



RANKING EXTREME AND NON-EXTREME EFFICIENT DECISION MAKING UNITS IN DATA ENVELOPMENT ANALYSIS

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Abstract: In evaluating decision making units (DMU) by using Data Envelopment Analysis (DEA) technique, it happens that more than one unit got efficiency score one. In such a case there should be some criterion for ranking these DMUs. Up to now, all of DEA model could rank only extreme efficient units. In this paper the authors proposed a method for ranking extreme and non extreme efficient units.

Key Words: Data Envelopment Analysis (DEA), Efficiency, Ranking.

1. INTRODUCTION

Data Envelopment Analysis (DEA) is a non parametric method for evaluating decision making units (DMU). It has been proposed by Charnes et al [8], in assessment of an educational center in USA and extended by Banker et al [5]. They used mathematical programming to reach this aim.

One of the main objectives of DEA is to measure the efficiency of Decision Making Units (DMUs) such as schools, public agencies, banks and etc. One of the ways for determining efficiency score of DMUs is to apply the Charnes et.al [8] model (CCR model) that deals with a ratio of multiple outputs and multiple inputs. One of the most interesting research subjects in DEA is to discriminate efficient units. In section 2 we summarized some ranking methods.

The objective of this work is to propose a method for ranking all efficient units, either extreme efficient or non extreme efficient DMUs. In section three the method is illustrated and in section four an example is used to explain our purpose. Our proposed method is based on using a ranking method; for example AP model which is proposed by Anderson and Peterson [4] or other ranking models; and finding strong efficient hyperplane which define production possibility set (PPS).

In evaluating the relative efficiency of each Decision making units (DMUs) by DEA models, we obtain efficiency scores between zero and one. In this way, usually more than one unit may be efficient in DEA models and their efficiency scores are 1. Also, pay attention that the number of efficient units in the Variable return to scale (VRS) models is not less than the Constant return to scale (CRS) models. Therefore, the researchers proposed some methods to discriminant these efficient units. This concept has named Ranking efficient units in DEA. There are lots of ranking methods and each of them has special quality and property to rank efficient units.

Charnes et al [6], counted the number of times that an efficient DMU play the role of benchmark unit for others, and used this norm to rank these units. Since, finding the reference set of a DMU is not easy, their model is not a suitable method. Charnes et al [7], proposed another method to find the benchmark DMUs. They changed the rate of outputs and evaluate the change of efficiency score. However, they didn't distinguish how they can do it.

Sexton et al [17], suggested the cross efficiency method. In this model they used the weights which they had obtained them by solving each of n -linear problems. They evaluate the efficiency of each DMU, n times and store these data in a matrix. Each row of this matrix contains the cross efficiency score of DMUs. They computed the average of these rows and store the result as a ranking measure. It seems this is an acceptable method, but it may has some difficulties. The biggest problem appears when the DEA models have alternative solutions.

Finally, it should be noted that there are some techniques and strategies in DEA which they affect on ranking. For example, Thompson et al [18], used the assurance regions. In their technique, the number of efficient DMUs may be decrease. But it isn't a suitable method because finding suitable weights isn't easy. Adler et al [2], proposed another method to difference between DMUs. In their model they decrease the number of inputs and outputs by component analysis. Therefore, the number of efficient DMUs is decreased. But in general this model couldn't use for a complete ranking.

Anderson and Petersen (AP model) [4], have ranked extreme efficient units by omitting them from Possibility Production Set (PPS), and then Mehrabian et al (MAJ) [15], have modified the AP model. In some circumstances, the mentioned models may be infeasible and specially the AP model may be unstable because of extreme sensitivity to small variations in data, where some DMUs have relatively small values for some of the inputs. Saati et al [16], have modified MAJ model and solved its infeasibility and Jahanshahloo et al [13], have changed the type of data normalization in order to receive a much better result. In order to remove the difficulties from AP and MAJ models, some mathematicians have used specific norms. For instance, Jahanshahloo et al [10], have practiced L_1 norm for ranking efficient units. Amirteimoori et al [3], have experienced L_1 norm to find the gap between evaluated efficient units and the new PPS. Gradient line and ellipsoid norms have been used by Jahanshahloo et al [14], in order to rank efficient units. Tone [19] and [20], has used SBM model in this way. To review ranking methods see also Adler et al [1].

