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Estimating Copula-Based Extension of Tail Value-at-Risk and Its Application in Insurance Claim

Khreshna Syuhada ^{1,*} , Oki Neswan ² and Bony Parulian Josaphat ¹¹ Statistics Research Division, Institut Teknologi Bandung, Bandung 40132, Indonesia; b_marbun@s.itb.ac.id² Analysis and Geometry Research Division, Institut Teknologi Bandung, Bandung 40132, Indonesia; oneswan@math.itb.ac.id

* Correspondence: khreshna@math.itb.ac.id

Abstract: Dependent Tail Value-at-Risk, abbreviated as DTVaR, is a copula-based extension of Tail Value-at-Risk (TVaR). This risk measure is an expectation of a target loss once the loss and its associated loss are above their respective quantiles but bounded above by their respective larger quantiles. In this paper, we propose nonparametric estimators for DTVaR and establish their property of consistency. Moreover, we also propose the variability measure around this expected value truncated by the quantiles, called the Dependent Conditional Tail Variance (DCTV). We use this measure for constructing confidence intervals of the DTVaR. Both parametric and nonparametric approaches for DTVaR estimations are explored. Furthermore, we assess the performance of DTVaR estimations using a proposed backtest based on the DCTV. As for the numerical study, we take an application in the insurance claim amount.

Keywords: Dependent TVaR (DTVaR); Dependent Conditional Tail Variance (DCTV); insurance claim; nonparametric estimators



Citation: Syuhada, Khreshna, Oki Neswan, and Bony Parulian Josaphat. 2022. Estimating Copula-Based Extension of Tail Value-at-Risk and Its Application in Insurance Claim. *Risks* 10: 113. <https://doi.org/10.3390/risks10060113>

Academic Editor: Mogens Steffensen

Received: 24 April 2022

Accepted: 24 May 2022

Published: 30 May 2022

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1. Introduction

In actuarial science, several risk measures have been proposed; the two most well-known are the Value-at-Risk (VaR) and the Tail Value-at-Risk (TVaR). Several authors have proposed (or compared) nonparametric estimators for VaR and TVaR; see [Chang et al. \(2003\)](#); [Brazauskas et al. \(2008\)](#); [Kaiser and Brazauskas \(2006\)](#); [Methni et al. \(2014\)](#); [Dutta and Biswas \(2018\)](#) and [Shen et al. \(2019\)](#).

In particular, [Chang et al. \(2003\)](#) introduced three types of VaR nonparametric estimation methods and their corresponding confidence intervals. [Brazauskas et al. \(2008\)](#) and [Kaiser and Brazauskas \(2006\)](#) proposed point and interval estimators for TVaR, as well as proved the consistency of the point estimator. [Methni et al. \(2014\)](#) combined nonparametric kernel methods with extreme-value statistics to find the estimator for TVaR. [Dutta and Biswas \(2018\)](#) compared the performance of nonparametric estimators of TVaR for varying p , namely the empirical estimator, kernel-based estimator, Brazauskas et al.'s estimator, tail-trimmed estimator by Hill, Yamai and Yoshiba's estimator and the filtered historical method. [Shen et al. \(2019\)](#) established empirical likelihood-based estimation with high-order precision for TVaR.

Several extensions of TVaR have also been developed. [Jadhav et al. \(2013\)](#); [Wang and Wei \(2020\)](#); [Bairakdar et al. \(2020\)](#) and [Bernard et al. \(2020\)](#) have modified TVaR by introducing a fixed boundary, instead of infinity, for values beyond the quantile (i.e., VaR). In particular, [Jadhav et al. \(2013\)](#) named the modified risk measure as Modified TVaR (MTVaR). Meanwhile, another extension of TVaR, called Copula TVaR (CTVaR), was suggested by [Brahim et al. \(2018\)](#), in which, they estimate a target loss¹ by involving another dependent or associated loss.

Motivated by the work of [Jadhav et al. \(2013\)](#) and [Brahim et al. \(2018\)](#); [Josaphat and Syuhada \(2021\)](#) proposed an alternative coherent risk measure that is not only “considering

a fixed upper bound of loss beyond VaR” but also “taking into account an associated loss”, called Dependent TVaR (DTV_aR). Moreover, Josaphat et al. (2021) proposed an optimization method for DTV_aR by applying two metaheuristic algorithms: Spiral Optimization (SpO) and Particle Swarm Optimization (PSO). When we calculate an MTV_aR estimate, it will subtract the number of losses beyond VaR and thus make this estimate smaller than the corresponding TVaR. This is a good feature in risk modeling. We argue that this estimate must also be accompanied by an associated risk since this risk scenario occurs in practice; see, for instance, Zhang et al. (2019) and Kang et al. (2019).

The DTV_aR can comprehend the connection between bivariate losses and help us to optimally position our investments and enlarge our financial risk protection (Josaphat and Syuhada 2021). In other words, employing the suggested risk measure will enable us to avoid non-essential additional capital allocation while not ignoring other risks associated with the target risk. In this paper, we propose two nonparametric estimators for the risk measure of DTV_aR by following the approaches of Brazauskas et al. (2008) and Jadhav et al. (2013). These estimators are proven to be consistent.

Although the DTV_aR serves crucial information on the tail distribution of the target loss, the necessity for other risk measures came up in competitive and unpredictable market environments. Principally, realizing that the DTV_aR, being the tail mean, is not able to capture the tail variability, we propose a second tail moment or variance in the tail distribution truncated by two pair of VaRs that is called Dependent Conditional Tail Variance (DCTV). This measure can concatenate the dissemination in the tail. Moreover, the DCTV can be considered a generalization of Conditional Tail Variance (CTV) proposed by Furman and Landsman (2006). Using DCTV, we are able to prove the asymptotic normality of DTV_aR and even derive confidence intervals for the DTV_aR estimators. Just as Righi and Ceretta (2015) used CTV for the backtesting of TVaR estimations using the bootstrap method, we also use DCTV for the backtesting of DTV_aR estimations.

The rest of the paper is organized as follows. In Section 2, we briefly explain the novel risk measure of dependent tail VaR. The nonparametric estimation of DTV_aR is discussed in Section 3, whereas the truncated variance, called the dependent conditional tail variance, is presented in Section 4. Section 5 presents the parametric estimate of DTV_aR in a Pareto case. The choice of the contraction parameters that appear in the definition of DTV_aR is considered in Section 6. Conclusions are discussed in Section 7. All mathematical proofs are deferred to Appendix A.

2. The Dependent Tail Value-at-Risk

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space, and L^1 be the set of real integrable random variables (i.e., random variables with finite means) defined on $(\Omega, \mathcal{F}, \mathbb{P})$. A risk measure is a functional $\rho : L^1 \rightarrow \mathbb{R}$.

Consider that X and Y are two random losses that are dependent and have marginal distribution functions F_X and F_Y . Provided a value $\alpha \in (0, 1)$, generally close to one, the VaR of X at a probability level α is the quantile Q_α of F_X for this level. Mathematically, the VaR is defined as follows:

$$Q_\alpha = F_X^{-1}(\alpha). \quad (1)$$

Based on this definition, we can note that the VaR does not consider information after the quantile of interest, only the point itself. Moreover, despite its simplicity and ease of implementation, VaR has the shortcoming of not being a coherent risk measure in the sense of Artzner et al. (1999). The TVaR at probability level α is then the expectation of X once X is above the VaR for this level, i.e., an extreme loss. Formally, Formulation (2) defines TVaR.

$$\text{TVaR}_\alpha(X) = E[X|X \geq Q_\alpha(X)] = \frac{1}{1-\alpha} \int_\alpha^1 Q_p(X) dp. \quad (2)$$

Note that $1 - \alpha$ in (2) is the significance level for TVaR.

As we state in Section 1; Josaphat and Syuhada (2021) proposed another risk measure as a generalization of TVaR that not only considers the information about the potential size of the loss X between two quantiles but also takes into account the excess of another loss Y that is associated with X . Formally, Formulation (3) defines DTVaR.

$$\text{DTV}\text{aR}_{(\alpha,a)}^{(\delta,d)}(X|Y) = E[X|Q_\alpha \leq X \leq Q_{\alpha_1}, Q_\delta \leq Y \leq Q_{\delta_1}], \quad (3)$$

where Y is another loss that is associated with X (or X depends on Y), $\alpha_1 = \alpha + (1 - \alpha)^{1+a}$, $\delta_1 = \delta + (1 - \delta)^{1+d}$ and $a, d \geq 0$. Here, α and δ denote the probability level and excess level, respectively. Moreover, X is called the target loss, whereas Y is called the associated loss. In the sequel, two lemmas related to the DTVaR are given.

Lemma 1 (Josaphat and Syuhada 2021). *Let X and Y be two random losses with a joint probability function $f_{X,Y}$. Let $\alpha, \delta \in (0, 1)$ and $a, d \geq 0$ be specified numbers. The Dependent Tail VaR (DTV aR) of X given values beyond its VaR up to a fixed value of losses and a random loss Y is given by*

$$\text{DTV}\text{aR}_{(\alpha,a)}^{(\delta,d)}(X|Y) = \frac{\int_{Q_\delta}^{Q_{\delta_1}} \int_{Q_\alpha}^{Q_{\alpha_1}} x f_{X,Y}(x,y) dx dy}{\int_{Q_\delta}^{Q_{\delta_1}} \int_{Q_\alpha}^{Q_{\alpha_1}} f_{X,Y}(x,y) dx dy}, \quad (4)$$

where $Q_\alpha = Q_\alpha(X)$, $Q_\delta = Q_\delta(Y)$, $\alpha_1 = \alpha + (1 - \alpha)^{a+1}$ and $\delta_1 = \delta + (1 - \delta)^{d+1}$.

In practice, a joint probability function is difficult to find unless a bivariate normal distribution is assumed. For the case of joint exponential distribution, we may refer to Kang et al. (2019) for Sarmanov's bivariate exponential distribution. In most cases, two or more dependent losses rely on a copula in order to have an explicit formula of its joint distribution.

Lemma 2 (Josaphat and Syuhada 2021). *Let X and Y be two random losses with a joint distribution function represented by a copula C . Let $\alpha, \delta \in (0, 1)$ and $a, d \geq 0$ be specified numbers. The Dependent Tail VaR (DTV aR) of X given values beyond its VaR up to a fixed value of losses and a random loss Y is given by*

$$\text{DTV}\text{aR}_{(\alpha,a)}^{(\delta,d)}(X|Y;C) = \frac{\int_{\alpha}^{\alpha_1} \int_{\delta}^{\delta_1} F_X^{-1}(u) c(u,v;\theta) dv du}{C(\alpha_1, \delta_1; \theta) - C(\alpha, \delta_1; \theta) - C(\alpha_1, \delta; \theta) + C(\alpha, \delta; \theta)}, \quad (5)$$

where F_X^{-1} denotes the quantile function of X , $u = F_X(x)$, $v = F_Y(y)$, $\alpha_1 = \alpha + (1 - \alpha)^{a+1}$ and $\delta_1 = \delta + (1 - \delta)^{d+1}$.

The following property applies to DTVaR. The property states that the DTVaR is a law-invariant convex risk measure.

Property 1. *The Dependent Tail VaR (DTV aR) is a law-invariant risk measure.*

3. The Estimation of DTVaR

When dealing with real data, it is not always easy for us to know the distribution of the data, even if we use software for fitting distribution. As a result, estimating DTVaR is also not easy. To avoid the difficulty of the parametric estimation of DTVaR, we propose a nonparametric one.

We propose two nonparametric estimators of the DTVaR. The first empirical estimator of $\text{DTV}\text{aR}_{(\alpha,a)}^{(\delta,d)}(X|Y;C)$ is defined as follows. Let $(X_1, Y_1), \dots, (X_m, Y_m)$ be a collection of

random vectors with size m , where $X_1, \dots, X_m$ and $Y_1, \dots, Y_m$ are independently and identically distributed (iid) random losses, respectively. Suppose that $F_{X,Y}^{m,m}$ denotes the corresponding empirical joint distribution function, which is given by

$$F_{X,Y}^{m,m}(x,y) = \frac{1}{m} \sum_{j=1}^m I(X_j \leq x, Y_j \leq y),$$

where $I(\cdot, \cdot)$ denotes the indicator function. In addition, suppose that F_X^m and F_Y^m denote the empirical marginal distribution functions of iid $X_1, \dots, X_m$ and iid $Y_1, \dots, Y_m$, which are given by

$$F_X^m(x) = \frac{1}{m} \sum_{j=1}^m I(X_j \leq x), \quad F_Y^m(y) = \frac{1}{m} \sum_{l=1}^m I(Y_l \leq y).$$

Suppose that F_X and F_Y denote the unknown distribution functions of X and Y .

