + \gamma(t))\mathrm{d}\overline{W}_S(s) + m(t) \right) \Big| \mathcal{F}_t^{S,r} \Bigg] - \frac{\sigma_{C_2}}{\sigma_{S_2}} \int_t^T D(t,s)\mathrm{d}s,$$

$$\begin{aligned} \hat{\pi}_K(t) = {} & \frac{\beta^{-\frac{1}{1-\delta}}}{\sigma_K(1-\delta)} e^{\int_0^t r(s)\mathrm{d}s} \Bigg\{ \left(\lambda_r + \frac{\delta\gamma\sigma_r(e^{-a(T-t)} - 1)}{a} \right) \widetilde{\mathbb{E}}\left[e^{\frac{\delta}{1-\delta}\int_0^T r(s)\mathrm{d}s} \zeta_T^{-\frac{1}{1-\delta}} \Big| \mathcal{F}_t^{S,r} \right] \\ & - \frac{\sigma_{S_1}}{\sigma_{S_2}} \widetilde{\mathbb{E}}\left[e^{\frac{\delta}{1-\delta}\int_0^T r(s)\mathrm{d}s} \zeta_T^{-\frac{1}{1-\delta}} \left(\int_t^T \phi(s)\phi^{-1}(t)(\sigma_\mu \rho + \gamma(t))\mathrm{d}\overline{W}_S(s) + m(t) \right) \Big| \mathcal{F}_t^{S,r} \right] \Bigg\} \\ & - \frac{\int_t^T \sigma_{S_2} D(t,s)(\sigma_{C_1} - f_2(s-t)\sigma_r)\mathrm{d}s - \sigma_{S_1}\sigma_{C_2} F(t,T)}{\sigma_{S_2}\sigma_K} \\ & + \int_T^w f(s)B(t,s)_{s-T}p_T \frac{\sigma_B(s-t)}{\sigma_K} \mathrm{d}s. \end{aligned}$$

5. Conclusions

In this paper, we investigate an optimal investment problem of a DC pension scheme under partial information. The fund manager is allowed to invest the wealth from the fund account into a financial market consisting of a risk-free account, a stock and a rolling bond. The drift of the stock price process is modeled by a mean-reverting stochastic process. In the model, we also take into account the minimum guarantee and stochastic contribution rate. The fund manager aims to maximize the expected utility of the terminal fund. We assume that the only information that can be observed by the fund manager is the stock price $S(t)$ and the stochastic interest rate $r(t)$, but that the drift term $\mu(t)$ in the stock price is unobservable. Obviously, the problem we consider is not self-financing, and we also require that the amount of the terminal fund should be greater than a minimum guarantee.

To overcome these difficulties, we first transform the original problem into a self-financing, unconstrained auxiliary problem, then use the martingale method and Clark's formula to obtain the expressions of the optimal investment strategy. In future work, we plan to continue to explore the optimal investment problem in DC pension schemes using the framework provided in this paper but operating under the assumption that both the drift and the volatility of the stock price are stochastic and unobservable. To obtain explicit expressions of the optimal trading strategy, we will combine the martingale method and the stochastic dynamic programming method to analyze the problem and consider some special cases.

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Appendix A

In this section, we provide the proof of Theorem 2. We define the gradient operator D acting on a subset of the class of functions of $\{\widetilde{W}(t),\ t \le T\}$ as $D_{1,1}$. For detailed explanations of the definitions of the space $D_{1,1}$ and the operator D, we refer to Ocone and Karatzas (1991) and Shigekawa (1980). We introduce Clark's formula (refer to Karatzas et al. (1991)). Clark's formula guarantees that, for any stochastic variable $A \in D_{1,1}$, we have the following expression:

$$\widetilde{\mathbb{E}}[A \mid \mathcal{F}_t^{S,r}] = \widetilde{\mathbb{E}}A + \int_0^t \widetilde{\mathbb{E}}[(D_u A)^{\mathrm{T}} \mid \mathcal{F}_u^{S,r}]\mathrm{d}\widetilde{W}(u). \tag{A1}$$

For an N-dimensional random variable $A \in (D_{1,1})^N$ and N-dimensional Brownian motion $B(t)$, define DA as a matrix with components $(DA)_{i,j} = D^i A_j, i,j = 1,2,\cdots,N$.