This paper is organized as follows: section 1 is introduction. Record of ranking will coming in section. We presented our method in section 3 and illustrated a numerical example in section 4. Finally, section 5 is conclusion.

2. DEA BACKGROUND

In this section we are going to summarize some DEA and ranking models. In this way consider n , DMUs with m inputs and s outputs. The input and output vectors of DMU_j ($j=1, \dots, n$) are $X_j = (x_{1j}, \dots, x_{mj})^t, Y_j = (y_{1j}, \dots, y_{sj})^t$ where $X_j \geq 0, X_j \neq 0, Y_j \geq 0, Y_j \neq 0$.

By using the variable return to scale, convexity and possibility postulates, the

non-empty production possibility set (PPS) is defined as follows:

$$T_v = \left\{ (X, Y) : X \geq \sum_{j=1}^n \lambda_j X_j, Y \leq \sum_{j=1}^n \lambda_j Y_j, \sum_{j=1}^n \lambda_j = 1, \lambda_j \geq 0, j = 1, \dots, n \right\}$$

By the above definition the BCC model proposed by Banker et al [7] is as follows:

$$\begin{aligned}
 \min \quad & \theta - \varepsilon \left[\sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+ \right] \\
 \text{S.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} + s_i^- = \theta x_{ip}, \quad i = 1, \dots, m \\
 & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{rp}, \quad r = 1, \dots, s \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0, \quad j = 1, \dots, n \\
 & s_i^- \geq 0, \quad i = 1, \dots, m \\
 & s_r^+ \geq 0, \quad r = 1, \dots, s
 \end{aligned} \tag{1}$$

and its dual is:

$$\begin{aligned}
 \max \quad & \sum_{r=1}^s u_r y_{rp} - u_0 \\
 \text{S.t.} \quad & \sum_{i=1}^m v_i x_{ip} = 1 \\
 & \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} - u_0 \leq 0, \quad j = 1, \dots, n \\
 & u_r \geq \varepsilon, \quad r = 1, \dots, s \\
 & v_i \geq \varepsilon, \quad i = 1, \dots, m
 \end{aligned} \tag{2}$$

Clearly, the evaluated DMU_p is efficient if and only if $\theta^* = 1$ and all slack variables in every optimal solution be zero in problem (1). Equivalently DMU_p is efficient if and only if there exists an optimal solution for problem (2) such that $(U^*, V^*) > 0$ and $\sum_{r=1}^s u_r y_{rp} - u_0 = 1$. Then the hyperplane $H = \{(X, Y) : U^* Y - V^* X - u_0 = 0\}$ is said to be strong efficient hyperplane when there exists an optimal solution (U^*, V^*, u_0^*) of problem (2) which $U^* > 0$ and $V^* > 0$ and $U^* Y - u_0^* = 1$.

Anderson and Peterson [4] by omission of efficient units have been ranked these DMUs. Their proposed model is as follows:

$$\begin{aligned}
 & \max \quad \sum_{r=1}^s u_r y_{rp} - u_0 \\
 & \text{S.t} \quad \sum_{i=1}^m v_i x_{ip} = 1 \\
 & \quad \quad \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} - u_0 \leq 0, \quad j = 1, \dots, n, j \neq p \\
 & \quad \quad u_r \geq \varepsilon, \quad r = 1, \dots, s \\
 & \quad \quad v_i \geq \varepsilon, \quad i = 1, \dots, m
 \end{aligned}
 \tag{3}$$

Clearly, the optimal objective value of AP model is grater than 1 for extreme efficient units and equal 1 for non extreme efficient units. Therefore, AP model does not have any suggestion for ranking non extreme efficient units, and it is the problem of all DEA modes.

3. PROPOSED METHOD FOR RANKING ALL EFFICIENT UNITS

According to the last section, ranking efficient units has been developed because of existence numbers of efficient units. One of the most important point in ranking models is that there are no DEA models for ranking non extreme efficient DMUs. Therefore, there are no difference between these units. For example by using super efficiency models such as AP and MAJ models, efficiency score 1 is obtained again for these units. In this article we are going to construct a method for ranking all efficient units either extreme or non extreme efficient units by using DEA methods. Before explaining this method it is important to say that Jahanshahloo et.al [12] used mont carlo method for ranking all efficient units. But it is not a DEA method.

