An Optimal Tail Selection in Risk Measurement

Małgorzata Just [1,*] and **Krzysztof Echaust** [2]

[1] Department of Finance and Accounting, Poznań University of Life Sciences, 60-637 Poznań, Poland
[2] Department of Operations Research and Mathematical Economics, Poznań University of Economics and Business, 61-875 Poznań, Poland; krzysztof.echaust@ue.poznan.pl
[*] Correspondence: malgorzata.just@up.poznan.pl

Abstract: The appropriate choice of a threshold level, which separates the tails of the probability distribution of a random variable from its middle part, is considered to be a very complex and challenging task. This paper provides an empirical study on various methods of the optimal tail selection in risk measurement. The results indicate which method may be useful in practice for investors and financial and regulatory institutions. Some methods that perform well in simulation studies, based on theoretical distributions, may not perform well when real data are in use. We analyze twelve methods with different parameters for forty-eight world indices using returns from the period of 2000–Q1 2020 and four sub-periods. The research objective is to compare the methods and to identify those which can be recognized as useful in risk measurement. The results suggest that only four tail selection methods, i.e., the Path Stability algorithm, the minimization of the Asymptotic Mean Squared Error approach, the automated Eyeball method with carefully selected tuning parameters and the Hall single bootstrap procedure may be useful in practical applications.

Keywords: optimal tail selection; threshold; extreme value theory; Value at Risk; Expected Shortfall

1. Introduction

Extreme value theory (EVT) is a branch of statistics dealing with extreme values. It focuses on the asymptotic behavior of extreme values of random variables instead of the whole distribution. Such an approach enables us to model extreme values more accurately. Nowadays, EVT is widely used in many disciplines such as hydrology, environmental sciences, engineering, insurance and finance (Akhundjanov and Chamberlain 2019; Embrechts et al. 1997; Embrechts and Schmidli 1994; Gilli and Këllezi 2006; Longin 2000; Loretan and Phillips 1994; Mcneil 1999; McNeil and Frey 2000; Roth et al. 2016; Smith 2003; Vilasuso and Katz 2000; Wang et al. 2010). One of the key problems in EVT is related to the search for the optimal tail fraction. The appropriate choice of the threshold (the beginning of a tail of distribution) is ambiguous but fundamental in the estimation of the tail index or parameters of a Generalized Pareto Distribution (GPD) and then in the calculation of tail-related risk measures, e.g., Value at Risk (VaR) or Expected Shortfall (ES).

While there are many concepts and approaches presented in the literature, none of them has been indicated to be suitable, and there is no single answer where the distribution tail begins. Searching for the tail of distribution is always a trade-off between bias and variance. If a threshold is specified too low, the method estimates indicate a high bias. The more the threshold is away from the tail, the more the empirical distribution of extrema deviates from the theoretical model. On the other hand, a too high threshold results in high variance of the model estimates, since not many data exceed the threshold (Coles 2001). Numerous authors apply a fixed percentile of the total sample size as the threshold, usually 10%, 7.5%, 5% or 1% of the upper statistics (Karmakar and Shukla 2014; Jones 2003; Bee et al. 2016; Fernandez 2005; Totić and Božović 2016; Gençay et al. 2003; Gençay and Selçuk 2004; Echaust and Just 2020a; Daníelsson and Morimoto 2000; Echaust 2021; Longin 2000; McNeil and Frey 2000). More sophisticated approaches to threshold selection are based

on graphical techniques. A frequently used procedure relies on the analysis of a mean excess plot, which represents the mean of the excesses of the threshold u. This method was applied in Aboura (2014), Cifter (2011), Gilli and Këllezi (2006), Łuczak and Just (2020) and Omari et al. (2017). The change in the pattern for a very high threshold is observed in this plot; therefore, the choice of a threshold is ambiguous. The second graphical procedure for threshold selection is to estimate parameters of the Generalized Pareto Distribution at a range of thresholds. Above a threshold u, at which the model is valid, the estimate of the shape parameter should be approximately constant, whereas the estimate of the scale parameter should be linear. Another very popular threshold selection technique involves a graphical representation of Hill (Beirlant et al. 2011; Hill 1975), Pickands (Pickands 1975) or Dekkers–Einmahl–de Haan estimators (Dekkers et al. 1989). However, the graphical-based threshold choice procedures require identification of stable regions in the graphs; therefore, they are highly subjective and difficult to apply in empirical studies. Finally, there are studies that use the choice of threshold based on optimization procedures (Caeiro and Gomes 2016; Danielsson et al. 2016; Echaust and Just 2020b; Scarrott and MacDonald 2012). An extensive overview of such methods is provided in Section 2.

This paper provides an empirical study on various methods of optimal tail selection in risk measurement. The results are to indicate which of them may be used in practice by investors and financial and regulatory institutions. According to the Amendment to Basel I (1996), market risk should be calculated as the Value at Risk using the 99th percentile confidence level. A number of weaknesses have been identified by using VaR for determining capital requirement during the global financial crisis (GFC) of 2007–2009. Therefore, in the Basel III document (2013) the Committee proposed to replace VaR with ES at the 97.5% confidence level. This confidence level is to provide a similar level of risk as the 99th percentile VaR while ensuring a number of benefits, including generally more stable risk estimates and often lesser sensitivity to extreme outlier observations. Moreover, in contrast to VaR, ES is a subadditive risk measure and thus does not hinder the appropriate risk measurement of investment portfolios. In our empirical study, we analyze twelve automatic methods of threshold selection. Unfortunately, some of these methods estimate a threshold at a very high level to estimate these risk measures at the recommended confidence levels. The research objective of this paper is to compare these methods and to identify these methods that can be recognized as useful in risk measurement. Some models that perform well in simulation studies, based on theoretical distributions, may not perform well when real data are in use (Danielsson et al. 2016). This paper sheds new light on the linkage between tail selection and financial applications.

This study contributes to the literature in two ways. Firstly, we provide a review of tail selection methods most popular in the literature. Secondly, we compare the results of the tail choice computed with a variety of methods. All threshold estimates are calculated in the tea R package (Ossberger 2020). We are able to specify which of them tend to produce systematically too high threshold levels to calculate risk measures at commonly accepted confidence levels. Therefore, our results may be helpful for all investors, risk managers and academics who apply tail models in practice.

The remainder of the paper is organized as follows. Section 2 provides an overview of optimal tail selection methods. Section 3 describes the data used in the empirical study. Section 4 presents empirical results and their discussion, while Section 5 concludes the study.

2. Optimization Approaches for Threshold Selection

In the empirical study we use twelve methods to choose the optimal number of data in the tail:

- The Mean Absolute Deviation Distance metric (MAD-Distance metric) method,
- The Kolmogorov–Smirnov Distance metric (KS-Distance metric) method,
- The Reiss and Thomas (RT) procedures,
- The Path Stability (PS) method,

- The automated Eyeball (Eyeball) method,
- The Guillou and Hall (GH) procedure,
- The minimization of the Asymptotic Mean Squared Error (dAMSE) method,
- The Hall and Welsh (HW) procedure,
- The single bootstrap (Hall) procedure proposed by Hall,
- The single bootstrap (Himp) procedure proposed by Caeiro and Gomes,
- The double bootstrap (Gomes) procedure proposed by Gomes, Figueiredo and Neves,
- The double bootstrap (Danielsson) procedure proposed by Danielsson, de Haan, Peng and de Vries.

2.1. Mean Absolute Deviation Distance Metric (MAD-Distance Metric) and Kolmogorov–Smirnov Distance Metric (KS-Distance Metric) Methods

The MAD-Distance metric (MAD Dis) and KS-Distance metric (KS Dis) methods presented by Danielsson et al. (2016) are based on minimizing the distance between the largest upper order statistics of the dataset, i.e., the distance between the empirical tail and the theoretical tail of a Pareto distribution. The tail index α of this distribution is estimated using the Hill estimator (Hill 1975):

$$\hat{\xi}_{k,n}^H = \frac{1}{\hat{\alpha}} = \frac{1}{k}\sum_{i=0}^{k-1}(\log(x_{n-i,n}) - \log(x_{n-k,n})),$$

where k is the number of upper order statistics used to estimate parameter α. The distance, which is measured in the quantile dimension, is minimized with respect to k. The optimal number of the upper order statistics k is called k_0. Determining k_0 is equivalent to specifying the number of extreme values (sample size in the tail). The MAD-Distance metric method uses the mean absolute deviation penalty functions:

$$Q_n^{MAD} = \frac{1}{T}\sum_{j=1}^{T}|x_{n-j,n} - q(j,k)|$$

and the KS-Distance metric takes the maximum absolute deviation penalty functions:

$$Q_n^{KS} = \inf_k\left[\sup_j|x_{n-j,n} - q(j,k)|\right], \ (j = 1,\dots,T),$$

where $T > k$ is the region over which the distance metric is measured, $x_{n-j,n}$ is the empirical quantile, and $q(j,k)$ is the quantile estimated by $x_{n-k+1,n}(k/j)^{1/\hat{\alpha}_k}$ with the Hill estimator $\hat{\alpha}_k$. In these approaches, the size of the upper tail is denoted by ts. The motivation for the selection of these methods is that the tail-related risk measures (e.g., VaR and ES) are concepts related to quantiles at a given probability level.

