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Pricing Options with Vanishing Stochastic Volatility

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Abstract: In the past years, there has been an extensive investigation of the class of stochastic volatility models for the evaluation of options and complex derivatives. These models have proven to be extremely useful in generalizing the classic Black–Scholes economy and accounting for discrepancies between observation and predictions in the simple log-normal, constant-volatility model. In this paper, we study the structure of an options market with a stochastic volatility that will eventually vanish (i.e., reaches zero) for very short periods of time with probability of one. We investigate the form of pricing measures in this situation, first in a simple binomial case, and then for a diffusion model, by constructing a weak approximation in discrete space and continuous time. The market described allows fleeting arbitrage opportunities, since a vanishing volatility prevents the construction of an equivalent measure, so that pricing contingent claims are, a priori, not obvious. Nevertheless, we can still produce a fair pricing equation. Let us note that this issue is not only of theoretical relevance, as the phenomenon of very low volatility has indeed been observed in the financial markets and the economy for quite a long time in the recent past.

Keywords: option pricing; stochastic volatility; arbitrage; degenerate diffusions; boundary conditions

Citation: Mastroeni, Loretta. 2022. Pricing Options with Vanishing Stochastic Volatility. *Risks* 10: 175. https://doi.org/10.3390/risks10090175

Academic Editor: Mogens Steffensen

Received: 10 June 2022
Accepted: 30 August 2022
Published: 5 September 2022

1. Introduction

In the past years there has been an extensive investigation of the class of stochastic volatility models for the evaluation of options and complex derivatives. These models have proven to be extremely useful in generalizing the classic Black–Scholes economy and accounting for discrepancies between observation and predictions in the simple log-normal, constant-volatility model.

Several different specifications of stochastic volatility models have been suggested. A review of the first and seminal stochastic volatility models can be found in Ball and Roma (1994). Among these, Wiggins (1987) studies a case in which the volatility (more precisely, its logarithm) follows an Ornstein—Uhlenbeck process with mean-reversion, while Hull and White, in their fundamental paper Hull and White (1987), exploit a Taylor expansion technique to solve their stochastic volatility option pricing problem in their framework, "the dynamics of the Hull–White stochastic volatility model predicts that both expectation and most likely value of instantaneous volatility converge to zero" Jaeckel (2022).

In Stein and Stein (1991), Stein and Stein assume that the volatility is driven by an arithmetic Ornstein–Uhlenbeck process and develop the analytic density function of stock returns to evaluate option prices. Schobel and Zhu (1999)'s model extends the stochastic volatility model of Stein and Stein, by using Fourier inversion techniques to allow for correlation between instantaneous volatilities and the underlying stock returns. They also derive a closed-form pricing solution for European options. One interesting aspect is that the dynamics of Stein and Stein's model, as well as that of Schoebl and Zhu, predict that volatility is most likely close to zero.

While all pre-1990 models, including Scott (1987) and Chesney and Scott (1989), necessitate wide use of numerical techniques since they have no closed-form solutions, Heston, in his fundamental paper (Heston 1993), in 1993 provides a new approach for deriving a closed-form solution for options with stochastic volatility. Although his model was not the first stochastic volatility model introduced in the option pricing literature and although it is now many years old, it has certainly assumed great importance within this field and is still used today as a benchmark for all stochastic volatility models. From our point of view, it is interesting to note that the dynamics of Heston's model predict that volatility can vanish (i.e., reach zero) and remain at zero for a while or remain very low or very high for long periods of time.

In Hagan et al. (2002), the authors derived so-called SABR models, i.e., stochastic volatility models trying to capture the volatility smile in derivatives markets. In particular, in SABR models, the forward value meets specific properties to overcome contradictions between the model and the market, such as those that emerged in the study of market smiles and skews using local volatility models. They write "market smiles and skews are usually managed by using local volatility models á la Dupire. We discover that the dynamics of the market smile predicted by local volatility models are opposite of observed market behavior: when the price of the underlying decreases, local volatility models predict that the smile shifts to higher prices; when the price increases, these models predict that the smile shifts to lower prices". Quoting Jaeckel (2022) again, "the dynamics of the Hagan model predict that the expectation of volatility is constant over time, that variance of instantaneous volatility grows without limit, and that the most likely value of instantaneous volatility converges to zero".

Another interesting paper dealing with option pricing under stochastic volatility is Carr and Sun (2007), in which the authors develop a new approach for pricing European-style contingent claims written on the spot price of an underlying asset whose volatility is stochastic. In He and Zhu (2016), an analytical approximation formula for European option pricing under a stochastic volatility model with regime-switching is derived, while in He and Chen (2021), the authors derive a closed-form pricing formula for European options under a stochastic volatility model with a stochastic long-term mean. The problem of option hedging and implied volatilities in the presence of stochastic volatility is addressed in Renault and Touzi (1996).

In Alghalith et al. (2020), the authors relax the assumption of constant volatility of volatility and therefore allow the volatility of volatility to vary over time. In this context, they propose novel nonparametric estimators for stochastic volatility and the volatility of volatility.

In Hoque et al. (2020), an interesting and relevant issue of option pricing is addressed. Specifically, the predictive power of implied volatility smirk to forecast foreign exchange return is investigated. In Le et al. (2021), the intraday implied volatility for pricing currency options is studied.

As for the issues relating to the estimation and computation of volatility, the reader can see, e.g., Liu et al. (2019), Wiggins 1987, Harvey (1998), Breidt et al. (1998), Poon and Granger (2005), Luo et al. (2008), He and Zhu (2016), Cuchiero et al. (2020), and references therein. In addition, for recent review papers, one can refer, e.g., to Taylor (1994), Shephard and Andersen (2009), Shephard (2005), Shin (2018), and to references therein.

As mentioned, most stochastic volatility models in the literature allow (although they do not deal with this issue) volatility to take on arbitrarily small values (or even vanish), albeit for such a *short time* that arbitrage opportunities are not created. It is also well known (and intuitive) that allowing the volatility to vanish will create arbitrage opportunities. This would seem to be a good reason to avoid this case from a financial point of view.