If the vectors $(X_1, Y_1), \dots, (X_m, Y_m)$ are rearranged by considering the ascending order of X_j , $j = 1, \dots, m$, then we obtain new vectors $(X_{m(1)}, Y'_{m(1)}), \dots, (X_{m(m)}, Y'_{m(m)})$. It is obvious that $X_{m(1)} \leq X_{m(2)} \leq \dots \leq X_{m(m)}$ are order statistics of $X_1, \dots, X_m$. However, the statistics $Y'_{m(j)}$, $j = 1, \dots, m$, are not necessarily order statistics of $Y_1, \dots, Y_m$. Furthermore, the quantiles $F_X^{-1}(\alpha)$, $F_X^{-1}(\alpha_1)$, $F_Y^{-1}(\delta)$ and $F_Y^{-1}(\delta_1)$, respectively, can be consistently estimated by

$$\begin{aligned} F_X^{m(-1)}(\alpha) &= X_{m(j)}, \quad \alpha \in \left(\frac{j-1}{m}, \frac{j}{m}\right], \\ F_X^{m(-1)}(\alpha_1) &= X_{m(j_1)}, \quad \alpha_1 \in \left(\frac{j_1-1}{m}, \frac{j_1}{m}\right], \\ F_Y^{m(-1)}(\delta) &= Y_{m(l)}, \quad \delta \in \left(\frac{l-1}{m}, \frac{l}{m}\right], \\ F_Y^{m(-1)}(\delta_1) &= Y_{m(l_1)}, \quad \delta_1 \in \left(\frac{l_1-1}{m}, \frac{l_1}{m}\right], \end{aligned}$$

where $j, l = 1, \dots, m$, and $j_1 > j$, $l_1 > l$. Hence, for $i = j, \dots, j_1$, the estimator $\widehat{\text{DTVaR}}_{(\alpha,a)}^{(\delta,d)(1)}(X|Y;C)$ is given by

$$\begin{aligned} &\widehat{\text{DTVaR}}_{(\alpha,a)}^{(\delta,d)(1)}(X|Y;C) \\ &= \frac{\int_{\delta}^{\delta_1} \int_{\alpha}^{\alpha_1} F_X^{m(-1)}(u) c(u,v) du dv}{P(F_X^{m(-1)}(\alpha) \leq X_{m(i)} \leq F_X^{m(-1)}(\alpha_1), F_Y^{m(-1)}(\delta) \leq Y'_{m(i)} \leq F_Y^{m(-1)}(\delta_1))} \\ &= \frac{1}{P(F_X^{m(-1)}(\alpha) \leq X_{m(i)} \leq F_X^{m(-1)}(\alpha_1), F_Y^{m(-1)}(\delta) \leq Y'_{m(i)} \leq F_Y^{m(-1)}(\delta_1))} \\ &\quad \times \iint_{\substack{(F_X^{m(-1)}(\alpha) \leq X_{m(i)} \leq F_X^{m(-1)}(\alpha_1)) \\ \times (F_Y^{m(-1)}(\delta) \leq Y'_{m(i)} \leq F_Y^{m(-1)}(\delta_1))}} x dF_{X,Y}^{m,m}(x,y). \end{aligned} \quad (6)$$

Consider a square $(0, 1]^2$ originating from two intervals $(0, 1]$. Subdivide each of both intervals $(0, 1]$ into m subintervals $(\frac{j-1}{m}, \frac{j}{m}]$ and m subintervals $(\frac{l-1}{m}, \frac{l}{m}]$, $j, l = 1, \dots, m$, so that we obtain m^2 small squares $(\frac{j-1}{m}, \frac{j}{m}] \times (\frac{l-1}{m}, \frac{l}{m}]$. When $\alpha \in (\frac{j-1}{m}, \frac{j}{m}]$ and $\alpha_1 = \alpha + (1 - \alpha)^{1+a} \in (\frac{j-1}{m}, \frac{j}{m}]$, then we have that $F_X^{m(-1)}(\alpha) = X_{m(j)}$ and $F_X^{m(-1)}(\alpha_1) = X_{m(j_1)}$.

Similarly, when $\delta \in (\frac{l-1}{m}, \frac{l}{m}]$ and $\delta_1 = \delta + (1 - \delta)^{1+d} \in (\frac{l-1}{m}, \frac{l}{m}]$, then we have that $F_Y^{m(-1)}(\delta) = Y_{m(l)}$ and $F_Y^{m(-1)}(\delta_1) = Y_{m(l_1)}$. Clearly, $F_X^m(F_X^{m(-1)}(\alpha)) = \frac{j}{m}$, $F_X^m(F_X^{m(-1)}(\alpha_1)) = \frac{j_1}{m}$, $F_Y^m(F_Y^{m(-1)}(\delta)) = \frac{l}{m}$ and $F_Y^m(F_Y^{m(-1)}(\delta_1)) = \frac{l_1}{m}$. Hence,

$$\begin{aligned} & \iint_{\substack{(F_X^{m(-1)}(\alpha) \leq X_{m(i)} \leq F_X^{m(-1)}(\alpha_1)) \\ \times (F_Y^{m(-1)}(\delta) \leq Y'_{m(i)} \leq F_Y^{m(-1)}(\delta_1))}} x \, dF_{X,Y}^{m,m}(x, y) \\ &= \frac{1}{m^2} \sum_{k=l}^{l_1} \sum_{i=j}^{j_1} X_{m(i)} \left\{ I(F_Y^m(Y'_{m(i)}) \leq F_Y^m(y_{m(k)})) - I(F_Y^m(Y'_{m(i-1)}) \leq F_Y^m(y_{m(k)})) \right. \\ & \quad \left. - I(F_Y^m(Y'_{m(i)}) \leq F_Y^m(y_{m(k-1)})) + I(F_Y^m(Y'_{m(i-1)}) \leq F_Y^m(y_{m(k-1)})) \right\}. \end{aligned} \quad (7)$$

Note that, in (7), we do not sum $Y'_{m(i)}$ but $X_{m(i)}$ paired with $Y'_{m(i)}$. To simplify the notation and computation, we sum $X_{m(i)}$ by applying the indicator function $I(y_{m(l)} \leq Y'_{m(i)} \leq y_{m(l_1)})$; thus, we obtain

$$\begin{aligned} & \iint_{\substack{(F_X^{m(-1)}(\alpha) \leq X_{m(i)} \leq F_X^{m(-1)}(\alpha_1)) \\ \times (F_Y^{m(-1)}(\delta) \leq Y'_{m(i)} \leq F_Y^{m(-1)}(\delta_1))}} x \, dF_{X,Y}^{m,m}(x, y) = \frac{1}{m} \sum_{i=j}^{j_1} X_{m(i)} I(y_{m(l)} \leq Y'_{m(i)} \leq y_{m(l_1)}). \end{aligned} \quad (8)$$

Next, note that

$$\begin{aligned} & P(F_X^{m(-1)}(\alpha) \leq X_{m(i)} \leq F_X^{m(-1)}(\alpha_1), F_Y^{m(-1)}(\delta) \leq Y'_{m(i)} \leq F_Y^{m(-1)}(\delta_1)) \\ &= P(x_{m(j)} \leq X_{m(i)} \leq x_{m(j_1)}, y_{m(l)} \leq Y'_{m(i)} \leq y_{m(l_1)}) = \frac{j_1 - j + 1 - r}{m}, \end{aligned} \quad (9)$$

where $x_{m(j)}$ and $y_{m(l)}$, respectively, denote the realizations of $X_{m(j)}$ and $Y_{m(l)}$, whilst,

$$\begin{aligned} r = \sum_{i=j}^{j_1} & \left[I(x_{m(j)} \leq X_{m(i)} \leq x_{m(j_1)}, Y'_{m(i)} < y_{m(l)}) \right. \\ & \left. + I(x_{m(j)} \leq X_{m(i)} \leq x_{m(j_1)}, Y'_{m(i)} > y_{m(l_1)}) \right]. \end{aligned}$$

From (8) and (9), we obtain

$$\widehat{\text{DTVaR}}_{(\alpha, a)}^{(\delta, d)(1)}(X|Y; C) = \frac{\sum_{i=j}^{j_1} X_{m(i)} I(y_{m(l)} \leq Y'_{m(i)} \leq y_{m(l_1)})}{j_1 - j + 1 - r}, \quad (10)$$

for all $\alpha \in (\frac{j-1}{m}, \frac{j}{m}]$, $\alpha_1 = \alpha + (1 - \alpha)^{1+a} \in (\frac{j-1}{m}, \frac{j}{m}]$, $\delta \in (\frac{l-1}{m}, \frac{l}{m}]$ and $\delta_1 = \delta + (1 - \delta)^{1+d} \in (\frac{l-1}{m}, \frac{l}{m}]$.

Note that, in a similar and simpler way, it can be shown that an estimator of MTVaR (proposed by Jadhav et al. 2013) is given by

$$\widehat{\text{MTVaR}}_{(\alpha, a)}(X) = \frac{\int_{\alpha}^{\alpha_1} F_X^{m(-1)}(u) du}{(1 - \alpha)^{1+a}} = \frac{1}{j_1 - j + 1} \sum_{i=j}^{j_1} X_{m(i)}, \quad (11)$$

for all $\alpha \in (\frac{j-1}{m}, \frac{j}{m}]$ and $\alpha_1 = \alpha + (1 - \alpha)^{1+a} \in (\frac{j_1-1}{m}, \frac{j_1}{m}]$. Note that $j_1 - j = \lceil m(1 - \alpha)^{1+a} \rceil$. If we adjust the index i in (11) to index q , then $\widehat{\text{MTVaR}}_{(\alpha,a)}(X)$ can be rewritten as (see [Jadhav et al. 2013](#))

$$\widehat{\text{MTVaR}}_{(\alpha,a)}^{(1)}(X) = \frac{\sum_{q=0}^{\lceil m(1-\alpha)^{1+a} \rceil} X_{(k(q))}}{\lceil m(1-\alpha)^{1+a} \rceil + 1}, \quad (12)$$

where $k(q) = \lceil m\alpha'_{(q)} \rceil$, $\alpha'_{(q)} = \alpha + \frac{q(1-\alpha)}{\lceil m(1-\alpha) \rceil}$, $q = 0, 1, \dots, \lceil (1-\alpha)^{1+a} \rceil$ and $\lceil x \rceil$ denotes the smallest integer that is larger than x .

Following the derivation method of the estimator $\widehat{\text{MTVaR}}_{(\alpha,a)}^{(1)}(X)$ in (12), we obtain

$$\begin{aligned} \widehat{\text{DTVaR}}_{(\alpha,a)}^{(\delta,d)(1)}(X|Y;C) &= \frac{1}{\lceil m(1-\alpha)^{1+a} \rceil + 1 - r} \\ &\times \sum_{q=0}^{\lceil m(1-\alpha)^{1+a} \rceil} X_{(k(q))} I(y_{m(l)} \leq Y'_{(k(q))} \leq y_{m(l_1)}), \end{aligned} \quad (13)$$

where $k(q) = \lceil m\alpha'_{(q)} \rceil$, $\alpha'_{(q)} = \alpha + \frac{q(1-\alpha)}{\lceil m(1-\alpha) \rceil}$, $q = 0, 1, \dots, \lceil (1-\alpha)^{1+a} \rceil$,

$$\begin{aligned} y_{m(l)} &= \hat{F}_Y^{m(-1)}(\delta), \quad \delta \in \left(\frac{l-1}{m}, \frac{l}{m}\right], \\ y_{m(l_1)} &= \hat{F}_Y^{m(-1)}(\delta_1), \quad \delta_1 \in \left(\frac{l_1-1}{m}, \frac{l_1}{m}\right], \\ \delta_1 &= \delta + (1-\delta)^{1+d}, \end{aligned}$$

r is the number of $X_{(k(q))}$ s paired with $Y'_{(k(q))}$ that does not satisfy $y_{m(l)} \leq Y'_{(k(q))} \leq y_{m(l_1)}$. The estimator given in (13) may be improved by considering a smoothed version, which is the second estimator, as follows:

$$\begin{aligned} \widehat{\text{DTVaR}}_{(\alpha,a)}^{(\delta,d)(2)}(X|Y;C) &= \frac{1}{\lceil m(1-\alpha)^{1+a} \rceil + 1 - r} \\ &\times \sum_{q=0}^{\lceil m(1-\alpha)^{1+a} \rceil} \{(1 - h_{k(q)})X_{(k(q))} + h_{k(q)}X_{(k(q)+1)}\} I(y_{m(l)} \leq Y'_{(k(q))} \leq y_{m(l_1)}), \end{aligned} \quad (14)$$

where $k(q) = \lceil m\alpha'_{(q)} \rceil$, $h_{k(q)} = \lceil m\alpha'_{(q)} \rceil - m\alpha'_{(q)}$.

In the following theorem, we prove the consistency of DTVaR estimators.

Theorem 1. The estimators $\widehat{\text{DTVaR}}_{(\alpha,0)}^{(\delta,0)(n)}(X|Y;C)$, $n = 1, 2$, given in (13) and (14) are consistent for every finite a , $d \geq 0$.

4. The Dependent Conditional Tail Variance and Confidence Intervals

In addition to the TVaR, some authors also consider the variability of the loss in the tail of the distribution. The notion is that, in spite of its practicality and desired properties, the TVaR only picks up the average loss in the tail and forsakes its variability, and thus it makes sense to concatenate the second tail moment or the variance in the tail distribution. In this regards, [Furman and Landsman \(2006\)](#) put forward the Tail Variance Premium (TVP) that contains Conditional Tail Variance (CTV),

$$\text{TVP}_\alpha(X) = \text{TVaR}_\alpha(X) + E[(X - \text{TVaR}_\alpha(X))^2 | X \geq Q_\alpha],$$

where the last term $E[(X - \text{TVaR}_\alpha(X))^2 | X \geq Q_\alpha]$ is called the CTV. Moreover, [Righi and Ceretta \(2015\)](#) used the square root of the CTV, instead of ordinary variance, for the backtesting of TVaR estimation using the bootstrap method.

Motivated by the CTV proposed by [Furman and Landsman \(2006\)](#), we propose a Dependent Conditional Tail Variance (DCTV) of target loss X associated with another loss Y ,

$$\text{DCTV}_{(\alpha,a)}^{(\delta,d)}(X|Y;C) = E[(X - \text{DTVaR}_{(\alpha,a)}^{(\delta,d)}(X|Y;C))^2 | Q_\alpha \leq X \leq Q_{\alpha_1}, Q_\delta \leq Y \leq Q_{\delta_1}]. \quad (15)$$

Thus, DCTV is a variability or dispersion measure around DTVaR truncated by the VaRs of target loss and associated loss. In addition, since DTVaR is a generalization of TVaR, then DCTV is also a generalization of CTV.

The DCTV of the target loss X computed under a fixed conditional probability $C(\alpha_1, \delta_1; \theta) - C(\alpha, \delta_1; \theta) - C(\alpha_1, \delta; \theta) + C(\alpha, \delta; \theta)$ with respect to the associated loss Y is given in the following lemma.