The following lemmas provide major steps for finding the optimal trading strategies under partial information. Since the proofs are closely related to those in Lakner (1998) and Ocone and Karatzas (1991), we omit them.

Lemma A1. *For every $u \in [0,T]$, $m(u) \in D_{1,1}$ and*

$$D_t m(u) = \phi(u)\phi^{-1}(t)(\sigma_\mu\rho + \gamma(t))1_{\{t\le u\}}.$$

Denote $F = f(\widetilde{W}_r(t), \widetilde{W}_S(t)) = (\lambda_r, m(t))$ and $DF = (D^1F, D^2F)$ with components

$$D^i F = \frac{\partial f}{\partial x^i}, \qquad i = 1,2,$$

where "∂" denotes a partial differential operator. Using Malliavin derivatives, we have

$$\begin{aligned} D_t\begin{pmatrix} \lambda_r \\ m(u) \end{pmatrix} &= \begin{pmatrix} \frac{\partial\lambda_r}{\partial\widetilde{W}_r(u)} & \frac{\partial\lambda_r}{\partial\widetilde{W}_S(u)} \\ \frac{\partial m(u)}{\partial\widetilde{W}_r(u)} & \frac{\partial m(u)}{\partial\widetilde{W}_S(u)} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & \phi(u)\phi^{-1}(t)(\sigma_\mu\rho + \gamma(t)) \end{pmatrix}, \end{aligned} \tag{A2}$$

also, by (13), we have

$$D_t \int_0^T r(u)\mathrm{d}u = \int_t^T D_t r(u)\mathrm{d}u = \int_t^T \begin{pmatrix} \frac{\partial r(u)}{\partial \widetilde{W}_r(u)} \\ \frac{\partial r(u)}{\partial \widetilde{W}_S(u)} \end{pmatrix} \mathrm{d}u \tag{A3}$$
$$= \begin{pmatrix} \int_t^T \left(- \int_t^u \sigma_r D_t e^{a(s-u)} \mathrm{d}\widetilde{W}_r(s) - \sigma_r e^{a(t-u)} \right) \mathrm{d}u \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} -\int_t^T \sigma_r e^{a(t-u)} \mathrm{d}u \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\sigma_r (e^{-a(T-t)} - 1)}{a} \\ 0 \end{pmatrix}.$$

Note that

$$m(u) \in D_{1,1},$$
$$m^2(u) \in D_{1,1},$$
$$(\lambda_r, m(u)) \in (D_{1,1})^2.$$

Lemma A2. *The following relations hold:*

$$\widetilde{\mathbb{E}}[\|(\lambda_r, m(u))\|^i] < \infty, \quad i = 1,2,3,4,$$
$$\widetilde{\mathbb{E}}\left[\int_0^T \|2m(u) D_t m(u)\|^2 \mathrm{d}u\right]^{\frac{1}{2}} < \infty,$$
$$\sup_{u \le T} \widetilde{\mathbb{E}}[\|(\lambda_r, m(u))\|^q] < \infty, \quad \textit{for every } u \in [0,T] \textit{ and some } q > 1,$$
$$\sup_{u,t \le T} \widetilde{\mathbb{E}}\left[\int_0^T \left| D_t^j \|(\lambda_r, m(u))\|^2 \right|^4 \mathrm{d}u\right] < \infty, \quad j = 1,2,$$
$$\widetilde{\mathbb{E}}\left[\int_0^T \|(\lambda_r, m(u))\|^2 \mathrm{d}u\right]^{\frac{1}{2}} < \infty,$$
$$\widetilde{\mathbb{E}}\zeta_T < \infty,$$

in which $||\cdot||$ is the Euclidean norm.

Introduce

$$V_1 = -\int_0^T (\lambda_r, m(u)) \mathrm{d}\widetilde{W}(u) \quad , \quad V_2 = \frac{1}{2}\int_0^T \left\| \begin{pmatrix} \lambda_r \\ m(u) \end{pmatrix} \right\|^2 \mathrm{d}u.$$

Lemma A3. *Both V_1 and V_2 are members of $D_{1,1}$, and*

$$D_t V_1 = -\int_t^T \left[D_t \begin{pmatrix} \lambda_r \\ m(u) \end{pmatrix} \right] \mathrm{d}\widetilde{W}(u) - \begin{pmatrix} \lambda_r \\ m(t) \end{pmatrix}, \tag{A4}$$

$$D_t V_2 = \int_t^T \left[\left(D_t \begin{pmatrix} \lambda_r \\ m(u) \end{pmatrix} \right) \begin{pmatrix} \lambda_r \\ m(u) \end{pmatrix} \right] \mathrm{d}u, \tag{A5}$$

where $D_t \begin{pmatrix} \lambda_r \\ m(u) \end{pmatrix}$ is given in (A2).