In practice, it is hardly possible to discriminate statistically between volatility processes that become arbitrarily small and processes that vanish altogether for a sufficiently small amount of time, so it might be useful to examine the structure of a model that has a volatility component effectively vanishing, with a nontrivial local time. The market described in this way allows fleeting arbitrage opportunities, so that pricing contingent claims are, a priori, not obvious.

Our paper aims at studying the structure of an options market with vanishing stochastic volatility. Specifically, we seek a pricing formula for contingent claims written on an asset with a volatility that follows a stochastic Markov process that will eventually vanish over very short time periods with probability of one. We investigate the form of pricing measures in this situation, first in a simple binomial case, and then for a diffusion model, by constructing a weak approximation in discrete space and continuous time. We deduce our main result, for the continuous time case, through a passage to the limit.

Although many papers deal with stochastic volatility and its applications, the author is not aware of any papers addressing arbitrage issues and looking for pricing formulas for contingent claims in the case of vanishing stochastic volatility.

Let us note that in the recent past, the phenomenon of very low volatility has been observed in the financial markets and the economy. As an instance, we all remember how, in 2016 and 2017, financial markets were characterized by very low volatility, raising the question of whether volatility measures may adequately reflect risks in financial markets; however, this phenomenon is not limited to 2017: volatility has been subdued for the majority of the recovery since early 2009. See, e.g., Hausman (2017), Guagliano and Ramella (2018).

With regard to literature on market arbitrage opportunities in this framework, in Jarrow and Protter (2005), the authors study hidden (i.e., almost not observable) arbitrage opportunities in markets where large traders affect the price process. The arbitrage opportunities are hidden because they occur on a small set of times (typically of Lebesgue measure zero).

It is important to point out that our model and that of Jarrow and Protter (2005) are different, in the sense that in the latter the price process is studied and the null local time directly affects this process, whereas in our case, we consider the stochastic volatility process, and the null local time directly affects the volatility process. The two processes obviously differ in their nature and, accordingly, the types of arbitrage are also different. Still, basically, the principle is the same: there is a very short time in which arbitrage opportunities may arise.

Another interesting paper on arbitrage occurrence is Osterrieder and Rheinlander (2006), in which the authors study "arbitrage opportunities in diverse markets (as introduced by Fernholz (1999)). By a change of measure technique they can generate a variety of diverse markets. The construction is based on an absolutely continuous but non-equivalent measure change which implies the existence of instantaneous arbitrage opportunities in diverse markets".

The content of the paper is organized as follows. In Section 2, we introduce our problem. Then, after a brief description of the notation in Section 3, Section 4 details the class of models under consideration. In Section 5 we discuss the general case: first, we study discrete-time models that shed some light on the pricing of options under these circumstances. Then, we deduce our main result, for the continuous time case, through a passage to the limit. Finally, after a brief explicit discussion of a simple model in Section 6, we devote Section 7 to possible interpretations and practical implications of our models. We draw our conclusions in Section 8.

2. The Problem

Consider a diffusion-type model for the evolution of the price of an asset:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t \quad t \geq u \tag{1}$$

$$S_u = s > 0. \tag{2}$$

Here, μ_t and σ_t are processes adapted to a filtration $\{\mathfrak{F}_t\}$, describing the information structure. Especially, assume that the volatility σ_t itself follows a stochastic Markov process:

$$d\sigma_t = b_t + c_t dB_t$$

where B_t is a Brownian motion assumed, for simplicity, to be independent of W_t.

We will look at the problem of pricing for a contingent claim written on this asset.

For convenience, we will often work with the logarithm of the price process, rather than the price itself. Thus, we rewrite our model as

$$\begin{aligned} dY_t &= \zeta(t, \sigma_t, Y_t)dt + \sigma_t dW_t \\ d\sigma_t &= b(t, \sigma_t)dt + c(t, \sigma_t)dB_t \\ Y_u &= y, \sigma_u = \sigma \end{aligned} \tag{3}$$

where Y is the logarithm of the price of the asset (or a vector of assets, if $Y \in \mathbb{R}^n$), and the volatility is itself a diffusion process. Again, W and B are independent standard Brownian motions. By Itô's rule, if $Y = \log S$, $\frac{dS_t}{S_t} = dY_t + \frac{\sigma_t^2}{2}dt$, and, under an equivalent martingale measure,

$$dY_t = -\frac{\sigma_t^2}{2}dt + \sigma_t dW_t \tag{4}$$

In general, μ_t is an adapted process, and the drift term for Y will be $\zeta_t = \mu_t - \frac{\sigma_t}{2}$. Under the martingale measure, the new drift will thus be $\hat{\zeta}_t = -\frac{\sigma_t^2}{2}$ and $\zeta_t - \hat{\zeta}_t = \mu_t$.

Remark 1. *In the following, we will resort to continuous time jump processes. A continuous time jump process could be defined with equations similar to* (3)*, where the Brownian driving noises are replaced by Poisson martingales, $N_\lambda(t) - \lambda t$. With appropriate rescaling, these SDEs can be made to converge weakly to* (3) *(see, for example, Ethier and Kurtz (1986), Ikeda and Watanabe (1981)).*

We should point out that σ_t is not in itself a *physical* quantity, since it is σ_t^2 that has an actual interpretation, as infinitesimal variance rate. It is thus advisable, if a model is written for σ instead of σ^2, that it avoids σ to change sign for definiteness, preventing it from turning negative. It is also well known (and intuitive) that allowing σ_t to vanish will allow arbitrage opportunities to arise (see., e.g., Duffie (1996)). This would seem a good reason to avoid this case from a financial point of view. However, most stochastic volatility models, as we said in the Introduction, will let σ_t take arbitrarily small values or even vanish, albeit for such a *short time* that no arbitrage opportunities actually arise (see, for instance, respectively, Bjork (2009) for a log-normal model, and Heston (1993) for a CIR-like model). Technically, this corresponds to the fact that the boundary $\sigma = 0$ acts as a natural or a no-exit entrance boundary for the volatility process (see, e.g., Revuz and Yor (1991)). In such a situation, no additional boundary condition can be imposed at $\sigma = 0$.

It is, however, possible to construct models that keep σ nonnegative by imposing an appropriate behavior at a boundary, for instance, through a reflecting boundary condition at $\sigma = 0$. In this case, the process displays a local time measuring, in a sense, the time spent at the boundary, and even if the Lebesgue measure of the set of ts where the process stays on the boundary is zero, it accumulates to a finite value.

