risks

MDPI

Article

Precise Large Deviations for Subexponential Distributions in a Multi Risk Model

Dimitrios G. Konstantinides

Department of Mathematics, University of the Aegean, Karlovassi, GR-83 200 Samos, Greece; konstant@aegean.gr

Received: 6 February 2018; Accepted: 24 March 2018; Published: 27 March 2018

check for updates

Abstract: The precise large deviations asymptotics for the sums of independent identical random variables when the distribution of the summand belongs to the class S^* of heavy tailed distributions is studied. Under mild conditions, we extend the previous results from the paper Denisov et al. (2010) to asymptotics that are valid uniformly over some time interval. Finally, we apply the main result on the multi-risk model introduced by Wang and Wang (2007).

Keywords: heavy tails; large deviations; multi-risk model

JEL Classification: 60F10; 60F05; 60G50

---

1. Introduction

In this paper, the precise large deviations for a random walk whose steps represent random variables with distribution F from a subclass $\mathcal{S}^*$ of the subexponential class $\mathcal{S}$ is studied. What that means is F has heavy tail and is regular enough in order to exist the limit

$$\lim_{x\to\infty} \frac{1}{\overline{F}(x)} \int_0^x \overline{F}(x-y)\,\overline{F}(y)\,dy\,.$$

Hence, the most popular distributions with heavy tails belong to the class $\mathcal{S}^*$, among others Pareto, Burr, Cauchy, Lognormal and Weibull. The inclusion of the class $\mathcal{S}^*$ in the class of subexponential distribution is proved proper, namely have been found subexponential distributions that do not belong to $\mathcal{S}^*$ (see Denisov et al. (2004)).

The topic of large deviations of non-random sums has already been well studied. Overviews are given in Nagaev (1973), Cline and Hsing (1991), Nagaev (1979). More insight in the field for non-random sums can be found in Heyde (1967a), Heyde (1967b), Heyde (1968), Nagaev (1969a), Nagaev (1969b), Wang et al. (2006). A general treatment of large deviations for subexponential distributions was presented in Pinelis (1985). Other contributions on precise large deviations are Konstantinides and Loukissas (2010), Loukissas (2012), Wang and Wang (2012). Papers Klüppelberg and Mikosch (1997), Tang et al. (2001) are studying large deviations of random sums. A review on large deviations for random sums is given in Mikosch and Nagaev (1998) and Mikosch and Nagaev (2001).

In case of independent r.v.'s, the large deviation has been established in Paulauskas and Skučaitė (2003) and Skučaitė (2004). Due to its importance in applications (see for example Issaka and SenGupta (2017), Habtemicael and SenGupta (2014)) this issue became very popular recently. Recent contributions in the topic are found in Baltrunas et al. (2004), Borovkov and Mogulskii (2006), Hult et al. (2005), Jelenkovic and Momcilovic (2004), Konstantinides and Mikosc (2005), Ng et al. (2004), Tang (2006), Chen et al. (2011), Yang et al. (2012), Gao et al. (2018), Zhang and Cheng (2017), Yang and Sha (2016).

Let us denote by S_n the sum of n independent identically distributed random variables $X_1 + \cdots + X_n$, with common distribution F. In the principle of one big jump, we get the intuition that indicates in the case of heavy tails, the most probable way that the event $\{S_n > x\}$ happens. Namely, only one of the random variables $X_1, \ldots, X_n$ becomes large while the others remain small. Asymptotically, as $x \to \infty$, we get $\mathbf{P}\left[S_n > x\right] \sim n\,\overline{F}(x)$, where by $\overline{F}$ is denoted the tail of the distribution F.

The multi-risk model was firstly introduced in Wang and Wang (2007) and has arisen from the following construction: Let $\{X_{i,j},\ j \geq 1\}$, $i = 1, 2, \ldots, k$ be i.i.d. non-negative random variables with common distribution function $F_i(x)$ and finite mean. Taking into account the notations

$$S_{n_i} = \sum_{j=1}^{n_i} X_{i,j}, \quad i = 1, 2, \ldots, k\,,$$

and

$$S(n_1, n_2, \ldots, n_k) = \sum_{i=1}^{k} S_{n_i} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} X_{ij}\,,$$

found in Wang and Wang (2007), we formulate the following result:

Let $\{X_{i,j},\ j \geq 1\}$ be i.i.d. non-negative random variables with common distribution function $F_i(x)$ and finite mean μ_i for any $i = 1, \ldots, k$ and let n_i, for any $i = 1, \ldots, k$ be a sequence of integers. Let us assume that $\{X_{i,j}, j \geq 1\}$, $i = 1, \ldots, k$ are mutually independent. If the distributions F_i are consistently varying ($F_i \in \mathcal{C}$, for the definition see below) for any $i = 1, \ldots, k$ then for any $\gamma > 0$

$$\mathbf{P}\left[S(n_1, n_2, \ldots, n_k) - \sum_{i=1}^{k} n_i\,\mu_i > x\right] \sim \sum_{i=1}^{k} n_i\,\overline{F}_i(x)\,,$$

holds, as $n_i \to \infty$, for any $i = 1, \ldots, k$, uniformly for all $x \geq \max\{\gamma n_1, \gamma n_2, \ldots, \gamma n_k\}$.

2. Preliminary Concepts

In this paper some sequence $\{X_n,\ n \geq 1\}$ of i.i.d. r.v.'s is considered, which represent claims in a risk model with common distribution function F and finite mean μ. Let us suppose that this sequence is independent from the integer counting process $\{N(t),\ t \geq 0\}$, representing the claim arrival process and denote by $\lambda(t) = EN(t) < \infty$ its mean value for any $0 \leq t < \infty$. We assume that $\lambda(t) \to \infty$, as $t \to \infty$. All limit relationships, unless otherwise stated, are for $n \to \infty$ or $t \to \infty$.

Let us call a distribution function F as heavy-tailed distribution, if it has no exponential moments, that means $E\left[e^{\epsilon X}\right] = \infty,\ \ \forall\,\epsilon > 0$. Next, we recall some useful facts from the following subclasses of heavy tailed distributions:

A distribution function F with support on $[0,\infty)$ belongs to $\mathcal{C}$ if the following asymptotic relation holds

$$\mathcal{C} = \left\{F\ :\ \lim_{y\searrow 1}\liminf_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)} = 1\right\}$$

or equivalently

$$\mathcal{C} = \left\{F\ :\ \lim_{y\nearrow 1}\limsup_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)} = 1\right\}.$$

For such a distribution function F, it is said to have a consistently varying tail.

