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Optimal Insurance with Heterogeneous Beliefs and Disagreement about Zero-Probability Events

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Abstract: In problems of optimal insurance design, Arrow's classical result on the optimality of the deductible indemnity schedule holds in a situation where the insurer is a risk-neutral Expected-Utility (EU) maximizer, the insured is a risk-averse EU-maximizer, and the two parties share the same probabilistic beliefs about the realizations of the underlying insurable loss. Recently, Ghossoub re-examined Arrow's problem in a setting where the two parties have different subjective beliefs about the realizations of the insurable random loss, and he showed that if these beliefs satisfy a certain compatibility condition that is weaker than the Monotone Likelihood Ratio (MLR) condition, then optimal indemnity schedules exist and are nondecreasing in the loss. However, Ghossoub only gave a characterization of these optimal indemnity schedules in the special case of an MLR. In this paper, we consider the general case, allowing for disagreement about zero-probability events. We fully characterize the class of all optimal indemnity schedules that are nondecreasing in the loss, in terms of their distribution under the insured's probability measure, and we obtain Arrow's classical result, as well as one of the results of Ghossoub as corollaries. Finally, we formalize Marshall's argument that, in a setting of belief heterogeneity, an optimal indemnity schedule may take "any"shape.

Keywords: optimal insurance; deductible contract; subjective probability; heterogeneous beliefs; mutual singularity

1. Introduction

The problem of optimal insurance design under uncertainty dates back to the seminal work of Arrow [1] who showed that when the insured, or Decision Maker (DM), is a risk-averse Expected-Utility (EU)-maximizer, the insurer is a risk-neutral EU-maximizer, the two parties assign the same distribution to the insurable random loss and the premium principle depends on the actuarial value (expected value) of the indemnity; then, full insurance above a deductible is optimal for the DM. In particular, the optimal indemnity schedule is a nondecreasing function of the insurable loss[1], and its distribution can be fully characterized. This is a fundamental and foundational result that has been extended in several directions, all the while maintaining the assumption of belief homogeneity. We refer to Gollier [3] and Schlesinger [4] for surveys.

Belief heterogeneity is pervasive in insurance markets. On a theoretical level, disagreements about (subjective) beliefs arise naturally in the De Finetti–Savage [5,6] framework from divergent

[1] The monotonicity of an insurance indemnity schedule is usually desired so as to eliminate *ex post* moral hazard issues that might arise from the DM's possible misreporting of the actual amount of the loss suffered (see Huberman, Mayers and Smith [2].

preferences over alternatives. Moreover, disagreement about (posterior) beliefs can be a direct consequence of relaxing the controversial and heavily-criticized common priors assumption in game theory [7,8]. On a practical level, primary and secondary insurance markets do display belief heterogeneity, more often than not: pooling and diversification effects give the insurer a different view of risks than the insured; advances in insurance analytics often lead to disagreements about beliefs. We refer to Ghossoub [9] for a more detailed description discussion.

Although belief heterogeneity is an important consideration in insurance markets, little work has been devoted to the systematic study of optimal insurance design when the insurer and the DM assign different distributions to the random loss, or more generally, entertain different subjective beliefs over the relevant state space. The first formal examination of this problem was done by Marshall [10] (hereafter, Marshall). In the setting of Marshall, the DM assigns a probability density function (pdf) $f(t)$ to the insurable loss, whereas the insurer attributes the pdf $g(t)$ to the loss. Marshall assumes that the DM is more optimistic than the insurer, in the sense of second-order stochastic dominance, and argues that at that level of generality, nothing can be said about the optimal contract: the optimal indemnity could take any shape (within the usual constraints on an indemnity schedule). Marshall then considers a special case in which the DM assigns a higher probability to the no-loss event than the insurer; but conditioning on the loss being non-zero, the two parties assign the same conditional distribution to the random loss. In this case, the probability of a zero loss can be seen as a proxy for the DM's optimism, and Marshall shows that if the insurer is risk-neutral, the optimal insurance indemnity is a deductible contract; and the deductible level increases with the DM's optimism. However, this is a rather restrictive approach to belief heterogeneity, since this heterogeneity is reduced only to the likelihood that each party attaches to the event of a zero loss. Huang et al. [11] examine a similar problem, but in their framework, risk-attitude and belief heterogeneity are intertwined; and it is not clear how to separate the effect of belief heterogeneity from the effect of risk-aversion on the shape of the optimal indemnity schedule. Jeleva [12], Jeleva and Villeneuve [13] and Anwar and Zheng [14] examine related problems, but their settings are too restrictive to yield either a distributional or an analytical characterization of the optimal indemnity. Recently, Ghossoub [9] re-examined Arrow's problem in a setting where the two parties have different subjective beliefs about the realizations of the insurable random loss, and he showed that if these beliefs satisfy a certain compatibility condition that is weaker than the Monotone Likelihood Ratio (MLR) condition[2], then optimal indemnity schedules exist and are nondecreasing in the loss. However, Ghossoub [9] only gave a characterization of these optimal indemnity schedules in the special case of an MLR: he showed that, in that case, the optimal indemnity schedule is a variable deductible schedule, with a state-contingent deductible that depends on the state of the world only through the likelihood ratio. Arrow's classical result was then obtained as a special case.

In this paper, we extend the analysis done in Ghossoub [9] by considering the general case of belief heterogeneity, hence allowing for disagreement about zero-probability events. We fully characterize the class of all optimal indemnity schedules that are nondecreasing in the loss, in terms of their distribution under the insured's probability measure, and we obtain Arrow's classical result, as well as one of the results of Ghossoub [9] as corollaries. Specifically, our contribution is three-fold:

(1) First, under a condition of compatibility between the subjective beliefs of the insurer and the DM introduced by Ghossoub [9], we characterize the optimal indemnity schedule $\mathcal{Y}^*$ in terms of its distribution under the DM's subjective probability measure. The importance of characterizing the distribution of an optimal indemnity schedule rather than its actual shape has been stressed by Gollier and Schlesinger [16]. Specifically, we show that $\mathcal{Y}^*$ has the same distribution (under the DM's subjective probability measure) as a function of the form:

[2] The MLR condition is a key property for obtaining the monotonicity of optimal contracts in classical contract theory with EU-maximizing agents (e.g., Milgrom [15]).

$$Z := \min\left[X, \max\left(0, X - \left[W_0 - \Pi - \left(u'\right)^{-1}\left(\lambda^* h\right)\right]\right)\right], \quad (1)$$

for some $\lambda^* \geqslant 0$ and a nonnegative measurable function h, which is entirely characterized from the subjective probabilities of both parties (Theorem 8). The function u is the DM's utility function; W_0 is the DM's initial level of wealth; and Π is the premium paid by the DM, as in Arrow's setting. The function Z can be written in the form $Z = \min(X, D)$, where $D = \max\left(X - d(h), 0\right)$ is a deductible indemnity schedule with a state-contingent deductible $d(h) = W_0 - \Pi - \left(u'\right)^{-1}(\lambda^* h)$ that depends on the state of the world only through the function h.

Moreover, when the beliefs of both parties coincide, the function h appearing in Equation (1) is the constant function equal to one, and hence, the function Z is simply a deductible contract, which is a nondecreasing function of the loss X. In this case, Corollary 9 states that the main result of this paper (Theorem 8) boils down to the classical result of Arrow (Theorem 5).

(2) Second, we formalize Marshall's argument that the optimal insurance indemnity schedule may take "any" form in general (within the usual constraints imposed on an indemnity schedule). Specifically, although Theorem 8 asserts the existence and monotonicity of an optimal indemnity schedule $\mathcal{Y}^*$ that has a given distribution for the DM's belief and that satisfies the constraint $0 \leqslant \mathcal{Y}^* \leqslant X$, Corollary 11 states that $\mathcal{Y}^*$ can take different shapes. Depending on the function h (seen as a proxy for belief heterogeneity) and/or the DM' s utility function u (seen as a proxy for the DM's risk aversion), $\mathcal{Y}^*$ may include a non-zero deductible or a disappearing deductible, whereby losses of high magnitude are fully insured. However, for losses of moderate magnitude, $\mathcal{Y}^*$ is of the form $f(X)$, where f is a nondecreasing, Borel-measurable and left-continuous function, such that $0 \leqslant f(t) \leqslant t$, for all t in the range of X. Nothing else can be said about the function f, in terms of concavity, convexity and inflection points, for instance.

(3) On a technical level, a third contribution of this paper is to present a methodology for dealing with disagreement about zero-probability events (i.e., singularity of measures) within an optimal insurance design problem. This methodology, however, can be easily extended to general contracting problems with heterogeneous beliefs that exhibit some singularity or disagreement about zero-probability events.

The rest of this paper is organized as follows. Section 2 introduces the formal model of insurance demand in the presence of belief heterogeneity, Section 3 presents the main results of this paper, and Section 4 concludes. Some proofs and related analyses are collected in Appendices A–F.

2. The Model

Let S denote the non-empty collection of states of the world. The DM faces a loss X, taken to be a mapping of S onto a closed interval $[0, M]$, against which she or he seeks insurance coverage. The information generated by observing the loss random variable is the σ-algebra $\sigma\{X\}$ of subsets of S generated by X.

Denote the σ-algebra $\sigma\{X\}$ by Σ; denote by $B(\Sigma)$ the Banach space (sup norm[3].) of all bounded, $\mathbb{R}$-valued and Σ-measurable functions on (S, Σ); and denote by $B^+(\Sigma)$ the collection of all $\mathbb{R}^+$-valued elements of $B(\Sigma)$. Then, for any $Y \in B(\Sigma)$, there exists a Borel-measurable map $\zeta : \mathbb{R} \to \mathbb{R}$, such that $Y = \zeta \circ X$ (e.g., Theorem 4.41 of [17]). For $C \subseteq S$, denote by $\mathbf{1}_C$ the indicator function of C. For any $A \subseteq S$ and for any $B \subseteq A$, denote by $A \backslash B$ the complement of B in A.

An insurance market gives the DM the possibility of purchasing insurance coverage for a premium $\Pi > 0$. The premium is paid ex ante by the DM in return for receiving the ex post indemnity $I(X(s))$ in the state of the world $s \in S$. The indemnity schedule is a Borel-measurable

[3] For any $Y \in B(\Sigma)$, the sup norm of Y is given by $\|Y\|_s := \sup\{|Y(s)| : s \in S\} < +\infty$

map $I : [0, M] \rightarrow [0, M]$. Then, $Y := I \circ X \in B^{+}(\Sigma)$, and hence, the collection of all indemnity schedules can be identified with the set $B^{+}(\Sigma)$.

Both the DM and the insurer have preferences over the elements of $B^{+}(\Sigma)$ (i.e., over indemnity schedules) that have a subjective expected-utility representation yielding:

(i) A utility utility function $u : \mathbb{R} \rightarrow \mathbb{R}$ and a probability measure P on the measurable space (S, Σ) for the DM;

(ii) A utility function $v : \mathbb{R} \rightarrow \mathbb{R}$ and a probability measure $Q \neq P$ on the measurable space (S, Σ) for the insurer.

Both u and v are unique up to a positive linear transformation (e.g., Theorem 14.1 of [18]). Moreover, as in Arrow's framework, we suppose that the DM is risk averse and that her or his utility function u satisfies the following.

Assumption 1. The DM's utility function u satisfies Inada's [19] conditions:

1. $u(0) = 0$;
2. u is strictly increasing and strictly concave;
3. u is continuously differentiable; and
4. $u'(0) = +\infty$ and $\lim_{x \rightarrow +\infty} u'(x) = 0$.

The DM has initial wealth $W_0 > \Pi$, and her or his total state-contingent wealth in each sate of the world $s \in S$ is given by:

$$W(s) := W_0 - \Pi - X(s) + Y(s).$$

We will also make the assumption that the random loss X has a nonatomic[4] law induced by the probability measure P, that the subjective probability measures P and Q are not mutually singular[5] and that the DM is almost certain that the random loss she or he will incur is not larger than her or his remaining wealth after the premium has been paid. Specifically:

Assumption 2. Assume that:

1. $P \circ X^{-1}$ is nonatomic (i.e., X is a continuous random variable for P);
2. $X \leqslant W_0 - \Pi$, P-a.s.In other words, $P\Big(\{s \in S : X(s) > W_0 - \Pi\}\Big) = 0$.
3. P and Q are not mutually singular.

Assumption 2 (1) is common (e.g., when it is assumed that a probability density function for X exists). Assumption 2 (2) simply states that the DM is well-diversified so that the particular loss exposure X against which she or he is seeking an insurance coverage is sufficiently small. Assumption 2 (3) means that the insurer and the DM do not have beliefs that are totally incompatible. However, this does not prevent the agents from assigning different probabilities to events, and they typically do not assign same likelihoods to the realizations of the uncertainty X. For instance, they might disagree on zero-probability events.

As in Arrow's model, we assume that the insurer is risk-neutral. Without loss of generality, the insurer's utility function v can then be taken to be the identity function. The insurer has initial wealth W_0^{ins}, and his or her total state-contingent wealth in each state of the world $s \in S$ is given by:

$$W^{ins}(s) := W_0^{ins} + \Pi - (1 + \rho) Y(s),$$

[4] A finite measure η on a measurable space $(\Omega, \mathcal{G})$ is said to be *nonatomic* if for any $A \in \mathcal{G}$ with $\eta(A) > 0$, there is some $B \in \mathcal{G}$, such that $B \subsetneq A$ and $0 < \eta(B) < \eta(A)$.

[5] Two finite nonnegative measures μ_1 and μ_2 on the measurable space (S, Σ) are said to be mutually singular, denoted by $\mu_1 \perp \mu_2$, if there is some $A \in \Sigma$, such that $\mu_1(S \backslash A) = \mu_2(A) = 0$; in other words, $\mu_1 \perp \mu_2$ if there is a Σ-partition $\{A, B\}$ of the set S of states of nature such that μ_1 is concentrated on A and μ_2 is concentrated on B.

where $\rho > 0$ is a loading factor meant to account for the cost associated with handling the insurance indemnity payment, as in the classical model.

The DM seeks an indemnity that will maximize her or his expected utility of wealth, subject to the insurer's participation constraint and to some constraints on the indemnity function:

Problem 3. For a given loading factor $\rho > 0$,

$$\sup_{Y \in B^+(\Sigma)} \left\{ \int u\Big(W_0 - \Pi - X + Y\Big)\, dP \right\} :$$
$$\begin{cases} 0 \leqslant Y \leqslant X \\ \int Y\, dQ \leqslant R := \frac{\Pi}{1+\rho} \end{cases}$$

The first constraint is standard and says that an indemnity is nonnegative and cannot exceed the loss itself. The latter requirement rules out situations where the DM has an incentive to create damage [2], which would result in ex post moral hazard. The second constraint is the risk-neutral insurer's participation constraint, restated as a premium constraint. Here, we do not impose an additional monotonicity constraint, and we show that an optimal indemnity will have this property (see Theorem 8).

Now, for any $Y \in B(\Sigma)$, which is feasible for Problem (3), one has $0 \leqslant Y \leqslant X$, and hence, $0 \leqslant \int Y\, dQ \leqslant \int X\, dQ$. It is easily seen that when $R \geqslant \int X\, dQ$, the solution to Problem (3) is $Y^* = X$, i.e., full insurance. In particular, the solution is comonotonic[6] with X and its distribution is the same as that of X, for both the DM and the insurer. Therefore, we will consider the remaining case; that is, we will make the following assumption all throughout:

Assumption 4. $0 < R < \int X\, dQ$.