In practice, it is hardly possible to discriminate statistically between volatility processes that become arbitrarily small and processes that vanish altogether for a sufficiently small amount of time, so it might be useful to examine the structure of a model that has a volatility component that effectively vanishes, with a nontrivial local time. In such a situation, pricing contingent claims are, a priori, not obvious, since the market allows arbitrage opportunities to arise briefly. In this paper, we address this question and look for a pricing formula for contingent claims. The case of American options with stochastic volatility (but no arbitrage) has been discussed, e.g., in Mastroeni (1998).

3. Notation

Hereafter, functions of a variable u are equivalently noted as in X_u or $X(u)$. Given a continuous process $\{U_s\}$, $\mathfrak{F}_t^U = \sigma\{U_u : u \leq t\}$ is the filtration generated by U, while $[U]_t$ is its quadratic variation process (see Karatzas and Shreve (1991)). All stochastic integrals are in the sense of Itô. Processes will be realized canonically in D (the space of right continuous paths with left-hand limits, RCLL) or C (the space of continuous paths). Weak convergence of measures over D (or C) will be denoted by $\rightharpoonup$.

4. The Model

Consider again (1). Assume that μ_t is a continuous process, adapted to $\{\mathfrak{F}_t\}$, and σ_t is a diffusion process, driven by a Brownian motion B_t, which, for simplicity, we will take to be independent of W_t.

Such an assumption simplifies proofs, but is not essential, as far as general theorems go (see, e.g., Chapter 14 in Bjork (2009)). It does affect the ability to compute explicit formulas in simple models. We assume, without further comment, that there is a unique strong solution for any set of initial conditions. Additional technical assumptions, satisfied by standard models, will be made later. For simplicity, we will assume a zero riskless borrowing rate, i.e., all prices are already discounted.

For convenience, we will often work with the logarithm of the price process, rather than the price itself, $Y_t = \log S_t$.

The solution to (1) is, in principle, adapted to the filtration $\mathfrak{F}_t = \sigma\{W_s, B_s : 0 \le s \le t\}$. Note that σ, in itself, has no *physical* meaning, since only σ^2 can be interpreted as infinitesimal variance and can be observed, at least in principle. In fact, $\int_u^t \sigma_s^2 ds = [Y]_t$, so that if $\sigma \ge 0$ with probability of one, it turns out that $\mathfrak{F}_t^Y$, even though not Markovian, still carries all the relevant information included in $\mathfrak{F}_t$ (in fact, for any t, σ_t^2 will be measurable with respect to $\cap_{\varepsilon>0}\mathfrak{F}_{t-\varepsilon,t}^Y \subseteq \mathfrak{F}_t$, where $\mathfrak{F}_{t-\varepsilon,t}^Y \equiv \sigma\{Y_s : t-\varepsilon \le s \le t\}$). On the other hand, a model where σ is allowed to change sign, and such that $\mathfrak{F}_t^{\sigma^2} \subset \mathfrak{F}_t^{\sigma}$ strictly, might have little practical significance. Thus, we will restrict ourselves to the case when $\sigma_t \ge 0$ *a.s.* $\forall\, t$.

It is well known (see, e.g., Bjork (2009)) that an *equivalent* martingale measure will exist if Girsanov's theorem can be applied to (1) and allow for an equivalent measure under which the price process S, discounted with the riskless rate, will be a martingale.

Taking into account Y_t, we rewrite our model as

$$\begin{aligned} dY_t &= \zeta(t, \sigma_t, Y_t)dt + \sigma_t dW_t \\ d\sigma_t &= b(t, \sigma_t)dt + c(t, \sigma_t)dB_t \\ Y_u &= y, \sigma_u = \sigma \end{aligned} \tag{5}$$

where, as said earlier, Y_t is the logarithm of the price of the asset (or a vector of assets, if $Y_t \in \mathbb{R}^n$), and the volatility is itself a diffusion process. Again, W_t and B_t are independent standard Brownian motions. By Itô's rule, $\frac{dS_t}{S_t} = dY_t + \frac{\sigma_t^2}{2}dt$, and, under an equivalent martingale measure, $dY_t = -\frac{\sigma_t^2}{2}dt + \sigma_t dW_t$. In general, μ_t is an adapted process, and the drift term for Y_t will be $\zeta_t = \mu_t - \frac{\sigma_t}{2}$. Under the martingale measure, the new drift will thus be $\hat{\zeta}_t = -\frac{\sigma_t^2}{2}$ and $\zeta_t - \hat{\zeta}_t = \mu_t$.

Girsanov's theorem will apply if the local martingale

$$\Lambda_t = \exp\left\{ \int_u^t H_s A_s^{-1} d\beta_s - \frac{1}{2}\int_u^t [H_s A_s^{-1}]^2 ds \right\} \tag{6}$$

is a martingale. Here,

$$\begin{aligned} A_t &= \begin{pmatrix} \sigma^2 & 0 \\ 0 & c^2(t, \sigma_t) \end{pmatrix} \\ H_t &= \begin{pmatrix} \mu_t \\ b(t, \sigma_t) - \varphi_t \end{pmatrix} \\ \beta_t &= \begin{pmatrix} W_t \\ B_t \end{pmatrix}. \end{aligned}$$

Here, φ_t is the *market price of risk* for the volatility, (i.e., a measure of the extra return, or risk premium, that investors demand to bear risk). Since this is not a traded asset, the choice for φ is essentially arbitrary, tied to the attitude towards the volatility risk of a given investor. This arbitrariness is tied to the incompleteness of the market and the non-uniqueness of any martingale measure. For simplicity, we will make a choice, e.g., $\varphi \equiv 0$, but our arguments do not essentially depend on any particular choice..

Remark 2. *Note that, as well known in the literature, in our model the financial market is not complete and, thus, even if an equivalent martingale measure does exist, it will not generally be unique. In fact, roughly speaking, we can change the stochastic volatility measure without changing the fact that the stock is a martingale; thus, we will have payoffs that have different values under different measures, i.e., the market is incomplete. We will not address this question and assume that a* market price of risk *for the volatility has been established. See Bjork (2009) for details.*

The martingale condition is equivalent to $\mathbf{E}[\Lambda_t] = 1$. A sufficient condition for this is, for instance, $\mathbf{E}\big[\exp\big\{\frac{1}{2}\int_u^t [H_s A_s^{-1}]^2 ds\big\}\big] < \infty$ (see, e.g., Karatzas and Shreve (1991)).