2.2. The Reiss and Thomas (RT) Procedures

Reiss and Thomas (2007) proposed alternative procedures to select the optimal number of data in the tail, k_0. These approaches select the lowest upper order statistic by minimizing the expression (RT1):

$$\frac{1}{k}\sum_{i=1}^{k} i^\beta \left|\hat{\xi}_i^H - \text{median}\left(\hat{\xi}_i^H,\dots,\hat{\xi}_k^H\right)\right|$$

or expression (RT2):

$$\frac{1}{k}\sum_{i=1}^{k} i^\beta \left(\hat{\xi}_i^H - \hat{\xi}_k^H\right)^2,$$

where $\hat{\xi}_i^H$ is the Hill estimator, and the tuning parameter satisfies $0 \leq \beta \leq 0.5$.

In practice, automated implementation of these procedures is unreliable for small k; thus, a minimum value of k, k_{min}, is usually used (Scarrott and MacDonald 2012.)

2.3. The Path Stability (PS) Method

The Path Stability method is a heuristic algorithm proposed by Caeiro and Gomes (2016). The algorithm looks for a stable region of the path stability, i.e., the plot of a tail index with respect to k. The optimal number of data in the tail, k_0, is identified by the algorithm that consists of six steps:

First step. Given an observed data $(x_1, \ldots, x_n)$, compute $T(k) = \hat{\xi}_{k,n}^H$ $(k = 1, \ldots, n-1)$ using the Hill estimator.

Second step. Take j_0 as a minimum value of j, a non-negative integer, such that the rounded values, to j decimal places, of the estimates in the first step are distinct. Define $a_k^{(T)}(j) = \text{round}(T(k), j)(k = 1, \ldots, n-1)$, the rounded values of $T(k)$ to j decimal places.

Third step. Consider the set of k values associated to equal consecutive values of $a_k^{(T)}(j_0)$ obtained in the second step. Set $k_{max}^{(T)}$ and $k_{min}^{(T)}$ the maximum and minimum values, respectively, of the set with the largest range. The largest run size is $l_T = k_{max}^{(T)} - k_{min}^{(T)}$.

Fourth step. Consider all those estimates, $T(k)$ $\left(k_{min}^{(T)} \leq k \leq k_{max}^{(T)}\right)$, now with two additional decimal places, i.e., compute $T(k) = a_k^{(T)}(j_0 + 2)$. Obtain the mode of $T(k)$ and denote $\mathcal{K}_T$ the set of k-values associated with this mode.

Fifth step. Obtain $\hat{k}_T$, the maximum value of $\mathcal{K}_T$.

Sixth step. Finally calculate $\hat{\xi}^{H|PS} = \hat{\xi}_{\hat{k}_T, n}^H$.

2.4. The Automated Eyeball (Eyeball) Method

Another heuristic algorithm is the automated Eyeball method proposed by Resnick and Stărică (1997) and explored in (Danielsson et al. 2016). The method finds a stable region in the Hill plot by defining a moving window. This method is based on a heuristic algorithm, which looks for a substantial drop in variance of the Hill plot as k is increased. The following estimator is used to select the optimal number of data in the tail:

$$k_0^{eye} = \min\left\{ k \in \{2, \ldots, n^+ - w\} \,\middle|\, h < \frac{1}{w} \sum_{i=1}^{w} I\{\hat{\alpha}(k) - \varepsilon < \hat{\alpha}(k+i) < \hat{\alpha}(k) + \varepsilon\} \right\},$$

where w is the size of the moving window (usually 1% of the full sample); n^+ is the number of positive data; ε is the size of the range in which estimates may vary (e.g., 0.3); and h is the percentage of data inside the moving window, which should occur in the tolerable range (usually around 90%).

2.5. The Guillou and Hall (GH) Procedure

The procedure for choosing the optimal threshold when fitting the Hill estimator of a tail exponent to extreme value data proposed by Guillou and Hall (2001) is based on bias reduction. The optimal number of data in the tail, k_0, in this method is identified by the procedure, which consists of three steps (e.g., see, Caeiro and Gomes 2016):

First step. Given an observed data $(x_1, \ldots, x_n)$, compute:

$$Q_n(k) = \left\{ \frac{1}{2\lfloor k/2 \rfloor + 1} \sum_{j=k-\lfloor k/2 \rfloor}^{k+\lfloor k/2 \rfloor} T_n^2(j) \right\}^{1/2} (1 \leq k \text{ and } k + \lfloor k/2 \rfloor < n),$$

where

$$T_n(k) = \sqrt{\frac{3}{k^3}} \frac{\sum_{i=1}^{k}(k - 2i + 1)U_i}{\frac{1}{k}\sum_{i=1}^{k} U_i}, \quad U_i = i \sum_{i=1}^{k}(\log x_{n-i+1:n} - \log x_{n-i:n})(1 \leq i \leq k < n).$$

Second step. Given $c_{crit} = 1.25$, consider the choice

$$k_0^{GH} = \inf\{k : |Q_n(j)| \geq c_{crit}, \forall j \geq k\}.$$

Third step. Next, obtain $\hat{\xi}^{GH} = \hat{\xi}^{GH}_{\hat{k}^{GH}_0,n}$.

[Guillou and Hall](2001) investigated a wider range of c_{crit}, but they suggest values between 1.25 and 1.5.

2.6. Minimization of the Asymptotic Mean Squared Error (dAMSE) Method

The minimization of the Asymptotic Mean Squared Error (dAMSE) method was presented by [Caeiro and Gomes](2016). The optimal number of data in the tail, k_0, in this method is identified by minimizing the Asymptotic Mean Squared Error (AMSE) of the Hill estimator with respect to k. This algorithm consists of five steps:

First step. Given an observed data $(x_1, \ldots, x_n)$, calculate for the tuning parameters $\tau = 0$ and $\tau = 1$, the values of $\hat{\rho}_\tau(k)$, that have the form:

$$\hat{\rho}_\tau(k) = -\left|3(W^{(\tau)}_{k,n} - 1)/(W^{(\tau)}_{k,n} - 3)\right|$$

dependent on the statistics:

$$W^{(\tau)}_{k,n} = \begin{cases} \dfrac{\left(M^{(1)}_{k,n}\right)^\tau - \left(M^{(2)}_{k,n}/2\right)^{\tau/2}}{\left(M^{(2)}_{k,n}/2\right)^{\tau/2} - \left(M^{(3)}_{k,n}/6\right)^{\tau/3}}, & \text{if } \tau \neq 0, \\[2mm] \dfrac{\log\left(M^{(1)}_{k,n}\right) - \log\left(M^{(2)}_{k,n}/2\right)/2}{\log\left(M^{(2)}_{k,n}/2\right)/2 - \log\left(M^{(3)}_{k,n}/6\right)/3}, & \text{if } \tau = 0, \end{cases}$$

where

$$M^{(j)}_{k,n} = \frac{1}{k}\sum_{i=1}^{k}(\log x_{n-i+1:n} - \log x_{n-k:n})^j \, (j = 1, 2, 3).$$

Second step. Consider $\mathcal{K} = (\lfloor n^{0.995} \rfloor, \lfloor n^{0.999} \rfloor)$. Calculate the median of $\{\hat{\rho}_\tau(k)\}_{k \in \mathcal{K}}$, denoted χ_τ, and compute $I_\tau = \sum_{k \in \mathcal{K}} (\hat{\rho}_\tau(k) - \chi_\tau)^2 \, (\tau = 0, 1)$.

Then, select the tuning parameter, $\tau^* = 0$, if $I_0 \leq I_1$, otherwise select $\tau^* = 1$.

Third step. Work with $\hat{\rho} \equiv \hat{\rho}_{\tau^*}(k) = \hat{\rho}_{\tau^*}(k_{01})$ and $\hat{\beta} \equiv \hat{\beta}_{\tau^*}(k) = \hat{\beta}_{\hat{\rho}_{\tau^*}}(k_{01})$, with $k_{01} = \lfloor n^{0.999} \rfloor$ and

$$\hat{\beta}_{\hat{\rho}}(k) = \left(\frac{k}{n}\right)^{\hat{\rho}} \frac{d_k(\hat{\rho})D_k(0) - D_k(\hat{\rho})}{d_k(\hat{\rho})D_k(\hat{\rho}) - D_k(2\hat{\rho})},$$

dependent on the estimator $\hat{\rho} = \hat{\rho}_{\tau^*}(k_{01})$, and where for any $\alpha \leq 0$,

$$d_k(\alpha) = \frac{1}{k}\sum_{i=1}^{k}(i/k)^{-\alpha}, \quad D_k(\alpha) = \frac{1}{k}\sum_{i=1}^{k}(i/k)^{-\alpha}U_i,$$

where

$$U_i = i\sum_{i=1}^{k}(\log x_{n-i+1:n} - \log x_{n-k:n})(1 \leq i \leq k < n),$$

the scaled log-spacings.

Fourth step. On the basis of estimators $\hat{\beta}$ and $\hat{\rho}$, calculate

$$\hat{k}^H_0 = \left\lfloor \left(\frac{(1-\hat{\rho})^2 n^{-2\hat{\rho}}}{-2\hat{\rho}\hat{\beta}^2}\right)^{1/(1-2\hat{\rho})} \right\rfloor.$$

Fifth step. Finally, compute $\hat{\xi}^H = \hat{\xi}^H_{\hat{k}^H_0,n}$.