A distribution function F with support on $[0,\infty)$ belongs to $\mathcal{S}$, if the following asymptotic formulas are valid

$$\mathcal{S} = \left\{F\ :\ \lim_{x\to\infty}\frac{\overline{F^{n*}}(x)}{\overline{F}(x)} = n\,,\quad \forall \quad n \geq 2\right\}.$$

For such a distribution function F, it is said to have a subexponential tail.

Let us denote by $\mathcal{S}^*$ the subclass of the subexponential distributions, which contains distributions with finite mean μ and the next limit exists

$$\mathcal{S}^* = \left\{ F \,:\, \lim_{x\to\infty} \int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)} \overline{F}(y) = 2\mu \right\}.$$

This class was firstly introduced by Klüppelberg (1988).

A distribution function F with support on $[0,\infty)$ belongs to $\mathcal{L}$ if the following asymptotic holds

$$\mathcal{L} = \left\{ F \,:\, \lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1\,, \quad \forall \quad y > 0 \right\}.$$

In this case the distribution function F is said to have long tail. For any long tail distribution there exist an non-decreasing function $h(x)$ such that $h(x) \to \infty$ as $x \to \infty$ and the following asymptotic relation holds

$$\overline{F}\,[x \pm h(x)] \sim \overline{F}(x)\,. \tag{1}$$

as $x \to \infty$ (see for example (Konstantinides 2017, Lemma 8.1)). It is well known the inclusions

$$\mathcal{C} \subseteq \mathcal{S}^* \subseteq \mathcal{S} \subseteq \mathcal{L}.$$

Remark 1. *In (Foss et al. 2011, Lemma 2.19) was established the following assertion: For any long tailed distribution F we can find an increasing function $l(x) < h(x)$ such that $l(x) \to \infty$, as $x \to \infty$, for which the following holds*

$$\overline{F}\,[x \pm l(x)] \sim \overline{F}(x)\,, \tag{2}$$

as $x \to \infty$. Let us denote by $l^{\leftarrow}$ the inverse function of l, which represents an increasing function and the limit relation $l^{\leftarrow}(x) \to \infty$, as $x \to \infty$, holds.

In case of distribution function with regularly varying tail, when $\lim_{x\to\infty} \overline{F}(t\,x)/\overline{F}(x) = t^{-\alpha}$, for some $\alpha > 0$, a possible choice of the function l is $l(x) = o(x)$. From (Denisov et al. 2008, Section 8) we obtain that for distribution with zero mean and finite variance, is possible the choice $l(x) = \sqrt{x}$. Further information related with heavy-tailed distributions can be found in Embrecht et al. (1997), Borovkov and Borovkov (2008), Foss et al. (2011), Konstantinides (2017).

Let us remind the following notations: for two positive functions $a(\cdot)$ and $b(\cdot)$ we write

$$a(x) \lesssim b(x) \qquad if \qquad \limsup_{x\to\infty} \frac{a(x)}{b(x)} \leq 1\,,$$

$$a(x) \gtrsim b(x) \qquad if \qquad \liminf_{x\to\infty} \frac{a(x)}{b(x)} \geq 1\,,$$

$$a(x) \sim b(x) \qquad if \qquad \lim_{x\to\infty} \frac{a(x)}{b(x)} = 1.$$

In the large deviations set-up, the asymptotic relation has the form:

$$\mathbf{P}\,[S_n > x] \sim n\,\overline{F}(x)\,,$$

and in the precise large deviations the corresponding asymptotic relation is the following:

$$\mathbf{P}\,[S_n - n\mu > x] \sim n\,\overline{F}(x)\,.$$

Both relations, hold uniformly for any $x \geq \alpha_n$ where α_n represents some non-negative sequence that tends to infinity.

3. Main result

The crucial step in our approach, comes from the following result by (Deniso et al. 2010, Theorem 5). For the sake of convenience we refer its short proof.

Theorem 1. *Let $\{X_k,\ k \geq 1\}$ be a non-negative independent and identically distributed sequence of random variables following the common distribution function $F \in \mathcal{S}^*$ with finite mean $\mu > 0$. Then the following asymptotic relation holds*

$$\mathbf{P}\left[S_n > x\right] \sim n\,\overline{F}(x)\,, \tag{3}$$

as $n \to \infty$, uniformly for any $x \geq l^{\leftarrow}(n)$.

Proof. The uniformity in (3) is understood in the following sense

$$\lim_{x\to\infty} \sup_{n \leq l(x)} \left| \frac{\mathbf{P}\left[S_n > x\right]}{n\,\overline{F}(x)} - 1 \right| = 0\,.$$

that means $\forall \epsilon > 0$ and $\forall n \leq l(x)$ there exists some $x(\epsilon) < \infty$ such that, if $x > x(\epsilon)$ the inequality

$$\left| \frac{\mathbf{P}\left[S_n > x]\right)}{n\,\overline{F}(x)} - 1 \right| < \epsilon\,,$$

holds, or equivalently $\forall \epsilon > 0$ and $\forall x \geq l^{\leftarrow}(n)$ there exists some $x'(\epsilon) = \lfloor x(\epsilon) \rfloor$, where $\lfloor \alpha \rfloor$ denotes the integer part of α, such that for any $n = \lfloor x \rfloor > x'(\epsilon)$, the inequality

$$\left| \frac{\mathbf{P}\left[S_n > x\right]}{n\,\overline{F}(x)} - 1 \right| < \epsilon$$

holds and so by the arbitrariness of the choice of $\varepsilon > 0$ we find

$$\lim_{n\to\infty} \sup_{x \geq l^{\leftarrow}(n)} \left| \frac{\mathbf{P}\left[S_n > x\right]}{n\,\overline{F}(x)} - 1 \right| = 0\,.$$

□

Now, we can examine the asymptotic relation of precise large deviations for a distribution $F \in \mathcal{S}^*$.