A simple computation will show that models where σ_t is allowed to vanish with probability of one may fail to satisfy Girsanov's condition. This is equivalent to violating the *no free lunch* condition and *almost* equivalent to violating the no-arbitrage condition (see, e.g., Duffie (1996)). This could be a reasonable motivation for considering only models that do not exhibit this behavior at all.

However, this restriction does not seem completely natural. For instance, it is very hard to check empirically, compared to a model with arbitrarily small volatility, such as log-normal or CIR-like models. Moreover, it seems worthwhile to examine how a market where the Girsanov theorem does not apply will actually behave.

In the following we will look at the pricing of European options for models where the volatility processes stay nonnegative, but are allowed to vanish.

Precisely, we will consider the case of a regular diffusion with reflection at 0. Much more general boundary conditions, constraining the volatility to the nonnegative axis but allowing for very different behaviors at zero, could be handled along the same lines.

Remark 3. *Asian options (where the payoff depends on $\int_u^T S_s ds$) present no further features, since they can be treated by simply adjoining an extra equation of the form $dX_t = S_t dt$. Though the resulting price process is degenerate, Girsanov's theorem still applies (see Albeverio et al. (1992)). Similarly, a wide class of exotic options falls within the realms of the following discussions.*

Assuming reflecting boundary conditions at $\sigma = 0$, it is easy to see that Girsanov's theorem will generally fail; hence, our model allows for arbitrage. Although this is very intuitive, we should note that the set $\mathfrak{Z} = \{t : \sigma_t = 0\}$ is very *thin*: for regular diffusion models, though uncountable, it will have Lebesgue measure zero. On the other hand, arbitrage is only possible at time $t \in \mathfrak{Z}$. The following discussion suggests that standard arguments can be extended to allow for *fair* pricing of a contingent claim, even in this case.

In order to achieve this, our argument will start from discrete-time and/or discrete-space approximating models. One could say that these models are, in some sense, more realistic descriptions of the market and that diffusion models are only convenient limits motivated by the enormous speed and volume of modern trading. If so, pricing for a diffusion model can be obtained by taking appropriate limits. In other words, since the lack of a paradigm for pricing under the present circumstances prevents direct pricing in a continuous-time, continuous-space model, we take the limit pricing from a discrete model as the definition of a fair price.

5. Arbitrage and Stochastic Volatility

5.1. A Simple Discrete-Time Model

To illustrate the main argument for pricing under vanishing volatility, we start first with the simplest possible situation. Consider a discrete-time price process

$$S_{k+1} = \left(1 + \mu_k \Delta t + \sigma_k \xi_{k+1} \sqrt{\Delta t}\right) S_k. \tag{7}$$

where $\{\xi_k\}$ is an iid Bernoulli sequence with values in $\{-1, +1\}$, and $\{\mu_k\}$ is a $\{\mathfrak{F}_k^{\xi}\}$ adapted process. Again, we assume a zero riskless return rate. Assume $\{\sigma_k\}$ to be a process, independent of $\{\xi_k\}$, taking values in some finite set $\{s_0 = 0, s_1, \ldots, s_m\}$ satisfying the following assumption:

Assumption 1. *$s_l > s_{l-1} > 0$, $l = 1, 2, \ldots, m$ and $\mu_j - s_1 \leq 0 \leq \mu_j$.*

The rationale here is that, as long as $\sigma_k > 0$, the riskless return rate falls between the upper and the lower return of the asset, and hence no arbitrage is possible, just as in the standard binomial model. As soon as $\sigma_k = 0$, however, the possible spread between μ_k and the riskless rate could produce an arbitrage opportunity. The possibility of arbitrage will depend on the assumed dependence of μ_k on σ_k, specifically when $\sigma_k = 0$.

Remark 4. *Note that we assume the market to be risk-averse so that a nonnegative risk premium is always assumed. The model allows this premium to persist even if the volatility vanishes. It is clear that, if the volatility is null, all assets have the same return as the riskless asset. Therefore, the risk premium is zero and there is no possibility of arbitrage. However, this has to be specifically stated in the model as in our setup μ_k can depend on σ_k, but need not vanish (equal the risk-free rate), when $\sigma_k = 0$. In case we had $\mu_t|_{\sigma_t=0} = 0$, there would be no risk premium when volatility vanishes; however, this is a choice of the model.*

The dynamics of the volatility is a separate issue: as an example, volatility might essentially be *reflected* at 0, i.e., it would return to the value s_1 at the immediately following time step. We assume a given stochastic model for σ_k, such that $\forall\, k : \sigma_k \geq 0$, and other appropriate restrictions on $\{\mu_k\}$, $\{\sigma_k\}$ to ensure that (7) is well posed.

A risk-neutral evolution for the price would thus be provided by

$$S_{k+1} = \left(1 + \sigma_k \xi_{k+1} \sqrt{\Delta t}\right) S_k. \tag{8}$$

We stress that, in (7) and (8), Δt should be small enough such that prices are positive. We also remark that, as already noted by Hull and White (1987), under the present circumstances we can work conditionally on a path $\{\sigma_k\}_{k=1}^{N}$ and then integrate on the volatility distribution. Clearly, positive (sub, super)martingales are preserved under this averaging procedure.