3. The Results

The difference between Problem (3) and the classical problem of Arrow is the fact that the probability measures Q and P differ. At this point, no assumption of absolute continuity is made, and the two parties can disagree about zero-probability events. Clearly, when $P = Q$, we recover the classical framework of Arrow:

Theorem 5 (Arrow). *If $P = Q$, then there exists some $d > 0$, such that $I_d \circ X$ is optimal for Problem (3), where I_d is a deductible indemnity schedule defined by:*

$$I_d(t) = \begin{cases} 0 \ \ if \ \ t < d \\ t - d \ \ if \ \ t \geqslant d \end{cases}$$

That is, an optimal solution for Problem (3) takes the form $Y^* = \max(0, X - d)$, for some $d > 0$.

Note that since $d > 0$, the optimal indemnity schedule can also be written as:

$$Y^* = \min\Big[X, \max(0, X - d)\Big].$$

[6] Two functions $Y_1, Y_2 \in B(\Sigma)$ are said to be comonotonic if $\Big[Y_1(s) - Y_1(s')\Big]\Big[Y_2(s) - Y_2(s')\Big] \geqslant 0$, for all $s, s' \in S$. For instance, any $Y \in B(\Sigma)$ is comonotonic with any $c \in \mathbb{R}$. Moreover, if $Y_1, Y_2 \in B(\Sigma)$ and if Y_2 is of the form $Y_2 = I \circ Y_1$, for some Borel-measurable function I, then Y_2 is comonotonic with Y_1 if and only if the function I is nondecreasing.

Unlike the classical and the vast majority of the subsequent insurance literature, the insurance model presented here allows for heterogeneity of beliefs. Ghossoub [9] shows that if the analysis is restricted to a class of beliefs Q that are compatible with the DM's belief P as per the definition below, then optimal indemnity schedules exist and are monotonic. This notion of belief compatibility is introduced in Ghossoub [9] and then extended to risk measures in Ghossoub [20] and to a setting with ambiguous beliefs in Amarante, Ghossoub and Phelps [21,22].

Definition 6 (Ghossoub [9]). The probability measure Q is said to be compatible with the probability measure P, or the insurer is said to be compatible, if for any two indemnity schedules $Y_1, Y_2 \in B^+(\Sigma)$, such that:

(i) Y_1 and Y_2 have the same distribution under P (i.e., $P \circ Y_1^{-1} = P \circ Y_2^{-1}$); and,
(ii) Y_2 and X are comonotonic,

we have $\int Y_2 \, dQ \leqslant \int Y_1 \, dQ$.

Clearly, the probability measure P is compatible with itself[7]. Therefore, the classical insurance setup of Arrow can be seen as a special case. Ghossoub [9] showed that when a likelihood ratio can be defined (as a ratio of probability density functions), then belief compatibility is a strictly weaker requirement than an MLR condition. Hence, the restriction on belief heterogeneity imposed by a condition of belief compatibility is general enough to encompass, for instance, cases where these heterogeneous beliefs induce a likelihood ratio that is monotone.

Theorem 7 (Ghossoub [9]). *If assumptions 1, 2 and 4 hold and if the insurer's subjective probability measure Q is compatible with the DM's subjective probability measure P, then there exists an optimal indemnity schedule Y^* which is a nondecreasing function of the loss X. Moreover, any other Z^* which is nondecreasing in X and which has the same distribution as Y^* under P is such that $Z^* = Y^*$, P-a.s. Finally, if the utility function u is strictly concave, then any solution Z^* to Problem (3) is such that $Z^* = Y^*$, P-a.s. In particular, any solution is nondecreasing in X, P-a.s.*

The above result shows the existence and monotonicity of optimal indemnity schedules, but it does not provide an analytical or distributional characterization of optima. The main complication in the setting where $Q \neq P$ is, precisely, dealing with the fact that the two parties can disagree about zero-probability events. One insight comes from Lebesgue's decomposition theorem (e.g., Theorem 4.3.1. of [23]).

By Lebesgue's decomposition theorem, there exists a unique pair (Q_{ac}, Q_s) of (nonnegative) finite measures on (S, Σ), such that $Q = Q_{ac} + Q_s$, $Q_{ac} \ll P$ and $Q_s \perp P$. That is, for all $B \in \Sigma$ with $P(B) = 0$, one has $Q_{ac}(B) = 0$, and there is some $A \in \Sigma$, such that $P(S \backslash A) = Q_s(A) = 0$. It then also follows that $Q_{ac}(S \backslash A) = 0$ and $P(A) = 1$. Note also that for all $Z \in B^+(\Sigma)$, $\int Z \, dQ = \int_A Z \, dQ_{ac} + \int_{S \backslash A} Z \, dQ_s$. Furthermore, by the Radon–Nikodým theorem (e.g., Theorem 4.2.2 of [23]), there exists a P-a.s. unique Σ-measurable and P-integrable function:

$$h : S \to [0, +\infty)$$

such that $Q_{ac}(C) = \int_C h \, dP$, for all $C \in \Sigma$.

The Lebesgue decomposition of Q with respect to P suggests a re-writing of the premium constraint appearing in Problem (3) as:

$$\Pi / (1 + \rho) \geqslant \int Y \, dQ = \int_A Y h \, dP + \int_{S \backslash A} Y \, dQ;$$

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[7] We refer to Amarante, Ghossoub, and Phelps [21,22] for several examples of compatibility.

and one can then re-write Problem (3) as follows: for a given loading factor $\rho > 0$,

$$\sup_{Y \in B^+(\Sigma)} \left\{ \int_A u\left(W_0 - \Pi - X + Y\right) dP + \int_{S \setminus A} u\left(W_0 - \Pi - X + Y\right) dP \right\} :$$
$$\begin{cases} 0 \leqslant Y\mathbf{1}_A + Y\mathbf{1}_{S\setminus A} \leqslant X\mathbf{1}_A + X\mathbf{1}_{S\setminus A} \\ \Pi / (1+\rho) \geqslant \int_A Yh\, dP + \int_{S\setminus A} Y\, dQ \end{cases}$$

This then suggests a splitting of Problem (3) into two problems. Each one of these problems is then solved separately, and the individuals solutions hence obtained are then combined appropriately so as to obtain a solution for Problem (3). All details are provided in Appendix C, but for now, consider heuristically the problems:

$$\sup_{Y \in B^+(\Sigma)} \left\{ \int_A u\left(W_0 - \Pi - X + Y\right) dP : Y\mathbf{1}_A \leqslant X\mathbf{1}_A, \int_A Yh\, dP = \beta \right\}, \tag{2}$$

for an appropriately chosen β; and,

$$\sup_{Y \in B^+(\Sigma)} \left\{ \int_{S\setminus A} u\left(W_0 - \Pi - X + Y\right) dP : Y\mathbf{1}_{S\setminus A} \leqslant X\mathbf{1}_{S\setminus A}, \int_{S\setminus A} Y\, dQ \leqslant \alpha \right\}, \tag{3}$$

for an appropriately chosen α. Since $P\left(S\setminus A\right) = 0$, any feasible Y for the problem given in Equation (3) is also optimal for that problem. Since $P\left(A\right) = 1$, the problem given in Equation (2) can be written as:

$$\sup_{Y \in B^+(\Sigma)} \left\{ \int u\left(W_0 - \Pi - X + Y\right) dP : Y\mathbf{1}_A \leqslant X\mathbf{1}_A, \int Yh\, dP = \beta \right\}. \tag{4}$$

Solving Problem (3) then boils down to solving the problem given in Equation (4). This is a considerably simpler problem than Problem (3), since dealing with the heterogeneity of beliefs appearing in Problem (3) has been reduced to dealing simply with the function h. The following theorem characterizes an optimal solution of Problem (3). Its proof is given in Appendix C.

Theorem 8. *Suppose that the previous assumptions hold, and for each $\lambda \geqslant 0$, define the function $Y^*_\lambda \in B^+(\Sigma)$ by:*

$$Y^*_\lambda := \min\left[X, \max\left(0, X - \left[W_0 - \Pi - \left(u'\right)^{-1}\left(\lambda h\right)\right]\right)\right]. \tag{5}$$

If the insurer's subjective probability measure Q is compatible with the DM's subjective probability measure P, then there exist:

- *some $\lambda^* \geqslant 0$; and*
- *an optimal solution $\mathcal{Y}^*$ to Problem (3), such that:*

  (1) $\mathcal{Y}^*$ is a nondecreasing function of the loss X and
  (2) $\mathcal{Y}^*$ has the same distribution as $Y^*_{\lambda^*}$ under P.

Moreover, any other $\mathcal{Z}^*$ which is nondecreasing in X and which has the same distribution as $Y^*_{\lambda^*}$ under P is such that $\mathcal{Z}^* = \mathcal{Y}^*$, P-a.s. Finally, if the utility function u is strictly concave, then any solution $\mathcal{Z}^*$ to Problem (3) is such that $\mathcal{Z}^* = \mathcal{Y}^*$, P-a.s. In particular, any solution is nondecreasing in X, P-a.s.

Theorem 8 characterizes a class of solutions to Problem (3) in terms of their distribution[8] for the DM, that is, for the probability measure P. Of course, when $P = Q$, so that there is perfect homogeneity of beliefs as in the classical model, then $h = 1$, and so:

$$Y_\lambda^* = \min\left[X, \max\left(0, X - \left[W_0 - \Pi - \left(u'\right)^{-1}(\lambda)\right]\right)\right] = \min\left[X, \left(X - d_\lambda\right)^+\right],$$

where $d_\lambda := W_0 - \Pi - \left(u'\right)^{-1}(\lambda)$. Since Y_λ^* is then a nondecreasing function of X; Theorem 8 simply says that there is some λ^*, such that an optimal indemnity schedule $\mathcal{Y}^*$ for the DM is such that $\mathcal{Y}^* = \min\left[X, \left(X - d_{\lambda^*}\right)^+\right]$, P-a.s., which is a result similar to Arrow's classical theorem (Theorem 5). This is stated below, and the proof is omitted.

Corollary 9. *Suppose that the previous assumptions hold, and suppose also that $P = Q$. For each $\lambda \geqslant 0$, let $d_\lambda := W_0 - \Pi - \left(u'\right)^{-1}(\lambda)$, and define the function $Y_\lambda^* \in B^+(\Sigma)$ by:*

$$Y_\lambda^* := \min\left[X, \max\left(0, X - d_\lambda\right)\right]. \tag{6}$$

Then, there exists a $\lambda^* \geqslant 0$ and an optimal solution $\mathcal{Y}^*$ to Problem (3), such that $\mathcal{Y}^* = Y_{\lambda^*}^*$, P-a.s.

As a second consequence of Theorem 8, we obtain one of the results of Ghossoub [9]. Namely, suppose that the probability measure Q is absolutely continuous with respect to the probability measure P, with a Radon–Nikodým derivative $h : S \to [0, +\infty)$, given by $\widetilde{h} = dQ/dP = \Psi \circ X$, for some Borel-measurable and $P \circ X^{-1}$-integrable map $\Psi : X(S) \to [0, +\infty)$. The function Ψ can be interpreted as a likelihood ratio. Ghossoub [9] showed that under an assumption of monotonicity on the likelihood ratio Ψ, the optimal indemnity schedule is a variable deductible schedule:

Corollary 10 (Ghossoub [9]). *If assumptions 1, 2 and 4 hold and if the function Ψ is nonincreasing, then Problem (3) admits a solution which is nondecreasing on the range of the random loss X, and an optimal such indemnity schedule for the DM takes the form:*

$$Y_{\lambda^*}^* = \min\left[X, \max\left(0, X - d(\widetilde{h})\right)\right],$$

where $d(\widetilde{h}) = \left[W_0 - \Pi - \left(u'\right)^{-1}\left(\lambda^* \widetilde{h}\right)\right]$ and $\lambda^* > 0$ is chosen, so that the premium constraint binds.

Corollary 10 is an immediate consequence of Theorem 8, since in this case, the optimal indemnity is a nondecreasing function of the loss X (by concavity of the utility function and by the fact that Ψ is nonincreasing). Corollary 10 states that monotonicity of the Radon–Nikodým derivative yields that the optimal indemnity schedule for the DM takes the form of a variable deductible schedule, with a state-contingent deductible $d(\widetilde{h})$ that depends on the sate of the world only through the Radon–Nikodým derivative $\widetilde{h}$. Therefore, when $P = Q$, $\widetilde{h}$ is the constant function that equals one, and hence, one recovers Arrow's result.

Theorem 8 above asserts the existence of an optimal indemnity schedule and shows that optimal indemnity schedules are nondecreasing functions of the loss X. It also characterizes a class of solutions to Problem (3) in terms of their distribution for the DM, that is, for the probability measure P. However, Theorem 8 does not give any indication as to what an optimal indemnity schedule looks

[8] The importance of characterizing the distribution of an optimal indemnity schedule, rather than its actual shape has been stressed by Gollier and Schlesinger [16].

like. It turns out that an optimal indemnity schedule might take "any" form, as long it has the same distribution as that specified in Theorem 8 and as long as it satisfies the imposed constraints. This can be seen as a formalisation of the results of Marshall [10].

Corollary 11 (**A Formalisation of Marshall's Argument).** *Under the previous assumptions and provided the insurer's subjective probability measure Q is compatible with the DM's subjective probability measure P, there exists an optimal solution $\mathcal{Y}^*$ to Problem (3), which is nondecreasing in the loss X and such that for P-a.a. $s \in S$,*

$$\mathcal{Y}^*(s) = \begin{cases} 0 \text{ iff } X(s) \in [0, a^*), \\ f(X(s)) \text{ iff } X(s) \in [a^*, b^*], \\ X(s) \text{ iff } X(s) \in (b^*, M], \end{cases} \tag{7}$$

for some a^* and b^*, such that $0 \leqslant a^* \leqslant b^* \leqslant M$, and a nondecreasing, left-continuous and Borel-measurable function $f : [0, M] \to [0, M]$, such that $0 \leqslant f(t) \leqslant t$ for each $t \in [a^*, b^*]$.

The proof of Corollary 11 is given in Appendix D. When $a^* > 0$, the indemnity schedule $\mathcal{Y}^*$ includes a deductible provision, whereby no indemnification is paid to the DM for a loss of amount less than a^*. Sufficient conditions for a^* appearing in Equation (7) to be strictly positive are given in Appendix E. When $b^* < M$, the indemnity schedule $\mathcal{Y}^*$ fully reimburses[9] losses of magnitude larger than b^*. Sufficient conditions for b^* appearing in Equation (7) to be strictly less than M are given in Appendix F. For losses of magnitude in the range $[a^*, b^*]$, $\mathcal{Y}^*$ is of the form $f(X)$, where f is a nondecreasing, Borel-measurable and left-continuous function such that $0 \leqslant f(t) \leqslant t$, for all t in the range of X. Nothing else can be said about the function f, in terms of concavity, convexity and inflection points, for instance. In this sense, the optimal indemnity schedule $\mathcal{Y}^*$ may take any form. This is reminiscent of the results of Marshall [10].

4. Conclusions

The classical approach to problems of optimal insurance design assumes that the insurer and the insured assign the same distribution to the insurable random loss. Recently, Ghossoub [9] considered a setting in which the two parties have different subjective beliefs about the realisations of the insurable loss. Under a requirement of compatibility between the insurer's and the insured's subjective beliefs that is weaker than the Monotone Likelihood Ratio (MLR) condition, Ghossoub [9] showed the existence and monotonicity of optimal indemnity schedules. However, Ghossoub [9] only provided an analytical characterization of the optimal indemnity in the special case of an MLR.