2.7. The Hall and Welsh (HW) Procedure

[Hall and Welsh](1985) proposed a procedure to identify the optimal number of upper order statistics for the Hill estimator by minimizing the AMSE criterion. This method is

based on a similar algorithm as that in the dAMSE method but with a different estimation of second-order parameters dependent upon three tuning parameters, $\sigma < \tau_1 < \tau_2$ (Caeiro and Gomes 2016). The Hall and Welsh procedure is quite sensitive to changes in the tuning parameters, but the parameter values proposed by Hall and Welsh ($\sigma = 0.5$, $\tau_1 = 0.9$, $\tau_2 = 0.95$) seem to work well (Gomes and Oliveira 2001). This method may be presented in five steps (e.g., see Gomes and Oliveira 2001; Hall and Welsh 1985):

First step. For $\sigma = 0.5$, $\tau_1 = 0.9$, $\tau_2 = 0.95$, obtain $s = [n^\sigma]$, $t_1 = [n^{\tau_1}]$, $t_2 = [n^{\tau_2}]$.

Second step. Obtain

$$\hat{\rho}_n = -\left| \log \left| \frac{\left(1/M^{(1)}_{t_1,n}\right) - \left(1/M^{(1)}_{s,n}\right)}{\left(1/M^{(1)}_{t_2,n}\right) - \left(1/M^{(1)}_{s,n}\right)} \right| / \log \frac{t_1}{t_2} \right|,$$

where $M^{(1)}_{k,n} = \frac{1}{k}\sum_{i=1}^{k}(\log x_{n-i+1:n} - \log x_{n-k:n})$ $(k = t_1,\ t_2,\ s)$.

Third step. Put

$$\hat{\lambda}_n = \left| \sqrt{-2\hat{\rho}_n} \left(\frac{n}{t_1}\right)^{-\hat{\rho}_n} \frac{M^{(1)}_{s,n} - M^{(1)}_{t_1,n}}{M^{(1)}_{t_1,n}} \right|^{-2/(1-2\hat{\rho}_n)}.$$

Fourth step. Compute

$$\hat{k}^{HW}_0 = \left[\hat{\lambda}_n^{-2\hat{\rho}_n/(1-2\hat{\rho}_n)} \right].$$

If $\hat{k}^{HW}_0 \notin [1,n)$, select another method, otherwise continue.

Fifth step. Finally, compute $\hat{\xi}^{HW} = \hat{\xi}^{HW}_{\hat{k}^{HW}_0,n}$.

The consistency of this procedure holds only for $\rho \in (-\sigma/2(1-\tau_1),\ -\sigma/2(1-\sigma))$.

2.8. The Hall Single Bootstrap (Hall) Procedure

In 1990, Hall introduced a bootstrap-based methodology for the estimation of the optimal tail fraction (see, e.g., Hall 1990; Gomes and Oliveira 2001). This method is applied to the Hall class of Pareto-type tails (Danielsson et al. 2016). Given a simple $\underline{X}_n = (X_1, \ldots, X_n)$ from an unknown model F, and the functional $\hat{\xi}^H_{k,n} = \phi_k(\underline{X}_n)$, $1 \leq k < n$, consider the bootstrap sample $\underline{X}^*_{n_1} = (X^*_1, \ldots, X^*_{n_1})$, where $n_1 \leq n$, from the model $F^*_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{\{X_i \leq x\}}$, the empirical d.f. associated with the original sample $\underline{X}_n$. Given an initial value k, i.e., k_{aux}, such that $\hat{\xi}^H_{k_{aux},n}$ is a consistent estimator of ξ, Hall proposes the minimization of the bootstrap estimate of the Mean Squared Error (*MSE*) of $\hat{\xi}^H_{k,n_1}$:

$$MSE^*(n_1,\ k) = E\left[\left\{ \hat{\xi}^{H*}_{k,n_1} - \hat{\xi}^H_{k_{aux},n} \right\}^2 | \underline{X}_n \right].$$

Next, the value $k^*_0(n_1)$ is selected so that it minimizes $MSE^*(n_1,\ k)$, and the tail fraction is determined from the formula:

$$\hat{k}^*_0(n) = k^*_0(n_1)(n/n_1)^{2/3}.$$

In the algorithm, the number of bootstrap repetitions is denoted by B.

Most often, k_{aux} and $n_1 = n^\varepsilon$ are taken as $2\sqrt{n}$ and $n^{0.955}$, respectively (Caeiro and Gomes 2016). Gomes and Oliveira (2001) noted that there is a disturbing sensitivity of the method to the initial value of k_{aux}; in turn, there is almost independence on n_1. These facts contributed to the search for an alternative bootstrap methodology.

2.9. The Single Bootstrap (Himp) Procedure Proposed by Caeiro and Gomes

An improved version of Hall bootstrap methodology was introduced in Caeiro and Gomes (2014). They proposed a single bootstrap procedure consisting of five steps:

First step. Given an observed sample $(x_1, \ldots, x_n)$, compute the estimates $\hat{\rho}$ and $\hat{\beta}$ of the second order parameters ρ and β as described in the dAMSE algorithm.

Second step. Consider a sub-sample size $n_1 = o(n)$. For l from 1 until B, generate independently B bootstrap samples $(x_1^*, \ldots, x_{n_1}^*)$, of sizes n_1, from the empirical distribution $F_n^*(x) = \frac{1}{n} \sum_{i=1}^n I_{\{X_i \leq x\}}$ associated with the observed data $(x_1, \ldots, x_n)$.

Third step. Denoting $T_{k,n_1}^{(1)*}$ the bootstrap counterpart of $T_{k,n_1}^{(1)} = \hat{\xi}_{k/2,n_1}^H - \hat{\xi}_{k,n_1}^H$, obtain $t_{k,n_1,l}^* (1 \leq l \leq B)$, the observed values of $T_{k,n_1}^{(1)}$. For $k = 2, \ldots, n_1 - 1$, compute

$$MSE^*(n_1, k) = \frac{1}{B} \sum_{l=1}^B \left(t_{k,n_1,l}^* \right)^2$$

and obtain

$$\hat{k}_{0|T}^*(n_1) = \text{argmin}_{1 < k < n_1} MSE^*(n_1, k).$$

Fourth step. Compute the threshold estimate

$$\hat{k}_0^*(n) = \left\lfloor \left(1 - 2^{\hat{\rho}} \right)^{2/(1-2\hat{\rho})} \hat{k}_{0|T}^*(n_1) / (n/n_1)^{-2\hat{\rho}/(1-2\hat{\rho})} \right\rfloor + 1.$$

If $\hat{k}_0^*(n) \notin [1, n)$ go back to the second step generating different bootstrap samples.

Fifth step. Finally, obtain $\hat{\xi}^* = \hat{\xi}_{\hat{k}_0^*(n), n}^H$.

2.10. The Double Bootstrap (Gomes) Procedure Proposed by Gomes, Figueiredo and Neves

The double bootstrap procedure described by Gomes et al. (2012) leads to an increased precision of the result with the same number B of bootstrap samples generated (Caeiro and Gomes 2016). This algorithm consists of five steps:

Let $\hat{\xi}_{k,n}^H$ denote the Hill estimator.

First step. Given an observed sample $(x_1, \ldots, x_n)$, compute the estimates $\hat{\rho}$ and $\hat{\beta}$ of the second order parameters ρ and β as described in the dAMSE algorithm.

Second step. Consider a sub-sample size $n_1 = o(n)$ and $n_2 = n_1^2/n + 1$. For l from 1 until B, generate independently B bootstrap samples $(x_1^*, \ldots, x_{n_2}^*)$ and $\left(x_1^*, \ldots, x_{n_2}^*, x_{n_2+1}^*, \ldots, x_{n_1}^* \right)$ of sizes n_2 and n_1, respectively, from the empirical distribution $F_n^*(x) = \frac{1}{n} \sum_{i=1}^n I_{\{X_i \leq x\}}$ associated with the observed data $(x_1, \ldots, x_n)$.

Third step. Denoting T_{k,n_i}^* the bootstrap counterpart of $T_{k,n_i} = \hat{\xi}_{\lfloor k/2 \rfloor, n_i}^H - \hat{\xi}_{k,n_i}^H$, obtain $t_{k,n_i,l}^* (1 \leq l \leq B, i = 1, 2)$, the observed values of $T_{k,n}$. For $k = 2, \ldots, n_i - 1$, and $i = 1.2$ compute

$$MSE^*(n_i, k) = \frac{1}{B} \sum_{l=1}^B \left(t_{k,n_i,l}^* \right)^2$$

and obtain

$$\hat{k}_{0|T}^*(n_i) = \text{argmin}_{1 < k < n_i} MSE^*(n_i, k).$$

Fourth step. Compute the threshold estimate

$$\hat{k}_0^*(n) = \left\lfloor \left(1 - 2^{\hat{\rho}} \right)^{2/(1-2\hat{\rho})} \left(\hat{k}_{0|T}^*(n_1) \right)^2 / \hat{k}_{0|T}^*(n_2) \right\rfloor + 1.$$

If $\hat{k}_0^*(n) \notin [1, n)$, go back to the second step generating different bootstrap samples.

Fifth step. Finally obtain $\hat{\xi}^* = \hat{\xi}_{\hat{k}_0^*(n), n}^H$.