Theorem 1. *A contingent claim, with underlying asset evolving according to (7), expiring at T, and with payoff Z_T (an $\mathfrak{F}_T$–measurable random variable) has a price that can be determined by taking conditional expectations with respect to a measure on path space, under which S_k evolves according to (8). This pricing measure is a martingale measure, but is not necessarily equivalent to the* physical *one.*

Proof. Conditional on a path for $\{\sigma_k\}$, the usual dynamic programming argument applies with no change, until we reach a $j : \sigma_j = 0$. In this case, $S_{j+1} = S_j(1 + \mu_j \Delta t)$ is perfectly known at time $j\Delta t$ (the σ–fields $\mathfrak{F}_k^S$ and $\mathfrak{F}_{k+1}^S$ coincide). If $Z_{j+1}(S_{j+1})$ is the payoff of a European option at time $(j+1)\Delta t$, this would be a known quantity at time $j\Delta t$. Hence, the claim is perfectly covered by allocating its (discounted) known payoff. Thus, $Z_j = Z_{j+1}$ and the pricing measure is a martingale measure, but not equivalent to the original. □

Remark 5. *The singularity of the pricing measure is connected to the occurrence of arbitrage possibilities: clearly, if, say, $\mu_j > 0$, borrowing S_j at rate 0 would reap a riskless profit of $S_j\mu_j$ at time $(j+1)\Delta t$. Coincidentally, the hedging portfolios are all made 100% of asset. In fact, the writer superreplicates the option, and the buyer has a riskless portfolio with a higher yield than the bond rate. All of this is* impossible *in the world of pricing measures, where a claim is valued by considering it a* fair bet *on the underlying asset.*

5.2. Continuous Time Models

5.2.1. A Discrete Space Model

It is more convenient to use a different model as an approximation to the continuous-time, continuous-space model as represented by (1) and (5), especially given our need to handle boundary behavior. We will use some results from Gerardi et al. (1984), which would allow the present discussion to be extended to much more general boundary conditions. Although less popular in the financial literature than discrete-time models, it might be of independent interest. A related model was discussed in Frey and Runggaldier (2001), where it was presented as a candidate for modeling liquid assets whose price changes are mostly due to frequent trades rather than occasional big price changes.

The (continuous time Markov) scheme follows the following evolution:

- State variables (S or Y and σ, in our case) move on a grid. It is a natural choice, given the special form of our models, to choose a logarithmic grid for S and, consequently, choose to work with $Y_t = \log S_t$ over a regular grid of mesh h. Eventually, we will let $h \to 0$.
- The joint process jumps between first neighbor grid points at exponential random times, where the mean jump time and the jump size depend only on the present state.
- Jump time and sizes come in two flavors: symmetric jumps by $\pm h$ (with equal probability) either in the Y or σ direction, with mean jump time proportional to h^2, and jumps in one direction only, with mean jump time proportional to h. Since we assume h to be small, *most* jumps are of the symmetric type, while asymmetric jumps are only occasional.
- Correspondingly, we assume a continuous time riskless rate process r_t. We keep our simplifying assumption that the rate is $r = 0$.

A possible interpretation could be the following: prices and volatility change, as trades occur, by fixed (small) amounts; prices (and, possibly, volatility) follow a trend, but trades in this dominating direction are interspersed between many more *noise* (random) trades. Note that this corresponds to a common interpretation of diffusion models for markets, related to *informed* vs. *noise* trading.

The specific choice of jump rates will be motivated by the continuous limit we are aiming at, although the process can be easily inverted: find an appropriate jump model and choose the continuous model as the limit obtained by letting the scale factor vanish.

Under broad conditions, a martingale problem can be defined using approximate generators in the following way. The basic idea is to look at the infinitesimal generator A of the continuous model and look for a sequence of jump generators that approximate A appropriately (several theorems are available: see Ethier and Kurtz (1986)).

We are aiming at a process involving a non-trivial boundary condition, specifically a Neumann reflection, at the boundary $\sigma = 0$ for the volatility component. Specifically, the motion of σ at $\sigma = 0$ will be prescribed consistently with the required boundary behavior of a limiting process: for a reflecting diffusion, this is given by (10) below. Again, we refer to Gerardi et al. (1984) for other allowable choices and their corresponding limits, which correspond to other possible boundary conditions that preserve the Markov property.

Considering the price and volatility processes, we approximate the derivation operators in their generator A, resulting in a generator A_h, i.e., an integral operator that generates the discrete space continuous time process. In general, such a generator has the form

$$A_h u(y,\sigma) = \lambda(y,\sigma) \int [u(y+z,\sigma+\rho) - u(y,\sigma)] m^h_{s,\sigma}(dz, d\rho)$$

where λ is the intensity of the jump process and m is the probability distribution of jumps of sizes z and ρ, conditional on the current state.

In short, we use a finite difference approximation to our generator A to create A_h, and approximate the Neumann boundary condition (i.e., in the continuous limiting case, $\sigma'(0) = 0$) which we are assuming for the σ component, that is an instantaneous jump by one mesh size if σ_t hits the 0 boundary: $A_h u(y,0) = \frac{u(h)-u(0)}{h^2}$ (see Gerardi et al. (1984)).

By making an appropriate choice we can ensure that, e.g., for every f in a core for A, we can find a sequence $\{f_h\}$, $f_h \in \mathcal{D}(A_h) : A_h f_h$ converges, in an appropriate sense, to Af, so that the associated jump process converges in law to the diffusion generated by A (see Kushner (1984)).

The choice for such an approximation is not unique for a given A, and some care has to be exercised to ensure that A_h does, in fact, generate a Markov jump process. A possible choice for the jump process, written in terms of average rates of change over small time intervals of size δ, and letting $\delta \to 0$, in our case, is given by

$$\begin{aligned}
\mathbf{P}[Y^h_{t+\delta} - Y^h_t = +h / Y^h_t, \sigma^h_t] &= \zeta_t \frac{\delta}{h} + \frac{(\sigma^h_t)^2}{2} \frac{\delta}{h^2} + o(\delta) && (9)\\
\mathbf{P}[Y^h_{t+\delta} - Y^h_t = -h / Y^h_t, \sigma^h_t] &= \frac{(\sigma^h_t)^2}{2} \frac{\delta}{h^2} + o(\delta) \\
\mathbf{P}[\sigma^h_{t+\delta} - \sigma^h_d = \mathrm{sign} b(\sigma^h_t) h / Y^h_t, \sigma^h_t, \sigma^h_t > 0] &= b(\sigma^h_t) \frac{\delta}{h} + \frac{(c(\sigma^h_t))^2}{2} \frac{\delta}{h^2} + o(\delta) \\
\mathbf{P}[\sigma^h_{t+\delta} - \sigma^h_d = -\mathrm{sign} b(\sigma^h_t) h / Y^h_t, \sigma^h_t, \sigma^h_t > 0] &= \frac{(c(\sigma^h_t))^2}{2} \frac{\delta}{h^2} + o(\delta) \\
\mathbf{P}[\sigma^h_{t+\delta} = h / Y^h_t, \sigma^h_t = 0] &= \frac{c(0)^2}{h^2} + o(\delta). && (10)
\end{aligned}$$

The arguments used in the previous section to handle the case of vanishing volatility can be repeated here.