In this paper, we extended the analysis of Ghossoub [9] to the general case of belief heterogeneity, allowing for bona fide disagreement about zero-probability events. We gave a characterization of the class of optimal indemnity schedules in terms of their distribution for the DM's belief, and we showed how Arrow's classical result on the optimality of a deductible contract can be obtained as a special case. We also showed that even though we can characterize the distribution of an optimal indemnity, we cannot give an exact characterization of its shape: this is a formalization of Marshall's [10] argument that in the general case, an optimal insurance schedule may take "any" form.

[9] The fact that losses of high magnitude are fully insured is a similar result to the recent one of Gollier [24] who studies the problem of optimal insurance design when the insured is ambiguity-averse in the sense of Klibanoff, Marinacci and Mukerji [25].

Appendix A. Two Useful Results

Lemma A1. *Let $(\Omega, \mathcal{F})$ be a given measurable space, and suppose that η is a finite non-negative measure on $(\Omega, \mathcal{F})$. Let Z be any $\mathbb{R}^+$-valued, bounded and $\mathcal{F}$-measurable function on Ω. If $A \in \mathcal{F}$ is such that $\eta(A) > 0$, then the following are equivalent:*

1. $\int_A Z \, d\eta = 0$
2. $Z = 0$, *η-a.s. on A.*

Proof. See Theorem 11.16–(3) of [17], for instance. □

Lemma A2. *Let (S, Σ, P) be a finite nonnegative measure space. If $\{A_n\}_n \subset \Sigma$ is such that $P(A_n) = P(S)$, for each $n \geqslant 1$, then $P\left(\bigcap_{n=1}^{+\infty} A_n\right) = P(S)$.*

Proof. See Lemma A.1 of [26], for instance. □

Appendix B. Equimeasurable Rearrangements and Supermodularity

The classical theory of monotone equimeasurable rearrangements of Borel-measurable functions on $\mathbb{R}$ dates back to the work of Hardy, Littlewood and Pólya [27], who gave the first integral inequalities involving functions and their rearrangements. Here, the idea of an equimeasurable rearrangement of any element Y of $B^+(\Sigma)$ with respect to the fixed underlying loss random variable X is discussed. All of the results in this Appendix are taken from Ghossoub [26,28] to which we refer the reader for proofs, additional results and additional references on this topic.

B1. The Nondecreasing Rearrangement Let $(S, \mathcal{G}, P)$ be a probability space, and let $X \in B^+(\mathcal{G})$ be a continuous random variable (i.e., $P \circ X^{-1}$ is nonatomic) with range $[0, M] := X(S)$, where $M := \sup\{X(s) : s \in S\} < +\infty$, i.e., X is a mapping of S onto the closed interval $[0, M]$. Denote by Σ the σ-algebra $\sigma\{X\}$, and denote by ϕ the law of X defined by:

$$\phi(B) := P\Big(\{s \in S : X(s) \in B\}\Big) = P \circ X^{-1}(B),$$

for any Borel subset B of $\mathbb{R}$.

Proposition B1. *For any Borel-measurable map $I : [0, M] \to [0, M]$, there exists a ϕ-a.s. unique Borel-measurable map $\widetilde{I} : [0, M] \to [0, M]$, such that:*

1. *$\widetilde{I}$ is left-continuous and nondecreasing;*
2. *$\widetilde{I}$ is ϕ-equimeasurable with I, in the sense that for any Borel set B,*

$$\phi\Big(\{t \in [0, M] : I(t) \in B\}\Big) = \phi\Big(\{t \in [0, M] : \widetilde{I}(t) \in B\}\Big);$$

3. *If $I_1, I_2 : [0, M] \to [0, M]$ are such that $I_1 \leqslant I_2$, ϕ-a.s., then $\widetilde{I}_1 \leqslant \widetilde{I}_2$; and,*
4. *If $Id : [0, M] \to [0, M]$ denotes the identity function, then $\widetilde{Id} \leqslant Id$.*

$\widetilde{I}$ will be called the nondecreasing ϕ-rearrangement of I. Now, define $Y := I \circ X$ and $\widetilde{Y} := \widetilde{I} \circ X$. Since both I and $\widetilde{I}$ are Borel-measurable mappings of $[0, M]$ into itself, it follows that $Y, \widetilde{Y} \in B^+(\Sigma)$. Note also that $\widetilde{Y}$ is nondecreasing in X, in the sense that if $s_1, s_2 \in S$ are such that $X(s_1) \leqslant X(s_2)$, then $\widetilde{Y}(s_1) \leqslant \widetilde{Y}(s_2)$, and that Y and $\widetilde{Y}$ are P-equimeasurable, that is, for any $\alpha \in [0, M]$, $P(\{s \in S : Y(s) \leqslant \alpha\}) = P\left(\{s \in S : \widetilde{Y}(s) \leqslant \alpha\}\right)$. The function $\widetilde{Y}$ will be called a nondecreasing P-rearrangement of Y with respect to X, and it will be denoted by $\widetilde{Y}_P$ to avoid confusion in case a different measure on $(S, \mathcal{G})$ is also considered. Note that $\widetilde{Y}_P$ is P-a.s. unique. Note also that if Y_1 and

Y_2 are P-equimeasurable and if $Y_1 \in L_1(S, \mathcal{G}, P)$, then $Y_2 \in L_1(S, \mathcal{G}, P)$ and $\int \psi(Y_1)\, dP = \int \psi(Y_2)\, dP$, for any measurable function ψ, such that the integrals exist.

Similarly to the previous construction, for a given a Borel-measurable $B \subseteq [0, M]$ with $\phi(B) > 0$, there exists a ϕ-a.s. unique (on B) nondecreasing, Borel-measurable mapping $\widetilde{I}_B : B \to [0, M]$, which is ϕ-equimeasurable with I on B, in the sense that for any $\alpha \in [0, M]$,

$$\phi\Big(\{t \in B : I(t) \leqslant \alpha\}\Big) = \phi\Big(\{t \in B : \widetilde{I}_B(t) \leqslant \alpha\}\Big).$$

$\widetilde{I}_B$ is called the nondecreasing ϕ-rearrangement of I on B. Since X is $\mathcal{G}$-measurable, there exists $A \in \mathcal{G}$, such that $A = X^{-1}(B)$, and hence, $P(A) > 0$. Now, define $\widetilde{Y}_A := \widetilde{I}_B \circ X$. Since both I and $\widetilde{I}_B$ are bounded Borel-measurable mappings, it follows that $Y, \widetilde{Y}_A \in B^+(\Sigma)$. Note also that $\widetilde{Y}_A$ is nondecreasing in X on A, in the sense that if $s_1, s_2 \in A$ are such that $X(s_1) \leqslant X(s_2)$, then $\widetilde{Y}_A(s_1) \leqslant \widetilde{Y}_A(s_2)$, and that Y and $\widetilde{Y}_A$ are P-equimeasurable on A, that is, for any $\alpha \in [0, M]$, $P(\{s \in S : Y(s) \leqslant \alpha\} \cap A) = P\Big(\{s \in S : \widetilde{Y}_A(s) \leqslant \alpha\} \cap A\Big)$. The function $\widetilde{Y}_A$ will be called a nondecreasing P-rearrangement of Y with respect to X on A, and it will be denoted by $\widetilde{Y}_{A,P}$ to avoid confusion in case a different measure on $(S, \mathcal{G})$ is also considered. Note that $\widetilde{Y}_{A,P}$ is P-a.s. unique. Note also that if $Y_{1,A}$ and $Y_{2,A}$ are P-equimeasurable on A and if $\int_A Y_{1,A}\, dP < +\infty$, then $\int_A Y_{2,A}\, dP < +\infty$ and $\int_A \psi(Y_{1,A})\, dP = \int_A \psi(Y_{2,A})\, dP$, for any measurable function ψ, such that the integrals exist.

Lemma B1. *Let $Y \in B^+(\Sigma)$, and let $A \in \mathcal{G}$ be such that $P(A) = 1$ and $X(A)$ is a Borel set*[10]*. Let $\widetilde{Y}_P$ be the nondecreasing P-rearrangement of Y with respect to X, and let $\widetilde{Y}_{A,P}$ be the nondecreasing P-rearrangement of Y with respect to X on A. Then $\widetilde{Y}_P = \widetilde{Y}_{A,P}$, P-a.s.*

B2. Supermodularity and Hardy–Littlewood–Pólya Inequalities A partially-ordered set (poset) is a pair $(T, \succcurlyeq)$ where $\succcurlyeq$ is a reflexive, transitive and antisymmetric binary relation on T. For any $x, y \in S$, denote by $x \vee y$ (resp. $x \wedge y$) the least upper bound (resp. greatest lower bound) of the set $\{x, y\}$. A poset $(T, \succcurlyeq)$ is called a lattice when $x \vee y, x \wedge y \in T$, for each $x, y \in T$. For instance, the Euclidean space $\mathbb{R}^n$ is a lattice for the partial order $\succcurlyeq$ defined as follows: for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, write $x \succcurlyeq y$, when $x_i \geqslant y_i$, for each $i = 1, \ldots, n$. It is then easy to see that $x \vee y = (\max(x_1, y_1), \ldots, \max(x_n, y_n))$ and $x \wedge y = (\min(x_1, y_1), \ldots, \min(x_n, y_n))$.

Definition B1. Let $(T, \succcurlyeq)$ be a lattice. A function $L : T \to \mathbb{R}$ is said to be supermodular if for each $x, y \in T$,

$$L(x \vee y) + L(x \wedge y) \geqslant L(x) + L(y). \tag{B1}$$

In particular, a function $L : \mathbb{R}^2 \to \mathbb{R}$ is supermodular if for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$, one has:

$$L(x_2, y_2) + L(x_1, y_1) \geqslant L(x_1, y_2) + L(x_2, y_1). \tag{B2}$$

Equation (B2) then implies that a function $L : \mathbb{R}^2 \to \mathbb{R}$ is supermodular if and only if the function $\eta(y) := L(x + h, y) - L(x, y)$ is nondecreasing on $\mathbb{R}$, for any $x \in \mathbb{R}$ and $h \geqslant 0$.

Example B1. The following are supermodular functions:

1. If $g : \mathbb{R} \to \mathbb{R}$ is concave and $a \in \mathbb{R}$, then the function $L_1 : \mathbb{R}^2 \to \mathbb{R}$ defined by $L_1(x, y) = g(a - x + y)$ is supermodular. Moreover, if g is strictly concave, then L_1 is strictly supermodular.

[10] For any event $A \in \sigma\{X\}$, there is some Borel set B, such that $A = X^{-1}(B)$, by the very definition of $\sigma\{X\}$. Then, $X(A) = B \cap X(S)$ (e.g., [29], p.7). Therefore, $X(A) = B \cap [0, M]$, which is a Borel subset of $[0, M]$.

2. If $\psi, \phi : \mathbb{R} \to \mathbb{R}$ are both nonincreasing or both nondecreasing functions, then the function $L_2 : \mathbb{R}^2 \to \mathbb{R}$ defined by $L_2(x, y) = \phi(x)\ \psi(y)$ is supermodular.

Lemma B1 (Hardy–Littlewood–Pólya inequalities)**.** *Let $Y \in B^+(\Sigma)$, and let $A \in \mathcal{G}$ be such that $P(A) > 0$ and $X(A)$ is a Borel set. Let $\widetilde{Y}_P$ be the nondecreasing P-rearrangement of Y with respect to X, and let $\widetilde{Y}_{A,P}$ be the nondecreasing P-rearrangement of Y with respect to X on A. If L is supermodular, then:*

1. $\int L\left(X, Y\right) dP \leqslant \int L\left(X, \widetilde{Y}_P\right) dP$*; and,*
2. $\int_A L\left(X, Y\right) dP \leqslant \int_A L\left(X, \widetilde{Y}_{A,P}\right) dP$*,*

provided the integrals exist. Moreover, if L is strictly supermodular, then equality holds in (1) *if and only if $Y = \widetilde{Y}_P$, P-a.s.*

Lemma B2. *Let $Y \in B^+(\Sigma)$, and let $A \in \mathcal{G}$ be such that $P(A) > 0$ and $X(A)$ is a Borel set. Let $\widetilde{Y}_P$ be the nondecreasing P-rearrangement of Y with respect to X, and let $\widetilde{Y}_{A,P}$ be the nondecreasing P-rearrangement of Y with respect to X on A. Then, the following hold:*

1. *If $0 \leqslant Y \leqslant X$, P-a.s., then $0 \leqslant \widetilde{Y}_P \leqslant X$; and,*
2. *If $0 \leqslant Y \leqslant X$, P-a.s. on A, then $0 \leqslant \widetilde{Y}_{A,P} \leqslant X$, P-a.s. on A.*

B3. Approximation of the Rearrangement

Lemma B1. *If f and f_n are $[0, +\infty)$-valued, Σ-measurable functions on S, such that the sequence $\{f_n\}_n$ converges pointwise P-a.s. to f monotonically downwards, then the sequence $\{\widetilde{f}_{n,P}\}_n$ converges pointwise P-a.s. to $\widetilde{f}_P$ monotonically downwards, where $\widetilde{f}_P$ is the nondecreasing P-rearrangement of f with respect to X and $\widetilde{f}_{n,P}$ is the nondecreasing P-rearrangement of f_n with respect to X, for each $n \in \mathbb{N}$.*

Lemma B2. *Let f and f_n be $[0, +\infty)$-valued, Σ-measurable functions on S. If $f_n \in B^+(\Sigma)$, for each $n \geqslant 1$, and if the sequence $\{f_n\}_n$ converges uniformly to $f \in B^+(\Sigma)$, then:*

1. *The functions $\widetilde{f}_P$ and $\widetilde{f}_{n,P}$ are in L_∞, for each $n \geqslant 1$, where $\widetilde{f}_P$ is the nondecreasing P-rearrangement of f with respect to X and $\widetilde{f}_{n,P}$ is the nondecreasing P-rearrangement of f_n with respect to X, for each $n \in \mathbb{N}$; and*
2. *The sequence $\{\widetilde{f}_{n,P}\}_n$ converges to $\widetilde{f}_P$ in the L_∞ norm.*

Appendix C. Proof of Theorem 8

As in Section 3, there exists a unique pair (Q_{ac}, Q_s) of (nonnegative) finite measures on (S, Σ), such that $Q = Q_{ac} + Q_s$, $Q_{ac} \ll P$ and $Q_s \perp P$. That is, for all $B \in \Sigma$ with $P(B) = 0$, one has $Q_{ac}(B) = 0$, and there is some $A \in \Sigma$, such that $P(S \backslash A) = Q_s(A) = 0$. It then also follows that $Q_{ac}(S \backslash A) = 0$ and $P(A) = 1$. In the following, the Σ-measurable set A on which P is concentrated and $Q_s(A) = 0$ is assumed to be fixed all throughout.