Lemma 3. Let X and Y be two random losses with a joint distribution function represented by a copula C . Let $\alpha, \delta \in (0, 1)$ and $a, d \geq 0$ be specified numbers. The Dependent Conditional Tail Variance (DCTV) of X given values beyond its VaR up to a fixed value of losses and a random loss Y is given by

$$\text{DCTV}_{(\alpha,a)}^{(\delta,d)}(X|Y;C) = \frac{\int_{\alpha}^{\alpha_1} \int_{\delta}^{\delta_1} (F_X^{-1}(u))^2 c(u, v; \theta) dv du}{C(\alpha_1, \delta_1; \theta) - C(\alpha, \delta_1; \theta) - C(\alpha_1, \delta; \theta) + C(\alpha, \delta; \theta)} - \left(\text{DTVaR}_{(\alpha,a)}^{(\delta,d)}(X|Y;C) \right)^2, \quad (16)$$

where $\text{DTVaR}_{(\alpha,a)}^{(\delta,d)}(X|Y;C)$ is given in (5).

4.1. The Estimation of DCTV

Following the derivation method of the estimators $\widehat{\text{DTVaR}}_{(\alpha,a)}^{(\delta,d)(1)}(X|Y;C)$ and $\widehat{\text{DTVaR}}_{(\alpha,a)}^{(\delta,d)(2)}(X|Y;C)$ in (13) and (14), we obtain two estimators for $\text{DCTV}_{(\alpha,a)}^{(\delta,d)}(X|Y;C)$, namely,

$$\widehat{\text{DCTV}}_{(\alpha,a)}^{(\delta,d)(1)}(X|Y;C) = \frac{1}{\lceil m(1-\alpha)^{1+a} \rceil + 1 - r} \sum_{q=0}^{\lceil m(1-\alpha)^{1+a} \rceil} X_{(k(q))}^2 \times I(y_{m(l)} \leq Y'_{(k(q))} \leq y_{m(l_1)}) - \left(\widehat{\text{DTVaR}}_{(\alpha,a)}^{(\delta,d)(1)}(X|Y;C) \right)^2, \quad (17)$$

and

$$\widehat{\text{DCTV}}_{(\alpha,a)}^{(\delta,d)(2)}(X|Y;C) = \frac{1}{\lceil m(1-\alpha)^{1+a} \rceil + 1 - r} \sum_{q=0}^{\lceil m(1-\alpha)^{1+a} \rceil} \{ (1 - h_{k(q)}) X_{(k(q))} + h_{k(q)} X_{(k(q)+1)} \}^2 I(y_{m(l)} \leq Y'_{(k(q))} \leq y_{m(l_1)}) - \left(\widehat{\text{DTVaR}}_{(\alpha,a)}^{(\delta,d)(2)}(X|Y;C) \right)^2, \quad (18)$$

where $k(q) = \lceil m\alpha'_{(q)} \rceil$, $h_{k(q)} = \lceil m\alpha'_{(q)} \rceil - m\alpha'_{(q)}$.

In the following theorem, we prove the consistency of DCTV estimators.

Theorem 2. The estimators $\widehat{\text{DCTV}}_{(\alpha,0)}^{(\delta,0)(n)}(X|Y;C)$, $n = 1, 2$, given in (17) and (18) are consistent for every finite $a, d \geq 0$.

4.2. Confidence Intervals for DTVaR

It is obvious that the estimators in (13) and (14) are point estimators for DTVaR. The next step is to construct point-wise confidence intervals for DTVaR. We derive the (point-wise) confidence intervals, whose construction is based on the following asymptotic result.

Theorem 3. Let $\alpha, \delta \in [0, 1]$ and contraction parameters a and d be fixed. Let the distribution function F_X be continuous at the points $F_X^{-1}(\alpha)$ and $F_X^{-1}(\alpha_1)$. Then, for $n = 1, 2$, we have

$$\sqrt{m}(\widehat{\text{DTVaR}}_{(\alpha,a)}^{(\delta,d)(n)}(X|Y;C) - \text{DTVaR}_{(\alpha,a)}^{(\delta,d)}(X|Y;C)) \rightarrow_d \mathcal{N}(0, \text{DCTV}_{(\alpha,a)}^{(\delta,d)}(X|Y;C)), \quad (19)$$

where $\text{DTVaR}_{(\alpha,a)}^{(\delta,d)}(X|Y;C)$ and $\text{DCTV}_{(\alpha,a)}^{(\delta,d)}(X|Y;C)$ are given in (5) and (16). In particular, statement (19) holds for any finite contraction parameters $a, d \geq 0$ if the distribution function F_X is continuous everywhere on the real line.

Using (19), we derive the following $(1 - \gamma)100\%$ level asymptotic confidence intervals for the DTVaR,

$$\widehat{\text{DTVaR}}_{(\alpha,a)}^{(\delta,d)(n)}(X|Y;C) \pm z_{\gamma/2} \sqrt{\frac{\text{DCTV}_{(\alpha,a)}^{(\delta,d)}(X|Y;C)}{m}},$$

where $z_{\gamma/2}$ is the $(1 - \gamma/2) \times 100\%$ percentile of the standard normal distribution. The truncated variance DCTV is unknown but has been estimated empirically from DCTV estimators given in (17) and (18). Hence, we have the following $(1 - \gamma)100\%$ level asymptotic confidence intervals for the DTVaR,

$$\widehat{\text{DTVaR}}_{(\alpha,a)}^{(\delta,d)(n)}(X|Y;C) \pm z_{\gamma/2} \sqrt{\frac{\widehat{\text{DCTV}}_{(\alpha,a)}^{(\delta,d)(n)}(X|Y;C)}{m}}. \quad (20)$$

Remark 1. We apply the confidence intervals (20) for DTVaR backtesting using the bootstrap method in Section 6.2.

5. Parametric Estimation under FGM Copula

In this section, we find parametric estimates for the DTVaR at any given $\alpha, \delta \in (0, 1)$ and specified contraction parameters $a, d \geq 0$ under a Pareto distribution by using the Maximum Likelihood Estimation (MLE) method of Pham et al. (2019). However, before calculating the estimates, we take the following steps:

1. Derive the expression of the DTVaR for the Pareto distribution;
2. Calculate the parametric estimates of the distribution parameters of random samples $X_1, \dots, X_m$ and $Y_1, \dots, Y_m$, each of which is assumed to be a Pareto distribution.

Moreover, we show that the DTVaR, when we consider the correlation (or dependence) between positive quadrant dependent (PQD) losses, is larger than the TVaR. That means, for $\alpha, \delta \in (0, 1)$, then

$$\text{DTVaR}_{(\alpha,0)}^{(\delta,0)}(X|Y;C) \geq \text{TVaR}_\alpha(X). \quad (21)$$

Note that, in the Negative-Quadrant-Dependent (NQD) losses, we have the reverse of Inequality (21). In particular, also note that $1 - \alpha - \delta + C(\alpha, \delta; \theta)$ is the joint significance level (j.s.l.) for the DTVaR in (21). We use the j.s.l. for assessing the performance of DTVaR estimation.

Now, we derive the expression for the DTVaR for a Pareto distributed loss associated with another loss joined by a Farlie–Gumbel–Morgenstern (FGM) copula that is defined as $C_{\text{FGM}}(u, v; \theta) = uv + \theta uv(1 - u)(1 - v)$, for $u, v \in [0, 1]$ and $\theta \in [-1, 1]$. We are aware that the FGM copula introduces only light dependence. However, it admits positive as well as negative dependence between a set of random variables. The FGM copula is often used in applications to describe dependence structures due to its tractability and simplicity (see, for instance, Bargès et al. 2009; Chadjiconstantinidis and Vrontos 2014; and Jiang and Yang 2016).

Suppose that X is a Pareto random loss with parameter (γ_1, β_1) . Suppose also that our dependent (associated) random loss Y is following Pareto distribution with parameter (γ_2, β_2) . The distribution function of X and Y are, respectively, $F_X(x) = 1 - (\beta_1/(x + \beta_1))^{\gamma_1}$ for $x \geq 0$, and $F_Y(y) = 1 - (\beta_2/(y + \beta_2))^{\gamma_2}$ for $y \geq 0$. Their inverses are easy to find and thus their VaRs are as well, which are $Q_\alpha(X) = \beta_1 [(1 - \alpha)^{-1/\gamma_1} - 1]$ and $Q_\delta(Y) = \beta_2 [(1 - \delta)^{-1/\gamma_2} - 1]$.

The risk measure DTVaR formula for X given Y , under the FGM copula, may be found by using Lemma 2.

Lemma 4. Let X and Y be two Pareto distributed random variables with parameters (γ_1, β_1) and with parameter (γ_2, β_2) . Suppose that the joint distribution of X and Y are defined by a bivariate FGM copula as follows:

$$F_{X,Y}(x, y) = C_{\text{FGM}}(F_X(x), F_Y(y); \theta),$$

with $\theta \in [-1, 1]$. Then, the DTVaR of X given Y at levels α and δ , $0 < \alpha, \delta < 1$, is

$$\text{DTVaR}_{(\alpha, a)}^{(\delta, d)}(X|Y; C) = \frac{\beta_1(A + 2\theta B - D)}{C(\alpha_1, \delta_1; \theta) - C(\alpha, \delta_1; \theta) - C(\alpha_1, \delta; \theta) + C(\alpha, \delta; \theta)}. \quad (22)$$

where θ denotes the dependence (or Copula) parameter between X and Y , the copula $C(p, q; \theta) = pq + \theta pq(1 - p)(1 - q)$,

$$\begin{aligned} A &= \frac{\gamma_1}{\gamma_1 - 1} \left[(1 - \alpha)^{\frac{\gamma_1 - 1}{\gamma_1}} - (1 - \alpha_1)^{\frac{\gamma_1 - 1}{\gamma_1}} \right] [(\theta + 1)(1 - \delta)^{d+1} - \theta(\delta_1^2 - \delta^2)], \\ B &= \frac{\gamma_i}{\gamma_1 - 1} \left[\delta_1^2 - \delta^2 - (1 - \delta)^{d+1} \right] \left\{ \alpha(1 - \alpha)^{\frac{\gamma_1 - 1}{\gamma_1}} - \alpha_1(1 - \alpha_1)^{\frac{\gamma_1 - 1}{\gamma_1}} + \frac{\gamma_1}{2\gamma_1 - 1} \right. \\ &\quad \times \left. \left[(1 - \alpha)^{\frac{2\gamma_1 - 1}{\gamma_1}} - (1 - \alpha_1)^{\frac{2\gamma_1 - 1}{\gamma_1}} \right] \right\}, \\ D &= (1 - \delta)^{d+1} \left[(1 + \theta)(1 - \alpha)^{a+1} - \theta(\alpha_1^2 - \alpha^2) \right] - \theta(\delta_1^2 - \delta^2) \left[(1 - \alpha)^{a+1} - \alpha_1^2 + \alpha^2 \right]. \end{aligned}$$

The corresponding parametric estimate of DTVaR in (22) is then found by replacing unknown parameters γ_1 , β_1 and θ with their respective estimates. That is, we have

$$\widehat{\text{DTVaR}}_{(\alpha, a)}^{(\delta, d)(p)}(X|Y; C) = \frac{\hat{\beta}_1(\hat{A} + 2\hat{\theta}\hat{B} - \hat{D})}{C(\alpha_1, \delta_1; \hat{\theta}) - C(\alpha, \delta_1; \hat{\theta}) - C(\alpha_1, \delta; \hat{\theta}) + C(\alpha, \delta; \hat{\theta})}. \quad (23)$$

Example 1. Let X_i , $i = 1, 2, 3$ have a Pareto distribution with parameters $\gamma_i = 3$ and $\beta_1 = 2500$, i.e., $\text{Pa}(3, 2, 500)$. Let Y be another Pareto random loss that also has a Pareto distribution and associates with X_i . Both Figures 1 and 2 present the $\text{DTVaR}_{(0.9, 0)}^{(\delta, 0)}$ estimates for various FGM copula parameters and $\delta \in (0.1, 1)$, along with the $\text{TVaR}_{0.9}$ estimates. Consider the bivariate losses (X_i, Y) , $i = 1, 2, 3$. For each couple (X_i, Y) , we set $\theta_1 = 1$, $\theta_2 = 0.5$ and $\theta_3 = 0.01$, respectively (see Figure 1a). The selection of parameters θ_i , $i = 1, 2, 3$ corresponds, respectively, to the strong, medium and weak dependences. In Figure 1a, the comparison of the riskiness of X_1 , X_2 and X_3 is presented. Notice that the risk measures of the TVaR of X_i at level α are the same in the three cases. Furthermore, note that DTVaR coincides with TVaR in the independence case ($\theta = 0$), whereas DTVaR is exactly the same as CTVaR when $a = d = 0$. The DTVaR of the loss X_1 is higher than those of X_2 and X_3 , respectively, i.e., X_1 is riskier than X_2 and X_3 . In Figure 1b, it is shown that both DTVaR and TVaR of X_1 are located above the VaR of X_1 for the same probability level α . We can see that the DTVaR estimates are always larger than the TVaR estimates when the copula parameters are positive, whereas the DTVaR estimates are always smaller than the TVaR estimates when the opposite occurs (see Figure 2). Therefore, these results are in accordance with the statement (21) and its reverse.

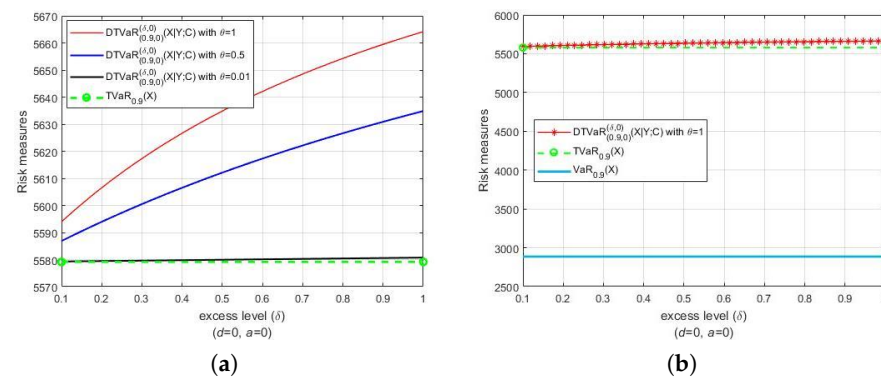


Figure 1. (a) DTVaR of the target loss X with associated loss Y for positive values of FGM copula parameter and $a = d = 0$ along with (b) its comparison with TVaR and VaR of X . Both X and Y are Pareto distributed ($\gamma_1 = 3, \beta_1 = 2500$).