2.11. The Double Bootstrap (Danielsson) Procedure Proposed by Danielsson, de Haan, Peng and de Vries

The procedure proposed in Danielsson et al. (2001) for selecting the optimal sample fraction in tail index estimation simulates the AMSE criterion of the Hill estimator using

an auxiliary statistic. For the AMSE, ξ is unknown. To solve this problem, the theoretical ξ in the AMSE formula is replaced with a control variate. Since a simple bootstrap is inconsistent in the tail area, a sub-sample bootstrap is used. Moreover, to be able to scale the sub-sample *MSE* back to the sample size, a second, even smaller sub-sample bootstrap is needed. The AMSE of the control variate is

$$T_{k_1,n_1} = \left(\left[M_{k_1,\,n_1}^{*(2)} - 2\left(\xi_{k_1,n_1}^{*H} \right)^2 \right]^2 \right),$$

where

$$M_{k_1,n_1}^{*(2)} = \frac{1}{k_1} \sum_{i=1}^{k_1} \left(\log\left(x_{n_1-i+1:n_1} \right) - \log\left(x_{n_1-k_1:n_1} \right) \right)^2 .$$

In this procedure, $n_1 = n^\varepsilon$ is the sub-sample for the bootstrap (the number of bootstrap repetitions is denoted by B). In the first step, the T function is minimized over two dimensions: n_1 and k_1. In the next step, given the optimal n_1 and k_1, a second bootstrap with a sub-sample size $n_2 = n_1^2/n$ is made to find k_2. Finally, the optimal number of order statistics is determined as follows:

$$\hat{k}_0^*(n) = \left\lfloor \frac{k_1^2}{k_2} \left[\frac{(\log(k_1))^2}{(2\log(n_1) - \log(k_1))^2} \right]^{(\log(n_1) - \log(k_1))/\log(n_1)} \right\rfloor + 1.$$

3. Data

We employed several approaches and compared the results of the optimal tail selection for forty-eight world indices computed using twelve methods with different parameters (56 different variants of tail estimation for each asset). The source of the price data is the Polish financial service stooq.pl. The country sample encompasses both developed and emerging markets and comes from different continents. We conducted the analysis for the period 2000–Q1 2020 and selected four sub-periods to demonstrate the robustness of our analysis with reference to the time period under consideration. The first sub-period covers 750 initial index quotes, the second sub-period covers the next 750 quotes, the third sub-period covers 750 index quotes starting in 2008 in the subprime crisis, and the fourth sub-period covers another 750 index quotes ending at the beginning of the COVID-19 crisis. This means that for each sub-period there are 749 log returns. In turn, the entire period ranges from 4520 to 5176 log-returns depending on the number of trading days in the market taken into account. The basic characteristics for the entire research period are given in Table 1. The average return ranges from −0.05% for Greece to 0.08% for Argentina. Higher volatility is found for the emerging markets, especially for Ukraine, Russia, Turkey and Argentina. Since our research period covers two crises, i.e., the global financial crisis in 2007–2009, the European debt crisis (EDC) in 2010–2012 and the beginning of the COVID-19 outbreak, almost all time-series of returns demonstrate left skewness. The Iceland index is characterized by distinctive kurtosis. Certainly, it is an effect of Iceland's systemic banking collapse in 2008.

Table 1. Summary statistics for the sample of countries.

Index	Min	Mean	Max	Sd. Dev.	Skewness	Ex. Kurtosis
ATX (Austria)	−0.1467	0.0001	0.1202	0.0143	−0.59	10.01
AEX (Netherlands)	−0.1138	−0.0001	0.1003	0.0140	−0.22	7.49
ATH (Greece)	−0.1771	−0.0005	0.1343	0.0189	−0.55	7.84
BEL20 (Belgium)	−0.1533	0.0000	0.0933	0.0126	−0.44	10.36
BET (Romania)	−0.1190	0.0006	0.1458	0.0150	−0.34	11.67
BUX (Hungary)	−0.1265	0.0003	0.1318	0.0149	−0.24	7.08
CAC (France)	−0.1310	−0.0001	0.1059	0.0144	−0.22	6.52
DAX (Germany)	−0.1305	0.0001	0.1080	0.0148	−0.17	5.99
FMIB (Italy)	−0.1854	−0.0002	0.1087	0.0154	−0.59	9.34
HEX (Finland)	−0.1740	−0.0001	0.1456	0.0173	−0.40	7.86
IBEX (Spain)	−0.1515	−0.0001	0.1348	0.0147	−0.32	8.13
ICEX (Iceland)	−1.0622	0.0000	0.0506	0.0186	−39.03	2164.64
MOEX (Russia)	−0.2066	0.0005	0.2523	0.0200	−0.25	15.99
OMXR (Latvia)	−0.1633	0.0004	0.1209	0.0140	−0.56	20.56
OMXS (Sweden)	−0.1117	0.0000	0.0987	0.0148	−0.08	4.37
OMXT (Estonia)	−0.1060	0.0004	0.1209	0.0105	−0.25	12.85
OMXV (Lithuania)	−0.1194	0.0004	0.1100	0.0099	−0.77	24.07
OSEAX (Norway)	−0.0983	0.0003	0.0919	0.0137	−0.69	6.76
PSI20 (Portugal)	−0.1038	−0.0002	0.1020	0.0119	−0.42	7.49
PX (Czechia)	−0.1619	0.0001	0.1236	0.0134	−0.54	13.73
RTS (Russia)	−0.2120	0.0003	0.2020	0.0215	−0.54	9.27
SAX (Slovakia)	−0.1481	0.0003	0.1188	0.0118	−0.69	14.81
SMI (Switzerland)	−0.1013	0.0000	0.1079	0.0117	−0.29	7.95
SOFIX (Bulgaria)	−0.1136	0.0003	0.0839	0.0123	−0.64	11.56
UKX (UK)	−0.1151	0.0000	0.0938	0.0119	−0.34	8.34
UX (Ukraine)	−0.3437	0.0005	0.3908	0.0260	0.85	53.49
WIG20 (Poland)	−0.1425	0.0000	0.0815	0.0150	−0.33	4.22
XU100 (Turkey)	−0.1998	0.0003	0.1777	0.0207	−0.10	7.91
AOR (Australia)	−0.1001	0.0001	0.0635	0.0098	−0.91	9.63
HSI (Hong Kong)	−0.1358	0.0001	0.1341	0.0146	−0.11	7.90
JCI (Indonesia)	−0.1095	0.0004	0.0970	0.0134	−0.63	7.18
KLCI (Malaysia)	−0.0998	0.0001	0.0663	0.0081	−0.79	11.20
KOSPI (South Korea)	−0.1280	0.0001	0.1128	0.0150	−0.57	7.16
NKX (Japan)	−0.1211	0.0000	0.1323	0.0150	−0.38	6.42
NZ50 (New Zealand)	−0.0789	0.0004	0.0694	0.0072	−0.72	10.19
PSEI (Philippines)	−0.1432	0.0002	0.1618	0.0131	−0.15	16.91
SET (Thailand)	−0.1606	0.0002	0.1058	0.0132	−0.93	11.77
SHBS (China)	−0.1029	0.0004	0.1840	0.0198	−0.02	7.12
SNX (India)	−0.1410	0.0003	0.1599	0.0147	−0.41	9.37
STI (Singapore)	−0.0909	0.0000	0.0753	0.0112	−0.40	6.79
TWSE (Taiwan)	−0.0994	0.0000	0.0652	0.0134	−0.29	3.89
BVP (Brazil)	−0.1599	0.0003	0.1368	0.0182	−0.38	6.64
DJI (US)	−0.1384	0.0001	0.1076	0.0119	−0.40	13.63
IPC (Mexico)	−0.0827	0.0003	0.1044	0.0128	−0.05	5.47
IPSA (Chile)	−0.1522	0.0002	0.1180	0.0104	−0.84	22.35
MRV (Argentina)	−0.4769	0.0008	0.1612	0.0232	−2.02	39.79
SPX (US)	−0.1277	0.0001	0.1096	0.0125	−0.40	11.37
TSX (Canada)	−0.1318	0.0001	0.1129	0.0114	−0.96	17.30

Note: Sd. Dev. means standard deviation; Ex. Kurtosis—excess kurtosis; BET—the beginning of trading 31 October 2000; SOFIX—the beginning of trading 26 November 2001; NZ50—the beginning of trading 3 January 2001. Symbols for indices are consistent with the symbols on the financial website, www.stooq.pl (Accessed on: 16 April 2020).

4. Results of the Empirical Study

Table 2 presents the results of an optimal threshold selection for the left and right tails. More precisely, it shows the percentile ranks for the entire period. Tables 3 and 4 show the results for the first and third sub-periods. For the sake of brevity, we do not report the results for the second and fourth sub-periods. Note that the results for these periods do not differ substantially from those presented in Tables 3 and 4.

Table 2. Estimates of the threshold under different methods in the entire research period.