C1. "Splitting" the Initial Problem The idea of splitting the problem into two sub-problems is inspired by the techniques used in Jin and Zhou [30] (although in a different context and for different purposes), but with some differences that are peculiar to the insurance problem examined here. Now, consider the following three problems:

Problem C1. For a given $\beta \in \left[0, \min\left(\Pi/(1+\rho), \int_A X\, dQ\right)\right]$,

$$\sup_{Y \in B^+(\Sigma)} \left\{ \int_A u\left(W_0 - \Pi - X + Y\right) dP \right\} :$$

$$\begin{cases} 0 \leqslant Y\mathbf{1}_A \leqslant X\mathbf{1}_A \\ \int_A Y\, dQ = \beta \end{cases}$$

Problem C2.

$$\sup_{Y \in B^{+}(\Sigma)} \left\{ \int_{S \backslash A} u\Big(W_0 - \Pi - X + Y\Big)\, dP \right\} :$$
$$\begin{cases} 0 \leqslant Y\mathbf{1}_{S\backslash A} \leqslant X\mathbf{1}_{S\backslash A} \\ \int_{S\backslash A} Y\, dQ \leqslant \min\left(\frac{\Pi}{1+\rho} - \beta, \int_{S\backslash A} X\, dQ\right), \text{ for the same } \beta \text{ as in Problem (C1)} \end{cases}$$

Problem C3.

$$\sup_{\beta} \left[F_A^{*}(\beta) + F_A^{*}\left(\frac{\Pi}{1+\rho} - \beta\right) : 0 \leqslant \beta \leqslant \min\left(\Pi/(1+\rho), \int_A X\, dQ\right) \right] :$$
$$\begin{cases} F_A^{*}(\beta) \text{ is the supremum value of Problem (C1), for a fixed } \beta \\ F_A^{*}\left(\frac{\Pi}{1+\rho} - \beta\right) \text{ is the supremum value of Problem (C2), for the same fixed } \beta \end{cases}$$

Note that since the utility function u is continuous (Assumption 1), it is bounded on every closed and bounded subset of $\mathbb{R}$. Therefore, since the range of X is closed and bounded, the supremum value of each of the above three problems is finite.

Lemma C1. *The feasibility sets of Problems* (C1) *and* (C2) *are non-empty.*

Proof. Since P and Q are not mutually singular, by Assumption 2, and since $P(S\backslash A) = 0$, it follows that $Q(A) > 0$. Since $Q(A) > 0$, $h \geqslant 0$ and $Q(A) = Q_{ac}(A) + Q_s(A) = Q_{ac}(A) = \int_A h\, dP$, it follows from Lemma A1 that there exists some $B \in \Sigma$, such that $B \subseteq A$, $P(B) > 0$ and $h > 0$ on B. There are three cases to consider:

(1) If $\int_A X\, dQ = \int_A Xh\, dP = 0$, then by Lemma A1, one has $Xh = 0$, P-a.s. on A. However, $h > 0$ on B. Thus, $X = 0$, P-a.s. on B. Consequently, there is some $C \in \Sigma$, with $C \subseteq B$ and $P(C) > 0$, such that $X = 0$ on C and $P(B\backslash C) = 0$. Therefore, $P(B) = P(C)$. Now, since $X(s) = 0$, for each $s \in C$, it follows that $C \subseteq \{s \in S : X(s) = 0\}$. Thus, by monotonicity of P, $P(C) \leqslant P(\{s \in S : X(s) = 0\}) = P \circ X^{-1}(\{0\})$. However, $P \circ X^{-1}(\{0\}) = 0$, by non-atomicity of $P \circ X^{-1}$ (Assumption 2). Therefore, $P(C) = 0$, a contradiction. Hence $\int_A X\, dQ > 0$. Now, for a given $\beta \in \left[0, \min\left(\Pi/(1+\rho), \int_A X\, dQ\right)\right]$, the function $Y_1 := \beta X / \int_A X\, dQ$ is feasible for Problem (C1) with parameter β.

(2) If $\int_{S\backslash A} X\, dQ = 0$, then $Y_2 := 0$ is feasible for Problem (C2).

(3) If $\int_{S\backslash A} X\, dQ > 0$, then $Y_3 := \alpha X / \int_{S\backslash A} X\, dQ$, with $\alpha := \min\left(\frac{\Pi}{1+\rho} - \beta, \int_{S\backslash A} X\, dQ\right) \Big/ 2$, is feasible for Problem (C2) with parameter β, for any given $\beta \in \left[0, \min\left(\Pi/(1+\rho), \int_A X\, dQ\right)\right]$.

□

Lemma C2. *If β^{*} is optimal for Problem* (C3)*, then $\beta^{*} > 0$.*

Proof. First note that, as in the proof of Lemma C1, $Q(A) > 0$ and there exists some $B \in \Sigma$ such that $B \subseteq A$, $P(B) > 0$ and $h > 0$ on B. Moreover, since $P(S\backslash A) = 0$, it follows that $\int_{S\backslash A} Z\, dP = 0$, for each $Z \in B(\Sigma)$, and so, $F_A^{*}\left(\frac{\Pi}{1+\rho} - \beta\right) = 0$, for each $\beta \in \left[0, \min\left(\Pi/(1+\rho), \int_A X\, dQ\right)\right]$. Consequently, $F_A^{*}(\beta) + F_A^{*}\left(\frac{\Pi}{1+\rho} - \beta\right) = F_A^{*}(\beta)$, for each $\beta \in \left[0, \min\left(\Pi/(1+\rho), \int_A X\, dQ\right)\right]$. Therefore, in particular, $F_A^{*}(\beta^{*}) + F_A^{*}\left(\frac{\Pi}{1+\rho} - \beta^{*}\right) = F_A^{*}(\beta^{*})$.

Now, suppose, by way of contradiction, that $\beta^* = 0$ is optimal for Problem (C3), and let Y_0 be optimal for Problem (C1) with parameter 0, so that $F_A^*(0) = \int_A u(W_0 - \Pi - X + Y_0)\, dP$. Since $\beta^* = 0$ is optimal for Problem (C3), one has $F_A^*(0) \geqslant F_A^*(\beta)$, for each $\beta \in \left[0, \min\left(\Pi/(1+\rho), \int_A X\, dQ\right)\right]$. Since Y_0 is feasible for Problem (C1) with parameter $\beta^* = 0$, one has $\int_A Y_0\, dQ = \int_A Y_0 h\, dP = \beta^* = 0$. Now, since $P(A) > 0$ and $Y_0 h \geqslant 0$, it follows from Lemma A1 that $Y_0 h = 0$, P-a.s. on A. Moreover, since $h > 0$ on B and $P(B) > 0$, it follows that $Y_0 = 0$, P-a.s. on B. Define the function Z by $Z := Y_0 \mathbf{1}_{A\setminus B} + \min\left(X, \Pi/(1+\rho)\right)\mathbf{1}_B$, and let $K_Z := \int_A Z\, dQ$. Then, $Z \in B^+(\Sigma)$, $0 \leqslant Z\mathbf{1}_A \leqslant X\mathbf{1}_A$, and $0 \leqslant K_Z \leqslant \min\left(\int_A X\, dQ, \Pi/(1+\rho)\right)$. Therefore, in particular, K_Z is feasible for Problem (C3) and Z is feasible for Problem (C1) with parameter K_Z. Moreover,

$$
\begin{aligned}
0 \leqslant K_Z &= \int_{A\setminus B} Y_0\, dQ + \int_B \min\left(X, \Pi/(1+\rho)\right) dQ \\
&= \int_B \min\left(X, \Pi/(1+\rho)\right) dQ = \int_B \min\left(X, \Pi/(1+\rho)\right) h\, dP.
\end{aligned}
$$

If $K_Z = 0$, then $\int_B \min\left(X, \Pi/(1+\rho)\right) h\, dP = 0$ and $\min\left(X, \Pi/(1+\rho)\right) h \geqslant 0$. Hence, by Lemma A1, $\min\left(X, \Pi/(1+\rho)\right) h = 0$, P-a.s. on B. However, $h > 0$ on B. Thus, $\min\left(X, \Pi/(1+\rho)\right) = 0$, P-a.s. on B. Since $\Pi > 0$, this yields $X = 0$, P-a.s. on B. Consequently, there is some $C \in \Sigma$, with $C \subseteq B$ and $P(C) > 0$, such that $X = 0$ on C and $P(B\setminus C) = 0$. Therefore, $P(B) = P(C)$. Now, since $X(s) = 0$, for each $s \in C$, it follows that $C \subseteq \{s \in S : X(s) = 0\}$. Thus, by monotonicity of P, $P(C) \leqslant P(\{s \in S : X(s) = 0\}) = P \circ X^{-1}(\{0\})$. However, $P \circ X^{-1}(\{0\}) = 0$, by non-atomicity of $P \circ X^{-1}$ (Assumption 2). Therefore, $P(C) = 0$, a contradiction. Hence, $K_Z > 0$. Finally,

$$
\begin{aligned}
F_A^*(K_Z) &\geqslant \int_A u(W_0 - \Pi - X + Z)\, dP \\
&= \int_{A\setminus B} u(W_0 - \Pi - X + Y_0)\, dP + \int_B u\left(W_0 - \Pi - X + \min\left(X, \Pi/(1+\rho)\right)\right) dP \\
&\geqslant \int_{A\setminus B} u(W_0 - \Pi - X + Y_0)\, dP + \int_B u(W_0 - \Pi - X)\, dP \\
&= \int_A u(W_0 - \Pi - X + Y_0)\, dP := F_A^*(0) = F_A^*(\beta^*).
\end{aligned}
$$

This contradicts the optimality of $\beta^* = 0$ for Problem (C3). Consequently, if β^* is optimal for Problem (C3), then $\beta^* > 0$. □

The following lemma shows how to combine the solutions of these three problems stated above to obtain a solution to the original problem (Problem (3)).

Lemma C3. *If β^* is optimal for Problem (C3), Y_3^* is optimal for Problem (C1) with parameter β^* and Y_4^* is optimal for Problem (C2) with parameter β^*, then $Y_2^* := Y_3^* \mathbf{1}_A + Y_4^* \mathbf{1}_{S\setminus A}$ is optimal for Problem (3).*

Proof. Feasibility of Y_2^* for Problem (3) is immediate. To show optimality of Y_2^* for Problem (3), let $\widetilde{Y}$ be any other feasible solution for Problem (3), and define $\alpha := \int_A \widetilde{Y}\, dQ$. Then, $\alpha = \int_A \widetilde{Y} h\, dP$ and $\int_A X\, dQ = \int_A X h\, dP$, since $Q_s(A) = 0$. Moreover, $\alpha \in \left[0, \min\left(\Pi/(1+\rho), \int_A X\, dQ\right)\right]$ since $\widetilde{Y}$ is feasible for Problem (3). Consequently, α is feasible for Problem (C3). Furthermore, $\widetilde{Y}\mathbf{1}_A$ (resp. $\widetilde{Y}\mathbf{1}_{S\setminus A}$) is feasible for Problem (C1) (resp. Problem (C2)) with parameter α. Hence,

$F_A^*(\alpha) \geqslant \int_A u\left(W_0 - \Pi - X + \widetilde{Y}\right) dP$ and $F_A^*\left(\frac{\Pi}{1+\rho} - \alpha\right) \geqslant \int_{S\setminus A} u\left(W_0 - \Pi - X + \widetilde{Y}\right) dP$. Now, since β^* is optimal for Problem (C3), it follows that:

$$F_A^*(\beta^*) + F_A^*\left(\frac{\Pi}{1+\rho} - \beta^*\right) \geqslant F_A^*(\alpha) + F_A^*\left(\frac{\Pi}{1+\rho} - \alpha\right).$$

However, $F_A^*(\beta^*) = \int_A u\left(W_0 - \Pi - X + Y_3^*\right) dP$ and $F_A^*\left(\frac{\Pi}{1+\rho} - \beta^*\right) = \int_{S\setminus A} u\left(W_0 - \Pi - X + Y_4^*\right) dP$. Therefore, $\int u\left(W_0 - \Pi - X + Y_2^*\right) dP \geqslant \int u\left(W_0 - \Pi - X + \widetilde{Y}\right) dP$. Hence, Y_2^* is optimal for Problem (3). □

The following lemma shows how to obtain monotonicity of an optimal indemnity schedule and how to characterize its distribution.

Lemma C4. *Let Y^* be an optimal solution for Problem (3), and suppose that Q is compatible P. Let $\widetilde{Y}_P^*$ be the nondecreasing P-rearrangement of Y^* with respect to X. Then:*

(1) $\widetilde{Y}_P^*$ is optimal for Problem (3); and
(2) $\widetilde{Y}_P^* = \widetilde{Y}_{P,A}^*$, P-a.s., where $\widetilde{Y}_{P,A}^*$ is the nondecreasing P-rearrangement of Y^* with respect to X on A. In particular, Y^* and $\widetilde{Y}_{P,A}^*$ are identically distributed under P.

Proof. Since the function $\mathcal{U} : \mathbb{R}^2 \to \mathbb{R}$ defined by $\mathcal{U}(x,y) := u(W_0 - \Pi - x + y)$ is supermodular (see Example B1 (1)), it follows from Lemma B1 that $\int u(W_0 - \Pi - X + \widetilde{Y}_P^*)\, dP \geqslant \int u(W_0 - \Pi - X + Y^*)\, dP$. Moreover, since $0 \leqslant Y^* \leqslant X$, it follows from Lemma B2 that $0 \leqslant \widetilde{Y}_P^* \leqslant X$. Finally, since Q is compatible with P, it follows that $\Pi/(1+\rho) \geqslant \int Y^*\, dQ \geqslant \int \widetilde{Y}_P^*\, dQ$, and so, $\widetilde{Y}_P^*$ is optimal for Problem (3). Now, let $\widetilde{Y}_{P,A}^*$ be the nondecreasing P-rearrangement of Y^* with respect to X on A. Since $P(A) = 1$, then by Lemma B1, one has that $\widetilde{Y}_P^* = \widetilde{Y}_{P,A}^*$, P-a.s. Therefore, $\widetilde{Y}_P^*$ and $\widetilde{Y}_{P,A}^*$ have the same distribution under P. Hence, form the equimeasurability of Y^* and $\widetilde{Y}_P^*$, it follows that Y^* and $\widetilde{Y}_{P,A}^*$ have the same distribution under P. □

The following lemma shows that a distributional characterization of an optimal indemnity schedule can be reduced to the problem of characterizing the distribution of the solution of Problem (C1).

Lemma C5. *Let an optimal solution for Problem (3) be given by:*

$$Y^* = Y_1^* \mathbf{1}_A + Y_2^* \mathbf{1}_{S\setminus A}, \tag{B1}$$

for some $Y_1^*, Y_2^* \in B^+(\Sigma)$. Let $\widetilde{Y}_P^*$ be the nondecreasing P-rearrangement of Y^* with respect to X, and let $Y_{1,P}^*$ be the nondecreasing P-rearrangement of Y_1^* with respect to X. Then, $\widetilde{Y}_P^* = \widetilde{Y}_{1,P}^*$, P-a.s., and hence, Y^* and $\widetilde{Y}_{1,P}^*$ have the same distribution under P.