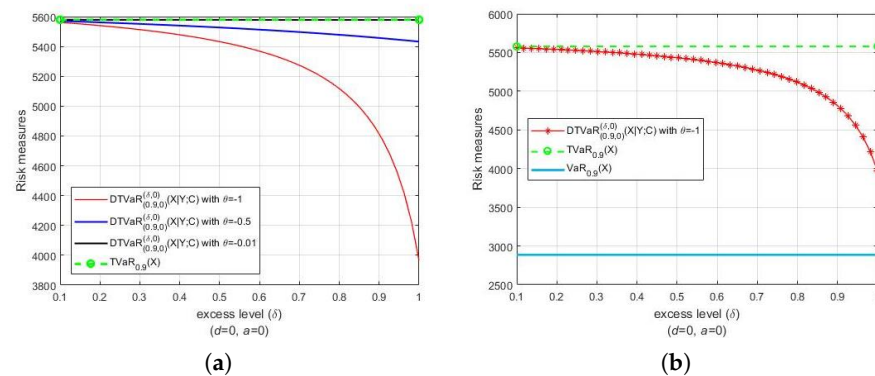


Figure 2. (a) DTVaR of the target loss X with associated loss Y for negative values of FGM copula parameter and $a = d = 0$ along with (b) its comparison with TVaR and VaR of X . Both X and Y are Pareto distributed ($\gamma_1 = 3, \beta_1 = 2500$).

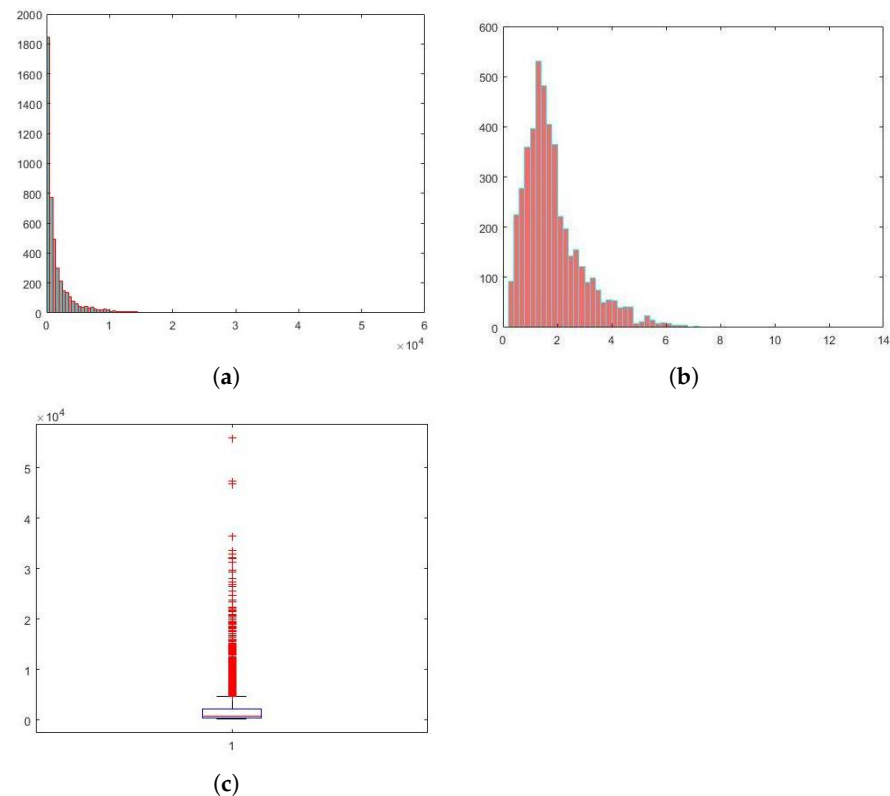
6. Data Analysis

We have used the data of one-year vehicle insurance policies from [Macquarie University \(2005\)](#). This data set is based on one-year vehicle insurance policies taken out in 2004 or 2005. There are 67,856 policies, of which, 4624 (6.8%) had at least one claim. To be clear, the vehicle values written in the source (data set) are values in USD 10,000 s. Out of 4624 policies, there are six observations whose vehicle value is 0. We do not include these six observations in the calculation of DTVaR estimations. Therefore, the data size is $m = 4618$. Suppose that the target loss X is the insurance claim amount and the associated loss Y is the vehicle value.

Table 1 provides summary statistics on the claim amount and vehicle value. We can find that both the claim amount and vehicle value have positive skewness, namely 5.0470 and 1.8614, respectively. Moreover, the respective kurtosis of the claim amount and vehicle value is significant when higher than 3. Kurtosis values above 3 (43.3102 and 9.9344) indicate that, relative to a normal distribution, more probability tends to be at points away from the mean than at points near the mean. This is confirmed by Figure 3. Moreover, Figure 3c shows the box plot of the data of the claim amount, depicting that there are 758 outliers in the right tail.

Table 1. Descriptive statistics.

Statistics	Claim Amount (X)	Vehicle Value (Y)
Sample number	4618	4618
Mean	2.0131×10^3	1.8616
Standard deviation	3.5480×10^3	1.1584
Skewness	5.0470	1.8614
Kurtosis	43.3102	9.9344

**Figure 3.** (a) Histogram of insurance claim amount; (b) histogram of vehicle value; (c) box plot of insurance claim amount.

In this section, we compute the parametric estimates of DTVaR. Before comparing the parametric and nonparametric (empirical) estimates of DTVaR, we compute the estimators of DTVaR suggested in Section 3.

Our empirical analysis supports the claim that the suggested risk measure does not underestimate or overestimate the actual risk for $a = 0$ and various δ . Thus, DTVaR is quite meaningful. However, the DTVaR does not necessarily estimate the actual risk properly when the contraction parameters $a > 0$ and $d > 0$. This is not surprising, because of the presence of the excess level δ (other than the probability level α), together with contraction parameter d , which also contributes to the DTVaR estimation. We employ the hypothesis testing procedure (backtesting) to verify this claim empirically. In the future, we need to concurrently estimate $a > 0$ and $d > 0$, which can optimize DTVaR using numerical optimization so that DTVaR estimates the actual risk properly.

6.1. Parametric Preliminary Results

By using the MLE method, we obtain the results that X and Y are both Pareto distributed with parameter estimates that are $\hat{\gamma}_1 = 2.0468$, $\hat{\beta}_1 = 2203.9$ for X , and $\hat{\gamma}_a = 295.12$, $\hat{\beta}_1 = 563.44$ for Y . In particular, for X , the estimation of parameters is very likely to be influenced by the large number of outliers in the claim amount. Again, by using the MLE method, we obtain the estimate of FGM copula parameter $\hat{\theta}_{\text{FGM}} = 0.0221$.

Consider the bivariate loss (X, Y) . For (X, Y) , we have $\hat{\theta}_{FGM} = 0.0221$. Furthermore, note again that DTVaR coincides with CTVaR when $a = d = 0$. In Figure 4a, it is evident that both $DTVaR_{(0.9,0)}^{(\delta,0)}$ and $DTVaR_{(0.9,0.01)}^{(\delta,0)}$ estimates are located between the estimates of VaR with different probability levels. It is interesting to note that $DTVaR_{(0.9,0.01)}^{(\delta,0)}$ is smaller than both $DTVaR_{(0.9,0)}^{(\delta,0)}$ and $TVaR_{0.9}$ (see Figure 4a–c). This fact indicates that DTVaR is much more flexible than TVaR and CTVaR, i.e., DTVaR can be set as equal to or less than CTVaR, or even less than TVaR, by carefully determining the parameters a and d . Furthermore, the results in Figure 4b,c show the same pattern as the results in Figure 1, i.e., the estimates of DTVaR are larger than those of TVaR, which is due to the positive copula parameter estimate.

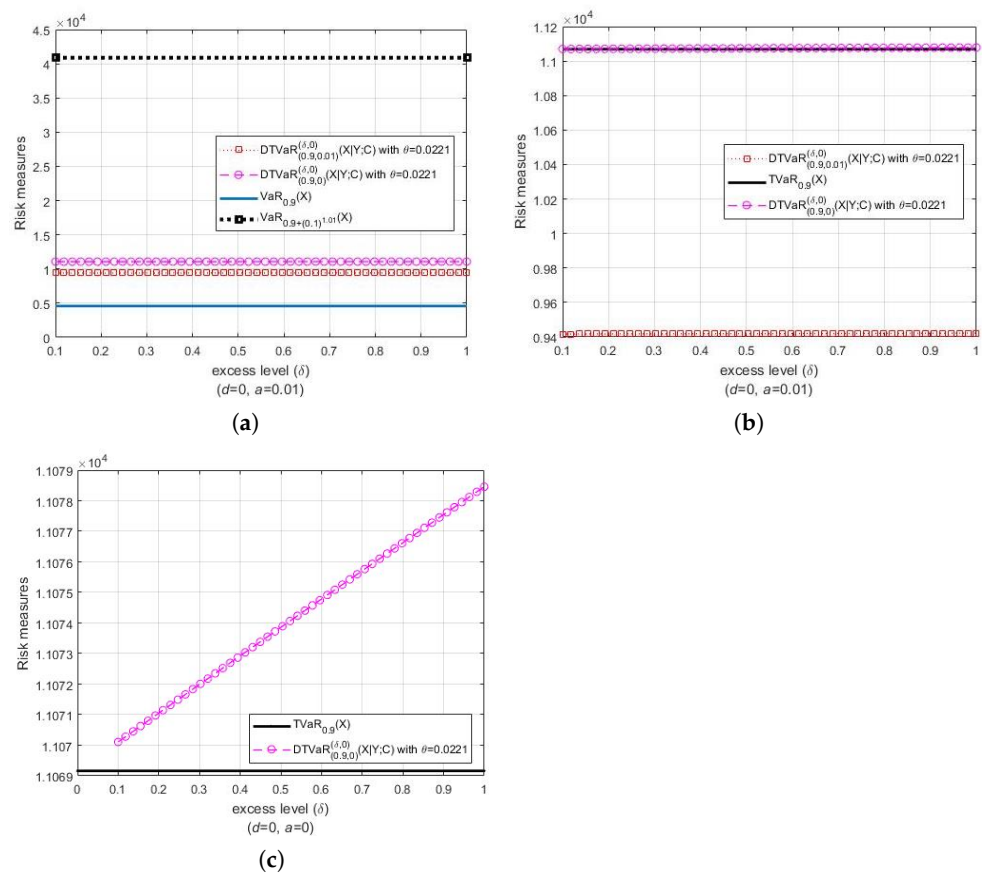


Figure 4. DTVaR of the target loss X with associated loss Y and its comparison with (a) VaR of X and (b–c) TVaR of X . Both X and Y are Pareto distributed with $\hat{\gamma}_1 = 2.0468$, $\hat{\beta}_1 = 2203.9$, whilst FGM copula parameter estimate is $\hat{\theta} = 0.0221$.

6.2. Backtesting

In the backtesting for the DTVaR, we are interested in the size of the discrepancy between the claims above the VaR estimate and the estimate of the DTVaR when VaRs violation occurs. A VaRs violation occurs when the actual loss is larger than the estimated figures at specified probability and excess levels. These discrepancies can be positive, negative or zero. We assume that these discrepancies (also called residuals) are iid, conditioned on claims that are larger than the VaRs estimates. We propose an adaptation (generalization) of the [Righi and Ceretta \(2015\)](#) procedure. This approach is based on series r , which represents the residual exceedances over the VaR, i.e., the violations standardized

by the DTVaR estimate and the DCTV estimate of claim X . Given a probability level α and an excess level δ , we can formally represent r by formulation

$$r = \begin{cases} \frac{X - \widehat{\text{DTVaR}}_{(\alpha,a)}^{(\delta,d)}}{\sqrt{\widehat{\text{DCTV}}_{(\alpha,a)}^{(\delta,d)}}}, & \text{if } X \in \{X | I(X \geq Q_\alpha, Y \geq Q_\delta) = 1\}, \\ 0, & \text{otherwise.} \end{cases} \quad (24)$$

It is clear that we consider the standard deviation truncated by the VaRs, which is the square root of the presented measure DCTV. Similar to [Righi and Ceretta \(2015\)](#), under the null hypothesis, r has a zero mean, against the alternative that the mean of r is positive or negative. This alternative hypothesis represents the real danger, which is underestimating or overestimating loss. Once there is a violation, we take into consideration the information regarding the quantile that the VaR is calculated rather than all of the distribution. Instead of using the p -value for the hypothesis, we use the confidence interval (CI) at a confidence level $1 - \gamma$, which is calculated based on 1000 bootstrap samples (see [Righi and Ceretta 2015](#) and [Jadhav et al. 2013](#)). We reject the null hypothesis when the resulting bootstrap CI contains 0.