Method	Lower Tail					Upper Tail				
	Min	Mean	Median	Max	Sd. Dev.	Min	Mean	Median	Max	Sd. Dev.
MAD Dis $ts = 0.25$	0.8970	0.9192	0.9078	0.9996	0.0312	0.8832	0.9132	0.9021	0.9992	0.0342
MAD Dis $ts = 0.20$	0.9092	0.9337	0.9213	0.9986	0.0288	0.9021	0.9319	0.9163	0.9998	0.0336
MAD Dis $ts = 0.15$	0.9148	0.9499	0.9400	0.9986	0.0243	0.8998	0.9505	0.9361	0.9998	0.0290
MAD Dis $ts = 0.10$	0.9412	0.9649	0.9575	0.9992	0.0163	0.9389	0.9634	0.9539	0.9961	0.0188
MAD Dis $ts = 0.05$	0.9517	0.9792	0.9789	0.9968	0.0100	0.9561	0.9808	0.9834	0.9968	0.0113
KS Dis $ts = 0.25$	0.9605	0.9961	0.9982	0.9996	0.0074	0.9620	0.9933	0.9963	0.9996	0.0078
KS Dis $ts = 0.20$	0.9605	0.9959	0.9982	0.9996	0.0074	0.9620	0.9934	0.9968	0.9996	0.0078
KS Dis $ts = 0.15$	0.9605	0.9956	0.9980	0.9996	0.0075	0.9620	0.9933	0.9965	0.9996	0.0079
KS Dis $ts = 0.10$	0.9605	0.9958	0.9980	0.9996	0.0073	0.9620	0.9935	0.9965	0.9994	0.0077
KS Dis $ts = 0.05$	0.9605	0.9958	0.9980	0.9996	0.0073	0.9620	0.9936	0.9966	0.9994	0.0078
RT1 $\beta = 0, k_{min} = 2$	0.9084	0.9762	0.9800	0.9998	0.0238	0.9311	0.9830	0.9920	0.9998	0.0205
RT1 $\beta = 0.1, k_{min} = 2$	0.9185	0.9821	0.9917	0.9998	0.0214	0.9339	0.9863	0.9946	0.9998	0.0178
RT1 $\beta = 0.2, k_{min} = 2$	0.9259	0.9861	0.9970	0.9998	0.0193	0.9344	0.9897	0.9978	0.9998	0.0166
RT1 $\beta = 0.3, k_{min} = 2$	0.9478	0.9941	0.9990	0.9998	0.0108	0.9421	0.9931	0.9992	0.9998	0.0132
RT1 $\beta = 0, k_{min} = 3$	0.9086	0.9738	0.9773	0.9998	0.0241	0.9313	0.9790	0.9881	0.9998	0.0212
RT1 $\beta = 0, k_{min} = 4$	0.9088	0.9726	0.9772	0.9998	0.0244	0.9013	0.9749	0.9793	0.9998	0.0236
RT1 $\beta = 0, k_{min} = 5$	0.8979	0.9672	0.9705	0.9998	0.0262	0.9015	0.9744	0.9795	0.9998	0.0230
RT1 $\beta = 0, k_{min} = 10$	0.8989	0.9662	0.9694	0.9998	0.0249	0.9025	0.9732	0.9765	0.9998	0.0221
RT1 $\beta = 0, k_{min} = 0.003n$	0.8997	0.9662	0.9687	0.9998	0.0241	0.9033	0.9741	0.9774	0.9998	0.0220
RT1 $\beta = 0, k_{min} = 0.005n$	0.9017	0.9664	0.9702	0.9998	0.0230	0.9053	0.9710	0.9720	0.9998	0.0226
RT2 $\beta = 0, k_{min} = 2$	0.9064	0.9816	0.9884	0.9998	0.0219	0.9352	0.9839	0.9923	0.9998	0.0196
RT2 $\beta = 0.1, k_{min} = 2$	0.9118	0.9835	0.9917	0.9998	0.0206	0.9352	0.9848	0.9933	0.9998	0.0189
RT2 $\beta = 0.2, k_{min} = 2$	0.9399	0.9855	0.9945	0.9998	0.0178	0.9414	0.9886	0.9972	0.9998	0.0160
RT2 $\beta = 0.3, k_{min} = 2$	0.9441	0.9876	0.9967	0.9998	0.0169	0.9414	0.9895	0.9984	0.9998	0.0155
RT2 $\beta = 0, k_{min} = 3$	0.9066	0.9805	0.9853	0.9998	0.0214	0.9354	0.9807	0.9898	0.9998	0.0201
RT2 $\beta = 0, k_{min} = 4$	0.9068	0.9806	0.9855	0.9998	0.0214	0.9019	0.9761	0.9817	0.9998	0.0228
RT2 $\beta = 0, k_{min} = 5$	0.9070	0.9767	0.9842	0.9998	0.0231	0.9021	0.9748	0.9793	0.9998	0.0220
RT2 $\beta = 0, k_{min} = 10$	0.9080	0.9723	0.9759	0.9998	0.0219	0.9031	0.9745	0.9783	0.9994	0.0213
RT2 $\beta = 0, k_{min} = 0.003n$	0.9090	0.9724	0.9759	0.9996	0.0212	0.9039	0.9753	0.9793	0.9998	0.0212
RT2 $\beta = 0, k_{min} = 0.005n$	0.9110	0.9731	0.9765	0.9996	0.0206	0.9059	0.9733	0.9735	0.9986	0.0203
PS $j = 1$	0.7651	0.8791	0.8873	0.9511	0.0485	0.7513	0.8813	0.8869	0.9571	0.0472
PS $j = 0$	0.7324	0.8272	0.8262	0.9013	0.0420	0.6409	0.7994	0.8051	0.8961	0.0476
Eyeball $w = 0.01, \varepsilon = 0.3, h = 0.9$	0.9910	0.9939	0.9940	0.9956	0.0008	0.9912	0.9933	0.9933	0.9952	0.0008
Eyeball $w = 0.02, \varepsilon = 0.3, h = 0.9$	0.9860	0.9895	0.9896	0.9915	0.0009	0.9858	0.9886	0.9887	0.9909	0.0009
Eyeball $w = 0.025, \varepsilon = 0.3, h = 0.9$	0.9838	0.9874	0.9874	0.9899	0.0009	0.9836	0.9862	0.9862	0.9887	0.0009
GH	0.9473	0.9871	0.9967	0.9994	0.0153	0.9488	0.9880	0.9932	0.9994	0.0143
dAMSE	0.9721	0.9738	0.9738	0.9770	0.0007	0.9704	0.9723	0.9724	0.9747	0.0007
HW	0.6482	0.8965	0.9086	0.9459	0.0500	0.7145	0.8788	0.8901	0.9327	0.0409
Hall $B = 10,000, \varepsilon = 0.955, k_{aux} = 2\sqrt{n^+}$	0.9187	0.9714	0.9751	0.9836	0.0129	0.9245	0.9731	0.9765	0.9848	0.0109
Hall $B = 10,000, \varepsilon = 0.995, k_{aux} = 2\sqrt{n^+}$	0.9085	0.9687	0.9723	0.9817	0.0143	0.9187	0.9711	0.9754	0.9828	0.0117
Hall $B = 10,000, \varepsilon = 0.9, k_{aux} = 2\sqrt{n^+}$	0.9289	0.9748	0.9782	0.9860	0.0112	0.9315	0.9757	0.9786	0.9863	0.0098
Hall $B = 10,000, \varepsilon = 0.955, k_{aux} = \sqrt{n^+}$	0.9385	0.9773	0.9819	0.9930	0.0151	0.8947	0.9771	0.9807	0.9941	0.0168
Hall $B = 10,000, \varepsilon = 0.955, k_{aux} = 3\sqrt{n^+}$	0.9464	0.9681	0.9713	0.9763	0.0078	0.9436	0.9661	0.9677	0.9746	0.0076
Hall $B = 1000, \varepsilon = 0.955, k_{aux} = 2\sqrt{n^+}$	0.9183	0.9714	0.9751	0.9837	0.0128	0.9269	0.9733	0.9770	0.9838	0.0106
Hall(r) $B = 10,000, \varepsilon = 0.955, k_{aux} = 2\sqrt{n^+}$	0.9187	0.9714	0.9748	0.9836	0.0127	0.9255	0.9731	0.9762	0.9842	0.0108
Himp $B = 10,000, \varepsilon = 0.955$	0.9640	0.9848	0.9863	0.9998	0.0072	0.9618	0.9826	0.9831	0.9988	0.0088
Himp $B = 10,000, \varepsilon = 0.995$	0.9604	0.9839	0.9856	0.9994	0.0078	0.9564	0.9812	0.9831	0.9986	0.0099
Himp $B = 10,000, \varepsilon = 0.9$	0.9692	0.9862	0.9861	0.9998	0.0062	0.9662	0.9840	0.9850	0.9992	0.0077
Himp $B = 1000, \varepsilon = 0.955$	0.9640	0.9852	0.9863	0.9998	0.0069	0.9620	0.9827	0.9841	0.9988	0.0088
Himp (r) $B = 10,000, \varepsilon = 0.955$	0.9640	0.9845	0.9860	0.9998	0.0072	0.9620	0.9826	0.9833	0.9988	0.0088
Gomes $B = 10,000, \varepsilon = 0.995$	0.9598	0.9840	0.9851	0.9980	0.0081	0.9550	0.9808	0.9826	0.9986	0.0102
Gomes $B = 10,000, \varepsilon = 0.955$	0.9586	0.9827	0.9846	0.9998	0.0085	0.9548	0.9806	0.9818	0.9988	0.0105
Gomes $B = 10,000, \varepsilon = 0.9$	0.9596	0.9838	0.9819	0.9998	0.0098	0.9540	0.9810	0.9812	0.9998	0.0120
Gomes $B = 1000, \varepsilon = 0.995$	0.9580	0.9838	0.9851	0.9975	0.0081	0.9554	0.9819	0.9833	0.9986	0.0098
Gomes (r) $B = 10,000, \varepsilon = 0.995$	0.9600	0.9840	0.9853	0.9976	0.0080	0.9560	0.9809	0.9828	0.9986	0.0100
Danielsson $B = 500, \varepsilon = 0.9$	0.9390	0.9859	0.9899	0.9998	0.0147	0.9398	0.9886	0.9934	0.9998	0.0130

Note: (r) means repeating the calculation; Sd. Dev.—standard deviation.