Theorem 2. *Let $\{X_k,\ k \geq 1\}$ be a non-negative, independent and identically distributed sequence of random variables with common distribution function $F \in \mathcal{S}^*$ with finite mean $\mu > 1$. Then holds*

$$\mathbf{P}\left[S_n - n\,\mu > x\right] \sim n\,\overline{F}(x)\,, \tag{4}$$

uniformly for $x \geq l^{\leftarrow}(n\,\mu)$.

Proof. From relation (3) we get

$$\mathbf{P}\left[S_n - n\,\mu > x\right] < \mathbf{P}\left[S_n > x\right] \sim n\,\overline{F}(x)\,.$$

On the other hand for $x \geq l^{\leftarrow}(n\,\mu)$ and from relations (3) and (1) we obtain

$$\mathbf{P}\left[S_n - n\,\mu > x\right) \sim n\,\overline{F}(x + n\,\mu] > n\,\overline{F}[x + l(x)] \sim n\,\overline{F}(x)\,.$$

□

The precise large deviations refers to a random walk of the type

$$S_{N(t)} = \sum_{k=1}^{N(t)} X_k, \quad t > 0 \tag{5}$$

where the asymptotic relation of precise large deviations is formulated as

$$\mathbf{P}[S_{N(t)} - \mu\,\lambda(t) > x] \sim \lambda(t)\,\overline{F}(x)\,, \tag{6}$$

as $t \to \infty$, uniformly for $x \geq f(t)$, with $f(t)$ representing a non-negative function, that tends to infinity.

Therefore, Let us consider the asymptotic relation (6) for random sums, when $F \in S^*$ under the following conditions on $N(t)$:

Assumption $\mathbf{N_1}$:

$$\frac{N(t)}{\lambda(t)} \xrightarrow{P} 1.$$

Assumption $\mathbf{N_2}$: For any $\epsilon > 0$ and for any $\delta > 0$ the following asymptotic relation holds

$$\sum_{k>(1+\delta)\lambda(t)} (1+\epsilon)^k\,\mathbf{P}[N(t) = k] = o(1)\,.$$

Remark 2. *From Klüppelberg (1988) we see that Assumption N_1 permits the following equivalent formulation. There exists some positive function $\epsilon(t)$, with $\epsilon(t) \to 0$, such that*

$$\mathbf{P}\left[|N(t) - \lambda(t)| \leq \epsilon(t)\,\lambda(t)\right] = o(1)\,.$$

Theorem 3. *Let $\{X_k,\ k \geq 1\}$ be a non-negative, independent and identically distributed sequence of random variables with common distribution function $F \in \mathcal{S}^*$ with finite mean $\mu > 1$. Let $\{N(t),\ t \geq 0\}$ be a non-negative and integer valued counting process. We assume that $\{X_k,\ k \geq 1\}$ and $\{N(t),\ t \geq 0\}$ are mutually independent. If $N(t)$ satisfies both assumptions N_1 and N_2 then the following asymptotic relation holds*

$$\mathbf{P}[S_{N(t)} - \mu\lambda(t) > x] \sim \lambda(t)\,\overline{F}(x)\,, \tag{7}$$

as $t \to \infty$, uniformly for any $x \geq l^{\leftarrow}[\mu\lambda(t)]$.

Proof. Let us use the decomposition, proposed in the proof of (Klüppelberg and Mikosch 1997, Theorem 3.1). We can state

$$\mathbf{P}[S_{N(t)} - \mu\lambda(t) > x] = \sum_{k=1}^{\infty} \mathbf{P}[S_k > x + \mu\lambda(t)]\,\mathbf{P}[N(t) = k]\,.$$

Further, let us split the sum in three parts

$$= \sum_{k<(1-\delta)\lambda(t)} + \sum_{(1-\delta)\lambda(t)\leq k\leq(1+\delta)\lambda(t)} + \sum_{k>(1+\delta)\lambda(t)} := I_1 + I_2 + I_3. \tag{8}$$

Now, we can see that

$$\begin{aligned} I_1 &= \sum_{k<(1-\delta)\lambda(t)} \mathbf{P}[S_k > x + \mu\lambda(t)]\,\mathbf{P}[N(t) = k] \\ &\leq \mathbf{P}\left[S_{(1-\delta)\lambda(t)} > x + \mu\lambda(t)\right] \sum_{k<(1-\delta)\lambda(t)} P[N(t) = k]\,. \end{aligned}$$

Let us observe that $l[x + \mu\,\lambda(t)] \geq l(x) \geq \mu\,\lambda(t) \geq (1-\delta)\,\lambda(t)$, hence using relation (3) and (2) we obtain

$$\begin{aligned} I_1 &\lesssim (1-\delta)\,\lambda(t)\,\overline{F}[x+\mu\lambda(t)]\,\mathbf{P}[N(t) < (1-\delta)\lambda(t] \\ &\lesssim (1-\delta)\,\lambda(t)\,\overline{F}(x)\,\mathbf{P}\left[N(t) - \lambda(t) < -\delta\lambda(t)\right]\,, \end{aligned}$$

so by Assumption N_1 and taking into account the Remark 2 it follows

$$I_1 = o[\lambda(t)\overline{F}(x)]\,, \tag{9}$$

as $x \geq l^{\leftarrow}(\mu\lambda(t))$.