Lemma A4. *The following relations hold:*

$$\widetilde{\mathbb{E}}\left(\int_0^T \|D_t V_1\|^4 \mathrm{d}t \right) < \infty \tag{A6}$$

and

$$\widetilde{\mathbb{E}}\left(\int_0^T \|D_t V_2\|^4 \mathrm{d}t\right) < \infty. \tag{A7}$$

By applying (A6) and (A7), we have the following result.

Lemma A5. *The random variable ζ_T in (29) is a member of $D_{1,1}$ and*

$$D_t\zeta_T = \zeta_T(D_t V_1 + D_t V_2). \tag{A8}$$

Furthermore, by considering (A2), (A4), (A5), (A8) and (49), we obtain that

$$\begin{aligned} D_t\zeta_T =& \zeta_T\left[-\int_t^T \begin{pmatrix} 0 & 0 \\ 0 & \phi(u)\phi^{-1}(t)(\sigma_\mu\rho+\gamma(t)) \end{pmatrix}\begin{pmatrix} \mathrm{d}\overline{W}_r(u) \\ \mathrm{d}\overline{W}_S(u) \end{pmatrix} - \begin{pmatrix} \lambda_r \\ m(t) \end{pmatrix}\right] \\ =& \zeta_T\begin{pmatrix} -\lambda_r \\ -\int_t^T \phi(u)\phi^{-1}(t)(\sigma_\mu\rho+\gamma(t))\mathrm{d}\overline{W}_S(u) - m(t) \end{pmatrix}. \end{aligned} \tag{A9}$$

So far, we have found the explicit expression of $D_t\zeta_T$. However, in addition to ζ_T, $I(\beta e^{-\int_0^T r(u)\mathrm{d}u}\zeta_T)$ and $e^{-\int_0^T r(u)\mathrm{d}u}$ also have random terms, so we still need to calculate $D_t I(\beta e^{-\int_0^T r(u)\mathrm{d}u}\zeta_T)$ and $D_t e^{-\int_0^T r(u)\mathrm{d}u}$. We first introduce the following auxiliary result when we prove Theorem 2.

Lemma A6. *Given ζ_T in (27), the function I as the inverse of the derivative of the utility function u, and with (50) and (51) for I and $-I'$, the following four relations hold:*

$$\widetilde{\mathbb{E}}\left[I(\beta e^{-\int_0^T r(u)\mathrm{d}u}\zeta_T)\right] < \infty, \tag{A10}$$

$$\begin{aligned} &\widetilde{\mathbb{E}}\bigg(\int_0^T \Big\|\beta\zeta_T I'(\beta e^{-\int_0^T r(u)\mathrm{d}u}\zeta_T)D_t e^{-\int_0^T r(u)\mathrm{d}u} \\ &\qquad + \beta e^{-\int_0^T r(u)\mathrm{d}u} I'(\beta e^{-\int_0^T r(u)\mathrm{d}u}\zeta_T)D_t\zeta_T\Big\|^2 \mathrm{d}t\bigg)^{\frac{1}{2}} < \infty, \end{aligned} \tag{A11}$$

$$\widetilde{\mathbb{E}}\left[e^{-\int_0^T r(u)\mathrm{d}u} I(\beta e^{-\int_0^T r(u)\mathrm{d}u}\zeta_T)\right] < \infty \tag{A12}$$

and

$$\begin{aligned} &\widetilde{\mathbb{E}}\bigg(\int_0^T \Big\|I(\beta e^{-\int_0^T r(u)\mathrm{d}u}\zeta_T)D_t e^{-\int_0^T r(u)\mathrm{d}u} \\ &\qquad + e^{-\int_0^T r(u)\mathrm{d}u} D_t I(\beta e^{-\int_0^T r(u)\mathrm{d}u}\zeta_T)\Big\|^2 \mathrm{d}t\bigg)^{\frac{1}{2}} < \infty. \end{aligned} \tag{A13}$$