Proof. Let $\widetilde{Y}_{P,A}^*$ be the nondecreasing P-rearrangement of Y^* with respect to X on A. Since $P(A) = 1$, then by Lemma B1, one has $\widetilde{Y}_P^* = \widetilde{Y}_{P,A}^*$, P-a.s. Similarly, let $\widetilde{Y}_{1,P,A}^*$ be the nondecreasing P-rearrangement of Y_1^* with respect to X on A. Then, $\widetilde{Y}_{1,P}^* = \widetilde{Y}_{1,P,A}^*$, P-a.s. Therefore, it suffices to show that $\widetilde{Y}_{P,A}^* = \widetilde{Y}_{1,P,A}^*$, P-a.s. Since both $\widetilde{Y}_{P,A}^*$ and $\widetilde{Y}_{1,P,A}^*$ are nondecreasing functions of X on A, then by the P-a.s. uniqueness of the nondecreasing rearrangement, it remains to show that they are P-equimeasurable with Y^* on A. Now, for each $t \in [0, M]$,

$$\begin{aligned} P\left(\{s \in A : \widetilde{Y}_{P,A}^*(s) \leqslant t\}\right) &= P\left(\{s \in A : Y^*(s) \leqslant t\}\right) = P\left(\{s \in A : Y_1^*(s) \leqslant t\}\right) \\ &= P\left(\{s \in A : \widetilde{Y}_{1,P,A}^*(s) \leqslant t\}\right), \end{aligned}$$

where the first equality follows from the definition of $\widetilde{Y}^*_{P,A}$ (equimeasurability), the second equality follows from Equation (B1) and the third equality follows from the definition of $\widetilde{Y}^*_{1,P,A}$ (equimeasurability). Therefore, $\widetilde{Y}^*_P = \widetilde{Y}^*_{1,P}$, P-a.s., and hence, $\widetilde{Y}^*_P$ and $\widetilde{Y}^*_{1,P}$ have the same distribution under P. Consequently, by equimeasurability of Y^* and $\widetilde{Y}^*_P$, it follows that Y^* and $\widetilde{Y}^*_{1,P}$ have the same distribution under P. □

Remark C1. By Lemmata C3, C4 and C5, if Q is compatible with P, β^* is optimal for Problem (C3), Y_1^* is optimal for Problem (C1) with parameter β^* and Y_2^* is optimal for Problem (C2) with parameter β^*, then $\widetilde{Y}^*_P$ is optimal for Problem (3) and $\widetilde{Y}^*_P = \widetilde{Y}^*_{1,P}$, P-a.s., where $\widetilde{Y}^*_P$ (resp. $\widetilde{Y}^*_{1,P}$) is the P-a.s. unique nondecreasing P-rearrangement of $Y^* := Y_1^* \mathbf{1}_A + Y_2^* \mathbf{1}_{S\backslash A}$ (resp. of Y_1^*) with respect to X. In particular, Y^* and $\widetilde{Y}^*_{1,P}$ have the same distribution under P.

Henceforth, we focus on solving each problem individually. The solutions can then be combined as per Remark C1.

C2. Solving Problem (C2) Since $P(S\backslash A) = 0$, it follows that, for all $Y \in B^+(\Sigma)$, one has

$$\int_{S\backslash A} u\left(W_0 - \Pi - X + Y\right)\, dP = 0.$$

Consequently, any Y, which is feasible for Problem (C2), with parameter β is also optimal for Problem (C2) with parameter β. For instance, define $Y_4^* := \min\left[X, \max\left\{0, X - \bar{d}_\beta\right\}\right]$, where $\bar{d}_\beta$ is chosen such that $\int_{S\backslash A} Y_4^*\, dQ \leqslant \min\left(\frac{\Pi}{1+\rho} - \beta, \int_{S\backslash A} X\, dQ\right)$. Then $Y_4^* \mathbf{1}_{S\backslash A}$ is optimal for Problem (C2).

Remark C1. The choice of $\bar{d}_\beta$ so that $\int_{S\backslash A} Y_4^*\, dQ \leqslant \min\left(\frac{\Pi}{1+\rho} - \beta, \int_{S\backslash A} X\, dQ\right)$ is justified by the following argument. Define the function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\phi(\alpha) = \int_{S\backslash A} Y_{4,\alpha}\, dQ$, where $Y_{4,\alpha} := \min\left[X, \max\{0, X - \alpha\}\right]$, for each $\alpha \geqslant 0$. Then, ϕ is a nonincreasing function of α. Moreover, by the continuity of the functions $\max(0, .)$ and $\min(x, .)$, and by Lebesgue's dominated convergence theorem (e.g., Theorem 2.4.4 of [23]), ϕ is a continuous function of the parameter α. Now, by the continuity of the functions max and min, $\lim\limits_{\alpha \to 0} Y_{4,\alpha} = X$ and $\lim\limits_{\alpha \to +\infty} Y_{4,\alpha} = 0$. Therefore, by continuity of the function ϕ in α, $\lim\limits_{\alpha \to 0} \phi(\alpha) = \int_{S\backslash A} X\, dQ$ and $\lim\limits_{\alpha \to +\infty} \phi(\alpha) = 0$. Consequently, ϕ is a continuous nonincreasing function of α, such that $\lim\limits_{\alpha \to +\infty} \phi(\alpha) = 0$ and $\lim\limits_{\alpha \to 0} \phi(\alpha) = \int_{S\backslash A} X\, dQ$. Thus, by the intermediate value theorem (e.g., Theorem 4.23 of [31]), one can always choose α, such that $\phi(\alpha) \leqslant \min\left(\frac{\Pi}{1+\rho} - \beta, \int_{S\backslash A} X\, dQ\right)$, for any $\beta \in \left[0, \min\left(\Pi/(1+\rho), \int_A X\, dQ\right)\right]$.

C3. Solving Problem (C1)**.** For a fixed parameter $\beta \in \left[0, \min\left(\Pi/(1+\rho), \int_A X\, dQ\right)\right]$, Problem (C1) will be solved "state-wise", as described below. Moreover, by Lemma C2, one can restrict the analysis to the case where $\beta \in \left(0, \min\left(\Pi/(1+\rho), \int_A X\, dQ\right)\right]$.

Lemma C1. *If $Y^* \in B^+(\Sigma)$ satisfies the following:*

(1) $0 \leqslant Y^*(s) \leqslant X(s)$, for all $s \in A$;
(2) $\int_A Y^* h\, dP = \beta$, for some $\beta \in \left(0, \min\left(\Pi/(1+\rho), \int_A X\, dQ\right)\right]$; and
(3) There exists some $\lambda \geqslant 0$, such that for all $s \in A\backslash\{s \in S : h(s) = 0\}$,

$$Y^*(s) = \underset{0 \leqslant y \leqslant X(s)}{\arg\max} \left[u\left(W_0 - \Pi - X(s) + y\right) - \lambda y h(s)\right],$$

then the function $Z^* := Y^* \mathbf{1}_{A\backslash\{s \in S: h(s)=0\}} + X \mathbf{1}_{A \cap \{s \in S: h(s)=0\}}$ solves Problem (C1) *with parameter β.*

Proof. Suppose that $Y^* \in B^+(\Sigma)$ satisfies (1), (2) and (3) above. Then, Z^* is clearly feasible for Problem (C1) with parameter β. To show optimality of Z^* for Problem (C1), note that for any other $Y \in B^+(\Sigma)$, which is feasible for Problem (C1) with parameter β, one has, for all $s \in A \backslash \{s \in S : h(s) = 0\}$,

$$\begin{aligned} & u\Big(W_0 - \Pi - X(s) + Z^*(s)\Big) - u\Big(W_0 - \Pi - X(s) + Y(s)\Big) \\ & = u\Big(W_0 - \Pi - X(s) + Y^*(s)\Big) - u\Big(W_0 - \Pi - X(s) + Y(s)\Big) \\ & \geqslant \lambda\Big[h(s)\,Y^*(s) - h(s)\,Y(s)\Big] = \lambda\Big[h(s)\,Z^*(s) - h(s)\,Y(s)\Big]. \end{aligned}$$

Furthermore, since u is increasing, since $0 \leqslant Y \leqslant X$ on A and since $Z^*(s) = X(s)$ for all $s \in \{s \in S : h(s) = 0\} \cap A$, it follows that for all $s \in \{s \in S : h(s) = 0\} \cap A$,

$$u\Big(W_0 - \Pi - X(s) + Z^*(s)\Big) = u\Big(W_0 - \Pi\Big) \geqslant u\Big(W_0 - \Pi - X(s) + Y(s)\Big).$$

Thus,

$$\int_{A \cap \{s \in S : h(s) = 0\}} u\Big(W_0 - \Pi - X + Z^*\Big)\, dP \; - \; \int_{A \cap \{s \in S : h(s) = 0\}} u\Big(W_0 - \Pi - X + Y\Big)\, dP \geqslant 0.$$

Consequently,

$$\begin{aligned} & \int_A u\Big(W_0 - \Pi - X + Z^*\Big)\, dP - \int_A u\Big(W_0 - \Pi - X + Y\Big)\, dP \\ & \geqslant \int_{A \backslash \{s \in S : h(s) = 0\}} u\Big(W_0 - \Pi - X + Z^*\Big)\, dP - \int_{A \backslash \{s \in S : h(s) = 0\}} u\Big(W_0 - \Pi - X + Y\Big)\, dP \\ & \geqslant \lambda\Big[\beta - \beta\Big] = 0, \end{aligned}$$

which completes the proof. □

Lemma C2. *For any $\lambda \geqslant 0$, the function given by:*

$$Y_\lambda^* := \min\left[X, \max\left(0, X - \left[W_0 - \Pi - \left(u'\right)^{-1}(\lambda h)\right]\right)\right] \tag{B2}$$

satisfies Conditions (1) *and* (3) *of Lemma C1.*

Proof. Fix $\lambda \geqslant 0$; fix $s \in A \backslash \{s \in S : h(s) = 0\}$; and consider the problem:

$$\max_{0 \leqslant y \leqslant X(s)} f(y) := \left[u\Big(W_0 - \Pi - X(s) + y\Big) - \lambda y h(s)\right]. \tag{B3}$$

Since u is strictly concave (by Assumption 1), so is f, as a function of y. In particular, $f'(y)$ is a (strictly) decreasing function. Hence the first-order condition on f yields a global maximum for f at $y^* := X(s) - \Big[W_0 - \Pi - \left(u'\right)^{-1}(\lambda h(s))\Big]$. If $y^* < 0$, then since f' is decreasing, it is negative on the interval $[0, X(s)]$. Therefore, f is decreasing on the interval $[0, X(s)]$ and hence attains a local maximum of $f(0)$ at $y = 0$. If $y^* > X(s)$, then since f' is decreasing, it is positive on the interval $[0, X(s)]$. Therefore, f is increasing on the interval $[0, X(s)]$ and hence attains a local maximum of $f(X(s))$ at $y = X(s)$. If $0 \leqslant y^* \leqslant X(s)$, then the local maximum of f on the interval $[0, X(s)]$ is its global maximum $f(y^*)$. Consequently, the function $y^{**} := \min\Big[X(s), \max(0, y^*)\Big]$ solves the problem appearing in Equation (B3). Since s and λ were chosen arbitrarily, this completes the proof of Lemma C2. □

Lemma C3. *For Y_λ^* defined in Equation (B2), the following holds:*

$$Y_\lambda^* \mathbf{1}_{A \setminus \{s \in S: h(s)=0\}} + X \mathbf{1}_{A \cap \{s \in S: h(s)=0\}} = Y_\lambda^* \mathbf{1}_A.$$

Therefore,

$$\int_A \left[Y_\lambda^* \mathbf{1}_{A \setminus \{s \in S: h(s)=0\}} + X \mathbf{1}_{A \cap \{s \in S: h(s)=0\}} \right] dQ = \int_A Y_\lambda^* \, dQ = \int_A Y_\lambda^* h \, dP.$$

Proof. Indeed, if $s \in \{s \in S : h(s) = 0\}$, then $(u')^{-1} (\lambda h(s)) = (u')^{-1}(0) = +\infty$, by Assumption 1. Thus, for each $s \in \{s \in S : h(s) = 0\}$ one has:

$$Y_\lambda^*(s) = \min \left[X(s), \max \left(0, X(s) - \left[W_0 - \Pi - (u')^{-1}(0) \right] \right) \right] = X(s).$$

The rest then easily follows. □

Lemma C4. *Define the function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ as follows: for each $\lambda \in \mathbb{R}^+$,*

$$\phi(\lambda) = \int_A \left[Y_\lambda^* \mathbf{1}_{A \setminus \{s \in S: h(s)=0\}} + X \mathbf{1}_{A \cap \{s \in S: h(s)=0\}} \right] dQ = \int_A Y_\lambda^* \, dQ = \int_A Y_\lambda^* h \, dP.$$

Then, ϕ is a continuous nonincreasing function of the parameter λ.

Proof. First, recall that:

$$Y_\lambda^* := \min \left[X, \max \left(0, X - \left[W_0 - \Pi - (u')^{-1}(\lambda h) \right] \right) \right].$$

Continuity of ϕ is a direct consequence of Lebesgue's dominated convergence theorem and of the continuity of each of the functions[11] $(u')^{-1}$, $\max(0, .)$ and $\min(x, .)$. The fact that ϕ is nonincreasing in λ results from the concavity of u, i.e., from the fact that u' is a nonincreasing function. □

Lemma C5. *Consider the function ϕ defined above. Then:*

$$\lim_{\lambda \to 0} \phi(\lambda) = \int_A X \, dQ \quad \textit{and} \quad \lim_{\lambda \to +\infty} \phi(\lambda) = 0.$$

Proof. By continuity of the functions $(u')^{-1}$, $\max(0, .)$ and $\min(x, .)$, it follows that for each $s \in S$, $\lim_{\lambda \to 0} Y_\lambda^*(s) = \min \left[X(s), \max \left(0, X(s) - \left[W_0 - \Pi - (u')^{-1}(0) \right] \right) \right]$. Moreover, as was shown above, $\min \left[X, \max \left(0, X - \left[W_0 - \Pi - (u')^{-1}(0) \right] \right) \right] = X$. Therefore, $\lim_{\lambda \to 0} Y_\lambda^*(s) = X(s)$, for each $s \in S$. Hence, by continuity of the function ϕ in λ, it follows that $\lim_{\lambda \to 0} \phi(\lambda) = \int_A X \, dQ$. Similarly, by continuity of the functions $(u')^{-1}$, $\max(0, .)$ and $\min(x, .)$, one has that for each $s \in S$, $\lim_{\lambda \to +\infty} Y_\lambda^*(s) = \min \left[X(s), \max \left(0, X(s) - \left[W_0 - \Pi - (u')^{-1}(+\infty) \right] \right) \right]$. However, by continuity of the function ϕ in λ, one has $\lim_{\lambda \to +\infty} \phi(\lambda) = \int_A \lim_{\lambda \to +\infty} Y_\lambda^* \, dQ$. However, by Assumption

[11] By Assumption 1, the function u is strictly concave and continuously differentiable. This implies that u' is both continuous and strictly decreasing, which, in turn, implies that $(u')^{-1}$ is continuous and strictly decreasing by the inverse function theorem (e.g., [31] pp. 221–223).

1, $(u')^{-1}(+\infty) = 0$, and by Assumption 2, $X \leqslant W_0 - \Pi$, P-a.s. Moreover, $P(A) = 1$. Therefore, $\int_A \lim_{\lambda\to+\infty} Y_\lambda^* \, dQ = \int_A \lim_{\lambda\to+\infty} Y_\lambda^* \, h \, dP = 0$. □

Remark C1. Hence, summing up, the function ϕ defined above is a nonincreasing continuous function of the parameter λ, such that $\lim_{\lambda\to 0} \phi(\lambda) = \int_A X \, dQ$ and $\lim_{\lambda\to+\infty} \phi(\lambda) = 0$. Therefore, $\phi(\lambda) \in \left[0, \int_A X \, dQ\right]$, and so, by the intermediate value theorem, for each $\beta \in \left(0, \min\left(\Pi/(1+\rho), \int_A X \, dQ\right)\right]$, one can chose $\overline{\lambda} = \overline{\lambda}_\beta \in [0, +\infty)$, such that:

$$\beta = \phi\left(\overline{\lambda}\right) = \int \left[Y_{\overline{\lambda}}^* \mathbf{1}_{A \setminus \{s \in S: h(s)=0\}} + X \mathbf{1}_{A \cap \{s \in S: h(s)=0\}}\right] h \, dP.$$

Therefore, by Lemmata C1 and C2, the function $Y_{\overline{\lambda}}^*$ defined above solves Problem (C1), with parameter β. Finally, let β^* be optimal for Problem (C3); let λ^* be chosen for β^* just as $\overline{\lambda}$ was chosen for β in Remark C1; and let $Y_{\lambda^*}^*$ be a corresponding optimal solution for Problem (C2) with parameter β^*. The rest then follows from Remark C1. The P-a.s. uniqueness part of Theorem 8 follows from the uniqueness property of the nondecreasing rearrangement. Finally, if the utility function u is strictly concave, then any solution $\mathcal{Z}^*$ to Problem (3) is such that $\mathcal{Z}^* = \mathcal{Y}^*$, P-a.s., by Lemma B1 and Example B1 (1). This concludes the proof of Theorem 8. □

Appendix D. Proof of Corollary 11

The idea behind the proof of Corollary 11 is to approximate the solution of Problem (3) characterized above by a sequence of functions that can be characterized. Taking limits then gives us a characterization of the solution of Problem (3).