Table 3. Estimates of the threshold under different methods in the first sub-period.

Method	Lower Tail					Upper Tail				
	Min	Mean	Median	Max	Sd. Dev.	Min	Mean	Median	Max	Sd. Dev.
MAD Dis ts = 0.25	0.8611	0.9070	0.8905	0.9973	0.0414	0.8438	0.8946	0.8805	0.9973	0.0395
MAD Dis ts = 0.20	0.8692	0.9173	0.9039	0.9973	0.0374	0.8665	0.9056	0.8992	0.9893	0.0248
MAD Dis ts = 0.15	0.8732	0.9359	0.9266	0.9987	0.0302	0.8772	0.9305	0.9232	0.9973	0.0265
MAD Dis ts = 0.10	0.9105	0.9521	0.9486	0.9920	0.0227	0.9079	0.9530	0.9466	0.9987	0.0227
MAD Dis ts = 0.05	0.9573	0.9739	0.9713	0.9987	0.0100	0.9519	0.9710	0.9693	0.9920	0.0101
KS Dis ts = 0.25	0.9399	0.9830	0.9880	0.9987	0.0140	0.9519	0.9791	0.9793	0.9973	0.0118
KS Dis ts = 0.20	0.9279	0.9828	0.9840	0.9987	0.0140	0.9439	0.9790	0.9806	0.9973	0.0129
KS Dis ts = 0.15	0.9479	0.9841	0.9866	0.9987	0.0117	0.9466	0.9812	0.9813	0.9973	0.0108
KS Dis ts = 0.10	0.9479	0.9843	0.9880	0.9987	0.0118	0.9359	0.9801	0.9806	0.9973	0.0126
KS Dis ts = 0.05	0.9680	0.9877	0.9893	0.9987	0.0078	0.9519	0.9844	0.9860	0.9987	0.0091
RT1 $\beta = 0, k_{min} = 2$	0.8024	0.9661	0.9907	0.9987	0.0440	0.8745	0.9801	0.9933	0.9987	0.0279
RT1 $\beta = 0.1, k_{min} = 2$	0.8772	0.9728	0.9920	0.9987	0.0340	0.8852	0.9810	0.9947	0.9987	0.0267
RT1 $\beta = 0.2, k_{min} = 2$	0.9092	0.9769	0.9947	0.9987	0.0307	0.9065	0.9854	0.9960	0.9987	0.0217
RT1 $\beta = 0.3, k_{min} = 2$	0.9092	0.9824	0.9947	0.9987	0.0259	0.9159	0.9865	0.9960	0.9987	0.0203
RT1 $\beta = 0, k_{min} = 3$	0.8037	0.9613	0.9766	0.9987	0.0432	0.8758	0.9782	0.9887	0.9987	0.0273
RT1 $\beta = 0, k_{min} = 4$	0.8051	0.9620	0.9780	0.9987	0.0428	0.8772	0.9752	0.9866	0.9987	0.0287
RT1 $\beta = 0, k_{min} = 5$	0.8064	0.9581	0.9713	0.9987	0.0438	0.8785	0.9742	0.9873	0.9987	0.0277
RT1 $\beta = 0, k_{min} = 10$	0.8064	0.9491	0.9646	0.9987	0.0492	0.8117	0.9704	0.9813	0.9987	0.0365
RT1 $\beta = 0, k_{min} = 0.003n$	0.8024	0.9661	0.9907	0.9987	0.0440	0.8745	0.9801	0.9933	0.9987	0.0279
RT1 $\beta = 0, k_{min} = 0.005n$	0.8037	0.9613	0.9766	0.9987	0.0432	0.8758	0.9782	0.9887	0.9987	0.0273
RT2 $\beta = 0, k_{min} = 2$	0.8825	0.9701	0.9907	0.9987	0.0353	0.8892	0.9802	0.9920	0.9987	0.0281
RT2 $\beta = 0.1, k_{min} = 2$	0.8825	0.9721	0.9933	0.9987	0.0334	0.8892	0.9811	0.9940	0.9987	0.0270
RT2 $\beta = 0.2, k_{min} = 2$	0.8825	0.9731	0.9933	0.9987	0.0334	0.8999	0.9826	0.9960	0.9987	0.0251
RT2 $\beta = 0.3, k_{min} = 2$	0.9105	0.9777	0.9940	0.9987	0.0285	0.8999	0.9835	0.9960	0.9987	0.0244
RT2 $\beta = 0, k_{min} = 3$	0.8838	0.9630	0.9760	0.9987	0.0359	0.8905	0.9794	0.9920	0.9987	0.0273
RT2 $\beta = 0, k_{min} = 4$	0.8852	0.9641	0.9773	0.9987	0.0357	0.8919	0.9768	0.9893	0.9987	0.0275
RT2 $\beta = 0, k_{min} = 5$	0.8865	0.9620	0.9780	0.9987	0.0356	0.8932	0.9772	0.9893	0.9987	0.0268
RT2 $\beta = 0, k_{min} = 10$	0.8131	0.9554	0.9660	0.9987	0.0425	0.8425	0.9724	0.9866	0.9987	0.0326
RT2 $\beta = 0, k_{min} = 0.003n$	0.8825	0.9701	0.9907	0.9987	0.0353	0.8892	0.9802	0.9920	0.9987	0.0281
RT2 $\beta = 0, k_{min} = 0.005n$	0.8838	0.9630	0.9760	0.9987	0.0359	0.8905	0.9794	0.9920	0.9987	0.0273
PS j = 1	0.7370	0.8457	0.8531	0.9506	0.0542	0.7130	0.8642	0.8665	0.9840	0.0646
PS j = 0	0.6088	0.7510	0.7557	0.8852	0.0632	0.6649	0.7623	0.7644	0.8798	0.0477
Eyeball w = 0.01, ε = 0.3, h = 0.9	0.9813	0.9866	0.9866	0.9933	0.0033	0.9733	0.9846	0.9853	0.9933	0.0044
Eyeball w = 0.02, ε = 0.3, h = 0.9	0.9693	0.9801	0.9800	0.9880	0.0036	0.9680	0.9780	0.9780	0.9893	0.0043
GH	0.8745	0.9638	0.9766	0.9960	0.0306	0.9079	0.9702	0.9780	0.9960	0.0238
dAMSE	0.8451	0.9400	0.9413	0.9479	0.0142	0.8198	0.9385	0.9439	0.9479	0.0234
HW	x	x	x	x	x	x	x	x	x	x
Hall B = 10,000, ε = 0.955, $k_{aux} = 2\sqrt{n^+}$	0.8331	0.9372	0.9473	0.9586	0.0271	0.8331	0.9462	0.9499	0.9653	0.0199
Hall B = 10,000, ε = 0.995, $k_{aux} = 2\sqrt{n^+}$	0.8198	0.9324	0.9439	0.9653	0.0294	0.8211	0.9420	0.9466	0.9546	0.0216
Hall B = 10,000, ε = 0.9, $k_{aux} = 2\sqrt{n^+}$	0.8505	0.9435	0.9506	0.9626	0.0239	0.8505	0.9518	0.9546	0.9693	0.0175
Hall B = 10,000, ε = 0.955 $k_{aux} = \sqrt{n^+}$	0.8892	0.9582	0.9713	0.9853	0.0233	0.7570	0.9631	0.9713	0.9813	0.0332
Hall B = 10,000, ε = 0.955, $k_{aux} = 3\sqrt{n^+}$	0.8438	0.9193	0.9246	0.9453	0.0219	0.8772	0.9252	0.9252	0.9413	0.0122
Hall B = 1000, ε = 0.955, $k_{aux} = 2\sqrt{n^+}$	0.8251	0.9377	0.9479	0.9613	0.0276	0.8278	0.9468	0.9493	0.9626	0.0200
Hall (r) B = 10,000, ε = 0.955, $k_{aux} = 2\sqrt{n^+}$	0.8344	0.9373	0.9459	0.9586	0.0269	0.8331	0.9461	0.9506	0.9653	0.0198
Himp B = 10,000, ε = 0.955	0.9226	0.9695	0.9706	0.9973	0.0177	0.9199	0.9785	0.9806	0.9987	0.0165
Himp B = 10,000, ε = 0.995	0.9146	0.9685	0.9693	0.9987	0.0205	0.9199	0.9774	0.9806	0.9987	0.0177
Himp B = 10,000, ε = 0.9	0.9332	0.9696	0.9706	0.9987	0.0156	0.9292	0.9801	0.9820	0.9987	0.0149
Himp B = 1000, ε = 0.955	0.9226	0.9696	0.9713	0.9973	0.0180	0.9212	0.9789	0.9820	0.9987	0.0167
Himp (r) B = 10,000, ε = 0.955	0.9239	0.9693	0.9700	0.9973	0.0178	0.9212	0.9785	0.9806	0.9987	0.0166
Gomes B = 10,000, ε = 0.995	0.9146	0.9684	0.9686	0.9987	0.0205	0.9226	0.9778	0.9806	0.9987	0.0177
Gomes B = 10,000, ε = 0.955	0.9132	0.9683	0.9693	0.9987	0.0209	0.9226	0.9769	0.9793	0.9987	0.0177
Gomes B = 10,000, ε = 0.9	0.9092	0.9655	0.9666	0.9987	0.0224	0.9212	0.9773	0.9786	0.9987	0.0183
Gomes B = 1000, ε = 0.995	0.9146	0.9684	0.9700	0.9987	0.0202	0.9065	0.9774	0.9800	0.9987	0.0186
Gomes (r) B = 10,000, ε = 0.995	0.9159	0.9682	0.9686	0.9987	0.0205	0.9105	0.9773	0.9806	0.9987	0.0186
Danielsson B = 500, ε = 0.9	0.9172	0.9842	0.9933	0.9987	0.0204	0.8665	0.9896	0.9980	0.9987	0.0218
Danielsson B = 500, ε = 0.955	0.9239	0.9815	0.9933	0.9987	0.0226	0.9332	0.9931	0.9987	0.9987	0.0117

Note: (r) means repeating the calculation; Sd. Dev.—standard deviation; x—cannot calculate, the method is not appropriate for the data.