Proof of Theorem 2. In this proof, we use Lemmas A1–A6 and Lemma A.1 in Ocone and Karatzas (1991). From Lemma A.1 in Ocone and Karatzas (1991) and (A10)–(A13), for every $\beta \in (0,\infty)$, we obtain four results:

$$\begin{aligned} & e^{-\int_0^T r(u)\mathrm{d}u} I(\beta e^{-\int_0^T r(u)\mathrm{d}u}\zeta_T) \in D_{1,1}, \\ & D_t e^{-\int_0^T r(u)\mathrm{d}u} I(\beta e^{-\int_0^T r(u)\mathrm{d}u}\zeta_T) \\ =& I(\beta e^{-\int_0^T r(u)\mathrm{d}u}\zeta_T)D_t e^{-\int_0^T r(u)\mathrm{d}u} + e^{-\int_0^T r(u)\mathrm{d}u} D_t I(\beta e^{-\int_0^T r(u)\mathrm{d}u}\zeta_T), \\ & I(\beta e^{-\int_0^T r(u)\mathrm{d}u}\zeta_T) \in D_{1,1} \end{aligned} \tag{A14}$$

and

$$D_t I(\beta e^{-\int_0^T r(u)\mathrm{d}u}\zeta_T) = \beta\zeta_T I'(\beta e^{-\int_0^T r(u)\mathrm{d}u}\zeta_T) D_t e^{-\int_0^T r(u)\mathrm{d}u} + \beta e^{-\int_0^T r(u)\mathrm{d}u} I'(\beta e^{-\int_0^T r(u)\mathrm{d}u}\zeta_T) D_t\zeta_T. \tag{A15}$$

By substituting $e^{-\int_0^T r(s)\mathrm{d}s} I(\beta e^{-\int_0^T r(s)\mathrm{d}s}\zeta_T)$ for A in (A1), we have

$$\widetilde{\mathbb{E}}[e^{-\int_0^T r(s)\mathrm{d}s} I(\beta e^{-\int_0^T r(s)\mathrm{d}s}\zeta_T) \mid \mathcal{F}_t^{S,r}] = \widetilde{\mathbb{E}}[e^{-\int_0^T r(s)\mathrm{d}s} I(\beta e^{-\int_0^T r(s)\mathrm{d}s}\zeta_T)] + \int_0^t \underbrace{\widetilde{\mathbb{E}}\left\{ \left[D_u\left(e^{-\int_0^T r(s)\mathrm{d}s} I\left(\beta e^{-\int_0^T r(s)\mathrm{d}s}\zeta_T\right)\right)\right]^{\mathrm{T}} \mid \mathcal{F}_u^{S,r} \right\}}_{C} \mathrm{d}\widetilde{W}(u). \tag{A16}$$

For the second item on the right-hand side of (A16), by (A3), (A9), (A14) and (A15), we have

$$\begin{aligned} C = \widetilde{\mathbb{E}}\Bigg[& e^{-\int_0^T r(s)\mathrm{d}s} I(\beta e^{-\int_0^T r(s)\mathrm{d}s}\zeta_T) \left(-\frac{\sigma_r(e^{-a(T-u)}-1)}{a}, 0 \right) \\ & + \beta e^{-2\int_0^T r(s)\mathrm{d}s}\zeta_T I'(\beta e^{-\int_0^T r(s)\mathrm{d}s}\zeta_T) \left(-\frac{\sigma_r(e^{-a(T-u)}-1)}{a}, 0 \right) \Bigg| \mathcal{F}_u^{S,r} \Bigg] \\ & + \widetilde{\mathbb{E}}\Big[\beta e^{-2\int_0^T r(s)\mathrm{d}s}\zeta_T I'(\beta e^{-\int_0^T r(s)\mathrm{d}s}\zeta_T) \times \\ & \left(-\lambda_r, -\int_u^T \phi(s)\phi^{-1}(u)(\sigma_\mu\rho + \gamma(u))\mathrm{d}\overline{W}_S(s) - m(u) \right) \Big| \mathcal{F}_u^{S,r} \Big]. \end{aligned}$$

Finally, by (39) and (A16), we complete our proof. □

Notes

1 Here the bold formatting represents a 2×1 zero vector.

2 Inequalities (50) and (51) are used to guarantee the inequalities (A10)–(A13) in the Appendix A hold.

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