Fix $\beta \in \left(0, \min\left(\Pi/(1+\rho), \int_A X \, dQ\right)\right]$, and let $\overline{\lambda}$ be the corresponding λ, chosen as in Remark C1. Since h is nonnegative, Σ-measurable and P-integrable, there is a sequence $\{h_n\}_n$ of nonnegative, P-simple and P-integrable functions on (S, Σ) that converges monotonically upwards and pointwise to h (e.g., Proposition 2.1.7 of [23]). Therefore, since $(u')^{-1}$ is continuous, the sequence $\{Y_{\overline{\lambda},n}\}_n$, defined by $Y_{\overline{\lambda},n} := X - W_0 + \Pi + (u')^{-1}\left(\overline{\lambda} h_n\right)$, for all $n \in \mathbb{N}$, converges pointwise to $Y_{\overline{\lambda}}$, defined by:

$$Y_{\overline{\lambda}} := X - W_0 + \Pi + (u')^{-1}\left(\overline{\lambda} h\right).$$

Since the sequence $\{h_n\}_n$ converges monotonically upwards and pointwise to h and since $(u')^{-1}$ is continuous and decreasing, it follows that the sequence $\{Y_{\overline{\lambda},n}\}_n$ converges monotonically downwards and pointwise to $Y_{\overline{\lambda}}$. Now, for each $n \in \mathbb{N}$, there is some $m_n \in \mathbb{N}$, a Σ-partition $\{B_{i,n}\}_{i=1}^{m_n}$ of S and some nonnegative real numbers $\alpha_{i,n} \geqslant 0$, for $i = 1, \ldots, m_n$, such that $h_n = \sum_{i=1}^{m_n} \alpha_{i,n} \mathbf{1}_{B_{i,n}}$. Since $X - W_0 + \Pi$ can be written as $\sum_{i=1}^{m_n} (X - W_0 + \Pi) \mathbf{1}_{B_{i,n}}$; it is then easy to see that:

$$Y_{\overline{\lambda},n} = \sum_{i=1}^{m_n} \left((u')^{-1}\left(\overline{\lambda}\,\alpha_{i,n}\right) + X - W_0 + \Pi\right) \mathbf{1}_{B_{i,n}}, \; \forall n \in \mathbb{N}.$$

Define $Y_{\overline{\lambda},n}^*$ by:

$$Y_{\overline{\lambda},n}^* := \min\left[X, \max\left(0, Y_{\overline{\lambda},n}\right)\right].$$

By the continuity of the functions $\max(0,.)$ and $\min(x,.)$ and since $\max(0,t)$ and $\min(X(s),t)$ are nondecreasing functions of t for each $s \in S$, it follows that the sequence $\{Y^*_{\overline{\lambda},n}\}_n$ converges monotonically downwards and pointwise to:

$$Y^*_{\overline{\lambda}} := \min\left[X, \max\left(0, X - \left[W_0 - \Pi - (u')^{-1}\left(\overline{\lambda}h\right)\right]\right)\right].$$

For each $n \in \mathbb{N}$, one can rewrite $Y^*_{\overline{\lambda},n}$ as:

$$Y^*_{\overline{\lambda},n} = \sum_{i=1}^{m_n} I^*_{\overline{\lambda},n,i}\, \mathbf{1}_{B_{i,n}},$$

where, for $i = 1, \ldots, m_n$, $I^*_{\overline{\lambda},n,i} := \min\left[X, \max\left(0, X - d_{\overline{\lambda},n,i}\right)\right]$ and $d_{\overline{\lambda},n,i} := W_0 - \Pi - (u')^{-1}\left(\overline{\lambda}\alpha_{i,n}\right)$.

Lemma D1. *For each $n \in \mathbb{N}$ and for each $i_0 \in \{1, 2, \ldots, m_n\}$, $I^*_{\overline{\lambda},n,i_0}$ is either a full insurance indemnity schedule or a deductible indemnity schedule (with a strictly positive deductible) on the set $B_{i_0,n}$.*

Proof. Fix $n \in \mathbb{N}$, and fix $i_0 \in \{1, 2, \ldots, m_n\}$. If $\alpha_{i_0,n} > 0$ and $\overline{\lambda} \leqslant u'(W_0 - \Pi)/\alpha_{i_0,n}$, then since u' is decreasing (u is concave), it follows that $(u')^{-1}\left(\overline{\lambda}\alpha_{i_0,n}\right) \geqslant W_0 - \Pi$. Therefore, $(u')^{-1}\left(\overline{\lambda}\alpha_{i_0,n}\right) - W_0 + \Pi + X \geqslant X \geqslant 0$, and so, $I^*_{\overline{\lambda},n,i_0} = X$, a full insurance indemnity schedule (on $B_{i_0,n}$). If $\alpha_{i_0,n} = 0$, then $I^*_{\overline{\lambda},n,i_0} = \min\left[X, \max\left(0, (u')^{-1}(0) + X - W_0 + \Pi\right)\right]$. However, $(u')^{-1}(0) = +\infty$, by Assumption 1. Therefore, $(u')^{-1}(0) - W_0 + \Pi + X \geqslant X \geqslant 0$, and so $I^*_{\overline{\lambda},n,i_0} = X$, a full insurance indemnity schedule (on $B_{i_0,n}$). If $\alpha_{i_0,n} > 0$ and $\overline{\lambda} > u'(W_0 - \Pi)/\alpha_{i_0,n}$, then since u' is strictly decreasing (u is strictly concave), it follows that $(u')^{-1}\left(\overline{\lambda}\alpha_{i_0,n}\right) < W_0 - \Pi$. Therefore, $0 < W_0 - \Pi - (u')^{-1}\left(\overline{\lambda}\alpha_{i_0,n}\right) = d_{\overline{\lambda},n,i_0}$, and so, $I^*_{\overline{\lambda},n,i_0} = \left(X - d_{\overline{\lambda},n,i_0}\right)^+$, a deductible insurance indemnity schedule (on $B_{i_0,n}$) with a strictly positive deductible, where for any $a, b \in \mathbb{R}$, $(a-b)^+ := \max(0, a-b)$. □

The following lemma is a direct consequence of Lemmata B1 and B1, and it is hence stated without a proof.

Lemma D2. *If $\widetilde{Y}^*_{\overline{\lambda},n,P}$ (resp. $\widetilde{Y}^*_{\overline{\lambda},P}$) denotes the nondecreasing P-rearrangement of $Y^*_{\overline{\lambda},n}$ (resp. $Y^*_{\overline{\lambda}}$) with respect to X, then $\{\widetilde{Y}^*_{\overline{\lambda},n,P}\}_n$ converges monotonically downwards and pointwise P-a.s. to $\widetilde{Y}^*_{\overline{\lambda},P}$. Moreover, $\widetilde{Y}^*_{\overline{\lambda},n,P} = \widetilde{Y}^*_{\overline{\lambda},n,A,P}$, P-a.s., where $\widetilde{Y}^*_{\overline{\lambda},n,A,P}$ denotes the nondecreasing P-rearrangement of $Y^*_{\overline{\lambda},n,P}$ with respect to X on A.*

Let $C_{2,n} := \left\{s \in S : Y^*_{\overline{\lambda},n}(s) = X(s)\right\}$. Then, $C_{2,n}$ is of the form[12] $C_{2,n} = B_{k_1,n} \cup \ldots \cup B_{k_N,n}$, for some $\{k_1, k_2, \ldots, k_N\} \subseteq \{1, 2, \ldots, m_n\}$. Therefore,

$$Y^*_{\overline{\lambda},n} = \sum_{j \in J}\left(X - d_{\overline{\lambda},n,j}\right)^+ \mathbf{1}_{B_{j,n}} + X\mathbf{1}_{C_{2,n}},$$

for $J = \{1, 2, \ldots, m_n\} \backslash \{k_1, k_2, \ldots, k_N\}$.

Lemma D3. *Fix $n \in \mathbb{N}$. If there exists some $i_0 \in \{1, 2, \ldots, m_n\}$ such that $\alpha_{i_0,n} = 0$ and $B_{i_0,n} \backslash \{s \in S : X(s) = 0\} \neq \varnothing$, then $C_{2,n} \backslash \{s \in S : X(s) = 0\} \neq \varnothing$.*

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12 Note that since the random loss X is a mapping of S onto the closed interval $[0, M]$, it follows that $\{s \in S : X(s) = 0\} \neq \varnothing$. Now, since $0 \leqslant Y_{\overline{\lambda},n} \leqslant X$, it follows that $\varnothing \neq \{s \in S : X(s) = 0\} \subseteq C_{2,n}$. Therefore, $C_{2,n} \neq \varnothing$.

Proof. As in the proof of Lemma D1. □

Lemma D4. *If P is not absolutely continuous with respect to Q, then for each $n \in \mathbb{N}$, there is some $i_0 \in \{1, 2, \ldots, m_n\}$, such that $\alpha_{i_0,n} = 0$.*

Proof. Suppose, by way of contradiction, that P is not absolutely continuous with respect to Q, but that there is some $n \in \mathbb{N}$, such that $\alpha_{i_0,n} > 0$, for each $i_0 \in \{1, 2, \ldots, m_n\}$. Then, $h_n = \sum_{i=1}^{m_n} \alpha_{i,n} \mathbf{1}_{B_{i,n}} > 0$. However, the sequence $\{h_n\}_n$ converges monotonically upwards and pointwise, to $h := dQ_{ac}/dP$. Hence, since $h_n > 0$, it follows that $h(s) \geqslant h_k(s) > 0$, for each $s \in S$ and for each $k \geqslant n$. Consequently, $h > 0$. Therefore, P and Q_{ac} are mutually absolutely continuous (i.e., equivalent – see p. 179 of [32]). Furthermore, the finite measures Q, Q_{ac} and Q_s are nonnegative, and hence, $Q_{ac} \ll Q$. Thus, $P \ll Q$, a contradiction. □

Remark D1. Lemmata D3 and D4 imply that if P is not absolutely continuous with respect to Q, then $C_{2,n} \backslash \{s \in S : X(s) = 0\} \neq \varnothing$, for each $n \in \mathbb{N}$.

Now, let $C_{1,n} := \left\{s \in S : Y^*_{\overline{\lambda},n}(s) = 0\right\}$. Then, $C_{1,n}$ is non-empty (since $0 \leqslant Y^*_{\overline{\lambda},n} \leqslant X$ and $\{s \in S : X(s) = 0\} \neq \varnothing$), and of the form $C_{1,n} = C^{(i)}_{1,n} \cup C^{(ii)}_{1,n}$, where $C^{(i)}_{1,n} \subseteq C_{2,n}$ and $C^{(ii)}_{1,n} \subseteq S \backslash C_{2,n}$. Indeed, since $\left\{s \in S : X(s) = 0\right\} \neq \varnothing$ and $0 \leqslant Y^*_{\overline{\lambda},n} \leqslant X$, it follows that for all $s \in \left\{s \in S : X(s) = 0\right\}$ one has $Y^*_{\overline{\lambda},n}(s) = X(s) = 0$. It is then easily verified that:

$$C^{(i)}_{1,n} := \left\{s \in C_{2,n} : Y^*_{\overline{\lambda},n}(s) = 0\right\} = \left\{s \in S : X(s) = 0\right\} \neq \varnothing.$$

Therefore, $C_{1,n} = \left\{s \in S : X(s) = 0\right\} \cup C^{(ii)}_{1,n}$. Moreover, one can write $C^{(ii)}_{1,n} = \bigcup_{j=k_{N+1}}^{k_Q} B_{j,n}$, for some $\{k_{N+1}, \ldots, k_Q\} \subseteq J$. Letting $J' := J \backslash \{k_{N+1}, \ldots, k_Q\}$, it follows that $0 < \left(X - d_{\overline{\lambda},n,j}\right)^+ = X - d_{\overline{\lambda},n,j} < X$, for each $j \in J'$. Therefore,

$$Y^*_{\overline{\lambda},n} = 0\mathbf{1}_{C_{1,n}} + \sum_{j \in J'} \left(X - d_{\overline{\lambda},n,j}\right) \mathbf{1}_{B_{j,n}} + X\mathbf{1}_{C_{2,n} \backslash \{s \in S : X(s) = 0\}}.$$

One can assume, without loss of generality, that $\alpha_{j,n} < \alpha_{k,n}$, for all $j, k \in J'$ such that $j < k$. Then, it is easily verified that $d_{\overline{\lambda},n,j} < d_{\overline{\lambda},n,k}$, because of the concavity of u.

Lemma D5. *Let $\widetilde{Y}^*_{\overline{\lambda},n,P}$ denote the nondecreasing P-rearrangement of $Y^*_{\overline{\lambda},n}$ with respect to X. Then, there exist $a_n, b_n \in [0, M]$, such that $a_n \leqslant b_n$, and for P-a.a. $s \in S$,*

$$\widetilde{Y}^*_{\overline{\lambda},n,P}(s) = \begin{cases} 0 \text{ if } X(s) \in [0, a_n) \\ f_n(X(s)) \text{ if } X(s) \in [a_n, b_n] \\ X(s) \text{ if } X(s) \in (b_n, M] \end{cases} \tag{D1}$$

where $f_n : [0, M] \to [0, M]$ is a nondecreasing and Borel-measurable function, such that $0 \leqslant f_n(t) \leqslant t$ for each $t \in [0, M]$, and for $P \circ X^{-1}$-a.a. $t \in [0, M]$, one has $f(t) > 0$ if $t > a_n$ and $f_n(t) < t$ if $0 < t < b_n$.

Proof. First note that $0 \leqslant \widetilde{Y}^*_{\overline{\lambda},n,P} \leqslant X$, by Lemma B2, since $0 \leqslant Y^*_{\overline{\lambda},n} \leqslant X$, by definition of $Y^*_{\overline{\lambda},n}$. Moreover, one has $Y^*_{\overline{\lambda},n} = I_{\overline{\lambda},n} \circ X$, for some Borel-measurable function $I_{\overline{\lambda},n} : [0, M] \to [0, M]$. Therefore, $\widetilde{Y}^*_{\overline{\lambda},n,P} = \widetilde{I}^*_{\overline{\lambda},n} \circ X$, where $\widetilde{I}^*_n$ is the nondecreasing $P \circ X^{-1}$-rearrangement of $I_{\overline{\lambda},n}$.