Next, we deal with term I_2. Let us write

$$\begin{aligned} I_2 &= \sum_{(1-\delta)\lambda(t)\leq k\leq(1+\delta)\lambda(t)} \mathbf{P}[S_k > x+\mu\lambda(t)]\,\mathbf{P}[N(t)=k] \\ &\geq \mathbf{P}[S_{(1-\delta)\lambda(t)} > x+\mu\lambda(t)]\,P\left[1-\delta \leq \frac{N(t)}{\lambda(t)} \leq 1+\delta\right]\,, \end{aligned}$$

and by Assumption N_1 follows

$$I_2 \gtrsim \mathbf{P}\left[S_{(1-\delta)\lambda(t)} > x+\mu\lambda(t)\right]\,,$$

since holds the inequality $x \geq l^{\leftarrow}(\mu\lambda(t))$. Further by relation (3) we find

$$I_2 \gtrsim (1-\delta)\,\lambda(t)\,\overline{F}[x+\mu\lambda(t)] > (1-\delta)\,\lambda(t)\,\overline{F}[x+l(x)]\,.$$

Thus, relation (1) implies

$$I_2 \gtrsim (1-\delta)\,\lambda(t)\,\overline{F}(x)\,.$$

Similarly we find the upper bound

$$\begin{aligned} I_2 &= \sum_{(1+\delta)\lambda(t)\leq k\leq(1+\delta)\lambda(t)} \mathbf{P}[S_k > x]\,\mathbf{P}[N(t)=k] \\ &\leq \mathbf{P}\left[S_{(1+\delta)\lambda(t)} > x+\mu\lambda(t)\right]\,\mathbf{P}\left[1-\delta \leq \frac{N(t)}{\lambda(t)} \leq 1+\delta\right]\,, \end{aligned}$$

and Assumption N_1 implies

$$I_2 \lesssim \mathbf{P}\left[S_{(1+\delta)\lambda(t)} > x+\mu\lambda(t)\right]\,,$$

as $t \to \infty$. From relations (3) and (2) and for small enough $\delta > 0$ we obtain the inequality $l[x + \mu\,\lambda(t)] \geq (1+\delta)\,\lambda(t)$ and hence we find

$$I_2 \lesssim (1+\delta)\,\lambda(t)\,\overline{F}(x)\,.$$

Letting $\delta \to 0$ we get

$$I_2 \sim \lambda(t)\,\overline{F}(x)\,, \tag{10}$$

as $t \to \infty$, for any $x \geq l^{\leftarrow}[\mu\,\lambda(t)]$.

At last, on

$$I_3 = \sum_{k>(1+\delta)\lambda(t)} \mathbf{P}[S_k > x + \mu\lambda(t)]\, \mathbf{P}[N(t) = k]\,,$$

we apply Kesten's inequality (see for example (Konstantinides 2017, Theorem 6.2)) for $F \in \mathcal{S}$. For any $\epsilon > 0$ there exists some constant $K = K(\epsilon)$ such that

$$\mathbf{P}[S_n > x] \leq K\,(1+\epsilon)^n\, \overline{F}(x)\,,$$

hence, we obtain

$$I_3 \leq \sum_{k>(1+\delta)\lambda(t)} K\,(1+\epsilon)^n\, \overline{F}(x)\, \mathbf{P}[N(t) = k]\,,$$

and by Assumption N_2 we get

$$I_3 \sim o\left[\lambda(t)\overline{F}(x)\right] \tag{11}$$

as $t \to \infty$, uniformly for any $x \geq l^{\leftarrow}(\mu\lambda(t))$.

Putting relations (9)–(11) in (8) we find the final result. □

Remark 3. *Lemma 2.1 from Klüppelberg and Mikosch (1997) implies that the Poisson counting process satisfies both Assumptions* N_1 *and* N_2*. Also, the renewal counting process satisfies both Assumptions* N_1 *and* N_2 *under certain condition.*

4. The Multi-Risk Model

Recent works on the multi-risk model are found in Wang and Wang (2013), Liu (2010) and Wang et al. (2014). We examine these results for the case $X_i \in S^*$.

Theorem 4. *Let* $\{X_{i,j}, j \geq 1\}$ *be i.i.d., non-negative random variables with common distribution function* $F_i(x)$ *and finite mean* μ_i*, for* $i = 1, 2, \ldots, k$*. Let* n_i*,* $i = 1, 2, \ldots, k$ *be a positive integer sequence and assume* $\{X_{i,j}, j \geq 1\}$*,* $i = 1, 2, \ldots, k$ *are independent of* $\{n_i\}$*. If* $F_i \in S^*$ *for any* $i = 1, 2, \ldots, k$ *then the asymptotic relation*

$$\mathbf{P}\left[S(n_1, n_2, \ldots, n_k) - \sum_{i=1}^{k} n_i\mu_i\right] \sim \sum_{i=1}^{k} n_i \overline{F}_i(x) \tag{12}$$

holds, as $n_i \to \infty$*, for* $i = 1, 2, \ldots, k$*, uniformly for all* x *satisfying the inequality* $x \geq \max\{l_1^{\leftarrow}(n_1\mu_1), l_2^{\leftarrow}(n_2\mu_2), \ldots, l_k^{\leftarrow}(n_k\mu_k)\}$.

Proof. Let us employ mathematical induction: In case $k = 2$ we can see:

$$\begin{aligned}
\mathbf{P}\left[S(n_1, n_2) - n_1\mu_1 - n_2\mu_2 > x\right] &= \mathbf{P}\left[S_{n_1} + S_{n_2} - n_1\mu_1 - n_2\mu_2 > x\right] \\
&\geq \mathbf{P}[\{S_{n_1} - n_1\mu_1 > x + l_1(x),\; S_{n_2} - n_2\mu_2 > -l_1(x)\} \\
&\quad \cup \{S_{n_2} - n_2\mu_2 > x + l_2(x),\; S_{n_1} - n_1\mu_1 > -l_2(x)\}] \\
&\geq \mathbf{P}[S_{n_1} - n_1\mu_1 > x + l_1(x)]\, \mathbf{P}[S_{n_2} - n_2\mu_2 > -l_1(x)] \\
&\quad + \mathbf{P}[S_{n_2} - n_2\mu_2 > x + l_2(x)]\, \mathbf{P}[S_{n_1} - n_1\mu_1 > -l_2(x)] \\
&\quad - \mathbf{P}[S_{n_1} - n_1\mu_1 > x + l_1(x)]\, \mathbf{P}[S_{n_2} - n_2\mu_2 > x + l_2(x)]\,,
\end{aligned}$$

and relations (4) and (1) imply the asymptotic relations

$$\mathbf{P}[S_{n_1} - n_1\mu_1 > x + l_1(x)] \sim n_1\,\overline{F}_1(x)\,,$$

and

$$\mathbf{P}\,[S_{n_2} - n_2\mu_2 > x + l_2(x)] \sim n_2\,\overline{F}_2(x)\,.$$

Further, through the strong law of large numbers we find

$$\mathbf{P}[S_{n_1} - n_1\,\mu_1 > -l_2(x)] = 1\,,$$

and

$$\mathbf{P}[S_{n_2} - n_2\mu_2 > -l_1(x)] = 1\,,$$

so we obtain the asymptotic inequalities

$$\begin{aligned} \mathbf{P}[S(n_1,\,n_2) > x] &\gtrsim n_1\,\overline{F}_1(x) + n_2\,\overline{F}_2(x) - n_1\,\overline{F}_1(x)\,n_2\,\overline{F}_2(x) \\ &\gtrsim n_1\,\overline{F}_1(x) + n_2\,\overline{F}_2(x) - o[n_1\,\overline{F}_1(x) + n_2\,\overline{F}_2(x)]\,, \end{aligned} \tag{13}$$

as $n \to \infty$, uniformly for all $x \geq \max\{l_1^{\leftarrow}(n_1\mu_1),\, l_2^{\leftarrow}(n_2\mu_2)\}$.