Theorem 2. *The pricing measure for model* (9) *is the distribution of the following process* (*recall that, for simplicity, we take the riskless rate to be zero*)*:*

$$\begin{aligned}
\mathbf{P}[Y^h_{t+\delta} - Y^h_t = +h / Y^h_t, \sigma^h_t > 0] &= \frac{(\sigma^h_t)^2}{2} \frac{\delta}{h} + \frac{(\sigma^h_t)^2}{2} \frac{\delta}{h^2} + o(\delta) && (11)\\
\mathbf{P}[Y^h_{t+\delta} - Y^h_t = +h / Y^h_t, \sigma^h_t = 0] &= o(\delta).
\end{aligned}$$

Proof. Compared with Equation (4), the martingale form of the evolution of Y_t has a "slow" jump "up" with mean time proportional to h, and a "fast" symmetric jump with mean time proportional to h^2, with the drift term reduced to be proportional to $\left(\sigma^h_t\right)^2$. In other words, conditionally on $\sigma^h_t > 0$, the pricing process follows a (discounted) martingale model with

$$\zeta_t = (\sigma^h_t)^2 / 2$$

and the usual dynamic programming argument applies with no change, until we reach $\sigma^h_t = 0$. As soon as $\sigma^h_t = 0$, the next value of the asset becomes perfectly known and the option does not change in value (except for discounting) until the volatility turns positive again. □

5.2.2. Continuous Space Limit

The continuous space model corresponding to the continuous time Markov chain model in the previous section is

$$\begin{aligned} dY_t &= \zeta_t dt + \sigma_t dw_t \\ d\sigma_t &= b(\sigma_t)dt + c(\sigma_t)dB_t + dl_t \\ (\sigma_t &\geq 0) \end{aligned} \tag{12}$$

where l_t is the appropriate local time magenta[1] for the volatility process, providing standard normal reflection conditions at $\sigma = 0$. Note that this problem (under standard assumptions on the coefficients) has a unique solution which we could, e.g., construct as follows:

1. Solve the Skorohod problem for the reflected process σ (see, e.g., Ikeda and Watanabe (1981)).
2. Conditional on σ, solve for Y.
3. By standard arguments, find the solution as a $\{\mathfrak{F}_t\}$-adapted process, where $\{\mathfrak{F}_t\}$ is the reference filtration.

It follows from general results (see, e.g., Ethier and Kurtz (1986) and Gerardi et al. (1984) for details of the application) that the pair (Y^h, σ^h) converges weakly to (Y, σ), the solution of (12) with reflecting boundary conditions. In terms of semigroups, the convergence speed is $O(h)$. In particular, a direct computation yields the boundary condition. We emphasize that, through an appropriate modification of (10), other choices for boundary behavior could be treated exactly in the same way.

Incidentally, this model is easy to implement as a Monte Carlo simulation. The convergence speed is $O(h)$ and it applies to expectations of bounded functions on the state space. Note that the speed of discrete-time Monte Carlo simulations is usually expressed in terms of L^2 convergence of the state variables, and hence these rates cannot be directly compared.

In order to state the main result of our paper, there are a few technical assumptions that have to be made for consistency in applying this scheme to the pricing problem. First of all, we assume that the coefficients in (12) and the payoff for our claim are such that $EZ_T < \infty$. A further condition is that Z_T is uniformly integrable under the measures Q_h. For simplicity, we will assume that Z_T is bounded, but this restriction can easily be lifted for any reasonable model.

Thus, the main result we obtain is the following.

Theorem 3. *Under the current assumptions, the pricing of a contingent claim is obtained by the following formula:*

$$\mathbf{E}^Q[Z_T] \tag{13}$$

where Q is the distribution of the process with the generator, on sufficiently smooth functions,

$$L = -\frac{\sigma^2}{2}\frac{\partial}{\partial y} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2} + b(\sigma)\frac{\partial}{\partial \sigma} + \frac{c^2(\sigma)}{2}\frac{\partial^2}{\partial \sigma^2}$$

The associated parabolic equation for $u(t, y, \sigma) = E^Q[Z_t]$: $\frac{\partial u}{\partial t} = Lu$ with the boundary conditions:

$$\begin{cases} \left.\dfrac{\partial u}{\partial \sigma}\right|_{\sigma=0} = 0 \\ Lu(y, \sigma) \text{ continuous on } \mathbb{R} \times [0, \infty). \end{cases}$$

Remark 6. *Other boundary behaviors are available for the volatility (the constraint is to choose conditions that will preserve the Markov property), including a natural boundary for σ (no boundary condition, as in log-normal or CIR-like models), but also nonsingular local time (producing a* sticky boundary *condition), etc.*

Proof. We resort to a standard argument (see Ethier and Kurtz (1986)). As a matter of fact, the approximating jump process corresponds to a strongly continuous semigroup. This follows by noting that the semigroup is given by $T_t f(y, \sigma) = \mathbf{E}_x f(Y_t, \sigma_t)$ and the process (Y, σ) satisfies a stochastic differential equation which has a unique strong solution. On the other hand, for smooth functions f in the domain of the generator of the limiting process (which constitutes a core), it is easy to carry out the following construction. In the first place, let L_h be the generator corresponding to (11). Note that f, restricted to the grid, say f_h, will be in the domain of L_h for each h and that $\|f_h - f\|_\infty \to 0$. It is now easy to check that $\|L_h f_h - Lf\|_\infty \to 0$, provided f satisfies $\frac{\partial f}{\partial \sigma}|_{\sigma=0}$, which proves our claim. For each h, the pricing measure Q_h is defined by (11). Thus, we look at the sequence $E^{Q_h} Z_T$. The weak limit for the sequence Q_h is well posed. Since $Q_h \rightharpoonup Q$, $E^{Q_h} Z_T \to E^Q Z_T$ by the assumptions we made. We see that again the pricing measure is a martingale measure not equivalent to the physical measure. □