Let $f_n := \widetilde{I}^*_{\overline{\lambda},n}$. Then, $0 \leqslant f_n(t) \leqslant t$, for each $t \in [0, M]$, and $f_n : [0, M] \to [0, M]$ is nondecreasing and Borel-measurable. Now, note that:

$$\begin{aligned} P\Big(\big\{s \in S : Y^*_{\overline{\lambda},n}(s) \leqslant 0\big\}\Big) &= P\Big(\big\{s \in S : Y^*_{\overline{\lambda},n}(s) = 0\big\}\Big) = P\left(C_{1,n}\right) \\ &= P\Big(\big\{s \in S : Y^*_{\overline{\lambda},n}(s) \leqslant 0, X(s) = 0\big\}\Big) \\ &\quad + P\Big(\big\{s \in S : Y^*_{\overline{\lambda},n}(s) \leqslant 0, X(s) > 0\big\}\Big) \\ &= P\Big(\big\{s \in S : X(s) = 0\big\}\Big) + P\left(C^{(ii)}_{1,n}\right) = P\left(C^{(ii)}_{1,n}\right), \end{aligned}$$

where the last equality follows form the non-atomicity of $P \circ X^{-1}$ (Assumption 2). Moreover, by equimeasurability, one has that $P\Big(\big\{s \in S : Y^*_{\overline{\lambda},n}(s) \leqslant 0\big\}\Big) = P\Big(\big\{s \in S : \widetilde{Y}^*_{\overline{\lambda},n,P}(s) \leqslant 0\big\}\Big)$. However,

$$\begin{aligned} P\Big(\big\{s \in S : \widetilde{Y}^*_{\overline{\lambda},n,P}(s) \leqslant 0\big\}\Big) &= P\Big(\big\{s \in S : \widetilde{Y}^*_{\overline{\lambda},n,P}(s) = 0\big\}\Big) \\ &= P\Big(\big\{s \in S : \widetilde{Y}^*_{\overline{\lambda},n,P}(s) \leqslant 0, X(s) = 0\big\}\Big) \\ &\quad + P\Big(\big\{s \in S : \widetilde{Y}^*_{\overline{\lambda},n,P}(s) \leqslant 0, X(s) > 0\big\}\Big) \\ &= P\Big(\big\{s \in S : X(s) = 0\big\}\Big) + P\Big(\big\{s \in S : \widetilde{Y}^*_{\overline{\lambda},n,P}(s) = 0, X(s) > 0\big\}\Big) \\ &= P\Big(\big\{s \in S : \widetilde{Y}^*_{\overline{\lambda},n,P}(s) = 0, X(s) > 0\big\}\Big), \end{aligned}$$

where the last equality follows form the non-atomicity of $P \circ X^{-1}$ (Assumption 2). Consequently,

$$P\left(C_{1,n}\right) = P\left(C^{(ii)}_{1,n}\right) = P\Big(\big\{s \in S :\ \widetilde{Y}^*_{\overline{\lambda},n,P}(s) = 0,\ X(s) > 0\big\}\Big).$$

Thus, if $P\left(C^{(ii)}_{1,n}\right) \neq 0$, then since $f_n : [0, M] \to [0, M]$ is nondecreasing, there exists $a_n > 0$, such that for P-a.a. $s \in S$, $\widetilde{Y}^*_{\overline{\lambda},n,P}(s) = 0$ if $X(s)$ belongs to $[0, a_n]$ or $[0, a_n)$, and $\widetilde{Y}^*_{\overline{\lambda},n,P}(s) > 0$ if $X(s) > a_n$. Therefore, $f_n(t) > 0$ if $t > a_n$, for $P \circ X^{-1}$-a.a. $t \in [0, M]$. If $P\left(C^{(ii)}_{1,n}\right) = 0$, then $P\Big(\big\{s \in S : \widetilde{Y}^*_{\overline{\lambda},n,P}(s) = 0,\ X(s) > 0\big\}\Big) = 0$, and so for P-a.a. $s \in S$, $\widetilde{Y}^*_{\overline{\lambda},n,P}(s) = 0$ if $X(s) = 0$, and $\widetilde{Y}^*_{\overline{\lambda},n,P}(s) > 0$ if $X(s) > 0$. Thus, with $a_n = 0$, $\widetilde{Y}^*_{\overline{\lambda},n,P}$ is P-a.s. of the form Equation (D1), with $f_n(t) > 0$ if $t > a_n = 0$, for $P \circ X^{-1}$-a.a. $t \in [0, M]$.

Similarly, by equimeasurability, one has that:

$$P\Big(\big\{s \in S : Y^*_{\overline{\lambda},n}(s) = X(s)\big\}\Big) = P\Big(\big\{s \in S : \widetilde{Y}^*_{\overline{\lambda},n,P}(s) = X(s)\big\}\Big).$$

However,

$$\begin{aligned} P\Big(\big\{s \in S : Y^*_{\overline{\lambda},n,P}(s) = X(s)\big\}\Big) &= P\Big(\big\{s \in S : Y^*_{\overline{\lambda},n,P}(s) = X(s), X(s) = 0\big\}\Big) \\ &\quad + P\Big(\big\{s \in S : Y^*_{\overline{\lambda},n,P}(s) = X(s), X(s) > 0\big\}\Big) \\ &= P\Big(\big\{s \in S : X(s) = 0\big\}\Big) \\ &\quad + P\Big(\big\{s \in S : \widetilde{Y}^*_{\overline{\lambda},n,P}(s) = X(s), X(s) > 0\big\}\Big) \\ &= P\Big(\big\{s \in S : \widetilde{Y}^*_{\overline{\lambda},n,P}(s) = X(s), X(s) > 0\big\}\Big) \\ &= P\Big(C_{2,n} \backslash \{s \in S : X(s) = 0\}\Big), \end{aligned}$$

where the second-to-last equality follows form the non-atomicity of $P \circ X^{-1}$ (Assumption 2).

Thus, if $P\left(C_{2,n}\backslash\{s\in S:X(s)=0\}\right)\neq 0$, then $P\left(\{s\in S:\widetilde{Y}^{*}_{\overline{\lambda},n,P}(s)=X(s),X(s)>0\}\right)>0$. Therefore, since $f_n:[0,M]\to[0,M]$ is nondecreasing, there exists $b_n>0$, such that for P-a.a. $s\in S$, $\widetilde{Y}^{*}_{\overline{\lambda},n,P}(s)=X(s)$ if $X(s)$ belongs to $[b_n,M]$ or $(b_n,M]$, and $\widetilde{Y}^{*}_{\overline{\lambda},n,P}(s)<X(s)$ if $0<X(s)<b_n$. Therefore, $f_n(t)<t$ if $0<t<b_n$, for $P\circ X^{-1}$-a.a. $t\in[0,M]$. If $P\left(C_{2,n}\backslash\{s\in S:X(s)=0\}\right)=0$, then $P\left(\{s\in S:\widetilde{Y}^{*}_{\overline{\lambda},n,P}(s)=X(s),X(s)>0\}\right)=0$, and so, for P-a.a. $s\in S$, such that $X(s)>0$, one has that $\widetilde{Y}^{*}_{\overline{\lambda},n,P}(s)<X(s)$. Thus, with $b_n=M$, $\widetilde{Y}^{*}_{\overline{\lambda},n,P}$ is P-a.s. of the form Equation (D1), with $f_n(t)<t$ if $0<t<b_n=M$, for $P\circ X^{-1}$-a.a. $t\in[0,M]$.

To show that $a_n\leqslant b_n$, suppose, by way of contradiction, that $b_n<a_n$. Since $b_n>0$, it follows that $0<b_n<a_n$. Choose $s_0\in S$, such that $0<b_n<X(s_0)<a_n$. Then, $\widetilde{Y}^{*}_{\overline{\lambda},n,P}(s_0)=0$, since $X(s_0)<a_n$. However, $\widetilde{Y}^{*}_{\overline{\lambda},n,P}(s_0)=X(s_0)$, since $t_0>b_n$, hence contradicting the fact that $X(s_0)>0$. Therefore, $a_n\leqslant b_n$. □

Remark D2. For each $n\geqslant 1$, let $E_n\in\Sigma$ be the event, such that $P(E_n)=1$ and $\widetilde{Y}^{*}_{\overline{\lambda},n,P}$ is of the form of Equation (D1) on E_n. Let $E:=\bigcap_{n=1}^{+\infty}E_n$. Then, $E\in\Sigma$ and, by Lemma A2, $P(E)=1$. Moreover, for each $s\in E$ and for each $n\geqslant 1$, $\widetilde{Y}^{*}_{\overline{\lambda},n,P}(s)$ is given by Equation (D1).

By Lemma D2, the sequence $\{\widetilde{Y}^{*}_{\overline{\lambda},m,P}\}_m$ defined by Equation (D1) converges pointwise P-a.s. to $\widetilde{Y}^{*}_{\overline{\lambda},P}$, the nondecreasing P-rearrangement of $Y^{*}_{\overline{\lambda}}$ with respect to X.

Now, let $Y^{*}_{4,\beta}$ be an optimal solution to Problem (C2) with parameter β, as defined previously, and for each $m\in\mathbb{N}$, let:

$$\widetilde{Y}^{*}_{m,\beta}:=\widetilde{Y}^{*}_{\overline{\lambda},m,P}\mathbf{1}_A+Y^{*}_{4,\beta}\mathbf{1}_{S\backslash A}.$$

Finally, let β^{*} be optimal for Problem (C3); let λ^{*} be chosen for β^{*} just as $\overline{\lambda}$ was chosen for β; and let $Y^{*}_{4,\beta^{*}}$ be a corresponding optimal solution for Problem (C2) with parameter β^{*}. For each $n\geqslant 1$, let:

$$\widetilde{Y}^{*}_{m,\beta^{*}}:=\widetilde{Y}^{*}_{\lambda^{*},m,P}\mathbf{1}_A+Y^{*}_{4,\beta^{*}}\mathbf{1}_{S\backslash A}.$$

Then, by Remark C1, the sequence $\{\widetilde{Y}^{*}_{m,\beta^{*}}\}_m$ converges pointwise P-a.s. to an optimal solution of the initial problem (Problem (3)), which is P-a.s. nondecreasing in the loss X. Henceforth, $\mathcal{Y}^{*}$ will denote that optimal solution. Then:

$$\mathcal{Y}^{*}\mathbf{1}_A=\widetilde{Y}^{*}_{\lambda^{*},P}\mathbf{1}_A. \tag{D2}$$

Now, recall that $\widetilde{Y}^{*}_{\lambda^{*},P}$ is the P-a.s. unique nondecreasing P-rearrangement of $Y^{*}_{\lambda^{*}}$ with respect to X, where:

$$Y^{*}_{\lambda^{*}}:=\min\left[X,\max\left(0,Y_{\lambda^{*}}\right)\right]\quad\text{and}\quad Y_{\lambda^{*}}:=X-W_0+\Pi+\left(u'\right)^{-1}\left(\lambda^{*}h\right).$$

Moreover, the sequence $\{Y_{\lambda^{*},m}\}_m$, defined by:

$$Y_{\lambda^{*},m}:=X-W_0+\Pi+\left(u'\right)^{-1}\left(\lambda^{*}h_m\right),$$

converges pointwise to $Y_{\lambda^{*}}$. Since the sequence $\{h_m\}_m$ converges monotonically upwards and pointwise to h and since $\left(u'\right)^{-1}$ is continuous and decreasing, it follows that the sequence $\{Y_{\lambda^{*},m}\}_m$ converges monotonically downwards and pointwise to $Y_{\lambda^{*}}$. Consequently, one can easily check that the sequence $\{Y^{*}_{\lambda^{*},m}\}_m$ converges monotonically downwards and pointwise to $Y^{*}_{\lambda^{*}}$, where for each $m\geqslant 1$,

$$Y^{*}_{\lambda^{*},m}:=\min\left[X,\max\left(0,Y_{\lambda^{*},m}\right)\right].$$

Remark D3. For each $m \geqslant 1$, let $\widetilde{Y}^{*}_{\lambda*,m,P}$ denote the P-a.s. unique nondecreasing P-rearrangement of $Y^{*}_{\lambda*,m}$ with respect to X. Then, by Lemma B1, the sequence $\{\widetilde{Y}_{\lambda*,m,P}\}_m$ converges monotonically downwards and pointwise P-a.s. to $\widetilde{Y}_{\lambda*,P}$. That is, there is some $A^* \in \Sigma$ with $A^* \subseteq A$ and $P(A^*) = 1$, such that for each $s \in A^*$, the sequence $\{\widetilde{Y}_{\lambda*,m,P}(s)\}_m$ converges monotonically downwards to $\widetilde{Y}_{\lambda*,P}(s)$.

Now, as in Lemma D5, for P-a.a. $s \in S$,

$$\widetilde{Y}^{*}_{\lambda,n,P}(s) = \begin{cases} 0 \text{ if } X(s) \in [0, a_n) \\ f_n(X(s)) \text{ if } X(s) \in [a_n, b_n] \\ X(s) \text{ if } X(s) \in (b_n, M] \end{cases}$$

for given $a_n, b_n \in [0, M]$, such that $a_n \leqslant b_n$ and $f_n : [0, M] \to [0, M]$, a nondecreasing and Borel-measurable function, such that $0 \leqslant f_n(t) \leqslant t$ for each $t \in [0, M]$, $f(t) > 0$ if $t > a_n$ for $P \circ X^{-1}$-a.a. $t \in [0, M]$, and $f_n(t) < t$ if $0 < t < b_n$ for $P \circ X^{-1}$-a.a. $t \in [0, M]$.

Lemma D6. *The sequences $\{a_m\}_m$ and $\{b_m\}_m$ are bounded and nondecreasing.*

Proof. Since $\{a_m\}_m \subset [0, M]$ and $\{b_m\}_m \subset [0, M]$, the boundedness of the sequences $\{a_m\}_m$ and $\{b_m\}_m$ is clear. We now show that they are nondecreasing. Fix $m \in \mathbb{N}$. Since the sequence $\{\widetilde{Y}_{\lambda*,m,P}\}_m$ is nonincreasing pointwise on A^* (as in Remark D3), one has $\widetilde{Y}_{\lambda*,m,P}(s) \geqslant \widetilde{Y}_{\lambda*,m+1,P}(s)$, for each $s \in A^*$. To show that $a_m \leqslant a_{m+1}$, first note that if $a_m = 0$, then $a_{m+1} \geqslant 0 = a_m$. If $a_m > 0$, let $E \in \Sigma$ be as in Remark D2, let $A^* \in \Sigma$ be as in Remark D3, and choose $s \in E \cap A^*$, such that $X(s) \in [0, a_m)$. Then $0 = \widetilde{Y}_{\lambda*,m,P}(s) \geqslant \widetilde{Y}_{\lambda*,m+1,P}(s) \geqslant 0$, and so, $\widetilde{Y}_{\lambda*,m+1,P}(s) = 0$. Consequently, $X(s) \in [0, a_{m+1}]$, and so $[0, a_m) \subseteq [0, a_{m+1}]$. Therefore, $0 < a_m \leqslant a_{m+1}$. Similarly, if $b_{m+1} = M$, then $b_m \leqslant M = b_{m+1}$. If $b_{m+1} < M$, choose $s \in E \cap A^*$ such that $X(s) \in (b_{n+1}, M]$. Then, $X(s) = \widetilde{Y}_{\lambda*,n+1,P}(s) \leqslant \widetilde{Y}_{\lambda*,n,P}(s) \leqslant X(s)$, and so, $\widetilde{Y}_{\lambda*,n,P}(s) = X(s)$. Consequently, $X(s) \in (b_n, M]$, and so, $(b_{n+1}, M] \subseteq (b_n, M]$, that is, $b_n \leqslant b_{n+1}$. □

Hence, the sequences $\{a_m\}_m$ and $\{b_m\}_m$ are bounded and monotone. Therefore, each has a limit. Let:

$$a := \lim_{m \to +\infty} a_m \text{ and } b := \lim_{m \to +\infty} b_m. \tag{D3}$$

Moreover, if there is some $n \geqslant 1$, such that $a_n > 0$, then for each $m \geqslant n$, one has $a_m \geqslant a_n > 0$. Furthermore, if there is some $n \geqslant 1$, such that $b_n < M$, then for each $m \leqslant n$, one has $b_m \leqslant b_n < M$

Lemma D7. *With a as defined in Equation (D3) above, one has $0 \leqslant a \leqslant b \leqslant M$, and $a > 0$ if there is some $n \geqslant 1$ with $a_n > 0$.*

Proof. Since $0 \leqslant a_m \leqslant M$ and $0 \leqslant b_m \leqslant M$, for each $m \geqslant 1$, it follows that $0 \leqslant a \leqslant M$ and $0 \leqslant b \leqslant M$. Moreover, since $a_m \leqslant b_m$, for each $m \geqslant 1$, it follows that $a \leqslant b$. Finally, if there is some $n \geqslant 1$, such that $a_n > 0$, then for each $m \geqslant n$, one has $a_m \geqslant a_n > 0$. Therefore, $a \geqslant a_m > 0$, for each $m \geqslant n$, and so, $a > 0$. □

Lemma D8. *There exist some a^* and b^*, such that $0 \leqslant a^* \leqslant b^* \leqslant M$ and such that for P-a.a. $s \in S$,*

$$\mathcal{Y}^*(s) = \begin{cases} 0 \textit{ iff } X(s) \in [0, a^*) \\ f(X(s)) \textit{ iff } X(s) \in [a^*, b^*] \\ X(s) \textit{ iff } X(s) \in (b^*, M] \end{cases}$$

for some nondecreasing, left-continuous and Borel-measurable function $f : [0, M] \to [0, M]$*, such that* $0 \leqslant f(t) \leqslant t$ *for each* $t \in [a^*, b^*]$.