On the other hand for any fixed $0 < \epsilon < 1/2$ we find

$$\begin{aligned} &\mathbf{P}\,[S(n_1,n_2) - n_1\mu_1 - n_2\mu_2 > x] \leq \mathbf{P}\,[S_{n_1} + S_{n_2} > x] \\ &\leq \mathbf{P}\,[\{S_{n_1} > a_1\} \cup \{S_{n_2} > a_2\} \cup \{S_{n_1} > \epsilon x - l_1(\epsilon x), S_{n_2} > \epsilon x - l_2(\epsilon x)\}] \\ &\leq \mathbf{P}[S_{n_1} > a_1] + \mathbf{P}[S_{n_2} > a_2] + \mathbf{P}[S_{n_1} > \epsilon x - l_1(\epsilon x)]\,\mathbf{P}[S_{n_2} > \epsilon x - l_2(\epsilon x)]\,, \end{aligned}$$

with $a_1 := x - l_1(x),\ a_2 := x - l_2(x)$, and from relations (3) and $F_i \in \mathcal{S}^* \subsetneq \mathcal{L},\ i = 1, 2$ we see that the distribution of the sum $S(n_1, n_2)$ belongs to $\mathcal{L}$. Therefore we have

$$\begin{aligned} &\mathbf{P}[S(n_1,n_2) > x] \sim \mathbf{P}[S(n_1,n_2) > x + n_1\mu_1 + n_2\mu_2] \lesssim n_1\,\mathbf{P}[X_1 > x - l_1(x)] \\ &+ n_2\,\mathbf{P}[X_2 > x - l_2(x)] + n_1\,\mathbf{P}[X_1 > \epsilon x - l_1(\epsilon x)]\,n_2\,\mathbf{P}[X_2 > \epsilon x - l_2(\epsilon x)] \\ &\lesssim n_1\,\overline{F}_1[x - l_1(x)] + n_2\,\overline{F}_2[x - l_2(x)] + n_1\,\overline{F}_1[\epsilon x - l_1(\epsilon x)]\,n_2\,\overline{F}_2[\epsilon x - l_2(\epsilon x)] \\ &\lesssim n_1\,\overline{F}_1(x) + n_2\,\overline{F}_2(x) + n_1\,\overline{F}_1(\epsilon x)\,n_2\,\overline{F}_2(\epsilon x)\,. \end{aligned}$$

But now, we can note that the following equality

$$\frac{n_1\,\overline{F}_1(\epsilon x)\,n_2\,\overline{F}_2(\epsilon x)}{n_1\,\overline{F}_1(x) + n_2\,\overline{F}_2(x)} = \frac{1}{\frac{1}{n_1\overline{F}_1(\epsilon x)} \cdot \frac{\overline{F}_2(x)}{\overline{F}_2(\epsilon x)} + \frac{1}{n_2\overline{F}_2(x)} \cdot \frac{\overline{F}_1(x)}{\overline{F}_1(\epsilon x)}}$$

holds, since

$$\frac{\overline{F}_i(x)}{\overline{F}_i(\epsilon x)} \leq \left(\frac{\overline{F}_i[x + l_i(x)]}{\overline{F}_i(x)}\right)^{-1} \sim 1\,,$$

and $n_i\overline{F}_i(\epsilon x) \to 0$ as $n_i \to \infty$ $i = 1, 2$, for all $x \geq \max\{l_1^{\leftarrow}(n_1\mu_1),\, l_2^{\leftarrow}(n_2\mu_2)\}$. Hence, we conclude

$$\mathbf{P}[S(n_1, n_2) - n_1\mu_1 - n_2\mu_2 > x]$$
$$\lesssim n_1\,\overline{F}_1(x) + n_2\,\overline{F}_2(x) + o[n_1\,\overline{F}_1(x) + n_2\,\overline{F}_2(x)]\,, \tag{14}$$

uniformly for all $x \geq \max\{l_1^{\leftarrow}(n_1\mu_1),\, l_2^{\leftarrow}(n_2\mu_2)\}$. Relations (13) and (14) show that (12) holds for $k = 2$.

Next, wet suppose that (12) holds for some $k - 1$ and we show that then it is valid for k too.

$$\mathbf{P}\left[S(n_1, n_2, ..., n_k) - \sum_{i=1}^{k} n_i\mu_i > x\right]$$
$$\geq \mathbf{P}\left[S_{n_1} + S_{n_2} + ... + S_{n_{k-1}} - \sum_{i=1}^{k-1} n_i\mu_i > x + \min_{1\leq i\leq k-1}\{l_i(x)\}\right]$$
$$\mathbf{P}\left[S_{n_k} - n_k\mu_k > -\min_{1\leq i\leq k-1}\{l_i(x)\}\right]$$
$$+\mathbf{P}\left[S_{n_1} + S_{n_2} + ... + S_{n_{k-1}} - \sum_{i=1}^{k-1} n_i\mu_i > x - l_k(x)\right]\,\mathbf{P}\left[S_{n_k} - n_k\mu_k > x + l_k(x)\right]$$
$$-\mathbf{P}\left[S_{n_1} + S_{n_2} + ... + S_{n_{k-1}} - \sum_{i=1}^{k-1} n_i\mu_i > x + \min_{1\leq i\leq k-1}\{l_i(x)\}\right]$$
$$\times\,\mathbf{P}\left[S_{n_k} - n_k\mu_k > x + l_k(k)\right]\,.$$