Remark 7. *Again, we have arbitrage any time $\sigma = 0$, but the opportunity exists only at points of increase of local time. They form a set of Lebesgue measure zero on the real line, so there will not be a great deal of time to make any profit. For a portfolio setup, borrowing money to purchase θ_t shares at rate r_t at every such time point, to be held as long as σ stays null, the net (sure) profit will be $\int \theta_t S_t(\mu_t - r_t) dl_t$ where l_t is the local time at $\sigma = 0$. The rationale here is that, as long as $\sigma_k > 0$, the riskless return rate falls between the upper and lower return of the asset, and hence no arbitrage is possible, just as in the standard binomial model. As soon as $\sigma_k = 0$, however, the possible spread between μ_k and the riskless rate could produce an arbitrage opportunity. The possibility of arbitrage will depend on the assumed dependence of μ_k on σ_k, specifically when $\sigma_k = 0$.*

Note that we assume the market to be risk-averse so that a nonnegative risk premium is always assumed. The model allows for this premium to persist even if the volatility vanishes. It is clear that if the volatility is null, all assets have the same return as the riskless asset. A market model could assume that in this circumstance the risk premium is zero and there is no possibility of arbitrage. However, this has to be specifically stated in the model, as in our setup μ_k can depend on σ_k, but need not vanish (equal the risk-free rate), when $\sigma_k = 0$. In case we had $\mu_t|_{\sigma_t=0} = 0$, there would be no risk premium when volatility vanishes; however, this is a choice of the model. To understand how the presence of local time can generate arbitrage opportunities, we refer to Jarrow and Protter (2005) and references therein.

Remark 8. *The non-equivalence creates a problem since $Z_0 = E^Q(Z_T)$ only for modulo null sets. Note that Z is a well-defined $\mathfrak{F}_T$-measurable random variable, but that, as far as pricing is concerned, we could change its value on a set of Q (though not necessarily of P) measure zero without affecting the result. Such a nasty set would be one involving the vanishing of σ, i.e., an arbitrage opportunity. In fact, these opportunities are* invisible *to Q. The point of the argument is that while these opportunities (that arise unpredictably and last for extremely short timespans) are available for traders, they do not affect the valuation of contingent claims. We suggest that this might be an acceptable model for some types of arbitrage situations, and in fact is implicit in the no-arbitrage paradigm.*

6. A Simple Explicit Example

Consider the simplest reflecting volatility model, i.e., reflecting Brownian motion. In this case, the process $\{\sigma_t\}$ would have the same distribution as $\{|B_t|\}$. For a simple conditional log-normal price model

$$dS_t = S_t(\mu dt + |B_t| dW_t) \tag{14}$$

with B_t Brownian motion independent of W_t, according to our discussion, the pricing equation would be given by (keeping a zero riskless rate)

$$dS_t = S_t |B_t| dW_t$$

which can easily be solved as

$$S_t = S_u \exp\left\{ \int_u^t |B_s| dW_s - \frac{1}{2} \int_u^t |B_s|^2 ds \right\}.$$

In this case, our result states that the Hull and White formula (8) in Hull and White (1987) still applies even with reflecting volatility. Explicitly, the exponent is a random variable, with distribution, conditional on a path $|B|$, equal to $N(-\frac{1}{2} \int_u^t |B_s|^2 ds, \int_u^t |B_s|^2 ds)$. Let us stress how the presence of a reflected stochastic volatility prevents the construction of an equivalent martingale measure, and offers the opportunity of arbitrage. Indeed, in Equation (14), $\mu > 0$ is a constant, so there will be a positive risk premium even when local time is increasing, and, for a time of Lebesgue measure 0, there is no risk. This zero-measure is a consequence of the idealized continuous time model. However, as sketched in Section 7, there is no arbitrage when conditioned on $\sigma_t > 0$, and this allows us to provide a *fair* pricing equation as listed in Section 5.

Denote by $\psi(x)$ the density of the distribution of $\int_0^T |B_s|^2 ds$, and by $\Psi_{m,\Sigma}$ the log-normal density corresponding to $N(m, \Sigma)$ and ϕ the normal density. It is clear that the following formula holds for the price of a European contingent claim with payoff $Z_T = K(S_T)$, written on this asset:

$$\begin{aligned} p = EK(S_T) &= \int \psi(x) \Phi_{-\frac{1}{2}x,x}(y) K(y) dx\, dy = \\ &= \int dx \psi(x) \int dy \Phi_{-\frac{1}{2}x,x}(y) K(y) = \int dx \psi(x) \int dy e^y K(y) \phi_{-\frac{1}{2}x,x}(y). \end{aligned}$$

The Laplace transform of ψ is known (see, e.g., Kirillov and Gvishiani (1982), or computed easily, by a Girsanov transform between Brownian motion and an Ornstein–Uhlenbeck process) and equal to

$$E \exp\left\{ -k^2 \int_0^t B_s^2 ds \right\} = \sqrt{\frac{\pi}{kt \sinh(kt)}}.$$

7. Conditionally Non-Vanishing Volatilities

In this section, we address some issues related to the evolution of stochastic volatility in different situations, possible model interpretations, and practical implications. In these different situations, the correct model is not (5) any longer: specifically, the volatility equation has to be modified appropriately. In particular, in this section we describe the corresponding occurrence of a breakdown in the availability of an equivalent martingale measure in the different situations.

We discussed in the previous sections how to handle options within a model where volatility is allowed to vanish in a way that allows arbitrage opportunities. Let τ denote the first hitting time at zero for the volatility process. Generally, even though $\tau < \infty$ almost surely, a reasonable model will give high probability to high values for this random variable. Specifically, since most trading occurs over a finite horizon, say T, it is reasonable to wonder what practical implications a finite but very large τ may have, since $\mathbf{P}[\tau < T]$ will be small. We could thus consider our model, conditioned on the event that no actual arbitrage opportunities will arise within this finite horizon. This would take us back into familiar territory, but the conditioned model will have a significantly different form from its unconditioned *parent*. Equivalently, we could work with the version of Girsanov's theorem for non-equivalent measures (see Lenglart (1977)), since, in fact, the distribution of the process conditioned to $\{\tau > T\}$ is absolutely continuous, albeit not equivalent, to the unconditioned distribution. The present argument could just as well be cast in the language used by Lenglart (1977).