6.3. Result Analysis

We have estimated the DTVaR and DCTV based on data of one-year vehicle insurance policies. Figures 5–7 and Tables 2–5 present estimates of the DTVaR and DCTV of the respective probability levels and excess levels for various values of a and d . In Tables 3–5, the abbreviations LCL and UCL denote the lower confidence level and upper confidence level of a CI at confidence level $\gamma = 0.95$. Note that, according to CIs (20), we have

$$\text{LCL}^{(n)} = \widehat{\text{DTVaR}}_{(\alpha,a)}^{(\delta,d)(n)}(X|Y;C) - z_{\gamma/2} \sqrt{\frac{\widehat{\text{DCTV}}_{(\alpha,a)}^{(\delta,d)(n)}(X|Y;C)}{m}}, \quad (25)$$

$$\text{UCL}^{(n)} = \widehat{\text{DTVaR}}_{(\alpha,a)}^{(\delta,d)(n)}(X|Y;C) + z_{\gamma/2} \sqrt{\frac{\widehat{\text{DCTV}}_{(\alpha,a)}^{(\delta,d)(n)}(X|Y;C)}{m}}, \quad (26)$$

where $\widehat{\text{DTVaR}}_{(\alpha,a)}^{(\delta,d)(n)}(X|Y;C)$ are given in (13) and (14), and $\widehat{\text{DCTV}}_{(\alpha,a)}^{(\delta,d)(n)}(X|Y;C)$ are given in (17) and (18).

In particular, Table 2 shows the number as well as the percentage of violations of DTVaR estimations, i.e., the assessment of accuracy for the DTVaR estimates. The assessment is carried out by first observing the joint significance level. For example, in Table 2 (first row, first column), a 0.95% joint significance level (j.s.l.) is lower than 10%. This means that the DTVaR estimates are quite accurate. In the second place, by calculating the number of violations against the $\widehat{\text{DTVaR}}_{(0.9,0)}^{(0.9,0)(1)}(X|Y;C)$, $\widehat{\text{DTVaR}}_{(0.9,0)}^{(0.9,0)(2)}(X|Y;C)$ and $\widehat{\text{DTVaR}}_{(0.9,0)}^{(0.9,0)(p)}(X|Y;C)$, we obtain the percentages of violations of 1.34%, 1.41% and 2.81%, respectively. The number 1.34% is obtained from the division between 62 and 4618, where 62 is the number of violations and 4618 is the total number of observations. Essentially, the number of violations is the number of observations located outside of the critical value, i.e., greater than the DTVaR estimate. These computations are shown for different α and δ . Note that, for various α and δ , the differences between j.s.l. and the percentage of violations for the parametric estimates are always greater than those for the two nonparametric ones. This result implies that we should look for another distribution that is more fit for the variable of the claim amount. We also obtain the fact that the smaller the excess level δ , the smaller the differences between j.s.l. and the percentage of violations. This implies that both nonparametric estimators accurately estimate the DTVaR at an excess level of $\delta = 0.9$.

In Figure 5, we can see that the estimates of DTVaR are relatively larger than those of TVaR. Those relatively large DTVaR estimates are highly probably influenced by the number of outliers (758 observations). In Figure 6a, it can be seen that both first and second estimates of DTVaR relatively nearly coincide. Furthermore, we can see in Figure 6b,c that both first and second estimates of DTVaR are larger than the corresponding estimates of DCTV.

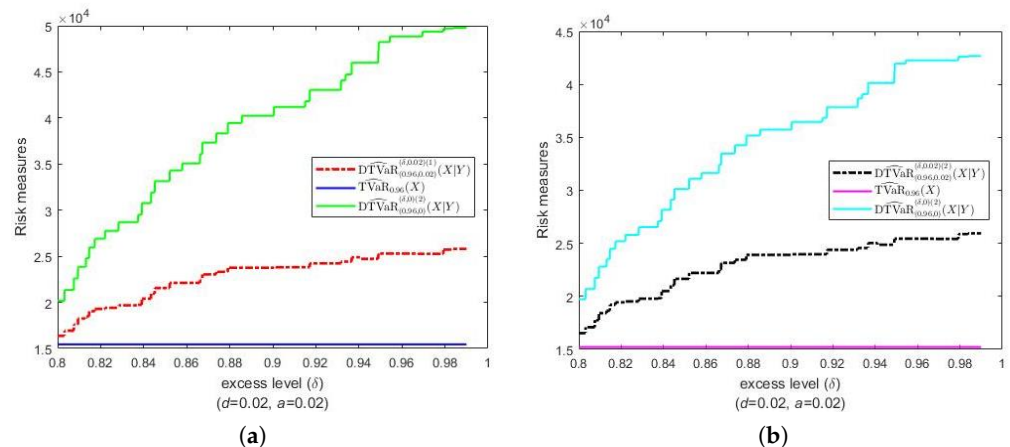


Figure 5. The estimates of $\text{DTVaR}_{(0.96,0.02)}^{(\delta,0.02)(n)}$ of claim amount associated with vehicle value, along with the estimates of $\text{DTVaR}_{(0.96,0)}^{(\delta,0)(n)}$ and $\text{TVaR}_{0.96}$ for (a) $n = 1$ and (b) $n = 2$.

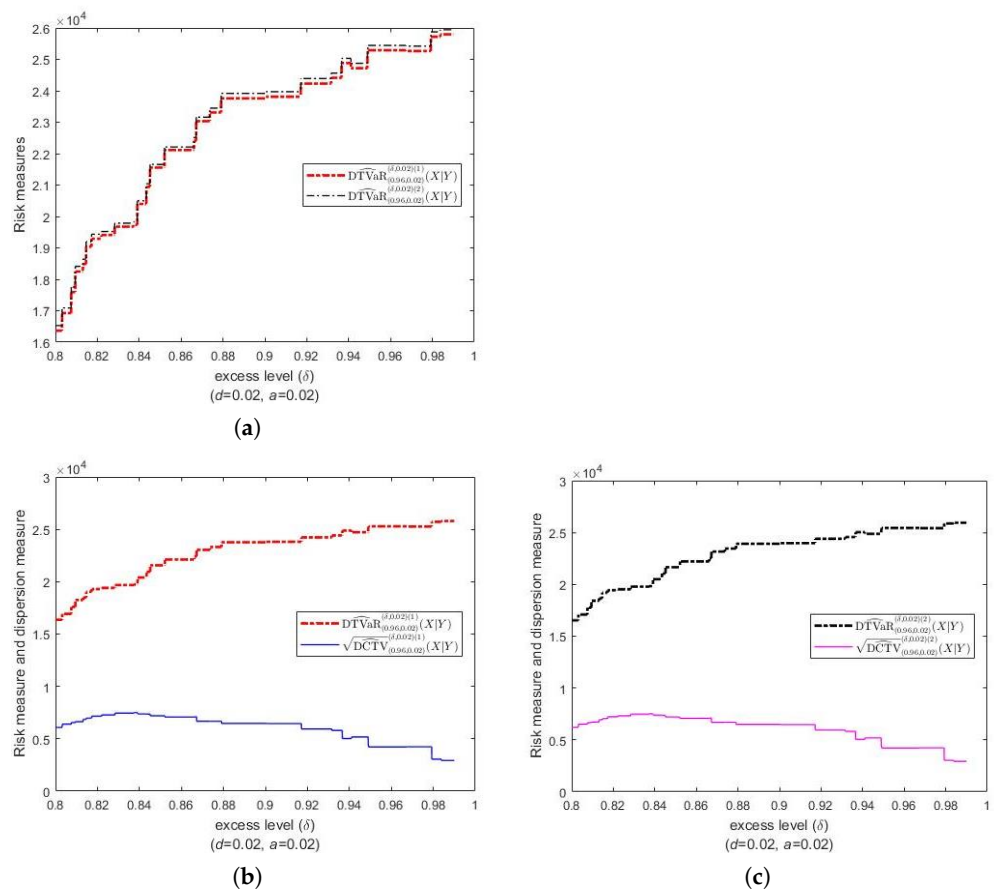


Figure 6. (a) The estimates of $\text{DTVaR}_{(\alpha,a)}^{(\delta,d)(n)}$, $n = 1, 2$, of claim value associated with vehicle value, along with their comparison with the estimates of $\text{DCTV}_{(\alpha,a)}^{(\delta,d)(n)}$ for (b) $n = 1$ and (c) $n = 2$.

From Table 3, we can see that, for $a = d = 0$, all bootstrap CIs contain 0. We can see similar results from Table 4, where, for $a = 0$, $d \neq 0$, all bootstrap CIs also contain 0. These results indicate that the null hypothesis cannot be rejected, which supports the suggested estimation method for $\text{DTVAr}_{(\alpha,0)}^{(\delta,0)}(X|Y;C)$ and $\text{DTVAr}_{(\alpha,0)}^{(\delta,d)}(S_N|Y;C)$, that is, there is no underestimation or overestimation of the target loss (claim amount). From both tables, as the values of probability and excess levels increase, estimates of the DTVaR also increase, which is quite obvious. It is interesting to note that, in Table 4, when $\delta = 0.92$, estimates of DTVaR are greater at $d = 0.025$ than at $d = 0.015$, but when $\delta = 0.98$, estimates of DTVaR are smaller at $d = 0.025$ than at $d = 0.015$.

Table 5 shows different results from Tables 3 and 4. Although we can see that the larger the values of probability and excess levels, the larger the estimates of the DTVaR, it is interesting to note that, for several pairs (a, d) , estimates of DTVaR fail the backtest. These results indicate that DTVaR estimation is complicated. To overcome this problem, we suggest in the future that the contraction parameters a and d be determined by performing DTVaR optimization so that DTVaR can properly estimate risk. Note that $\text{LCL}^{(1)}$, $\text{LCL}^{(2)}$, $\text{UCL}^{(1)}$ and $\text{UCL}^{(2)}$ in Tables 2–5 are calculated using Formulas (25) and (26).

Figure 7 presents two different results regarding the difference between the parametric estimates of the $\text{DTVAr}_{(0.96,a)}^{(0.96,d)}$ and its nonparametric estimates. When the contraction parameter $a = d = 0.01$, we can see in Figure 7a that the differences between the two are relatively large, and even very large for δ values approaching 1. However, for $a = d = 0.1$, the differences between the two estimates are relatively small (see Figure 7b).

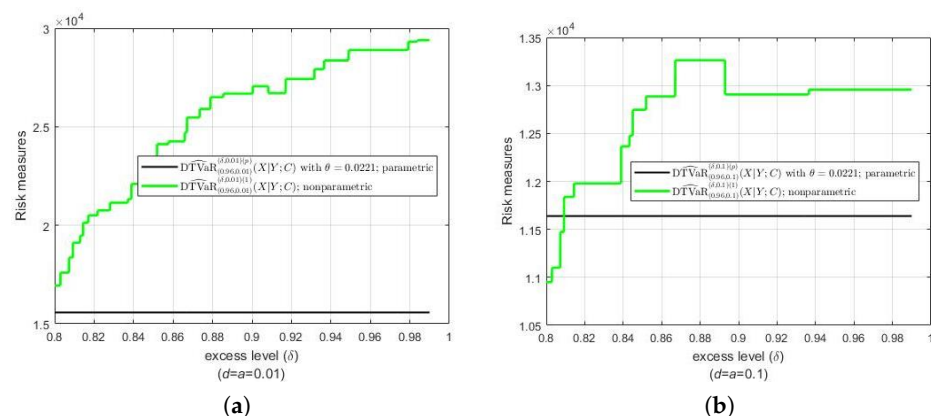


Figure 7. Parametric estimates of $\text{DTVAr}_{(0.96,a)}^{(0.96,d)}$ of claim amount associated with vehicle value, in comparison with nonparametric estimates for (a) $a = d = 0.01$ and (b) $a = d = 0.1$.

Table 2. Joint significance level and number of violations of nonparametric estimates of $\text{DTVAr}_{(\alpha,0)}^{(\delta,0)}$ and parametric estimates through FGM copula with $\hat{\theta}_{\text{FGM}} = -0.0645$.

Method of Estimations	Estimators	$\alpha = 0.9$ j.s.l. ² (%)	No. viol. ¹ (%)	Estimators	$\alpha = 0.92$ j.s.l. (%)	No. viol. (%)	Estimators	$\alpha = 0.94$ j.s.l. (%)	No. viol. (%)	Estimators	$\alpha = 0.96$ j.s.l. (%)	No. viol. (%)
$\delta = 0.9$												
Nonparametric	DTVAr ⁽¹⁾ (15,601)	0.95	62 (1.34)	DTVAr ⁽¹⁾ (18,216) ³	0.76	44 (0.95)	DTVAr ⁽¹⁾ (20,880)	0.57	29 (0.63)	DTVAr ⁽¹⁾ (23,693)	0.38	19 (0.41)
	DTVAr ⁽²⁾ (14,890)		65 (1.41)	DTVAr ⁽²⁾ (17,420)		49 (1.06)	DTVAr ⁽²⁾ (20,002)		33 (0.71)	DTVAr ⁽²⁾ (22,785)		20 (0.43)
Parametric (Pareto, FGM Copula)	DTVAr ^(p) (10,557)		130 (2.81)	DTVAr ^(p) (12,132)		103 (2.23)	DTVAr ^(p) (14,423)		65 (1.41)	DTVAr ^(p) (18,229)		41 (0.89)
$\delta = 0.92$												
Nonparametric	DTVAr ⁽¹⁾ (15,920)	0.76	59 (1.28)	DTVAr ⁽¹⁾ (18,744)	0.61	40 (0.87)	DTVAr ⁽¹⁾ (21,468)	0.45	26 (0.56)	DTVAr ⁽¹⁾ (24,143)	0.30	18 (0.39)
	DTVAr ⁽²⁾ (15,029)		65 (0.76)	DTVAr ⁽²⁾ (17,741)		47 (1.02)	DTVAr ⁽²⁾ (20,366)		33 (0.69)	DTVAr ⁽²⁾ (23,011)		20 (0.43)
Parametric (Pareto, FGM Copula)	DTVAr ^(p) (11,052)		130 (2.81)	DTVAr ^(p) (12,586)		103 (2.23)	DTVAr ^(p) (14,825)		65 (1.41)	DTVAr ^(p) (18,565)		41 (0.89)
$\delta = 0.94$												
Nonparametric	DTVAr ⁽¹⁾ (16,715)	0.57	52 (1.12)	DTVAr ⁽¹⁾ (19,184)	0.45	36 (0.78)	DTVAr ⁽¹⁾ (22,811)	0.34	20 (0.43)	DTVAr ⁽¹⁾ (25,298)	0.23	17 (0.37)
	DTVAr ⁽²⁾ (15,554)		63 (1.36)	DTVAr ⁽²⁾ (17,902)		47 (1.02)	DTVAr ⁽²⁾ (21,391)		27 (0.58)	DTVAr ⁽²⁾ (23,864)		18 (0.39)
Parametric (Pareto, FGM Copula)	DTVAr ^(p) (11,052)		130 (2.81)	DTVAr ^(p) (12,585)		103 (2.23)	DTVAr ^(p) (14,824)		65 (1.41)	DTVAr ^(p) (18,564)		41 (0.89)

Table 2. Cont.