Proof. Let $a^* := a$ and $b^* := b$, where a and b are as in Equation (D3). Then, $0 \leqslant a^* \leqslant b^* \leqslant M$. Let $E \in \Sigma$ be as in Remark D2; let $A^* \in \Sigma$ be as in Remark D3; and let $E^* := E \cap A^*$. Since $P(E) = P(A^*) = 1$, it follows form Lemma A2 that $P(E^*) = 1$. Suppose that there exists some $s_1 \in E^*$, such that $X(s_1) \in [0, a^*)$ but $\mathcal{Y}^*(s_1) > 0$. Then, for each $m \geqslant 1$ one has $\widetilde{Y}_{\lambda*,m,P}(s_1) > 0$, since the sequence $\{\widetilde{Y}_{\lambda*,m,P}\}_m$ converges monotonically downwards and pointwise on E^* to $\widetilde{Y}_{\lambda*,P}$ and $\mathcal{Y}^* \mathbf{1}_{E^*} = \widetilde{Y}^*_{\lambda*,P} \mathbf{1}_{E^*}$, by definition of $\mathcal{Y}^*$. Consequently, $X(s_1) \geqslant a_m$, for each $m \geqslant 1$. Therefore, $X(s_1) \geqslant a^* = a = \lim_{m \to +\infty} a_m$, a contradiction. Hence, for each $s \in E^*$, $X(s) \in [0, a^*) \Rightarrow \mathcal{Y}^*(s) = 0$.

Now, suppose that there exists some $s_2 \in E^*$, such that $X(s_2) \in (b, M]$, but $\mathcal{Y}^*(s_2) < X(s_2)$. Let $\varepsilon := X(s_2) - \mathcal{Y}^*(s_2)$. Since the sequence $\{\widetilde{Y}_{\lambda*,n,P}(s_2)\}_n$ converges monotonically downwards to $\widetilde{Y}_{\lambda*,P}(s_2)$, there is some $n^* \in \mathbb{N}$, such that for each $n \geqslant n^*$ one has $|\widetilde{Y}_{\lambda*,n,P}(s_2) - \widetilde{Y}_{\lambda*,P}(s_2)| = \widetilde{Y}_{\lambda*,n,P}(s_2) - \widetilde{Y}_{\lambda*,P}(s_2) < \varepsilon/2$. Fix some $n_0 \geqslant n^*$, and let $\delta := \widetilde{Y}_{\lambda*,n_0,P}(s_2) - \widetilde{Y}_{\lambda*,P}(s_2) < \varepsilon/2$. Then:

$$\begin{aligned}|X(s_2) - \widetilde{Y}_{\lambda*,n_0,P}(s_2)| = X(s_2) - \widetilde{Y}_{\lambda*,n_0,P}(s_2) &= \Big(X(s_2) - \widetilde{Y}_{\lambda*,P}(s_2)\Big) + \Big(\widetilde{Y}_{\lambda*,P}(s_2) - \widetilde{Y}_{\lambda*,n_0,P}(s_2)\Big) \\ &= \varepsilon - \delta > \varepsilon - \varepsilon/2 = \varepsilon/2 > 0.\end{aligned}$$

Therefore, $\widetilde{Y}_{\lambda*,n_0,P}(s_2) < X(s_2)$, and so $X(s_2) \leqslant b_{n_0} \leqslant b$, a contradiction. Consequently[13], for each $s \in E^*$, $X(s) \in (b, M] \Rightarrow \mathcal{Y}^*(s) = X(s)$.

Moreover, $\widetilde{Y}^*_{\lambda*,P} = \widetilde{I} \circ X$, for some bounded, nonnegative, nondecreasing, left-continuous and Borel-measurable function $\widetilde{I}$ on the range $[0, M]$ of X (see Section B). Let $f := \widetilde{I}$. One then has, for each $s \in E^*$, $\mathcal{Y}^*(s) = f(X(s))$ if $X(s) \in [a^*, b^*]$. Furthermore, since $0 \leqslant \widetilde{Y}^*_{\lambda*,P} \leqslant X$, it follows that $0 \leqslant f(t) \leqslant t$, for each $t \in [0, M]$. In particular, $f(0) = 0$. This completes the proof of Corollary 11. □

Appendix E. Sufficient Conditions for $a^* > 0$

This section gives some sufficient conditions for the a^* appearing in Corollary 11 (Equation (7) on p. 9), or Lemma D8, to be strictly positive. First, note that if there is some $n \geqslant 1$, such that $a_n > 0$ (where a_n is defined in Equation (D1)), then $a > 0$ by Lemma D7, where a is defined in equation (D3), and hence, it follows from the definition of a^* that $a^* > 0$.

Lemma E1. *There exists an event* $E^* \in \Sigma$*, such that* $P(E^*) = 1$*, and* $a^* > 0$ *whenever* $P(\mathcal{D}_{E*}) \neq 0$*, where:*

(i) $\mathcal{D}_{E*} := \left\{ s_0 \in E^* : X(s_0) > 0,\ h(s_0) > 0,\ \int_{E*} \mathcal{Y}^* \, h \, dP < \overline{L}(s_0) \right\}$*; and,*

(ii) $\overline{L}(s_0) := \int_{E*} \min\left[X, \max\left(0, X - \left[W_0 - \Pi - (u')^{-1}\left(\frac{u'(W_0 - \Pi - X(s_0))}{h(s_0)}\, h\right)\right]\right)\right] h \, dP.$

Finally, there exists $\kappa \in \mathbb{R}^+$*, such that* $a^* > 0$ *whenever* $P(\mathcal{E}_{E*}) \neq 0$*, where:*

$$\begin{aligned}\mathcal{E}_{E*} := \Big\{ s_0 \in E^* :\ & h(s_0) > 0,\ \kappa\, h(s_0) > u'(W_0 - \Pi), \\ & 0 < X(s_0) < W_0 - \Pi - (u')^{-1}(\kappa\, h(s_0)) \Big\}.\end{aligned} \tag{D1}$$

[13] Note that by definition of $\mathcal{Y}^*$, for each $s \in A$, we have $\mathcal{Y}^*(s) \leqslant X(s)$.

Proof. Let $E \in \Sigma$ be as in Remark D2; let $A^* \in \Sigma$ be as in Remark D3; and let $E^* := E \cap A^*$, as above. Then, $P(E^*) = 1$, by Lemma A2. For each $s_0 \in E^*$, define $\overline{L}(s_0)$ by:

$$\overline{L}(s_0) := \int_{E^*} \min\left[X, \max\left(0, X - \left[W_0 - \Pi - (u')^{-1}\left(\frac{u'(W_0 - \Pi - X(s_0))}{h(s_0)}\, h\right)\right]\right)\right] h\, dP.$$

Then:

$$\overline{L}(s_0) = \int_{A} \min\left[X, \max\left(0, X - \left[W_0 - \Pi - (u')^{-1}\left(\frac{u'(W_0 - \Pi - X(s_0))}{h(s_0)}\, h\right)\right]\right)\right] h\, dP.$$

Now, let:

$$\mathcal{D}_{E*} := \left\{ s_0 \in E^* : X(s_0) > 0,\ h(s_0) > 0,\ \int_{E^*} Y^*\, h\, dP < \overline{L}(s_0) \right\}.$$

Then:

$$\mathcal{D}_{E*} = \left\{ s_0 \in E^* : X(s_0) > 0,\ h(s_0) > 0,\ \int_{A} Y^*\, h\, dP < \overline{L}(s_0) \right\}.$$

Suppose that $P(\mathcal{D}_{E*}) \neq 0$. Then, in particular, $\mathcal{D}_{E*} \neq \varnothing$. Fix some $s_0 \in \mathcal{D}_{E*}$. Then, $X(s_0) > 0$, $h(s_0) > 0$ and $\int_A \mathcal{Y}^*\, h\, dP < \overline{L}(s_0)$. In other words,

$$\beta^* = \phi(\lambda^*) = \int_A \mathcal{Y}^*\, h\, dP < \phi\Big(u'(W_0 - \Pi - X(s_0)) \Big/ h(s_0)\Big) = \overline{L}(s_0).$$

Therefore, $\lambda^* \geqslant u'(W_0 - \Pi - X(s_0)) \Big/ h(s_0)$, since ϕ is a nonincreasing function. Consequently, $X(s_0) \leqslant W_0 - \Pi - \left(u'\right)^{-1}(\lambda^* h(s_0))$, and so:

$$Y^*_{\lambda*}(s_0) = \min\left[X(s_0), \max\left(0, X(s_0) - \left[W_0 - \Pi - \left(u'\right)^{-1}(\lambda^* h(s_0))\right]\right)\right] = 0.$$

Hence, for each $s_0 \in \mathcal{D}_{E*}$, one has $X(s_0) > 0$ and $Y^*_{\lambda*}(s_0) = 0$. Since $P(\mathcal{D}_{E*}) \neq 0$ by hypothesis, it follows that:

$$P\left(\left\{ s \in E^* :\ X(s) > 0,\ Y^*_{\lambda*}(s) = 0 \right\}\right) \neq 0.$$

Thus, the fact that in this case one has $a^* > 0$ follows from the properties of the equimeasurable rearrangement (recall Equation (D2) and the proof of Lemma D5).

Now, let $\kappa = \lambda^*$, and define the set $\mathcal{E}_{E*}$ as follows:

$$\mathcal{E}_{E*} := \Bigg\{ s_0 \in E^* :\ h(s_0) > 0,\ \kappa\, h(s_0) > u'(W_0 - \Pi),$$
$$0 < X(s_0) < W_0 - \Pi - \left(u'\right)^{-1}(\kappa\, h(s_0)) \Bigg\}.$$

Suppose that $P(\mathcal{E}_{E*}) \neq 0$. Then, in particular, $\mathcal{E}_{E*} \neq \varnothing$. Fix some $s_0 \in \mathcal{E}_{E*}$. Then, $h(s_0) > 0$, $\lambda^* > u'(W_0 - \Pi)/h(s_0)$, $X(s_0) > 0$ and $X(s_0) < W_0 - \Pi - \left(u'\right)^{-1}(\lambda^* h(s_0))$. Since the sequence $\{h_n\}_n$ of nonnegative, P-simple functions on (S, Σ) previously defined converges pointwise to h, one can choose n large enough, so that $h_n(s_0)$ is close enough to $h(s_0)$ and the following hold:

$$h_n(s_0) > 0, \quad \lambda^* > u'(W_0 - \Pi)/h_n(s_0), \quad \text{and} \quad 0 < X(s_0) < W_0 - \Pi - \left(u'\right)^{-1}(\lambda^* h_n(s_0)).$$

Therefore, from the proof of Lemma D1, one has $X(s_0) > 0$ and $Y^*_{\lambda*,n}(s_0) = 0$. Since $P(\mathcal{E}_{E*}) \neq 0$ by hypothesis, it follows that $P\left(\{ s \in E^* :\ X(s) > 0,\ Y^*_{\lambda*,n}(s) = 0, \text{ for some } n \geqslant 1\}\right) \neq 0$. Thus, there exists $n^* \geqslant 1$, such that $P\left(\left\{s \in E^* : \quad X(s) > 0,\ Y^*_{\lambda^*,n^*}(s) = 0\right\}\right) \neq 0$. For such n^*, one has $a_{n^*} > 0$ by properties of the equimeasurable rearrangement (as in the proof of Lemma D5) and by definition of the function $\widetilde{Y}^*_{\lambda^*,n^*,P}$ given in Equation (D1). This then yields $a > 0$ (by Lemma D7), and so, $a^* > 0$. $\square$

Appendix F. Sufficient Conditions for $b^* < M$

This section gives some sufficient conditions for the b^* appearing in Corollary 11 (Equation (7) on p. 9), or Lemma D8, to be strictly less than M.

Lemma F1. *Let b^* and E^* be defined as in the proof of Lemma D8. If P is not absolutely continuous with respect to Q and if there exists some $s_0 \in E^*$, such that $X(s_0) = M$, then $b^* < M$.*

Proof. Suppose that there exists some $s_0 \in E^*$, such that $X(s_0) = M$. Suppose also that P is not absolutely continuous with respect to Q. Then, $b_n < M$, for each $n \geqslant 1$, by Lemmata D3 and D4. To show that, in this case, $b^* < M$, suppose, by way of contradiction, that $b^* = M$. Then, in particular, $(b^*, M] = \varnothing$, and hence, $\widetilde{Y}^*_{\lambda^*,P}(s_0) < X(s_0)$ and $b_n < M = X(s_0) = b^*$, for each $n \geqslant 1$. Let $\varepsilon := X(s_0) - \widetilde{Y}^*_{\lambda^*,P}(s_0) > 0$. Since the sequence $\{\widetilde{Y}_{\lambda^*,n,P}(s_0)\}_n$ converges monotonically downwards to $\widetilde{Y}_{\lambda^*,P}(s_0)$, there is some $n_0 \in \mathbb{N}$, such that for each $n \geqslant n_0$, one has $|\widetilde{Y}_{\lambda^*,n,P}(s_0) - \widetilde{Y}_{\lambda^*,P}(s_0)| = \widetilde{Y}_{\lambda^*,n,P}(s_0) - \widetilde{Y}_{\lambda^*,P}(s_0) < \varepsilon/2$. Let $\delta_n := \widetilde{Y}_{\lambda^*,n,P}(s_0) - \widetilde{Y}_{\lambda^*,P}(s_0)$. Then, $\delta_n < \varepsilon/2$, for each $n \geqslant n_0$. Therefore, for each $n \geqslant n_0$, one has:

$$\begin{aligned} |X(s_0) - \widetilde{Y}_{\lambda^*,n,P}(s_0)| &= X(s_0) - \widetilde{Y}_{\lambda^*,n,P}(s_0) = \left(X(s_0) - \widetilde{Y}_{\lambda^*,P}(s_0)\right) + \left(\widetilde{Y}_{\lambda^*,P}(s_0) - \widetilde{Y}_{\lambda^*,n,P}(s_0)\right) \\ &= \varepsilon - \delta_n > \varepsilon - \varepsilon/2 = \varepsilon/2 > 0. \end{aligned}$$

Thus, $X(s_0) > \widetilde{Y}_{\lambda^*,n,P}(s_0)$, and hence, $X(s_0) \leqslant b_n$. Consequently, $X(s_0) < M$, a contradiction. Hence, $b^* < M$. $\square$

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