From the induction assumption and relations (4) and (1) follow the asymptotic relations

$$\mathbf{P}\left[S_{n_1} + S_{n_2} + ... + S_{n_{k-1}} - \sum_{i=1}^{k-1} n_i\mu_i > x + \min_{1\leq i\leq k-1}\{l_i(x)\}\right] \sim$$
$$\sum_{i=1}^{k-1} n_i\,\overline{F}_i\left[x + \min_{1\leq i\leq k-1}\{l_i(x)\}\right] \gtrsim \sum_{i=1}^{k-1} n_i\,\overline{F}_i[x + l_i(x)] \sim \sum_{i=1}^{k-1} n_i\overline{F}_i(x)\,,$$

and

$$\mathbf{P}\left[S_{n_k} - n_k\mu_k > x + l_k(x)\right] \sim n_k\overline{F}_k(x)\,.$$

Hence, the following asymptotic inequality

$$\mathbf{P}\left[S(n_1,\, n_2,\, ...,\, n_k) - \sum_{i=1}^{k} n_i\mu_i > x\right] \gtrsim \sum_{i=1}^{k} n_i\overline{F}_i(x) - o\left(\sum_{i=1}^{k} n_i\overline{F}_i(x)\right)\,, \tag{15}$$

holds, uniformly for any $x \geq \max\{l_1^{\leftarrow}(n_1\mu_1),\, l_2^{\leftarrow}(n_2\mu_2),\, ...,\, l_k{}^{\leftarrow}(n_k\mu_k)\}$.

On the other hand for any $0 < \epsilon < \frac{1}{2}$ we obtain

$$\mathbf{P}\left[S(n_1, n_2, ..., n_k) - \sum_{i=1}^{k} n_i\mu_i > x\right] \leq \mathbf{P}\left[S_{n_1} + S_{n_2} + ... + S_{n_k} > x\right]$$

$$\leq \mathbf{P}\left[S_{n_1} + S_{n_2} + ... + S_{n_{k-1}} > x - \max_{1\leq i\leq k-1}\{l_i(x)\}\right] + \mathbf{P}\left[S_{n_k} > x - l_k(x)\right]$$

$$+\mathbf{P}\left[S_{n_1} + S_{n_2} + ... + S_{n_{k-1}} > \epsilon x - \max_{1\leq i\leq k-1}\{l_i(\epsilon x)\}\right] \mathbf{P}\left[S_{n_k} > \epsilon x - l_k(\epsilon x)\right].$$

Now, by the induction assumption and through relations (3) and (1) we get

$$\mathbf{P}\left(S_{n_1} + S_{n_2} + ... + S_{n_{k-1}} > x - \max_{1\leq i\leq k-1}\{l_i(x)\}\right)$$

$$\sim \sum_{i=1}^{k-1} n_i\overline{F}_i\left(x - \max_{1\leq i\leq k-1}\{l_i(x)\}\right) \gtrsim \sum_{i=1}^{k-1} n_i\,\overline{F}_i[x - l_i(x)] \sim \sum_{i=1}^{k-1} n_i\,\overline{F}_i(x)\,.$$

Thus we find the inverse asymptotic relation

$$\mathbf{P}[S(n_1, n_2, ..., n_k) > x] \quad \lesssim \quad \sum_{i=1}^{k} n_i\overline{F}_i(x) + o\left(\sum_{i=1}^{k} n_i\overline{F}_i(x)\right), \tag{16}$$

as $n_i \to \infty$, for $i = 1, 2, \ldots, k$ uniformly for $x \geq \max\{l_1^{\leftarrow}(n_1\mu_1), l_2^{\leftarrow}(n_2\mu_2), ..., l_k^{\leftarrow}(n_k\mu_k)\}$. □

Further, we show the corresponding asymptotic relation for the case of random sums. Let us recall the notation

$$S_{N_i(t)} = \sum_{j=1}^{N_i(t)} X_{ij}, \quad i = 1, 2, 3, ..., k\,,$$

and

$$S(k, t) = \sum_{i=1}^{k} S_{N_i(t)}.$$

Corollary 1. *Let $\{X_{i,j}, j \geq 1\}$, $i = 1, 2, \ldots, k$ be i.i.d. non-negative random variables with common distribution function $F_i(x)$ and finite mean μ_i. Let $N_i(t)$ be a sequence of stochastic processes and assume that $\{X_{i,j},\ j \geq 1\}$ and $N_i(t)$ $i = 1, 2, \ldots, k$ are mutually independent. If $F_i \in S^*$ and $N_i(t)$ satisfy the assumptions N_1 and N_2 for all $i = 1, 2, \ldots, k$ then holds*

$$\mathbf{P}\left[S(k, t) - \sum_{i=1}^{k} \mu_i\lambda_i(t)\right] \sim \sum_{i=1}^{k} \lambda_i(t)\overline{F}_i(x) \tag{17}$$

as $t \to \infty$, uniformly for any $x \geq \max\{l_1^{\leftarrow}[\mu_1\lambda_1(t)],\ l_2^{\leftarrow}[\mu_2\lambda_2(t)],\ \ldots,\ l_k^{\leftarrow}[\mu_k\lambda_k(t)]\}$.