Method of Estimations	Estimators	$\alpha = 0.9$ j.s.l. ² (%)	No. viol. ¹ (%)	Estimators	$\alpha = 0.92$ j.s.l. (%)	No. viol. (%)	Estimators	$\alpha = 0.94$ j.s.l. (%)	No. viol. (%)	Estimators	$\alpha = 0.96$ j.s.l. (%)	No. viol. (%)
$\delta = 0.96$												
Nonparametric	DTV _a R ⁽¹⁾ (17,159)	0.38	51 (1.10)	DTV _a R ⁽¹⁾ (19,622)	0.30	34 (0.74)	DTV _a R ⁽¹⁾ (24,876)	0.23	17 (0.37)	DTV _a R ⁽¹⁾ (26,802)	0.15	15 (0.32)
	DTV _a R ⁽²⁾ (15,472)		63 (1.36)	DTV _a R ⁽²⁾ (17,744)		47 (1.02)	DTV _a R ⁽²⁾ (22,720)		20 (0.43)	DTV _a R ⁽²⁾ (24,638)		18 (0.39)
Parametric (Pareto, FGM Copula)	DTV _a R ^(p) (11,051)		130 (2.81)	DTV _a R ^(p) (12,584)		103 (2.23)	DTV _a R ^(p) (14,824)		65 (1.41)	DTV _a R ^(p) (18,564)		41 (0.89)
$\delta = 0.98$												
Nonparametric	DTV _a R ⁽¹⁾ (13,143)	0.19	93 (2.01)	DTV _a R ⁽¹⁾ (14,053)	0.15	74 (1.60)	DTV _a R ⁽¹⁾ (16,639)	0.11	53 (1.15)	DTV _a R ⁽¹⁾ (18,459)	0.08	41 (0.89)
	DTV _a R ⁽²⁾ (13,253)		92 (1.99)	DTV _a R ⁽²⁾ (14,184)		74 (1.60)	DTV _a R ⁽²⁾ (16,790)		52 (1.12)	DTV _a R ⁽²⁾ (18,656)		40 (0.87)
Parametric (Pareto, FGM Copula)	DTV _a R ^(p) (11,050)		130 (2.81)	DTV _a R ^(p) (12,584)		103 (2.23)	DTV _a R ^(p) (14,823)		65 (1.41)	DTV _a R ^(p) (18,564)		41 (0.89)

¹ No. viol. states the number of violations. ² j.s.l. states the joint significance level expressed in percent. ³ The numbers in parentheses in columns 4, 6, 8, 10 and 12 indicate the percentages of violations to the data size ($m = 4618$).

Table 3. $\text{DTVaR}_{(\alpha,0)}^{(\delta,0)}(X|Y;C)$ estimates and bootstrap confidence intervals.

	Estimators	$\alpha = 0.90$	$\alpha = 0.92$	$\alpha = 0.94$	$\alpha = 0.96$	$\alpha = 0.98$
$\delta = 0.9$	$\text{DTVaR}^{(1)}$	15,601	18,216	20,880	23,693	28,982
	$\sqrt{\text{DCTV}}^{(1)}$	12,826	13,189	13,233	13,057	11,630
	$\text{LCL}^{(1)}$	−0.3261	−0.3585	−0.3877	−0.4408	−0.5043
	$\text{UCL}^{(1)}$	0.3567	0.3937	0.4476	0.4860	0.5720
	$\text{DTVaR}^{(2)}$	14,890	17,420	20,002	22,785	28,059
	$\sqrt{\text{DCTV}}^{(2)}$	11,908	12,250	12,270	12,079	10,468
	$\text{LCL}^{(2)}$	−0.2784	−0.3119	−0.3598	−0.4069	−0.4719
	$\text{UCL}^{(2)}$	0.4462	0.4820	0.5410	0.5867	0.7214
$\delta = 0.92$	$\text{DTVaR}^{(1)}$	15,920	18,744	21,468	24,143	29,369
	$\sqrt{\text{DCTV}}^{(1)}$	13,366	13,789	13,841	13,672	12,346
	$\text{LCL}^{(1)}$	−0.3435	−0.3869	−0.4406	−0.4784	−0.5490
	$\text{UCL}^{(1)}$	0.3895	0.4422	0.4776	0.5228	0.6402
	$\text{DTVaR}^{(2)}$	15,029	17,741	20,366	23,011	28,218
	$\sqrt{\text{DCTV}}^{(2)}$	12,278	12,691	12,732	12,565	11,103
	$\text{LCL}^{(2)}$	−0.3017	−0.3428	−0.3844	−0.4296	−0.4988
	$\text{UCL}^{(2)}$	0.5095	0.5619	0.6233	0.6564	0.7848
$\delta = 0.94$	$\text{DTVaR}^{(1)}$	16,715	19,184	22,811	25,298	30,409
	$\sqrt{\text{DCTV}}^{(1)}$	14,148	14,530	14,567	14,291	12,954
	$\text{LCL}^{(1)}$	−0.3840	−0.4233	−0.4862	−0.5222	−0.6004
	$\text{UCL}^{(1)}$	0.4388	0.4971	0.5523	0.5970	0.6761
	$\text{DTVaR}^{(2)}$	15,554	17,902	21,391	23,864	28,987
	$\sqrt{\text{DCTV}}^{(2)}$	12,881	13,280	13,346	13,098	11,684
	$\text{LCL}^{(2)}$	−0.3358	−0.3568	−0.4303	−0.4519	−0.5566
	$\text{UCL}^{(2)}$	0.5700	0.6423	0.7151	0.7514	0.8861
$\delta = 0.96$	$\text{DTVaR}^{(1)}$	17,159	19,622	24,876	26,802	31,595
	$\sqrt{\text{DCTV}}^{(1)}$	15,074	15,541	15,540	15,205	13,923
	$\text{LCL}^{(1)}$	−0.4296	−0.4775	−0.5689	−0.6046	−0.6908
	$\text{UCL}^{(1)}$	0.5107	0.5495	0.6413	0.6918	0.8115
	$\text{DTVaR}^{(2)}$	15,472	17,744	22,720	24,638	29,412
	$\sqrt{\text{DCTV}}^{(2)}$	13,337	13,848	13,987	13,708	12,490
	$\text{LCL}^{(2)}$	−0.3640	−0.4029	−0.4832	−0.5196	−0.5904
	$\text{UCL}^{(2)}$	0.6988	0.7635	0.9066	0.9198	1.0359
$\delta = 0.98$	$\text{DTVaR}^{(1)}$	13,143	14,053	16,639	18,459	20,468
	$\sqrt{\text{DCTV}}^{(1)}$	6773.1	6644.3	5665.9	4318.5	1769.9
	$\text{LCL}^{(1)}$	−0.6585	−0.6870	−0.8201	−0.9703	−0.9887
	$\text{UCL}^{(1)}$	0.6671	0.6957	0.7464	0.7341	0.8679
	$\text{DTVaR}^{(2)}$	13,253	14,184	16,790	18,656	20,637
	$\sqrt{\text{DCTV}}^{(2)}$	6849.2	6,16.6	5715.9	4316.7	1711.9
	$\text{LCL}^{(2)}$	−0.6502	−0.7013	−0.8250	−1.0162	−1.1208
	$\text{UCL}^{(2)}$	0.6390	0.6518	0.7331	0.6889	0.7987

Table 4. $\text{DTV}\text{VaR}_{(\alpha,0)}^{(\delta,d)}(X|Y;C)$ estimates and bootstrap confidence intervals.

		Estimators	α				
			0.90	0.92	0.94	0.96	0.98
$\delta = 0.92$	$d = 0.015$	$\text{DTV}\text{VaR}^{(1)}$	15,910	19,014	22,145	25,388	30,139
		$\sqrt{\text{DCTV}}^{(1)}$	13,783	14,325	14,403	14,114	12,694
		$\text{LCL}^{(1)}$	−0.3248	−0.3862	−0.4613	−0.5435	−0.5979
		$\text{UCL}^{(1)}$	0.3840	0.4108	0.4380	0.4291	0.5628
		$\text{DTV}\text{VaR}^{(2)}$	14,939	17,907	20,928	24,151	28,936
		$\sqrt{\text{DCTV}}^{(2)}$	12,657	13,207	13,297	13,033	11,482
		$\text{LCL}^{(2)}$	−0.2875	−0.3471	−0.4074	−0.5003	−0.5566
		$\text{UCL}^{(2)}$	0.4912	0.5206	0.5557	0.5757	0.7132
	$d = 0.025$	$\text{DTV}\text{VaR}^{(1)}$	16,242	19,801	22,439	26,026	31,517
		$\sqrt{\text{DCTV}}^{(1)}$	14,197	14,767	14,829	14,509	12,651
		$\text{LCL}^{(1)}$	−0.3512	−0.4359	−0.4773	−0.5801	−0.6906
		$\text{UCL}^{(1)}$	0.3453	0.3377	0.4072	0.3706	0.4660
		$\text{DTV}\text{VaR}^{(2)}$	15,189	18,594	21,139	24,711	30,312
		$\sqrt{\text{DCTV}}^{(2)}$	13,049	13,649	13,739	13,474	11,457
		$\text{LCL}^{(2)}$	−0.2909	−0.3844	−0.4107	−0.5234	−0.6754
		$\text{UCL}^{(2)}$	0.4593	0.4547	0.5226	0.5022	0.5964
$\delta = 0.96$	$d = 0.015$	$\text{DTV}\text{VaR}^{(1)}$	17,580	20,330	26,482	28,849	31,595
		$\sqrt{\text{DCTV}}^{(1)}$	15,440	15,908	15,573	14,912	13,923
		$\text{LCL}^{(1)}$	−0.4515	−0.5125	−0.6770	−0.7624	−0.6662
		$\text{UCL}^{(1)}$	0.4726	0.4952	0.5424	0.5672	0.7846
		$\text{DTV}\text{VaR}^{(2)}$	15,801	18,342	24,243	26,648	29,412
		$\sqrt{\text{DCTV}}^{(2)}$	13,712	14,252	14,113	13,507	12,490
		$\text{LCL}^{(2)}$	−0.3748	−0.4290	−0.5916	−0.6701	−0.5855
		$\text{UCL}^{(2)}$	0.6643	0.7088	0.7794	0.8211	1.0668
	$d = 0.025$	$\text{DTV}\text{VaR}^{(1)}$	17,308	20,218	27,083	29,875	33,249
		$\sqrt{\text{DCTV}}^{(1)}$	15,909	16,553	16,419	15,676	14,387
		$\text{LCL}^{(1)}$	−0.4237	−0.4849	−0.6829	−0.7833	−0.7663
		$\text{UCL}^{(1)}$	0.4719	0.4938	0.4801	0.4499	0.6182
		$\text{DTV}\text{VaR}^{(2)}$	15,375	18,031	24,598	27,461	30,994
		$\sqrt{\text{DCTV}}^{(2)}$	14,097	14,846	15,054	14,430	13,158
		$\text{LCL}^{(2)}$	−0.3363	−0.3979	−0.5754	−0.6828	−0.6794
		$\text{UCL}^{(2)}$	0.6686	0.7002	0.6862	0.6770	0.8527

Table 5. $\text{DTVaR}_{(\alpha, d)}^{(\delta, d)}(X|Y; C)$ estimates and bootstrap confidence intervals.