This new method is based on AP model and omission of efficient units from production possibility set (Using AP model is arbitrary and one may use other ranking models, especially when AP model occurs infeasibility). Also, strong efficient hyperplane which are binding on extreme efficient units play an important role in this method. To gain this aim the following computational steps should be done:

- Use AP model for evaluating all units and suppose it is feasible for all DMUs. Three sets of units may be considered:

$$SE = \{DMU_j = (x_j, y_j) : \theta_j^* > 1\}$$

$$E' = \{DMU_j = (x_j, y_j) : \theta_j^* = 1\}$$

$$F = \{DMU_j = (x_j, y_j) : 0 < \theta_j^* < 1\}$$

θ_j^* is the optimal objective value obtained by AP model in evaluating DMU_j .

By above definition SE is the set of extreme efficient units, E' is the set of non extreme

efficient units and finally, F is the set of inefficient DMUs. Suppose that $E' \neq \emptyset$ to continue.

- Find all strong efficient hyperplane which defines PPS and are binding on members of SE. Then construct a subset of SE which exactly m+s strong efficient hyperplane are binding on them and name the new set by \overline{SE} . Rearrange members of \overline{SE} by their descending ranks. Suppose \overline{SE} be in following format:

$\overline{E} = \{DMU_{i_1}, \dots, DMU_{i_k}\}$, where DMU_{i_1} has the best rank in this set. May be there is no such DMU which m+s strong efficient hyperplane are binding on it. In this case order members of SE by their ranks and continue the process.

At the end of this step note that for finding strong efficient hyperplane the method which is proposed by Huang and Rousseau [9], may be used and it is as follows:

$$\begin{aligned}
 & \min \quad u_0 \\
 & S.t \quad \sum_{r=1}^s u_r y_{rp} - \sum_{i=1}^m v_i x_{ip} = u_0, \\
 & \quad \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq u_0, \quad j = 1, \dots, n \\
 & \quad \sum_{i=1}^m v_i + \sum_{r=1}^s u_r = 1 \\
 & \quad u_r \geq \varepsilon, \quad r = 1, \dots, s \\
 & \quad v_i \geq \varepsilon, \quad i = 1, \dots, m
 \end{aligned} \tag{4}$$

where $DMU_p = (X_p, Y_p) \in SE$.

- Consider DMU_{i_1} . Define a subset of E' which its members are binding on strong efficient hyperplane of DMU_{i_1} and omit it from E'. Define $D_1 = E' - E_1'$ where E_1' is the set of binding hyperplane on DMU_{i_1} and D is the set of the hyperplanes which are not binding on DMU_{i_1} . Then consider a subset of D_1 such that its members are binding on strong efficient hyperplane of DMU_{i_2} and name it E_2' and then define $D_2 = D_1 - E_2'$. Continue this process while for some k, $D_k = \emptyset$. After above partitioning go to 4.

- Omit DMU_{i_1} from SE and name the remaining DMUs by SE_{i_1} . Add one of the members of E_1' to SE_{i_1} and use AP model for evaluating this non extreme efficient unit under new data set and store its super efficiency score. Do this process for other members of E_1' and store their scores. Then omit DMU_{i_2} from SE and name remaining DMUs by SE_{i_2} . Add one of the members of E_2' to SE_{i_2} and use AP model for

evaluating this unit in new data set and store its super efficiency score and do the above for other members of E_2' . Continue these process for other members of \overline{SE} and store all super efficiency scores.

- Rank all efficient units by using super efficiency scores of extreme efficient units obtained by step 1 and super efficiency scores of non extreme efficient obtained by the last step.

By using the above procedure all efficient DMUs can be ranked.

4. EXAMPLE

In this section we constructed 3 examples to use our proposed method for ranking all efficient units. In the first example AP model is used for our proposal an i has one input and one output. L_1 norm method is used in second example because of infeasibility of AP model. The third example consists of 2 inputs and 2 outputs.