Consider an option to be exercised at (no later than) time T. We might want to consider its value depending on whether $\tau > T$ or not. Again, for simplicity, let us fix $n = 1$ and assume that the second equation in (5) allows $\mathbf{P}[\tau < \infty] > 0$ where $\tau = \inf\{t : \sigma_t = 0\}$, while $\sigma_t \geq 0$.

Consider a stopping time $\tau' < \tau$ (e.g., $\inf\{t : \sigma_t \leq \delta\}$ for some small δ) and the process $X_t^{\tau'} = X_{t \wedge \tau'}$. Denote the distribution of $X^{\tau'}$ by $P^{\tau'}$. Note that, on $\{t < \tau'\}$, $X = X^{\tau'}$. Applying Girsanov's theorem to $X^{\tau'}$, we thus see that there exists (at least) a probability measure $Q^{\tau'}$ on $\mathfrak{F}_{\tau'}$ such that, under $Q^{\tau'}$, on $\mathfrak{F}_{\tau'}$, $dY_t = -\frac{\sigma_t^2}{2}dt + \sigma_t dW_t$. A trader will have access to the present value of σ_t, but has no way of predicting when (and if) σ will actually touch zero. Until such an event arises, a trader might assume that the market does not allow for arbitrage at all. In fact, she/he would be working conditional on $\tau' > T$, where T is the time horizon under which she/he is trading. Most reasonable volatility models will indeed imply $\mathbf{P}[\tau' > T] = \varepsilon$ where ε is a small number. We could thus rewrite the model under $P^{\tau'}$ and a pricing formula can be established on this σ-field using $Q^{\tau'}$.

Note that

$$\mathbf{P}[\cdot] = \mathbf{P}[\cdot|\tau' > T](1-\varepsilon) + \mathbf{P}[\cdot|\tau' \leq T]\varepsilon \tag{15}$$

and, for sufficiently small ε, $P^{\tau'}$ will be *almost* the *right* measure for the financial market.

However, working under this conditional measure, clearly, the correct model is not (5) any longer; specifically, the volatility equation has to be modified appropriately.

To illustrate the situation quickly, consider the simple case of reflecting Brownian motion. For Brownian motion starting at σ (see, e.g., Ito and McKean (1974)),

$$\mathbf{P}[\tau > T] = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\sigma T^{-1/2}} e^{-x^2/2} dx\,.$$

For fixed σ, probability is quickly going to one as $T \to \infty$ and $\mathbf{E}\tau = \infty$. A realistic model for the volatility will presumably have even lower values, since it might include features such as mean reversion towards a positive value. If we now condition on $\{\tau > T\}$, Brownian motion, starting at 0 and remaining positive for $0 < t < T$, will be turned into a *Brownian meander* (see Durrett et al. (1977), Revuz and Yor (1991), and Pitman and Yor (1996); see also Ito and McKean (1974)).

These considerations suggest the following description of the evolution of the stochastic volatility and the corresponding occurrence of a breakdown in the availability of an equivalent martingale measure:

1. The time axis is split into two sets, depending on whether σ_t is positive or vanishes.
2. The set $\mathfrak{Z} = \{t : \sigma(t) = 0\}$ is a zero Lebesgue measure, uncountable, set with no isolated points. Levy's local time l_t (see Ito and McKean (1974)) provides a time scale that serves as a *clock* of the timespan over which arbitrage can occur. Note that the $\int \theta_t \mu_t dl_t$, for an adapted process θ_t, represents the arbitrage profit that can be realized by a trading strategy θ_t, to take advantage of the difference between the riskless lending rate (which we are setting at zero) and the (certain, while $\sigma_t = 0$) return rate of the underlying asset.
3. The complement of $\mathfrak{Z}$, $\mathfrak{Z}^c$, is a union of open intervals. The distribution of the lengths of these intervals has been extensively studied (see Jeanblanc et al. (1996)).
4. Over each component of $\mathfrak{Z}^c$, σ_t performs an *excursion*. As known, a Brownian excursion process is a stochastic process that is closely related to a Brownian motion. In particular, a Brownian excursion process is a Brownian motion conditioned to be positive and to take the value 0 at time 1 (assuming it is normalized). Alternatively, it is a Brownian bridge process conditioned to be positive. More generally, we can think of the path of σ_t as a sequence of *bridges* over some random time intervals. Over these, equivalent martingale measures can be used for pricing, conditional on a non-vanishing volatility during the lifetime of the option.
5. Upon reaching a zero for the volatility (often an event of minimal probability for finite time horizons), the pricing measure *breaks down*, in the sense that it is no longer equivalent to the physical measure. The fact is that it acts as if the second term in (15)

could be safely ignored since it is singular with respect to the corresponding measure. Of course, the relevance of this problem depends on **P**, i.e., it reduces to a subjective judgment call.

6. Using the reflecting volatility model allows for tractable formulas, at least in the simplest cases.

8. Conclusions

In this paper, we look at the problem of pricing for a contingent claim written on an asset with a volatility σ_t that follows a stochastic Markov process itself that will eventually vanish for very short periods with probability of one. Specifically, we look at the pricing of European options for models where the volatility processes stay nonnegative, but are allowed to vanish. Precisely, we consider the case of a regular diffusion with reflection at 0.

We investigate the form of pricing measures in this situation, first in a simple binomial case, and then for a diffusion model, by constructing a weak approximation in discrete space and continuous time.

In such a situation, pricing contingent claims are, a priori, not obvious, since the market allows arbitrage opportunities to arise briefly. Nevertheless, we can still produce a fair pricing equation.

Although many papers deal with stochastic volatility and its applications, the author is not aware of any papers addressing arbitrage issues and looking for pricing formulas for contingent claims in the case of vanishing stochastic volatility.

Let us note that, in the recent past, the phenomenon of very low volatility has actually been observed in the financial markets and the economy. As an instance, we all recall how, in 2016 and 2017, financial markets were characterized by very low volatility, raising the question of whether volatility measures adequately reflect risks in financial markets, but this phenomenon is not limited to 2017: volatility has been subdued for the majority of the recovery since early 2009.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

Note

[1] A local time is a stochastic process associated with semimartingale processes such as Brownian motion, that characterizes the amount of time a particle has spent at a given level.

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