Estimators		$\alpha = 0.92$			$\alpha = 0.96$			$\alpha = 0.98$		
		$a = 0.015$	$a = 0.020$	$a = 0.025$	$a = 0.015$	$a = 0.020$	$a = 0.025$	$a = 0.015$	$a = 0.020$	$a = 0.025$
$\delta = 0.015$	$\text{DTVaR}^{(1)}$	12,500	12,500	11,406	18,301	18,301	18,301	20,468	20,468	20,468
	$\sqrt{\text{DCTV}}^{(1)}$	6665.3	6665.3	6116.2	4615.1	4615.1	4615.1	1769.9	1769.9	1769.9
	$\text{LCL}^{(1)}$	−0.0647	−0.0719	0.1294	−0.1676	−0.1319	−0.1520	0.8529	0.9921	0.9070
	$\text{UCL}^{(1)}$	2.3908	2.3684	2.7870	4.0913	4.1357	4.0711	12.3001	12.6703	12.5987
	$\text{DTVaR}^{(2)}$	12,595	12,595	11,505	18,476	18,476	18,476	20,637	20,637	20,637
	$\sqrt{\text{DCTV}}^{(2)}$	6748.8	6748.8	6223.1	4654.9	4654.9	4654.9	1711.9	1711.9	1711.9
	$\text{LCL}^{(2)}$	−0.0804	−0.0469	0.1060	−0.1869	−0.1846	−0.1944	0.8545	0.7986	0.7832
	$\text{UCL}^{(2)}$	2.3226	2.3294	2.6779	4.0551	4.0480	4.0487	12.6727	12.7271	12.6842
	$\text{DTVaR}^{(1)}$	11,481	11,481	10,123	17,459	17,459	17,459	20,068	20,068	20,068
	$\sqrt{\text{DCTV}}^{(1)}$	6243.5	6243.5	5222.3	4804.0	4804.0	4804.0	1880.4	1880.4	1880.4
	$\text{LCL}^{(1)}$	0.1043	0.1172	0.3885	0.0013	−0.0240	−0.0054	1.297	1.1467	1.1327
	$\text{UCL}^{(1)}$	2.6871	2.7063	10,222	4.1230	4.1619	4.1322	118,503	12.0218	11.8399
$\delta = 0.96$	$\text{DTVaR}^{(2)}$	11,576	11,576	5362.1	17,653	17,653	17,653	20,264	20,264	20,264
	$\sqrt{\text{DCTV}}^{(2)}$	6341.7	6341.7	4867.9	4867.9	4867.9	4867.9	1830.7	1830.7	1830.7
	$\text{LCL}^{(2)}$	0.0759	0.1068	0.3889	−0.0120	−0.0516	−0.0019	1.0705	0.8565	0.9883
	$\text{UCL}^{(2)}$	2.6477	2.6221	3.3928	4.0887	4.0430	4.0177	12.2409	12.0540	12.0647
	$\text{DTVaR}^{(1)}$	11,481	11,481	10,123	17,459	17,459	17,459	20,068	20,068	20,068
	$\sqrt{\text{DCTV}}^{(1)}$	6243.5	6243.5	5222.3	4804.0	4804.0	4804.0	1880.4	1880.4	1880.4
	$\text{LCL}^{(1)}$	0.0971	0.0969	0.3918	−0.0014	−0.0212	−0.0049	1.1327	1.0817	1.0736
	$\text{UCL}^{(1)}$	2.6883	2.7121	3.4545	4.0688	4.0664	4.1201	11.8503	11.8451	11.8207
	$\text{DTVaR}^{(2)}$	11,576	11,576	10,222	17,653	17,653	17,653	20,264	20,264	20,264
	$\sqrt{\text{DCTV}}^{(2)}$	6341.7	6341.7	5362.1	4867.9	4867.9	4867.9	1830.7	1830.7	1830.7
	$\text{LCL}^{(2)}$	0.1104	0.1038	0.3510	−0.0625	−0.0243	−0.0441	0.9883	0.9955	1.0935
	$\text{UCL}^{(2)}$	2.6320	2.6449	3.4162	4.0382	3.9866	4.0767	12.2752	11.9624	11.9624
$\delta = 0.98$	$\text{DTVaR}^{(1)}$	14,572	14,572	13,277	20,468	20,468	20,468	20,468	20,468	20,468
	$\sqrt{\text{DCTV}}^{(1)}$	6950.0	6950.0	6679.9	1769.9	1769.9	1769.9	1769.9	1769.9	1769.9
	$\text{LCL}^{(1)}$	−0.7380	−0.7253	−0.5678	−3.5026	−3.5026	−3.5026	−0.9887	−0.9887	−0.9887
	$\text{UCL}^{(1)}$	0.5712	0.5648	0.8019	0.6560	0.6560	0.6560	0.8679	0.8679	0.8679
	$\text{DTVaR}^{(2)}$	14,701	14,701	13,417	20,678	20,678	20,678	20,637	20,637	20,637
	$\sqrt{\text{DCTV}}^{(2)}$	7030.0	7030.0	6790.9	1687.6	1687.6	1687.6	1711.9	1711.9	1711.9
	$\text{LCL}^{(2)}$	−0.7490	−0.7425	−0.5666	−3.7977	−3.7977	−3.7977	−1.1208	−1.1208	−1.1208
	$\text{UCL}^{(2)}$	0.5444	0.5659	0.7660	0.5639	0.5639	0.5639	0.7987	0.7987	0.7987
	$\text{DTVaR}^{(1)}$	14,572	14,572	13,277	20,468	20,468	20,468	20,468	20,468	20,468
	$\sqrt{\text{DCTV}}^{(1)}$	6950.0	6950.0	6679.9	1769.9	1769.9	1769.9	1769.9	1769.9	1769.9
	$\text{LCL}^{(1)}$	−0.7264	−0.7496	−0.5777	−3.5026	−3.5026	−3.5026	−0.9887	−0.9887	−0.9887
	$\text{UCL}^{(1)}$	0.5789	0.5870	0.8008	0.6560	0.6560	0.6560	0.8679	0.8679	0.8679
	$\text{DTVaR}^{(2)}$	14,701	14,701	13,417	20,678	20,678	20,678	20,637	20,637	20,637
	$\sqrt{\text{DCTV}}^{(2)}$	7030.0	7030.0	6790.9	1687.6	1687.6	1687.6	1711.9	1711.9	1711.9
	$\text{LCL}^{(2)}$	−0.7534	−0.7474	−0.5846	−3.7977	−3.7977	−3.7977	−1.1208	−1.1208	−1.1208
	$\text{UCL}^{(2)}$	0.5714	0.5659	0.7533	0.5639	0.5639	0.5639	0.7987	0.7987	0.7987
$d = 0.025$	$\text{DTVaR}^{(1)}$	14,572	14,572	13,277	20,468	20,468	20,468	20,468	20,468	20,468
	$\sqrt{\text{DCTV}}^{(1)}$	6950.0	6950.0	6679.9	1769.9	1769.9	1769.9	1769.9	1769.9	1769.9
	$\text{LCL}^{(1)}$	−0.7504	−0.7370	−0.5672	−3.5026	−3.5026	−3.5026	−0.9887	−0.9887	−0.9887
	$\text{UCL}^{(1)}$	0.5906	0.6010	0.8118	0.5803	0.6560	0.6560	0.8679	0.8679	0.8679
	$\text{DTVaR}^{(2)}$	14,701	14,701	13,417	20,678	20,678	20,678	20,637	20,637	20,637
	$\sqrt{\text{DCTV}}^{(2)}$	7030.0	7030.0	6790.9	1687.6	1687.6	1687.6	1711.9	1711.9	1711.9
	$\text{LCL}^{(2)}$	−0.7474	−0.7294	−0.5914	−3.7977	−3.7977	−3.7977	−1.1208	−1.1208	−1.1208
	$\text{UCL}^{(2)}$	0.5572	0.5752	0.7590	0.5639	0.5639	0.5639	0.7987	0.7987	0.7987

7. Conclusions

In this paper, we study a recent coherent risk measure called Dependent Tail Value-at-Risk (DTVaR) initially proposed by [Josaphat and Syuhada \(2021\)](#), and suggest the estimators. We have proven the consistency of the estimators. Moreover, we also derive a parametric estimate of DTVaR for Pareto distribution under an FGM copula. For the backtesting of DTVaR estimation, we have also suggested a novel variability measure called Dependent Conditional Tail Variance (DCTV), instead of an ordinary variance of the target loss, along with the estimators for DCTV. Additionally, using DCTV, we establish the asymptotic normality of DTVaR estimators and construct confidence intervals for DTVaR.

We found that the nonparametric estimators are more accurate at estimating DTVaR than the parametric estimator. This result implies that we should look for other distributions that are more fit for the variable of the claim amount. Moreover, we will find the DTVaR formulas of the claim amount for exponential and lognormal distributions since the application of both distributions covers actuarial science. Then, we will again compare the accuracy of the DTVaR parametric estimators for exponential and lognormal distributions

to the counterpart nonparametric estimators. In the empirical results, the bootstrap CIs in the backtesting procedure have also confirmed that the estimates of the DTVaR do not underestimate or overestimate the actual loss when $a = d = 0$ or $a = 0$. However, the DTVaR does not necessarily estimate the actual risk properly when the contraction parameters $a > 0$ and $d > 0$. The limitation in our research is that the data of the claim amount contain a large number of outliers, i.e., 16.41% of all observations. This situation may be the reason for why $\text{DTVaR}_{(\alpha,0)}^{(\delta,0)}$ estimates are relatively much larger than both $\text{DTVaR}_{(\alpha,a)}^{(\delta,d)}$ and TVaR estimates. In this case, if the risk measure of $\text{DTVaR}_{(\alpha,0)}^{(\delta,0)}$ is employed to the insurance company, this will push the company to prepare a very large extra fund, which is not necessary. In the future, we will apply Archimedean copulas, such as Clayton and Gumbel, to DTVaR. Archimedean copulas are broadly used in implementations due to their easy form, a diversity of dependence structures and other “nice” properties (Brahim et al. 2018).

Author Contributions: Conceptualization, K.S.; methodology, K.S., O.N. and B.P.J.; software, B.P.J.; validation, K.S.; formal analysis, K.S. and B.P.J.; investigation, B.P.J.; resources, K.S.; data curation, B.P.J.; writing—original draft preparation, K.S., O.N. and B.P.J.; writing—review and editing, K.S. and B.P.J.; visualization, K.S. and B.P.J.; supervision, K.S. and O.N.; project administration, K.S. and O.N.; funding acquisition, K.S. and O.N. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Institut Teknologi Bandung (ITB), Indonesia, under the grant of Riset PPMI KK 2021.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data analyzed are referenced in this article.

Acknowledgments: The authors are grateful to the academic editor’s and reviewers’ comments, careful reading, and numerous suggestions that greatly improved this paper.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Proof for Property 1. According to Cheung et al. (2014), a risk measure is called a law-invariant convex if it satisfies all four properties, namely monotonicity, translation invariance, law invariance and convexity. The first three properties are easy to verify. Now, we prove that DTVaR satisfies convexity. Suppose that X and Z denote two different target losses and Y denotes another loss associated, respectively, with the target losses. To prove convexity, we follow the proof of the subadditivity of DTVaR (Josaphat and Syuhada 2021).

Suppose that $F_{\lambda X}$ is a distribution function of λX and define quantile- α of λX as $Q_{\alpha}(\lambda X) = F_{\lambda X}^{-1}(\alpha)$ for specified probability level $\alpha \in (0,1)$, and quantile- δ of $(1 - \lambda)Z$ as $Q_{\delta}((1 - \lambda)Z) = F_{(1-\lambda)Z}^{-1}(\delta)$ for arbitrary excess level $\delta \in (0,1)$. Suppose that $S_2 = \lambda X + (1 - \lambda)Z$. Then,

$$\begin{aligned}
& (1 - \delta)^{d+1} + C(\alpha, \delta) - C(\alpha, \delta_1) \left\{ \lambda \text{DTV}\text{aR}_{(\alpha, a)}^{(\delta, d)}(X|Y; C) \right. \\
& \quad \left. + (1 - \lambda) \text{DTV}\text{aR}_{(\alpha, a)}^{(\delta, d)}(Z|Y; C) - \text{DTV}\text{aR}_{(\alpha, a)}^{(\delta, d)}(S_2|Y; C) \right\} \\
& = E \left[\lambda X \left(I_{\{Q_\alpha^2 \leq S_2 \leq Q_{\alpha_1}^2, Q_\delta \leq Y \leq Q_{\delta_1}\}} - I_{\{Q_\alpha \leq \lambda X \leq Q_{\alpha_1}, Q_\delta \leq Y \leq Q_{\delta_1}\}} \right) \right. \\
& \quad \left. + (1 - \lambda) Z \left(I_{\{Q_\alpha^2 \leq S_2 \leq Q_{\alpha_1}^2, Q_\delta \leq Y \leq Q_{\delta_1}\}} - I_{\{Q_\alpha \leq \lambda X \leq Q_{\alpha_1}, Q_\delta \leq Y \leq Q_{\delta_1}\}} \right) \right] \\
& \geq Q_\alpha E \left[I_{\{Q_\alpha^2 \leq S_2 \leq Q_{\alpha_1}^2, Q_\delta \leq Y \leq Q_{\delta_1}\}} - I_{\{Q_\alpha \leq \lambda X \leq Q_{\alpha_1}, Q_\delta \leq Y \leq Q_{\delta_1}\}} \right] \\
& \quad + Q_\alpha^1 E \left[I_{\{Q_\alpha^2 \leq S_2 \leq Q_{\alpha_1}^2, Q_\delta \leq Y \leq Q_{\delta_1}\}} - I_{\{Q_\alpha^1 \leq (1 - \lambda) Z \leq Q_{\alpha_1}^1, Q_\delta \leq Y \leq Q_{\delta_1}\}} \right] \\
& = Q_\alpha \{ C(\alpha_1, \delta_1) - C(\alpha, \delta_1) - C(\alpha_1, \delta) + C(\alpha, \delta) - C(\alpha_1, \delta_1) + C(\alpha, \delta_1) \\
& \quad + C(\alpha_1, \delta) - C(\alpha, \delta) \} + Q_\alpha^1 \{ C(\alpha_1, \delta_1) - C(\alpha, \delta_1) - C(\alpha_1, \delta) + C(\alpha, \delta) \\
& \quad - C(\alpha_1, \delta_1) + C(\alpha, \delta_1) + C(\alpha_1, \delta) - C(\alpha, \delta) \} \\
& = 0,
\end{aligned}$$

where $C(p, q) = C(p, q; \theta)$.

In the above inequality, we use the following fact:

(*) If $\lambda X < Q_\alpha$, then

$$I_{\{Q_\alpha^2 \leq S_2 \leq Q_{\alpha_1}^2, Q_\delta \leq Y \leq Q_{\delta_1}\}} - I_{\{Q_\alpha \leq \lambda X \leq Q_{\alpha_1}, Q_\delta \leq Y \leq Q_{\delta_1}\}} \geq 0;$$

(**) If $Q_\alpha \leq \lambda X \leq Q_{\alpha_1}$, then

$$I_{\{Q_\alpha^2 \leq S_2 \leq Q_{\alpha_1}^2, Q_\delta \leq Y \leq Q_{\delta_1}\}} - I_{\{Q_\alpha \leq \lambda X \leq Q_{\alpha_1}, Q_\delta \leq Y \leq Q_{\delta_1}\}} \leq 0.$$

This proves that DTVaR follows the law-invariant convex property. $\square$

Proof for Theorem 1. According to Property 1, DTVaR is a law-invariant convex risk measure. By the result of Theorem 2.6 of Krättschmer et al. (2014), the first nonparametric estimator $\widehat{\text{DTV}\text{aR}}_{(\alpha, a)}^{(\delta, d)(1)}(X|Y; C)$ is consistent.