Table 4. Estimates of the threshold under different methods in the third sub-period.

Method	Lower Tail					Upper Tail				
	Min	Mean	Median	Max	Sd. Dev.	Min	Mean	Median	Max	Sd. Dev.
MAD Dis ts = 0.25	0.8678	0.9095	0.8945	0.9920	0.0355	0.8411	0.9279	0.9152	0.9987	0.0480
MAD Dis ts = 0.20	0.8732	0.9249	0.9159	0.9933	0.0365	0.8505	0.9363	0.9399	0.9960	0.0412
MAD Dis ts = 0.15	0.8959	0.9458	0.9426	0.9933	0.0297	0.8665	0.9460	0.9533	0.9933	0.0336
MAD Dis ts = 0.10	0.9212	0.9579	0.9559	0.9933	0.0194	0.9052	0.9519	0.9519	0.9933	0.0278
MAD Dis ts = 0.05	0.9533	0.9690	0.9686	0.9960	0.0109	0.9519	0.9723	0.9693	0.9973	0.0136
KS Dis ts = 0.25	0.9332	0.9811	0.9826	0.9987	0.0135	0.8985	0.9753	0.9826	0.9973	0.0217
KS Dis ts = 0.20	0.9386	0.9820	0.9853	0.9987	0.0138	0.8985	0.9744	0.9833	0.9973	0.0226
KS Dis ts = 0.15	0.9386	0.9831	0.9873	0.9987	0.0138	0.8985	0.9733	0.9793	0.9973	0.0221
KS Dis ts = 0.10	0.9386	0.9826	0.9853	0.9987	0.0140	0.9079	0.9723	0.9786	0.9973	0.0213
KS Dis ts = 0.05	0.9519	0.9869	0.9893	0.9987	0.0092	0.9533	0.9787	0.9826	0.9973	0.0137
RT1 β = 0, k_{min} = 2	0.7904	0.9759	0.9973	0.9987	0.0395	0.8238	0.9603	0.9826	0.9987	0.0475
RT1 β = 0.1, k_{min} = 2	0.7997	0.9816	0.9973	0.9987	0.0362	0.8251	0.9685	0.9947	0.9987	0.0431
RT1 β = 0.2, k_{min} = 2	0.8131	0.9855	0.9973	0.9987	0.0318	0.8825	0.9837	0.9960	0.9987	0.0254
RT1 β = 0.3, k_{min} = 2	0.9212	0.9910	0.9973	0.9987	0.0170	0.8825	0.9843	0.9960	0.9987	0.0253
RT1 β = 0, k_{min} = 3	0.7917	0.9702	0.9880	0.9987	0.0390	0.8251	0.9547	0.9733	0.9987	0.0477
RT1 β = 0, k_{min} = 4	0.7931	0.9620	0.9766	0.9987	0.0428	0.8264	0.9498	0.9539	0.9987	0.0475
RT1 β = 0, k_{min} = 5	0.7944	0.9630	0.9780	0.9987	0.0425	0.8278	0.9438	0.9426	0.9987	0.0483
RT1 β = 0, k_{min} = 10	0.8011	0.9628	0.9713	0.9987	0.0404	0.8344	0.9305	0.9319	0.9987	0.0479
RT1 β = 0, k_{min} = 0.003n	0.7904	0.9759	0.9973	0.9987	0.0395	0.8238	0.9603	0.9826	0.9987	0.0475
RT1 β = 0, k_{min} = 0.005n	0.7917	0.9702	0.9880	0.9987	0.0390	0.8251	0.9547	0.9733	0.9987	0.0477
RT2 β = 0, k_{min} = 2	0.7664	0.9819	0.9973	0.9987	0.0394	0.8425	0.9696	0.9947	0.9987	0.0408
RT2 β = 0.1, k_{min} = 2	0.7690	0.9829	0.9973	0.9987	0.0381	0.8425	0.9696	0.9947	0.9987	0.0408
RT2 β = 0.2, k_{min} = 2	0.7690	0.9829	0.9973	0.9987	0.0381	0.8838	0.9799	0.9953	0.9987	0.0275
RT2 β = 0.3, k_{min} = 2	0.8024	0.9841	0.9973	0.9987	0.0332	0.8838	0.9818	0.9960	0.9987	0.0265
RT2 β = 0, k_{min} = 3	0.7677	0.9756	0.9933	0.9987	0.0392	0.8438	0.9640	0.9820	0.9987	0.0422
RT2 β = 0, k_{min} = 4	0.7690	0.9689	0.9880	0.9987	0.0424	0.8451	0.9595	0.9800	0.9987	0.0427
RT2 β = 0, k_{min} = 5	0.7704	0.9687	0.9880	0.9987	0.0420	0.8465	0.9516	0.9633	0.9987	0.0451
RT2 β = 0, k_{min} = 10	0.7770	0.9649	0.9700	0.9987	0.0390	0.8531	0.9428	0.9453	0.9973	0.0420
RT2 β = 0, k_{min} = 0.003n	0.7664	0.9819	0.9973	0.9987	0.0394	0.8425	0.9696	0.9947	0.9987	0.0408
RT2 β = 0, k_{min} = 0.005n	0.7677	0.9756	0.9933	0.9987	0.0392	0.8438	0.9640	0.9820	0.9987	0.0422
PS j = 1	0.7370	0.8600	0.8611	0.9479	0.0552	0.7343	0.8629	0.8645	0.9666	0.0537
PS j = 0	0.6435	0.7971	0.8071	0.8985	0.0565	0.6128	0.7728	0.7697	0.8959	0.0489
Eyeball w = 0.01, ε = 0.3, h = 0.9	0.9773	0.9868	0.9873	0.9947	0.0034	0.9826	0.9878	0.9880	0.9933	0.0027
Eyeball w = 0.02, ε = 0.3, h = 0.9	0.9706	0.9802	0.9800	0.9893	0.0040	0.9760	0.9813	0.9813	0.9866	0.0027
GH	0.8932	0.9682	0.9820	0.9960	0.0270	0.8798	0.9527	0.9706	0.9973	0.0402
dAMSE	0.8264	0.9386	0.9426	0.9573	0.0216	0.8158	0.9386	0.9413	0.9546	0.0184
HW	x	x	x	x	x	x	x	x	x	x
Hall B = 10,000, ε = 0.955, k_{aux} = $2\sqrt{n^+}$	0.8505	0.9443	0.9493	0.9640	0.0189	0.8665	0.9289	0.9332	0.9626	0.0248
Hall B = 10,000, ε = 0.995, k_{aux} = $2\sqrt{n^+}$	0.8425	0.9406	0.9466	0.9626	0.0202	0.8585	0.9240	0.9312	0.9586	0.0263
Hall B = 10,000, ε = 0.9, k_{aux} = $2\sqrt{n^+}$	0.8678	0.9497	0.9539	0.9693	0.0170	0.8838	0.9350	0.9393	0.9653	0.0220
Hall B = 10,000, ε = 0.955 k_{aux} = $\sqrt{n^+}$	0.7837	0.9556	0.9633	0.9826	0.0336	0.8558	0.9384	0.9453	0.9826	0.0362
Hall B = 10,000, ε = 0.955, k_{aux} = $3\sqrt{n^+}$	0.8665	0.9227	0.9239	0.9559	0.0163	0.8665	0.9173	0.9212	0.9519	0.0204
Hall B = 1000, ε = 0.955, k_{aux} = $2\sqrt{n^+}$	0.8505	0.9439	0.9479	0.9653	0.0190	0.8665	0.9283	0.9332	0.9626	0.0249
Hall (r) B = 10,000, ε = 0.955, k_{aux} = $2\sqrt{n^+}$	0.8491	0.9436	0.9486	0.9653	0.0192	0.8678	0.9289	0.9332	0.9613	0.0248
Himp B = 10,000, ε = 0.955	0.9186	0.9682	0.9720	0.9987	0.0187	0.9292	0.9640	0.9633	0.9987	0.0162
Himp B = 10,000, ε = 0.995	0.9306	0.9691	0.9706	0.9987	0.0187	0.9226	0.9611	0.9613	0.9973	0.0178
Himp B = 10,000, ε = 0.9	0.9266	0.9697	0.9733	0.9987	0.0163	0.9372	0.9640	0.9640	0.9987	0.0131
Himp B = 1000, ε = 0.955	0.9172	0.9694	0.9713	0.9987	0.0189	0.9266	0.9631	0.9633	0.9987	0.0172
Himp(r) B = 10,000, ε = 0.955	0.9186	0.9678	0.9706	0.9987	0.0188	0.9292	0.9632	0.9633	0.9987	0.0158
Gomes B = 10,000, ε = 0.995	0.9279	0.9687	0.9706	0.9987	0.0190	0.9212	0.9606	0.9599	0.9960	0.0178
Gomes B = 10,000, ε = 0.955	0.9105	0.9651	0.9693	0.9987	0.0209	0.9199	0.9614	0.9619	0.9987	0.0192
Gomes B = 10,000, ε = 0.9	0.9052	0.9645	0.9680	0.9987	0.0219	0.9079	0.9585	0.9593	0.9987	0.0186
Gomes B = 1000, ε = 0.995	0.9252	0.9690	0.9720	0.9987	0.0192	0.9226	0.9616	0.9613	0.9987	0.0183
Gomes (r) B = 10,000, ε = 0.995	0.9279	0.9687	0.9713	0.9987	0.0191	0.9226	0.9607	0.9613	0.9987	0.0180
Danielsson B = 500, ε = 0.9	0.8465	0.9839	0.9953	0.9987	0.0261	0.8892	0.9720	0.9773	0.9987	0.0275
Danielsson B = 500, ε = 0.955	0.8344	0.9849	0.9987	0.9987	0.0274	0.9012	0.9727	0.9780	0.9987	0.0266