Proof. We establish relation (17) by using Theorem 3 and employing mathematical induction as was done in the Theorem 4. In case $k = 2$ we see that

$$
\begin{aligned}
&\mathbf{P}\left[S_{N_1(t)}+S_{N_2(t)}-\mu_1\lambda_1(t)-\mu_2\lambda_2(t)>x\right] \\
&\geq \mathbf{P}[\{S_{N_1(t)}-\mu_1\lambda_1(t)>x+l_1(x),\ S_{N_2(t)}-\mu_2\lambda_2(t)>-l_1(x)\}\cup \\
&\qquad \{S_{N_2(t)}-\mu_2\lambda_2(t)>x+l_2(x),\ S_{N_1(t)}-\mu_1\lambda_1(t)>-l_2(x)\}] \\
&\geq \mathbf{P}\left[S_{N_1(t)}-\mu_1\lambda_1(t)>x+l_1(x)\right]\mathbf{P}\left[S_{N_2(t)}-\mu_2\lambda_2(t)>-l_1(x)\right] \\
&+\mathbf{P}\left[S_{N_2(t)}-\mu_2\lambda_2(t)>x+l_2(x)\right]\mathbf{P}\left[S_{N_1(t)}-\mu_1\lambda_1(t)>-l_2(x)\right] \\
&-\mathbf{P}\left[S_{N_1(t)}>x+l_1(x)\right]\mathbf{P}\left[S_{N_2(t)}>x+l_2(x)\right].
\end{aligned}
$$

From relations (7) and (1) follows the asymptotics

$$
\mathbf{P}\left[S_{N_1(t)}-\mu_1\lambda_1(t)>x+l_1(x)\right]\sim\lambda_1(t)\overline{F}_1(x)\,,
$$

and

$$
\mathbf{P}\left[S_{N_2(t)}-\mu_2\lambda_2(t)>x+l_1(x)\right]\sim\lambda_2(t)\overline{F}_2(x)\,.
$$

Hence, by the strong law of large numbers for random sums we find that

$$
\mathbf{P}\left[S_{N_1(t)}-\mu_1\lambda_1(t)>-l_2(x)\right]=1\,,
$$

and

$$
\mathbf{P}\left[S_{N_2(t)}-\mu_2\lambda_2(t)>-l_1(x)\right]=1.
$$

So we obtain

$$
\begin{aligned}
&\mathbf{P}\left[S_{N_1(t)}+S_{N_2(t)}-\mu_1\lambda_1(t)-\mu_2\lambda_2(t)>x\right]\gtrsim \\
&\lambda_1(t)\overline{F}_1(x)+\lambda_2(t)\overline{F}_2(x)-o[\lambda_1(t)\overline{F}_1(x)+\lambda_2(t)\overline{F}_2(x)]
\end{aligned}
\tag{18}
$$

as $t\to\infty$, uniformly for any $x\geq\max\{l_1^{\leftarrow}[\mu_1\lambda_1(t)],\,l_2^{\leftarrow}[\mu_2\lambda_2(t)]\}$. On the other hand, for any $0<\epsilon<1/2$ the following inequality holds

$$
\begin{aligned}
&\mathbf{P}[S_{N_1(t)}+S_{N_2(t)}-\mu_1\lambda_1-\mu_2\lambda_2>x] \\
&\leq\{\mathbf{P}[S_{N_1(t)}>x-l_1(x)]+\mathbf{P}[S_{N_2(t)}>x-l_2(x)] \\
&+\mathbf{P}[S_{N_1(t)}>\epsilon x+l_1(\epsilon x)]\,\mathbf{P}[S_{N_2(t)}>\epsilon x-l_2(\epsilon x)]\,.
\end{aligned}
$$

By relations (7) and (1) we find

$$
\begin{aligned}
&\mathbf{P}(S_{N_1(t)}+S_{N_2(t)}>x)\lesssim \\
&\lambda_1(t)\overline{F}_1(x)+\lambda_2(t)\overline{F}_2(x)+o[\lambda_1(t)\overline{F}_1(x)+\lambda_2(t)\overline{F}_2(x)]
\end{aligned}
\tag{19}
$$

as $t\to\infty$, uniformly for any $x\geq\max\{l_1^{\leftarrow}[\mu_1\lambda_1(t)],\,l_2^{\leftarrow}[\mu_2\lambda_2(t)]\}$. From (18) and (19) we get the required result (17) for the case $k=2$. Next, we continue the induction following the same lines from the proof of Theorem 4. □

Acknowledgments: The author is grateful to F. Loukissas for numerous discussions and advices and to the referees for careful reading and several substantial comments.

Conflicts of Interest: The author declares no conflict of interest.

References

Baltrũnas, Aleksandras, Daryl J. Daley, and Claudia Klüppelberg. 2004. Tail behaviour of the busy period of a GI/GI/1 queue with subexponential service times. *Stochastic Processes and their Applications* 111: 237–58.

Borovkov, Aleksandr A., and Anatolii A. Mogulskii. 2006. Integro-local and integral theorems for sums of random variables with semi-exponential distributions. *Siberian Mathematical* 47: 1218–57.

Borovkov, Aleksandr A., and Konstantin A. Borovkov. 2008. *Asymptotic Analysis of Random Walks: Heavy-Tailed Distributions.* New York: Cambridge University Press.

Chen, Yiqing, Kam Yuen, and Kai W. Ng. 2011. Precise large deviations of random sums in presence of negative dependence and consistent variation. *Methodology and Computing in Applied Probability* 13: 821–33.

Cline, Daren B. H., and Tailen Hsing. 1991. *Large Deviation Probabilities for Sums and Maxima or Rv's With Heavy or Subexponential Tails.* College Station: Texas AM University, preprint.

Denisov, Denis, Sergey Foss, and Dmitry Korshunov. 2004. Tail asymptotics for the supremum of a random walk when the mean is not finite. *Queueing Systems* 46: 15–33.

Denisov, Denis, Sergey Foss, and Dmitry Korshunov. 2010. Asymptotics of randomly stopped sums in the presence of heavy tails. *Bernoulli* 16: 971–994.

Denisov, Denis, Ton B. Dieker, and Seva Shneer. 2008. Large deviations for random walks under subexponentiality: The big-jump domain. *Annals of Probability* 36: 1946–91.

Embrechts, Paul, Claudia Klüppelberg, and Thomas Mikosch. 1997. *Modelling Extremal Events for Insurance and Finance.* Berlin: Springer.

Foss, Sergey, Dmitry Korshunov, and Stan Zachary. 2011. *An Introduction to Heavy-Tailed and Subexponential Distributions.* Berlin: Springer.

Gao, Miaomiao, Kaiyong Wang, and Lamei Chen. 2018. Precise large deviations for widely orthant dependent random variables with different distributions. *Journal of Inequalities and Applications* 2018: 21, doi:10.1186/s13660-018-1613-2.

Habtemicael, Semere, and Indranil SenGupta. 2014. Ornstein-Uhlenbeck processes for geophysical data analysis. *Physica A: Statistical Mechanics and its Applications* 399: 147–56.