For the estimator $\widehat{\text{DTV}\text{aR}}_{(\alpha, a)}^{(\delta, d)(2)}(X|Y; C)$ given in (14), we observe that $\alpha'_{(q)} \rightarrow \alpha$ as $m \rightarrow \infty$, and thus results in $h_{k(q)} \approx 0$ (compare Jadhav et al. 2013, p. 83). Therefore,

$$\widehat{\text{DTV}\text{aR}}_{(\alpha, a)}^{(\delta, d)(2)}(X|Y; C) \approx \widehat{\text{DTV}\text{aR}}_{(\alpha, a)}^{(\delta, d)(1)}(X|Y; C),$$

and thus the consistency property is also followed for $\widehat{\text{DTV}\text{aR}}_{(\alpha, a)}^{(\delta, d)(2)}(X|Y; C)$. $\square$

Proof for Lemma 3. We assume first that $x \leq Q_{p_1}(X)$. We obtain

$$P(X \leq x | Q_p \leq X \leq Q_{p_1}, Q_\delta \leq Y \leq Q_{\delta_1}) = \frac{P(Q_p \leq X \leq x, Q_\delta \leq Y \leq Q_{\delta_1})}{P(Q_p \leq X \leq Q_{p_1}, Q_\delta \leq Y \leq Q_{\delta_1})},$$

where the denominator may be written as follows:

$$P(Q_p \leq X \leq Q_{p_1}, Q_\delta \leq Y \leq Q_{\delta_1}) = C(p_1, \delta_1; \theta) - C(p, \delta_1; \theta) - C(p_1, \delta; \theta) + C(p, \delta; \theta).$$

Thus,

$$\begin{aligned}
& P(X \leq x | Q_p \leq X \leq Q_{p_1}, Q_\delta \leq Y \leq Q_{\delta_1}) \\
& = \frac{1}{C(p_1, \delta_1; \theta) - C(p, \delta_1; \theta) - C(p_1, \delta; \theta) + C(p, \delta; \theta)} \int_{Q_\delta}^{Q_{\delta_1}} \int_{Q_p}^x \frac{\partial^2 C(F_X(x), F_Y(y))}{\partial x \partial y} dx dy.
\end{aligned}$$

For fixed level $p = \alpha$ and specified a and d , the DCTV of X associated with Y is given by

$$\begin{aligned} \text{DCTV}_{(\alpha,a)}^{(\delta,d)}(X|Y;C) &= \frac{1}{C(p_1, \delta_1; \theta) - C(p, \delta_1; \theta) - C(p_1, \delta; \theta) + C(p, \delta; \theta)} \\ &\times \int_{Q_\delta}^{Q_{\delta_1}} \int_{Q_\alpha}^{Q_{\alpha_1}} x \frac{\partial^2 C(F_X(x), F_Y(y))}{\partial x^2 \partial y} dx dy \\ &- \left(\text{DTVAr}_{(\alpha,a)}^{(\delta,d)}(X|Y;C) \right)^2. \end{aligned}$$

We suppose that the densities of F_X and F_Y are f_X and f_Y , respectively. Thus,

$$\begin{aligned} \text{DCTV}_{(\alpha,a)}^{(\delta,d)}(X|Y;C) &= \frac{\int_{\delta}^{\delta_1} \int_{\alpha}^{\alpha_1} (F_X^{-1}(u))^2 c(u, v) du dv}{C(\alpha_1, \delta_1; \theta) - C(\alpha, \delta_1; \theta) - C(\alpha_1, \delta; \theta) + C(\alpha, \delta; \theta)} \\ &- \left(\text{DTVAr}_{(\alpha,a)}^{(\delta,d)}(X|Y;C) \right)^2. \end{aligned}$$

□

Theorem A1 (Glivenko–Cantelli Theorem). Suppose that $X_1, \dots, X_m$ are i.i.d. random variables from a distribution with distribution function F_X . For each m , let F_X^m be the empirical distribution function given by

$$F_X^m(u) = \frac{1}{m} \sum_{i=1}^m I(X_i \leq u).$$

Then, we have

$$\|F_X^m - F_X\|_\infty = \sup_{X \in \mathbb{R}} |F_X^m(x) - F_X(x)| \rightarrow_{a.s.} 0. \quad (\text{A1})$$

Proof for Theorem 2. Before proving the theorem, we state the Glivenko–Cantelli theorem.

The proof is similar to the proof for Theorem 1. The statement of Theorem 2 is almost surely equivalent to the convergence of $\int_{\delta}^{\delta_1} \int_{\alpha}^{\alpha_1} (F_X^{m(-1)}(u))^2 d(u, v)$ to $\int_{\delta}^{\delta_1} \int_{\alpha}^{\alpha_1} (F_X^{-1}(u))^2 d(u, v)$. This latter convergence is followed (even uniformly over all $\alpha \in [0, 1]$) if the statement

$$\int_0^1 \int_0^1 |(F_X^{m(-1)}(u))^2 - (F_X^{-1}(u))^2| d(u, v) \rightarrow_{a.s.} 0 \quad (\text{A2})$$

holds.

We now provide proof in the following steps:

- **Step 1.** Assuming that the random variables $X_1, \dots, X_m \in \mathbb{R}$ are i.i.d. with distribution function F_X , Brazauskas et al. (2008) argued that the bi-implication—the statement (A3) below—

$$\int_0^1 |F_X^{m(-1)}(u) - F_X^{-1}(u)| du \rightarrow_{a.s.} 0 \quad (\text{A3})$$

is true if and only if the following two statements $F_X^m \Rightarrow F_X$ (weak convergence) and $\int |x| dF_X^m(x) \rightarrow \int |x| dF_X(x)$ hold. The first statement follows from the classical Glivenko–Cantelli theorem, which says that the supremum distance between F_X^m and F_X converges almost surely to 0.

- **Step 2.** Similarly to Brazauskas et al. (2008), we argue that the statement (A2) is true if the following two statements $(F_X^m)^2 \Rightarrow (F_X)^2$ and $\int |x|^2 dF_X^m(x) \rightarrow \int |x|^2 dF_X(x)$ almost surely hold. However, previously, we know the fact that $F_X^m \Rightarrow F_X$ (weak

convergence) and $\int |x| dF_X^m(x) \rightarrow \int |x| dF_X(x)$ almost surely hold from Step 1. Then, we have

$$\begin{aligned} \|(F_X^m)^2 - (F_X)^2\|_\infty &= \sup_{x \in \mathbb{R}} |(F_X^m(x))^2 - (F_X(x))^2| \\ &= \sup_{x \in \mathbb{R}} (F_X^m(x) + F_X(x)) \times |F_X^m(x) - F_X(x)| \\ &\leq \sup_{x \in \mathbb{R}} (F_X^m(x) + F_X(x)) \times \sup_{x \in \mathbb{R}} |F_X^m(x) - F_X(x)| \\ &= 2 \cdot \sup_{x \in \mathbb{R}} |F_X^m(x) - F_X(x)| \rightarrow_{a.s.} 0. \end{aligned}$$

Hence, the statement (A2) holds. Thus, the estimator $\widehat{\text{DCTV}}_{(\alpha,a)}^{(\delta,d)(1)}$ is consistent for $\text{DCTV}_{(\alpha,a)}^{(\delta,d)(1)}$.

For the estimator $\widehat{\text{DCTV}}_{(\alpha,a)}^{(\delta,d)(2)}$ given in (18), we observe that $\alpha'_{(q)} \rightarrow \alpha$ as $m \rightarrow \infty$, and, thus, results in $h_{k(q)} \approx 0$. Therefore,

$$\widehat{\text{DCTV}}_{(\alpha,a)}^{(\delta,d)(2)} \approx \widehat{\text{DCTV}}_{(\alpha,a)}^{(\delta,d)(1)},$$

and thus the consistency property also follows for $\widehat{\text{DCTV}}_{(\alpha,a)}^{(\delta,d)(2)}$. This finishes the proof of Theorem 2. $\square$

Proof for Theorem 3. In the sequel, we have followed the proof of the asymptotic property of the TVaR estimator, originally given in Brazauskas et al. (2008), to prove the asymptotic property of the DTVaR. We start the proof of Theorem 3 with the representation

$$\widehat{\text{DTVaR}}_{(\alpha,a)}^{(\delta,d)(n)}(X|Y;C) - \text{DTVaR}_{(\alpha,a)}^{(\delta,d)}(X|Y;C) = \frac{\int_{\delta}^{\delta_1} \int_{\alpha}^{\alpha_1} (F_X^{m(-1)}(u) - F_X^{-1}(u)) d(u,v)}{P(Q_\alpha \leq X \leq Q_{\alpha_1}, Q_\delta \leq Y \leq Q_{\delta_1})}. \quad (\text{A4})$$

Our next step is to extract a sum of random variables from the right-hand side of (A4). To understand how to perform this well, we shall now look at the integral below:

$$\int_{\delta}^{\delta_1} \int_{\alpha}^{\alpha_1} (F_X^{m(-1)}(u) - F_X^{-1}(u)) d(u,v). \quad (\text{A5})$$

Note that the integral (A5) can be approximated as follows (compare Brazauskas et al. (2008)):

$$\int_{\delta}^{\delta_1} \int_{\alpha}^{\alpha_1} (F_X^{m(-1)}(u) - F_X^{-1}(u)) d(u,v) \approx \int_{F_Y^{-1}(\delta)}^{F_Y^{-1}(\delta_1)} \int_{F_X^{-1}(\alpha)}^{F_X^{-1}(\alpha_1)} (F_X^m(x) - F_X(x)) d(x,y).$$

Hence, for every fixed $\alpha, \delta \in (0, 1)$, as well as $a, d \geq 0$, we have that

$$\begin{aligned} & \sqrt{m} \left(\widehat{\text{DTVAr}}_{(\alpha, a)}^{(\delta, d)(n)}(X|Y; C) - \text{DTVAr}_{(\alpha, a)}^{(\delta, d)}(X|Y; C) \right) \\ & \quad \frac{\int_{F_Y^{-1}(\delta)}^{F_Y^{-1}(\delta_1)} \int_{F_X^{-1}(\alpha)}^{F_X^{-1}(\alpha_1)} \sqrt{m} (F_X^m(x) - F_X(x)) \, d(x, y)}{P(Q_\alpha \leq X \leq Q_{\alpha_1}, Q_\delta \leq Y \leq Q_{\delta_1})} \\ & = \frac{1}{\sqrt{m}} \sum_{i=1}^m H(X_i, Y_i; \alpha, a, \delta, d), \end{aligned}$$

where

$$H(X_i, Y_i; \alpha, a, \delta, d) = - \frac{\int_{F_Y^{-1}(\delta)}^{F_Y^{-1}(\delta_1)} \int_{F_X^{-1}(\alpha)}^{F_X^{-1}(\alpha_1)} (I(X_i \leq x, Y_i \leq y) - F_X(x)) \, d(x, y)}{P(F_X^{-1}(\alpha) \leq X \leq F_X^{-1}(\alpha_1), F_Y^{-1}(\delta) \leq Y \leq F_Y^{-1}(\delta_1))}.$$

For every fixed $\alpha, \delta \in [0, 1]$, as well as for $a, d \geq 0$, the random variables $H(X_i, Y_i; \alpha, a, \delta, d)$, $1 \leq i \leq m$, are centered, i.i.d., and have variances $\text{DCTV}_{(\alpha, a)}^{(\delta, d)}(X|Y; C)$. The variance $\text{DCTV}_{(\alpha, a)}^{(\delta, d)}(X|Y; C)$ is finite for every finite $a, d \geq 0$ if the second moment of X is finite. This completes the proof of Theorem 3. $\square$

Proof for Lemma 4. To begin with the DTVar calculation, we compute the numerator as follows:

$$\begin{aligned} & \int_{\alpha}^{\alpha_1} \int_{\delta}^{\delta_1} F_X^{-1}(u) c(u, v) \, dv \, du \\ & = \beta_1 \int_{\alpha}^{\alpha_1} \int_{\delta}^{\delta_1} [(1-u)^{-1/\gamma_1} - 1] (\theta_1 - 2u\theta_1 - 2v\theta + 4uv\theta + 1) \, dv \, du \\ & = \beta_1 \int_{\alpha}^{\alpha_1} (1-u)^{-1/\gamma_1} \, du \times \int_{\delta}^{\delta_1} (\theta - 2v\theta + 1) \, dv + 2\beta_1 \theta \int_{\alpha}^{\alpha_1} u(1-u)^{-1/\gamma_1} \, du \\ & \quad \times \int_{\delta}^{\delta_1} (2v-1) \, dv - \beta_1 \int_{\alpha}^{\alpha_1} \int_{\delta}^{\delta_1} (\theta - 2u\theta - 2v\theta + 4uv\theta + 1) \, dv \, du \\ & = \beta_1 (A + 2\theta_1 B - C), \end{aligned}$$

where $A = \left(\int_{\alpha}^{\alpha_1} (1-u)^{-1/\gamma_1} \, du \right) \left(\int_{\delta}^{\delta_1} (\theta - 2v\theta + 1) \, dv \right)$, and B and D are as follows:

$B = \left(\int_{\alpha}^{\alpha_1} u(1-u)^{-1/\gamma_1} \, du \right) \left(\int_{\delta}^{\delta_1} (2v-1) \, dv \right)$, $D = \int_{\alpha}^{\alpha_1} \int_{\delta}^{\delta_1} (\theta - 2u\theta - 2v\theta + 4uv\theta + 1) \, dv \, du$. Thus, we obtain

$$\text{DTVAr}_{(\alpha, a)}^{(\delta, d)}(X|Y; C) = \frac{\beta_1 (A + 2\theta B - D)}{C(\alpha_1, \delta_1; \theta) - C(\alpha, \delta_1; \theta) - C(\alpha_1, \delta; \theta) + C(\alpha, \delta; \theta)}, \quad (\text{A6})$$

where the copula $C(p, q; \theta) = pq + \theta pq(1-\alpha)(1-\delta)$. $\square$

Note

¹ In description we use the terms loss(es) and risk(s) interchangeably.

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