Note: (r) means repeating the calculation; Sd. Dev.—standard deviation; x—cannot calculate, the method is not appropriate for the data.

Not all methods perform well in finite samples. Some that perform well in simulation studies based on theoretical distributions may not be suitable in financial applications. We can distinguish several methods that produce very high threshold estimates and pick a small number of data in the tails. They include the MAD Dis, KS Dis, RT1, RT2, Eyeball, GH, Himp, Gomes and Danielsson approaches. These methods tend to produce thresholds above the 99th percentile (max. above 0.99); thus, they prevent an estimation of VaR or ES at commonly accepted confidence levels. In particular, the KS Dis, RT1 and RT2 for k_{min} = 2,

Eyeball with the tuning parameter $w = 0.01$, GH and Danielsson methods systematically produce high threshold estimates, since the median exceeds the 98th percentile. The findings support the results of Danielsson et al. (2016), who argued that the Eyeball and KS Dis methods tend to pick the threshold close to the maximum of distribution. For shorter time-series (in sub-periods), the RT1, RT2 for $k_{min} = 2$ and Danielsson methods are the most restrictive and systematically set the threshold at too high levels to be used in financial applications. This finding is in line with the observation by Scarrott and MacDonald (2012), who pointed out that the RT1 approach is unreliable for a small k despite the weighting by i^β. In turn, Reiss and Thomas (2007) suggested using alternative distance metrics or weighting schemes when we deal with limited data. Methods based on minimizing the asymptotic *MSE*, especially the bootstrap-based methods, do not perform well in empirical studies (Danielsson et al. 2016). Similarly, Ferreira et al. (2003) noted that these methods do not give satisfactory results for samples of size under approximately 2000.

The other methods estimate the threshold in a more conservative way, lower than the 99th quantile. Such a choice of threshold guarantees enough data in the tail to calculate Value at Risk at the 99th confidence level. These are the HW, dAMSE, Hall (excluding parameters satisfying $B = 10,000$, $\varepsilon = 0.955$, $k_{aux} = \sqrt{n^+}$), Eyeball ($w = 0.025$, $h = 0.9$, $\varepsilon = 0.3$) and PS approaches. However, only two of them, i.e., the PS and HW algorithms, satisfy the Basel III requirements with reference to the Expected Shortfall, since their maxima are below the 97.5th percentile. Although HW performs well for long time-series, it fails in shorter series. In sub-periods, the optimizing procedure does not converge for several assets either for the upper or lower tails. Two algorithms, i.e., PS and HW, exhibit a very high range and standard deviation. It means that the threshold is highly volatile and cannot be substituted by a fixed percentile of the dataset as a threshold in practical applications. On the other hand, the Eyeball and dAMSE approaches produce less volatile estimations. The dAMSE method establishes the threshold close to the 95th percentile, while the Eyeball method is even more stable, but it indicates a much higher threshold, i.e., at the 98th percentile. Thus, we cannot decide which method is the optimal choice for financial applications. However, we can indicate those approaches which may be useful in practice. A majority of the analyzed methods, i.e., MAD Dis, KS Dis, PS, GH, HW, Eyeball, dAMSE, Gomes and Himp, indicate a higher left threshold than the right one. High returns might be perceived as extreme returns when they are positive, but they do not have to be recognized as extreme when they are negative. This result suggests asymmetry between the left and right tails. This result seems to be general since it holds in the entire period as well as the sub-periods.

Having specified the distribution of tails, the next step in our study comprises a comparison of the methods recognized as applicable in a risk measurement. Two different distances between threshold estimates are compared. The results for the entire research period are presented in Tables 5 and 6. Average absolute differences are shown in Table 5 and root mean squared differences in Table 6. As can be seen, the methods may be divided into two groups. The Eyeball, dAMSE and Hall methods show a relatively large deviation from the two other methods. Note that all the three above-mentioned methodologies utilize a low fraction (median above the 97th and below the 99th percentiles) of the total sample. The other group consists of the PS and HW methods. The distance between the threshold estimates is relatively close, and they both pick the threshold far away from the maximum (median below the 90th percentile). Danielsson et al. (2016) documented the similarity in choosing optimal sample fractions between the KS Distance and Eyeball methods and between the approach proposed by Drees and Kaufmann (1998) and the Danielsson method. However, they cast doubt on the applicability of the latter methods for real-world empirical estimations. The results for the right tails and the left tails do not indicate any significant differences, but approximately the same relative differences are preserved between the methodologies.

Table 5. The mean absolute differences (MAE) of the threshold estimates in the left tail (below diagonal) and right tail (above diagonal) for five methods.

Method	PS	Eyeball	dAMSE	HW	Hall 10,000	Hall 1000
PS	0	0.1148	0.1023	0.0478	0.1043	0.1044
Eyeball	0.1184	0	0.0139	0.1147	0.0171	0.0168
dAMSE	0.1060	0.0136	0	0.1020	0.0107	0.0105
HW	0.0718	0.1032	0.0918	0	0.1044	0.1046
Hall 10,000	0.1069	0.0206	0.0130	0.0930	0	0.0018
Hall 1000	0.1070	0.0206	0.0129	0.0930	0.0013	0

Note: PS—$j = 1$; Eyeball—$w = 0.025$, $h = 0.9$, $\varepsilon = 0.3$; Hall 1000—$B = 1000$, $\varepsilon = 0.955$, $k_{aux} = 2\sqrt{n^+}$; Hall 10,000—$B = 10,000$, $\varepsilon = 0.955$, $k_{aux} = 2\sqrt{n^+}$.

Table 6. The root mean squared differences (RMSE) of the threshold estimates in the left tail (below diagonal) and right tail (above diagonal) for five methods.

Method	PS	Eyeball	dAMSE	HW	Hall 10,000	Hall 1000
PS	0	0.1148	0.1023	0.0478	0.1043	0.1044
Eyeball	0.1184	0	0.0139	0.1147	0.0171	0.0168
dAMSE	0.1060	0.0136	0	0.1020	0.0107	0.0105
HW	0.0718	0.1032	0.0918	0	0.1044	0.1046
Hall 10,000	0.1069	0.0206	0.0130	0.0930	0	0.0018
Hall 1000	0.1070	0.0206	0.0129	0.0930	0.0013	0

Note: PS—$j = 1$; Eyeball—$w = 0.025$, $h = 0.9$, $\varepsilon = 0.3$; Hall 1000—$B = 1000$, $\varepsilon = 0.955$, $k_{aux} = 2\sqrt{n^+}$; Hall 10,000—$B = 10,000$, $\varepsilon = 0.955$, $k_{aux} = 2\sqrt{n^+}$.

5. Conclusions

The selection of the threshold which separates the tail from the middle part of a return distribution is crucial in the estimation of tail-related risk measures. Unfortunately, the right threshold is unknown in empirical applications. This paper presents evidence for twelve different optimal tail selection methods in risk measurement. We selected the optimal tail fraction for daily return time-series for forty-eight world indices. We found that many methods tend to set the optimal tail above the 99th percentile; therefore, their applicability in risk management is very limited. The methods that perform well in long and relatively short time-series include the minimization of the Asymptotic Mean Squared Error (dAMSE), the single bootstrap procedure (Hall), Eyeball with carefully selected tuning parameters and Path Stability (PS). The methods can be divided into two categories. The first three methods produce threshold estimates around the 98th percentile for the entire research period and their estimates are close to one another. This level of the threshold allows the estimation of the Value at Risk at the 99th confidence level, but not the Expected Shortfall at the 97.5th confidence level recommended by the Basel Committee. In the sub-periods, the estimated thresholds are lower, and for the dAMSE and Hall (excluding parametrization: $B = 10,000$, $\varepsilon = 0.955$, $k_{aux} = \sqrt{n^+}$) methods, their maximum is below the 97.5th percentile. The PS method sets the optimal threshold much lower than the other methodologies (below the 95th percentile for the entire research period and sub-periods). The estimates of PS are relatively volatile, suggesting a difference from the fixed percentage of the total sample size, commonly presented in the literature. The PS method is based on a rather simple algorithm; thus, it can be easily implemented in a risk management process.

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