Heyde, Chris C. 1967a. A contribution to the theory of the large deviations for sums of independent random variables. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* 7: 303–8.

Heyde, Chris C. 1967b. On large deviation problems for sums of random variables which are not attracted to the normal law. *The Annals of Mathematical Statistics* 38: 1575–78.

Heyde, Chris C. 1968. On large deviation probabilities in the case of attraction to a non-normal stable law. *Sankhya A* 30: 253–58.

Hult, Henrik, Filip Lindskog, Thomas Mikosch, and Gena Samorodnitsky. 2005. Functional large deviations for multivariate regularly varying random walks. *Annals of Applied Probability* 15: 2651–80.

Issaka, Aziz, and Indranil SenGupta. 2017. Feynman path integrals and asymptotic expansions for transition probability densities of some Lévy driven financial markets. *Journal of Applied Mathematics and Computing* 54: 159–82.

Jelenković, Predrag R., and Petar Momčilović. 2004. Large deviations of a square root insensitive random walks. *Mathematics of Operations Research* 29: 398–406.

Klüppelberg, Claudia. 1988. Subexponential distributions and integrated tails. *Journal of Applied Probability* 25: 132–41.

Klüppelberg, Claudia, and Thomas Mikosch. 1997. Large deviation probabilities of heavy-tailed random sums with applications in insurance and finance. *Journal of Applied Probability* 34: 293–308.

Konstantinides, Dimitrios G. 2017. *Risk Theory: A Heavy Tail Approach.* New Jersey: World Scientific.

Konstantinides, Dimitrios G., and Fotis Loukissas. 2010. Precise large deviations for consistently varying-tailed distributions in the compound renewal risk model. *Lithuanian Mathematical Journal* 50: 391–400.

Konstantinides, Dimitrios G., and Thomas Mikosch. 2005. Large deviations and ruin probabilities for solutions of stochastic recurrence equations with heavy-tailed innovations. *Annals of Probability* 33: 1992–2035.

Loukissas, Fotis. 2012. Precise large deviations for long-tailed distributions. *Journal of Theoretical Probability* 25: 913–24.

Lu, Dawei. 2012. Lower bounds of large deviation for sums of long-tailed claims in a multi-risk model. *Statistics & Probability Letters* 82: 1242–50.

Mikosch, Thomas, and Aleksandr V. Nagaev. 1998. Large deviation of heavy-tailed sums with applications in insurance. *Extremes* 1: 81–110.

Mikosch, Thomas, and Aleksandr V. Nagaev. 2001. Rates in Approximations to Ruin Probabilities for Heavy-Tailed Distributions. *Extremes* 4: 67–78.

Nagaev, Aleksandr V. 1969a. Integral limit theorems for large deviations when Cramer's condition is not fulfilled I. *Theory of Probability & Its Applications* 14: 51–64.

Nagaev, Aleksandr V. 1969b. Integral limit theorems for large deviations when Cramer's condition is not fulfilled II. *Theory of Probability & Its Applications* 14: 193–208.

Nagaev, Sergey V. 1973. Large deviations for sums of independent random variables. In *Transactions of the Sixth Prague Conference on Information Theory, Random Processes and Statistical Decision Functions.* Academia Prague: Chechoslovakia, pp. 657–74.

Nagaev, Sergey V. 1979. Large deviation of sums of independent random variables. *Annals of Probability* 7: 745–89.

Ng, Kai W., Qihe Tang, Jia-An Yan, and Hailing Yang. 2004. Precise large deviations for sums of random variables with consistently varying tails. *Journal of Applied Probability* 41: 93–107.

Paulauskas, Vygantas, and Aldona Skučaitė. 2003. Some asymptotic results for one-sided large deviation probabilities. *Lithuanian Mathematical Journal* 43: 318–26.

Pinelis, Iosif F. 1985. Asymptotic equivalence of the probabilities of large deviations for sums and maximum of independent random variables. *Trudy Matematicheskogo Instituta* 5: 144–73. Novosibirsk, 'Nauka'.

Skučaitė, Aldona. 2004. Large deviations for sums of independent heavy-tailed random variables. *Lithuanian Mathematical Journal* 44: 198–208.

Tang, Qihe. 2006. Insensitivity to negative dependence of the asymptotic behavior of precise large deviations. *Electronic Journal of Probability* 11: 107–20.

Tang, Qihe, Chun Su, Tao Jiang, and Jinsong Zhang. 2001. Large deviation for heavy-tailed random sums in compound renewal model. *Statistics & Probability Letters* 52: 91–100.

Wang, Shijie, and Wensheng Wang. 2007. Precise large deviations for sums of random variables with consistently varying tails in multi-risk models. *Journal of Applied Probability* 44: 889–900.

Wang, Shijie, and Wensheng Wang. 2012. Extended Precise large deviation of Random sums in the Presence of END Structure and Consistent Variation. *Journal of Applied Mathematics* 2012: 436531, doi:10.1155/2012/436531.

Wang, Shijie, and Wensheng Wang. 2013. Precise large deviation for sums of random variables with consistent variation in dependent multi-risk models. *Communications in Statistics—Theory and Methods* 42: 4444–59.

Wang, Shijie, Xuejun Wang, and Wensheng Wang. 2014. Precise large deviations of aggregate claims with dominated variation in Dependent multi-risk models. *Abstract and Applied Analysis* 2014: 972029, doi:10.1155/2014/972029.

Wang, Yuebao, Kaiyong Wang, and Dongya Cheng. 2006. Precise large deviations for sums of negatively associated random variables with common dominatedly varying tails. *Acta Mathematica Sinica* 22: 1725–34.

Yang, Yang, and Liwei Sha. 2016. Precise large deviations for aggregate claims. *Communications in Statistics—Theory and Methods* 45: 2801–9.

Yang, Yang, Remigijus Leipus, and Jonas Šiaulys. 2012. Precise large deviations for compound random sums in the presence of dependence structures. *Computers & Mathematics with Applications* 64: 2074–83.

Zhang, Qiuying, and Fengyang Cheng. 2017. Precise local large deviations for heavy-tailed random sums with applications to risk models. *Scandinavian Actuarial Journal* 2017: 1–14, doi:10.1080/03461238.2017.1377106.