4.1 Example1:Ap Model Is Feasible

Data are summarized in table 1 and are shown in figure 1:

Table 1: Data

DMUs	A	B	C	D	E	F	G	H	I	J	K	L	M
Input	1	1.5	2	3	4	5	6	7	8	1	3	5	8
Output	2	3	4	5	6	7	7.5	8	8	1	2	4	7

The results of ranking by AP model are shown in table 2:

Table 2: The results of ranking

Eff Units	A	B	C	D	E	F	G	H
Score	1.25	1	1.125	1	1	1.0666	1	1.1428
Rank	1	-	3	-	-	4	-	2

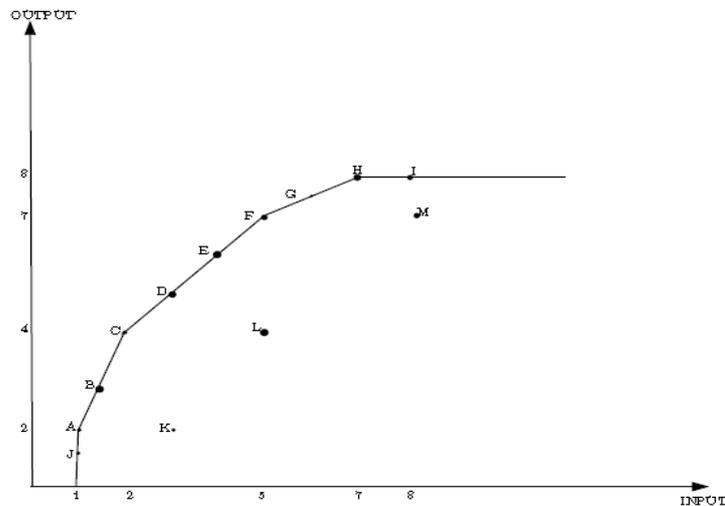


Fig 1: Data set in BCC model

In this example $SE = \{A, C, F, H\}$ and $SE' = \{B, D, E, G\}$ and $F = \{I, J, K, L, M\}$. Then the set of extreme efficient units which exactly 2 strong efficient hyperplane are binding on them is: $\overline{SE} = \{C, F\}$, where the members of \overline{SE} are ordered by their rank. After that it is evident from figure 1 that $E_1' = \{B, D, E\}$ and $E_2' = \{G\}$ where the members of E_1' are the set of all non extreme efficient units which are binding on strong efficient hyperplane of unit B and the member of E_2' is the set of non extreme efficient units which are binding on strong efficient hyperplane of unit F and are not in E_1' . Then omit DMU C from SE and name the remaining DMUs by SE_C . Add unit B to SE_C and use AP model for evaluating this unit under new data set. Again add unit D and E to SE_C separately and use AP model for evaluating them. At the end omit unit f from SE and name the remaining units by SE_F . Add unit G to SE_F and use AP model for this unit under new data set. The super efficiency scores for these units are; $\theta_B^* = 1.1111, \theta_D^* = 1.0555, \theta_E^* = 1.050, \theta_G^* = 1.041$. The results of new ranking are summarized in table 3:

Table 3: Results of ranking obtained by proposed model

DMUs	A	B	C	D	E	F	G	H
New ranking	1	4	3	6	7	5	8	2

Follow to the last table non extreme efficient unit B has better rank than extreme efficient unit F.

4.2 Example2: Ap Model Is Infeasible

Consider the following data set. It is evident that by using AP model that AP model is ifeasible for DMU H. Therefore, we use L_1 norm [10], and it is as ollows:

$$\begin{aligned}
 \min \quad & \Gamma_c^o(X, Y) = \sum_{i=1}^m x_i - \sum_{r=1}^s y_r + \alpha \\
 \text{S.t} \quad & \sum_{j=1, j \neq o}^n \lambda_j x_{ij} \leq x_i, \quad i = 1, \dots, m \\
 & \sum_{j=1, j \neq o} \lambda_j y_{rj} \geq y_r, \quad r = 1, \dots, s, \\
 & x_i \geq x_{io}, \quad i = 1, \dots, m \\
 & y_r \leq y_{ro}, \quad r = 1, \dots, s \\
 & \lambda_j \geq 0, \quad j = 1, \dots, n, \quad j \neq o
 \end{aligned} \tag{5}$$

where $\alpha = \sum_{r=1}^s y_{ro} - \sum_{i=1}^m x_{io}$ is a constant value. The table of data and result of

ranking is as follows:

Table 4: Data

DMUs	A	B	C	D	E	F	G	H	J	K	L	M
Input	1	1.5	2	3	4	5	6	7	1	3	5	8
Output	2	3	4	5	6	7	7.5	8	1	2	4	7
L_1 score	0.7142	0	0.03571	0	0	0.03125	0	0.0625	0	0	0	0

clearly, DMUB, DMUD, DMUE and DMUG are non-extreme efficient units. By using the mentioned method the following scores are obtained: $\theta_B^* = 0.04228$, $\theta_D^* = 0.05714$, $\theta_E^* = 0.02851$ and $\theta_G^* = 0.03750$. By these new scores the rank of extreme efficient units are as follows:

Table 5: The result of ranking

DMUs	A	B	C	D	E	F	G	H
L_1 rank	1	4	6	3	8	7	5	2

4.3 Example3: Multiple Inputs And Outputs

This example consists of 2 inputs and 2 outputs. Table of data and the result of ranking are shown in the following table:

Table 6: Data

DMUs	Input1	Input2	Output1	Output2	AP score	Ranking
1	586	581	46928	0.6514	1.001328	6
2	475	558	42879	0.5529	1.015965	4
3	201	600	43576	1.3500	1.848087	1
4	299	609	45673	0.7290	0.988677	-
5	318	613	40990	0.3190	0.926382	-
6	265	558	39079	0.5150	1.019981	3
7	467	580	38455	0.3184	0.941097	-
8	583	625	54291	1.7158	1.012007	5
9	347	535	34514	0.4512	1.042991	2
10	296	650	41984	1.2195	0.908321	-
11	600	740	43249	0.9205	0.772496	-
12	575	775	43291	0.5825	0.732652	-
13	351	888	46444	1.0400	0.863951	-
14	283	727	41841	0.3210	0.802734	-
15	431	695	40221	0.2365	0.801175	-
16	590	626	54291	1.7158	0.998403	-
17	583	625	54291	1.7150	1.000000	Non extreme eff

By using model 4 the efficient hyperplanes which are binding on efficient units is shown in the following table. Note that these hyperplanes are in the following format:

$H = \{(X, Y) : U^t Y - V^t X = u_0\}$ where U and V are the output and input weights vectors obtained by model 4.

Table 7: Coefficients of defining hyperplanes

No	u_o	V_1	V_2	U_1	U_2
1	9.135734636E+1	0.00001000	0.28946661	0.00162732	0.70889607
2	1.227411668E+1	0.00001000	0.02796313	0.00006522	0.97196165
3	1.3059359742E+3	0.96555695	0.00001000	0.03442305	0.00001000
4	1.548190233E+1	0.00244179	0.03625191	0.00012535	0.96118094
5	5.4289359207E+4	0.00001000	0.00001000	0.99997000	0.00001000
6	7.563800029406	0.00021017	0.01547687	0.00001000	0.98430295

Clearly, DMU17 is a non-extreme efficient unit and it is binding on hyperplane no. 3, 4, 5 and 6. Note that there is no extreme efficient unit which it is binding on exactly $m+s$ efficient hyperplane. Therefore, we do the proposed model for DMU3, because it has the best rank in this example. By doing this the new score of DMU17 is: 1.012000 .

5. CONCLUSION

In this paper we proposed a super efficiency method for ranking extreme and non extreme efficient units. AP model and the proposed model by Huang and Rousseau are used for ranking all efficient units. It is very important to note that in this method discriminant of efficient units is considered. For example unit B is a non extreme efficient units which its super efficiency score obtained by AP model is again 1. Follow to figure 1, unit B is better than extreme efficient units F and H; and it is not considered in AP model. But discriminant of unit B with units F and G is considered in our proposed model. Finally, note that this method may be threatened when non extreme efficient unit is very similar to extreme efficient units. For feature views the authors can remove this weakness and develop it for other ranking models.

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